Session 2

Definition of Complex Number, Conjugate Complex Numbers, Representation of a Complex Number in Various Forms

Definition of Complex Number

A number of the form $a + ib$, where $a, b \in R$ and $i = \sqrt{-1}$, is called a **complex number.** It is denoted by z i.e. $z = a + ib$.

A complex number may also be defined as an ordered pair of real numbers; and may be denoted by the symbol (a, b). If we write $z = (a, b)$, then a is called the real part and b is the imaginary part of the complex number z and may be denoted by Re (z) and Im (z), respectively i.e., $a = \text{Re}(z)$ and $b = \text{Im}(z)$.

Two complex numbers are said to be equal, if and only if their real parts and imaginary parts are separately equal.

Thus, $$a + ib = c + id$$
$$\Leftrightarrow \quad a = c \text{ and } b = d$$
where, $a, b, c, d \in R$ and $i = \sqrt{-1}$.

i.e. $$z_1 = z_2$$
$$\Leftrightarrow \quad \text{Re}(z_1) = \text{Re}(z_2) \text{ and } \text{Im}(z_1) = \text{Im}(z_2)$$

Important Properties of Complex Numbers

1. The complex numbers do not possess the property of order, i.e., $(a + ib) >$ or $< (c + id)$ is not defined. *For example*, $9 + 6i > 3 + 2i$ makes no sense.
2. A real number a can be written as $a + i \cdot 0$. Therefore, every real number can be considered as a complex number, whose imaginary part is zero. Thus, the set of real numbers (R) is a proper subset of the complex numbers (C) i.e. $R \subset C$. Hence, the complex number system is $N \subset W \subset I \subset Q \subset R \subset C$
3. A complex number z is said to be purely real, if $\text{Im}(z) = 0$; and is said to be purely imaginary, if $\text{Re}(z) = 0$. The complex number $0 = 0 + i \cdot 0$ is both purely real and purely imaginary.
4. In real number system, $a^2 + b^2 = 0 \Rightarrow a = 0 = b$.

   But if z_1 and z_2 are complex numbers, then $z_1^2 + z_2^2 = 0$ does not imply $z_1 = z_2 = 0$.

   For example, $z_1 = 1 + i$ and $z_2 = 1 - i$

   Here, $z_1 \neq 0, z_2 \neq 0$

   But $z_1^2 + z_2^2 = (1 + i)^2 + (1 - i)^2 = 1 + i^2 + 2i + 1 + i^2 - 2i$
   $$= 2 + 2i^2 = 2 - 2 = 0$$

   However, if product of two complex numbers is zero, then atleast one of them must be zero, same as in case of real numbers.

   If $z_1 z_2 = 0$, then $z_1 = 0, z_2 \neq 0$ or $z_1 \neq 0, z_2 = 0$

   or $z_1 = 0, z_2 = 0$

Algebraic Operations on Complex Numbers

Let two complex numbers be $z_1 = a + ib$ and $z_2 = c + id$, where $a, b, c, d \in R$ and $i = \sqrt{-1}$.

1. Addition $z_1 + z_2 = (a + ib) + (c + id)$
$$= (a + c) + i(b + d)$$

2. Subtraction $z_1 - z_2 = (a + ib) - (c + id)$
$$= (a - c) + i(b - d)$$

3. Multiplication $z_1 \cdot z_2 = (a + ib) \cdot (c + id)$
$$= ac + iad + ibc + i^2 bd$$
$$= ac + i(ad + bc) - bd$$
$$= (ac - bd) + i(ad + bc)$$

4. Division $\dfrac{z_1}{z_2} = \dfrac{(a + ib)}{(c + id)} \cdot \dfrac{(c - id)}{(c - id)}$

[multiplying numerator and denominator by $c - id$ where atleast one of c and d is non-zero]

$$= \frac{ac - iad + ibc - i^2 bd}{(c)^2 - (id)^2} = \frac{ac + i(bc - ad) + bd}{c^2 - i^2 d^2}$$
$$= \frac{(ac + bd) + i(bc - ad)}{c^2 + d^2} = \frac{(ac + bd)}{(c^2 + d^2)} + i\frac{(bc - ad)}{(c^2 + d^2)}$$

Remark

$\dfrac{1 + i}{1 - i} = i$ and $\dfrac{1 - i}{1 + i} = -i$, where $i = \sqrt{-1}$.

Properties of Algebraic Operations on Complex Numbers

Let z_1, z_2 and z_3 be any three complex numbers. Then, their algebraic operations satisfy the following properties :

Properties of Addition of Complex Numbers

(i) **Closure law** $z_1 + z_2$ is a complex number.

(ii) **Commutative law** $z_1 + z_2 = z_2 + z_1$

(iii) **Associative law** $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$

(iv) **Additive identity** $z+0=z=0+z$, then 0 is called the additive identity.

(v) **Additive inverse** $-z$ is called the additive inverse of z, i.e. $z+(-z)=0$.

Properties of Multiplication of Complex Numbers

(i) **Closure law** $z_1 \cdot z_2$ is a complex number.

(ii) **Commutative law** $z_1 \cdot z_2 = z_2 \cdot z_1$

(iii) **Associative law** $(z_1 \cdot z_2) z_3 = z_1 (z_2 \cdot z_3)$

(iv) **Multiplicative identity** $z \cdot 1 = z = 1 \cdot z$, then 1 is called the multiplicative identity.

(v) **Multiplicative inverse** If z is a non-zero complex number, then $\frac{1}{z}$ is called the multiplicative inverse of z i.e. $z.\ \frac{1}{z}=1=\frac{1}{z}\cdot z$

(vi) **Multiplication is distributive with respect to addition** $z_1(z_2+z_3)=z_1 z_2+z_1 z_3$

Conjugate Complex Numbers

The complex numbers $z=(a,b)=a+ib$ and $\bar{z}=(a,-b)=a-ib$, where a and b are real numbers, $i=\sqrt{-1}$ and $b\neq 0$, are said to be complex conjugate of each other (here, the complex conjugate is obtained by just changing the sign of i).

Note that, sum $=(a+ib)+(a-ib)=2a$, which is real.

And product $=(a+ib)(a-ib)=a^2-(ib)^2$

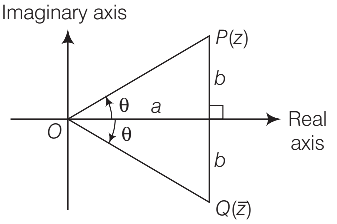
$$=a^2-i^2 b^2=a^2-(-1)b^2$$

$$=a^2+b^2, \text{ which is real.}$$

Geometrically, $\bar{z}$ is the mirror image of z along real axis on argand plane.

Remark

Let $z=-a-ib, a>0, b>0=(-a,-b)$ (III quadrant)



Then, $\bar{z}=-a+ib=(-a,b)$ (II quadrant). Now,

(i) If z lies in I quadrant, then $\bar{z}$ lies in IV quadrant and *vice-versa.*

(ii) If z lies in II quadrant, then $\bar{z}$ lies in III quadrant and *vice-versa.*

Properties of Conjugate Complex Numbers

Let z, z_1 and z_2 be complex numbers. Then,

(i) $\overline{(\bar{z})}=z$

(ii) $z+\bar{z}=2\,\text{Re}(z)$

(iii) $z-\bar{z}=2\,\text{Im}(z)$

(iv) $z+\bar{z}=0 \Rightarrow z=-\bar{z} \Rightarrow z$ is purely imaginary.

(v) $z-\bar{z}=0 \Rightarrow z=\bar{z} \Rightarrow z$ is purely real.

(vi) $\overline{z_1 \pm z_2}=\bar{z}_1 \pm \bar{z}_2$ Ingeneral,

$\overline{z_1 \pm z_2 \pm z_3 \pm \ldots \pm z_n}=\overline{z_1}\pm\overline{z_2}\pm\overline{z_3}\pm\ldots\pm\overline{z_n}$

(vii) $\overline{z_1 \cdot z_2}=\overline{z_1}\cdot\overline{z_2}$

In general, $\overline{z_1 \cdot z_2 \cdot z_3 \ldots z_n}=\overline{z_1}\cdot\overline{z_2}\cdot\overline{z_3}\ldots\overline{z_n}$

(viii) $\overline{\left(\frac{z_1}{z_2}\right)}=\frac{\bar{z}_1}{\bar{z}_2}, z_2\neq 0$

(ix) $\overline{z^n}=(\bar{z})^n$

(x) $z_1\overline{z_2}+\overline{z_1}z_2=2\,\text{Re}(z_1\overline{z_2})=2\,\text{Re}(\overline{z_1}z_2)$

(xi) If $z=f(z_1,z_2)$, then $\bar{z}=f(\overline{z_1},\overline{z_2})$

Example 18. If $\frac{x-3}{3+i}+\frac{y-3}{3-i}=i$, where $x, y\in R$ and $i=\sqrt{-1}$, find the values of x and y.

Sol. $\because \frac{x-3}{3+i}+\frac{y-3}{3-i}=i$

$\Rightarrow (x-3)(3-i)+(y-3)(3+i)=i(3+i)(3-i)$

$\Rightarrow (3x-xi-9+3i)+(3y+yi-9-3i)=10i$

$\Rightarrow (3x+3y-18)+i(y-x)=10i$

On comparing real and imaginary parts, we get

$3x+3y-18=0$

$\Rightarrow x+y=6$ ...(i)

and $y-x=10$ ...(ii)

On solving Eqs. (i) and (ii), we get

$x=-2,\ y=8$

Example 19. ***If*** $(a+ib)^5=p+iq$, ***where*** $i=\sqrt{-1}$, ***prove that*** $(b+ia)^5=q+ip$.

Sol. $\because (a+ib)^5=p+iq$

$\therefore \overline{(a+ib)^5}=\overline{p+iq} \Rightarrow (a-ib)^5=(p-iq)$

$\Rightarrow (-i^2a-ib)^5=(-i^2p-iq)$ $[\because i^2=-1]$

$\Rightarrow (-i)^5(b+ia)^5=(-i)(q+ip)$

$\Rightarrow (-i)(b+ia)^5=(-i)(q+ip)$

$\therefore (b+ia)^5=(q+ip)$

Example 20. Find the least positive integral value of n, for which $\left(\frac{1-i}{1+i}\right)^n$, where $i=\sqrt{-1}$, is purely imaginary with positive imaginary part.

Sol. $\left(\frac{1-i}{1+i}\right)^n=\left(\frac{1-i}{1+i}\times\frac{1-i}{1-i}\right)^n=\left(\frac{1+i^2-2i}{2}\right)^n=\left(\frac{1-1-2i}{2}\right)^n$

$$=(-i)^n=\text{Imaginary}$$

$\Rightarrow \quad n=1,3,5,\ldots$ for positive imaginary part $n=3$.

Example 21. If the multiplicative inverse of a complex number is $(\sqrt{3}+4i)/19$, where $i=\sqrt{-1}$, find complex number.

Sol. Let z be the complex number.

Then, $z\cdot\left(\frac{\sqrt{3}+4i}{19}\right)=1$

or $\quad z=\frac{19}{(\sqrt{3}+4i)}\times\frac{(\sqrt{3}-4i)}{(\sqrt{3}-4i)}$

$$=\frac{19(\sqrt{3}-4i)}{19}=(\sqrt{3}-4i)$$

Example 22. ***Find real*** θ***, such that*** $\frac{3+2i\sin\theta}{1-2i\sin\theta}$, ***where*** $i=\sqrt{-1}$***, is***

(i) purely real. (ii) purely imaginary.

Sol. Let $z=\frac{3+2i\sin\theta}{1-2i\sin\theta}$

On multiplying numerator and denominator by conjugate of denominator,

$$z=\frac{(3+2i\sin\theta)(1+2i\sin\theta)}{(1-2i\sin\theta)(1+2i\sin\theta)}=\frac{(3-4\sin^2\theta)+8i\sin\theta}{(1+4\sin^2\theta)}$$

$$=\frac{(3-4\sin^2\theta)}{(1+4\sin^2\theta)}+i\frac{(8\sin\theta)}{(1+4\sin^2\theta)}$$

(i) For purely real, $\text{Im}(z)=0$

$\Rightarrow \quad \frac{8\sin\theta}{1+4\sin^2\theta}=0$ or $\sin\theta=0$

$\therefore \quad \theta=n\pi, n\in I$

(ii) For purely imaginary, $\text{Re}(z)=0$

$\Rightarrow \quad \frac{(3-4\sin^2\theta)}{(1+4\sin^2\theta)}=0$ or $3-4\sin^2\theta=0$

or $\quad \sin^2\theta=\frac{3}{4}=\left(\frac{\sqrt{3}}{2}\right)^2=\left(\sin\frac{\pi}{3}\right)^2$

$\therefore \quad \theta=n\pi\pm\frac{\pi}{3}, n\in I$

Example 23. Find real values of x and y for which the complex numbers $-3+ix^2y$ and x^2+y+4i, where $i=\sqrt{-1}$, are conjugate to each other.

Sol. Given, $\quad -3+ix^2y=x^2+y+4i$

$\Rightarrow \quad -3-ix^2y=x^2+y+4i$

On comparing real and imaginary parts, we get

$$x^2+y=-3 \qquad \ldots\text{(i)}$$

and $\quad -x^2y=4 \qquad \ldots\text{(ii)}$

From Eq. (ii), we get $x^2=-\frac{4}{y}$

Then, $\quad -\frac{4}{y}+y=-3 \quad \left[\text{putting } x^2=-\frac{4}{y} \text{ in Eq. (i)}\right]$

$$y^2+3y-4=0 \Rightarrow (y+4)(y-1)=0$$

$\therefore \quad y=-4,1$

For $\quad y=-4, x^2=1 \Rightarrow x=\pm 1$

For $\quad y=1, x^2=-4 \quad$ [impossible]

$\therefore \quad x=\pm 1, y=-4$

Example 24. If $x=-5+2\sqrt{-4}$, find the value of

$$x^4+9x^3+35x^2-x+4.$$

Sol. Since, $\quad x=-5+2\sqrt{-4} \Rightarrow x+5=4i$

$\Rightarrow \quad (x+5)^2=(4i)^2 \Rightarrow x^2+10x+25=-16$

$\therefore \quad x^2+10x+41=0 \qquad \ldots\text{(i)}$

Now,

$$\begin{array}{r|l}
x^2+10x+41 & x^4+9x^3+35x^2-x+4 \quad | \; x^2-x+4 \\
 & x^4+10x^3+41x^2 \\
 & -\quad\; -\quad\;\; - \\ \hline
 & -x^3-6x^2-x+4 \\
 & -x^3-10x^2-41x \\
 & +\quad\;\; +\quad\;\; + \\ \hline
 & 4x^2+40x+4 \\
 & 4x^2+40x+164 \\
 & -\quad\; -\quad\; - \\ \hline
 & -160
\end{array}$$

$\therefore \quad x^4+9x^3+35x^2-x+4$

$$=(x^2+10x+41)(x^2-x+4)-160$$

$$=0-160=-160 \qquad \text{[from Eq. (i)]}$$

Example 25. Let z be a complex number satisfying the equation $z^2-(3+i)z+\lambda+2i=0$, where $\lambda\in R$ and $i=\sqrt{-1}$. Suppose the equation has a real root, find the non-real root.

Sol. Let α be the real root. Then,

$$\alpha^2-(3+i)\alpha+\lambda+2i=0$$

$\Rightarrow \quad (\alpha^2 - 3\alpha + \lambda) + i(2-\alpha) = 0$

On comparing real and imaginary parts, we get

$$\alpha^2 - 3\alpha + \lambda = 0 \quad \text{...(i)}$$

$$\Rightarrow \quad 2 - \alpha = 0 \quad \text{...(ii)}$$

From Eq. (ii), $\alpha = 2$

Let other root be β.

Then, $\alpha + \beta = 3 + i \Rightarrow 2 + \beta = 3 + i$

$\therefore \quad \beta = 1 + i$

Hence, the non-real root is $1 + i$.

Representation of a Complex Number in Various Forms

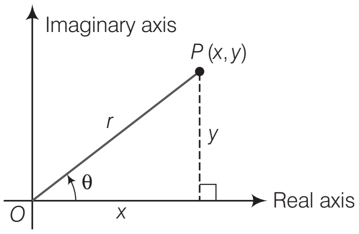
Cartesian Form

(Geometrical Representation)

Every complex number $z = x + iy$, where $x, y \in R$ and $i = \sqrt{-1}$, can be represented by a point in the cartesian plane known as complex plane (Argand plane) by the ordered pair (x, y).

Modulus and Argument of a Complex Number

Let $z = x + iy = (x, y)$ for all $x, y \in R$ and $i = \sqrt{-1}$.



The length OP is called modulus of the complex number z denoted by $|z|$,

i.e. $OP = r = |z| = \sqrt{(x^2 + y^2)}$

and if $(x, y) \neq (0, 0)$, then θ is called the argument or amplitude of z,

i.e. $\theta = \tan^{-1}\left(\dfrac{y}{x}\right)$ [angle made by OP with positive X-axis]

or $\arg(z) = \tan^{-1}(y/x)$

Also, argument of a complex number is not unique, since if θ is a value of the argument, so also is $2n\pi + \theta$, where $n \in I$. But usually, we take only that value for which $0 \le \theta < 2\pi$. Any two arguments of a complex number differ by $2n\pi$.

Argument of z will be $\theta, \pi - \theta, \pi + \theta$ and $2\pi - \theta$ according as the point z lies in I, II, III and IV quadrants respectively, where $\theta = \tan^{-1}\left|\dfrac{y}{x}\right|$.

Example 26. Find the arguments of $z_1 = 5 + 5i$, $z_2 = -4 + 4i$, $z_3 = -3 - 3i$ and $z_4 = 2 - 2i$, where $i = \sqrt{-1}$.

Sol. Since, z_1, z_2, z_3 and z_4 lies in I, II, III and IV quadrants respectively. The arguments are given by

$$\arg(z_1) = \tan^{-1}\left|\frac{5}{5}\right| = \tan^{-1} 1 = \pi/4$$

$$\arg(z_2) = \pi - \tan^{-1}\left|\frac{4}{-4}\right| = \pi - \tan^{-1} 1 = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$$

$$\arg(z_3) = \pi + \tan^{-1}\left|\frac{-3}{-3}\right| = \pi + \tan^{-1} 1 = \pi + \frac{\pi}{4} = \frac{5\pi}{4}$$

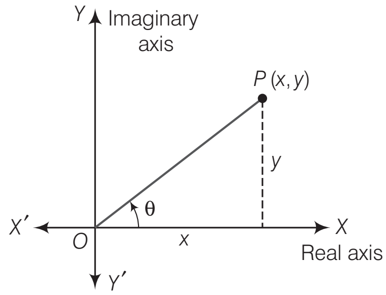
and $\arg(z_4) = 2\pi - \tan^{-1}\left|\dfrac{-2}{2}\right|$

$$= 2\pi - \tan^{-1} 1 = 2\pi - \frac{\pi}{4} = \frac{7\pi}{4}$$

Principal Value of the Argument

The value θ of the argument which satisfies the inequality $-\pi < \theta \le \pi$ is called the **principal value** of the argument.

If $z = x + iy = (x, y), \forall x, y \in R$ and $i = \sqrt{-1}$, then $\arg(z) = \tan^{-1}\left(\dfrac{y}{x}\right)$ always gives the principal value. It depends on the quadrant in which the point (x, y) lies.



(i) $(x, y) \in$ first quadrant $x > 0, y > 0$.

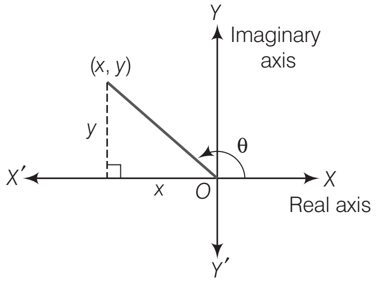
The principal value of $\arg(z) = \theta = \tan^{-1}\left(\dfrac{y}{x}\right)$

It is an acute angle and positive.

(ii) $(x, y) \in$ second quadrant $x < 0, y > 0$.

The principal value of $\arg(z) = \theta$

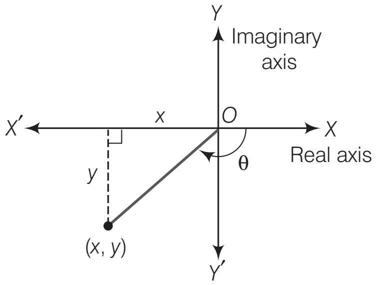
$$= \pi - \tan^{-1}\left(\frac{y}{|x|}\right)$$



It is an obtuse angle and positive.

(iii) $(x, y) \in$ third quadrant $x < 0, y < 0$.

The principal value of $\arg(z) = \theta = -\pi + \tan^{-1}\left(\dfrac{y}{x}\right)$

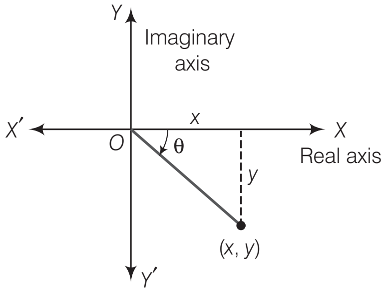


It is an obtuse angle and negative.

(iv) $(x, y) \in$ fourth quadrant $x > 0, y < 0$.

The principal value of $\arg(z) = \theta$

$$= -\tan^{-1}\left(\frac{|y|}{x}\right)$$



It is an acute angle and negative.

Example 27. Find the principal values of the arguments of $z_1 = 2 + 2i$, $z_2 = -3 + 3i$, $z_3 = -4 - 4i$ and $z_4 = 5 - 5i$, where $i = \sqrt{-1}$.

Sol. Since, z_1, z_2, z_3 and z_4 lies in I, II, III and IV quadrants respectively. The principal values of the arguments are given by

$$\tan^{-1}\left(\frac{2}{2}\right),\ \pi - \tan^{-1}\left(\frac{3}{|-3|}\right), -\pi + \tan^{-1}\left(\frac{-4}{-4}\right),$$
$$-\tan^{-1}\left(\frac{|-5|}{5}\right)$$

or $\tan^{-1} 1, \pi - \tan^{-1} 1, -\pi + \tan^{-1} 1, -\tan^{-1} 1$

or $\dfrac{\pi}{4}, \pi - \dfrac{\pi}{4}, -\pi + \dfrac{\pi}{4}, -\dfrac{\pi}{4}$ or $\dfrac{\pi}{4}, \dfrac{3\pi}{4}, -\dfrac{3\pi}{4}, -\dfrac{\pi}{4}$

Hence, the principal values of the arguments of z_1, z_2, z_3 and z_4 are $\dfrac{\pi}{4}, \dfrac{3\pi}{4}, -\dfrac{3\pi}{4}, -\dfrac{\pi}{4}$, respectively.

Remark

1. Unless otherwise stated, amp z implies principal value of the argument.
2. Argument of the complex number 0 is not defined.
3. If $z_1 = z_2 \Leftrightarrow |z_1| = |z_2|$ and $\arg(z_1) = \arg(z_2)$.
4. If $\arg(z) = \pi/2$ or $-\pi/2$, z is purely imaginary.
5. If $\arg(z) = 0$ or π, z is purely real.

Example 28. Find the argument and the principal value of the argument of the complex number $z = \dfrac{2+i}{4i + (1+i)^2}$, where $i = \sqrt{-1}$.

Sol. Since, $z = \dfrac{2+i}{4i + (1+i)^2} = \dfrac{2+i}{4i + 1 + i^2 + 2i} = \dfrac{2+i}{6i} = \dfrac{1}{6} - \dfrac{1}{3}i$

$\therefore$ z lies in IV quadrant.

Here, $\theta = \tan^{-1}\left|\dfrac{-\frac{1}{3}}{\frac{1}{6}}\right| = \tan^{-1} 2$

$\therefore$ $\arg(z) = 2\pi - \theta = 2\pi - \tan^{-1} 2$

Hence, principal value of $\arg(z) = -\theta = -\tan^{-1} 2$.

Properties of Modulus

(i) $|z| \ge 0 \Rightarrow |z| = 0$, iff $z = 0$ and $|z| > 0$, iff $z \ne 0$

(ii) $-|z| \le \text{Re}(z) \le |z|$ and $-|z| \le \text{Im}(z) \le |z|$

(iii) $|z| = |\bar{z}| = |-z| = |-\bar{z}|$

(iv) $z\bar{z} = |z|^2$

(v) $|z_1 z_2| = |z_1||z_2|$

In general, $|z_1 z_2 z_3 \dots z_n| = |z_1||z_2||z_3| \dots |z_n|$

(vi) $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|} (z_2 \ne 0)$

(vii) $|z_1 \pm z_2| \le |z_1| + |z_2|$

In general, $|z_1 \pm z_2 \pm z_3 \pm \dots \pm z_n| \le |z_1| + |z_2| + |z_3| + \dots + |z_n|$

(viii) $|z_1 \pm z_2| \ge ||z_1| - |z_2||$

(ix) $|z^n| = |z|^n$

(x) $||z_1| - |z_2|| \le |z_1 + z_2| \le |z_1| + |z_2|$

Thus, $|z_1|+|z_2|$ is the greatest possible value of $|z_1+z_2|$ and $||z_1|-|z_2||$ is the least possible value of $|z_1+z_2|$.

(xi) $|z_1 \pm z_2|^2 = (z_1 \pm z_2)(\bar{z}_1 \pm \bar{z}_2) = |z_1|^2 + |z_2|^2 \pm (z_1 \overline{z_2} + \overline{z_1} z_2)$

or $|z_1|^2 + |z_2|^2 \pm 2\,\text{Re}\,(z_1 \bar{z}_2)$

(xii) $z_1\bar{z}_2 + \bar{z}_1 z_2 = 2|z_1||z_2|\cos(\theta_1 - \theta_2)$, where $\theta_1 = \arg(z_1)$ and $\theta_2 = \arg(z_2)$

(xiii) $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 \Leftrightarrow \dfrac{z_1}{z_2}$ is purely imaginary.

(xiv) $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2\{|z_1|^2 + |z_2|^2\}$

(xv) $|az_1 - bz_2|^2 + |bz_1 + az_2|^2 = (a^2 + b^2)(|z_1|^2 + |z_2|^2)$, where $a, b \in R$

(xvi) Unimodular i.e., unit modulus

If z is unimodular, then $|z| = 1$. In case of unimodular, let $z = \cos\theta + i\sin\theta, \theta \in R$ and $i = \sqrt{-1}$.

Remark

1. If $f(z)$ is unimodular, then $|f(z)| = 1$ and let $f(z) = \cos\theta + i\sin\theta$, $\theta \in R$ and $i = \sqrt{-1}$.
2. $\dfrac{z}{|z|}$ is always a unimodular complex number, if $z \neq 0$.

(xvii) The multiplicative inverse of a non-zero complex number z is same as its reciprocal and is given by $\dfrac{1}{z} = \dfrac{\bar{z}}{z\bar{z}} = \dfrac{\bar{z}}{|z|^2}$.

Example 29. If $\theta_i \in [0, \pi/6]$, $i = 1, 2, 3, 4, 5$ and $\sin\theta_1 z^4 + \sin\theta_2 z^3 + \sin\theta_3 z^2 + \sin\theta_4 z + \sin\theta_5 = 2$, show that $\dfrac{3}{4} < |z| < 1$.

Sol. Given that,

$$\sin\theta_1 z^4 + \sin\theta_2 z^3 + \sin\theta_3 z^2 + \sin\theta_4 z + \sin\theta_5 = 2$$

or $2 = |\sin\theta_1 z^4 + \sin\theta_2 z^3 + \sin\theta_3 z^2 + \sin\theta_4 z + \sin\theta_5|$

$2 \leq |\sin\theta_1 z^4| + |\sin\theta_2 z^3| + |\sin\theta_3 z^2| + |\sin\theta_4 z| + |\sin\theta_5|$ [by property (vii)]

$\Rightarrow 2 \leq |\sin\theta_1||z^4| + |\sin\theta_2||z^3| + |\sin\theta_3||z^2| + |\sin\theta_4||z| + |\sin\theta_5|$ [by property (v)]

$\Rightarrow 2 \leq |\sin\theta_1||z|^4 + |\sin\theta_2||z|^3 + |\sin\theta_3||z|^2 + |\sin\theta_4||z| + |\sin\theta_5|$ [by property (ix)] ...(i)

But given, $\theta_i \in [0, \pi/6]$

$\therefore \quad \sin\theta_i \in \left[0, \dfrac{1}{2}\right]$,

i.e. $\quad 0 \leq \sin\theta_i \leq \dfrac{1}{2}$

$\therefore$ Inequality Eq. (i) becomes,

$$2 \leq \frac{1}{2}|z|^4 + \frac{1}{2}|z|^3 + \frac{1}{2}|z|^2 + \frac{1}{2}|z| + \frac{1}{2}$$

$\Rightarrow \quad 3 \leq |z|^4 + |z|^3 + |z|^2 + |z|$

$\Rightarrow \quad 3 \leq |z| + |z|^2 + |z|^3 + |z|^4 < |z| + |z|^2 + |z|^3 + |z|^4 + \ldots + \infty$

$\Rightarrow \quad 3 < |z| + |z|^2 + |z|^3 + |z|^4 + \ldots + \infty$

$\Rightarrow \quad 3 < \dfrac{|z|}{1 - |z|}$ [here, $|z| < 1$]

$\Rightarrow \quad 3 - 3|z| < |z| \Rightarrow 3 < 4|z|$

$\therefore \quad |z| > \dfrac{3}{4}$

Hence, $\quad \dfrac{3}{4} < |z| < 1$ [$\because |z| < 1$]

Example 30. ***If $|z - 2 + i| \leq 2$, find the greatest and least values of $|z|$, where $i = \sqrt{-1}$.***

Sol. Given that, $|z - 2 + i| \leq 2$ ...(i)

$\because \quad |z - 2 + i| \geq ||z| - |2 - i||$ [by property (x)]

$\therefore \quad |z - 2 + i| \geq ||z| - \sqrt{5}|$ ...(ii)

From Eqs. (i) and (ii), we get

$$||z| - \sqrt{5}| \leq |z - 2 + i| \leq 2$$

$\therefore \quad ||z| - \sqrt{5}| \leq 2$

$\Rightarrow \quad -2 \leq |z| - \sqrt{5} \leq 2$

$\Rightarrow \quad \sqrt{5} - 2 \leq |z| \leq \sqrt{5} + 2$

Hence, greatest value of $|z|$ is $\sqrt{5} + 2$ and least value of $|z|$ is $\sqrt{5} - 2$.

Example 31. ***If z is any complex number such that $|z + 4| \leq 3$, find the greatest value of $|z + 1|$.***

Sol. $\because |z + 1| = |(z + 4) - 3|$

$= |(z + 4) + (-3)| \leq |z + 4| + |-3|$

$= |z + 4| + 3$

$\leq 3 + 3 = 6$ [$\because |z + 4| \leq 3$]

$\therefore \quad |z + 1| \leq 6$

Hence, the greatest value of $|z + 1|$ is 6.

Example 32. If $|z_1|=1, |z_2|=2, |z_3|=3$ and $|9z_1z_2+4z_3z_1+z_2z_3|=6$, find the value of $|z_1+z_2+z_3|$.

Sol. $\because \quad |z_1|=1 \Rightarrow |z_1|^2=1$

$$\Rightarrow \quad z_1\bar{z}_1=1 \Rightarrow \frac{1}{z_1}=\bar{z}_1$$

$$|z_2|=2 \Rightarrow |z_2|^2=4 \Rightarrow z_2\bar{z}_2=4$$

$$\Rightarrow \quad \frac{4}{z_2}=\bar{z}_2 \text{ and } |z_3|=3 \Rightarrow |z_3|^2=9$$

$$\Rightarrow \quad z_3\bar{z}_3=9 \Rightarrow \frac{9}{z_3}=\bar{z}_3$$

and given $|9z_1z_2+4z_3z_1+z_2z_3|=6$

$$\Rightarrow \quad |z_1z_2z_3|\left|\frac{9}{z_3}+\frac{4}{z_2}+\frac{1}{z_1}\right|=6$$

$$\Rightarrow \quad |z_1||z_2||z_3||\bar{z}_3+\bar{z}_2+\bar{z}_1|=6$$

$$\left[\because \frac{1}{z_1}=\bar{z}_1, \frac{4}{z_2}=\bar{z}_2 \text{ and } \frac{9}{z_3}=\bar{z}_3\right]$$

$$\Rightarrow \quad 1\cdot 2\cdot 3\,|\overline{z_1+z_2+z_3}|=6$$

$$\therefore \quad |z_1+z_2+z_3|=1 \qquad [\because |\bar{z}|=|z|]$$

Example 33. *Prove that*

$$|z_1|+|z_2|=\left|\frac{1}{2}(z_1+z_2)+\sqrt{z_1z_2}\right|+\left|\frac{1}{2}(z_1+z_2)-\sqrt{z_1z_2}\right|.$$

Sol. $\text{RHS}=\left|\frac{1}{2}(z_1+z_2)+\sqrt{z_1z_2}\right|+\left|\frac{1}{2}(z_1+z_2)-\sqrt{z_1z_2}\right|$

$$=\left|\frac{z_1+z_2+2\sqrt{z_1z_2}}{2}\right|+\left|\frac{z_1+z_2-2\sqrt{z_1z_2}}{2}\right|$$

$$=\frac{1}{2}\{|\sqrt{z_1}+\sqrt{z_2}|^2+|\sqrt{z_1}-\sqrt{z_2}|^2\}$$

$$=\frac{1}{2}\cdot 2\{|\sqrt{z_1}|^2+|\sqrt{z_2}|^2\} \qquad \text{[by property (xiv)]}$$

$$=|z_1|+|z_2|=\text{LHS}$$

Example 34. z_1 and z_2 are two complex numbers, such that $\dfrac{z_1-2z_2}{2-z_1\cdot\overline{z_2}}$ is unimodular, while z_2 is not unimodular. Find $|z_1|$.

Sol. Here, $\left|\dfrac{z_1-2z_2}{2-z_1\bar{z}_2}\right|=1$

$$\Rightarrow \quad \left|\frac{z_1-2z_2}{2-z_1\bar{z}_2}\right|=1 \qquad \text{[by property (vi)]}$$

$$\Rightarrow \quad |z_1-2z_2|=|2-z_1\overline{z_2}|$$

$$\Rightarrow \quad |z_1-2z_2|^2=|2-z_1\overline{z_2}|^2$$

$$\Rightarrow \quad (z_1-2z_2)(\overline{z_1-2z_2})=(2-z_1\overline{z_2})(\overline{2-z_1\bar{z}_2})$$

[by property (iv)]

$$\Rightarrow \quad (z_1-2z_2)(\bar{z}_1-2\bar{z}_2)=(2-z_1\overline{z_2})(2-\bar{z}_1z_2)$$

$$\Rightarrow \quad z_1\overline{z_1}-2z_1\overline{z_2}-2z_2\overline{z_1}+4z_2\overline{z_2}$$
$$=4-2\overline{z_1}z_2-2z_1\overline{z_2}+z_1\overline{z_1}z_2\overline{z_2}$$

$$\Rightarrow \quad |z_1|^2+4|z_2|^2=4+|z_1|^2|z_2|^2$$

$$\Rightarrow \quad |z_1|^2-|z_1|^2\cdot|z_2|^2+4|z_2|^2-4=0$$

$$\Rightarrow \quad (|z_1|^2-4)(1-|z_2|^2)=0$$

But $|z_2|\neq 1$ [given]

$\therefore \quad |z_1|^2=4$

Hence, $|z_1|=2$

Properties of Arguments

(i) $\arg(z_1z_2)=\arg(z_1)+\arg(z_2)+2k\pi, k\in I$

In general, $\arg(z_1z_2z_3\ldots z_n)$

$=\arg(z_1)+\arg(z_2)+\arg(z_3)+\ldots+\arg(z_n)+2k\pi$, $k\in I$.

(ii) $\arg\left(\dfrac{z_1}{z_2}\right)=\arg(z_1)-\arg(z_2)+2k\pi, k\in I$

(iii) $\arg\left(\dfrac{z}{\bar{z}}\right)=2\arg(z)+2k\pi, k\in I$

(iv) $\arg(z^n)=n.\arg(z)+2k\pi, k\in I$, where proper value of k must be chosen, so that RHS lies in $(-\pi,\pi]$.

(v) If $\arg\left(\dfrac{z_2}{z_1}\right)=\theta$, then $\arg\left(\dfrac{z_1}{z_2}\right)=2n\pi-\theta$, where $n\in I$.

(vi) $\arg(\bar{z})=-\arg(z)$

Example 35. If $\arg(z_1)=\dfrac{17\pi}{18}$ and $\arg(z_2)=\dfrac{7\pi}{18}$, find the principal argument of z_1z_2 and (z_1/z_2).

Sol. $\arg(z_1z_2)=\arg(z_1)+\arg(z_2)+2k\pi$

$$=\frac{17\pi}{18}+\frac{7\pi}{18}+2k\pi$$

$$=\frac{4\pi}{3}+2k\pi$$

$$=\frac{4\pi}{3}-2\pi=-\frac{2\pi}{3} \qquad [\text{for } k=-1]$$

and $\arg\left(\dfrac{z_1}{z_2}\right)=\arg(z_1)-\arg(z_2)+2k\pi$

$$=\frac{17\pi}{18}-\frac{7\pi}{18}+2k\pi=\frac{10\pi}{18}+2k\pi$$

$$=\frac{5\pi}{9}+0=\frac{5\pi}{9} \qquad [\text{for } k=0]$$

Example 36. If z_1 and z_2 are conjugate to each other, find the principal argument of $(-z_1 z_2)$.

Sol. $\because$ z_1 and z_2 are conjugate to each other i.e., $z_2 = \overline{z_1}$, therefore, $z_1 z_2 = z_1 \overline{z_1} = |z_1|^2$

$$\therefore \arg(-z_1 z_2) = \arg(-|z_1|^2) = \arg \quad \text{[negative real number]}$$
$$= \pi$$

Example 37. Let z be any non-zero complex number, then find the value of $\arg(z) + \arg(\bar{z})$.

Sol. $\arg(z) + \arg(\bar{z}) = \arg(z\bar{z})$

$$= \arg(|z|^2) = \arg \quad \text{[positive real number]}$$
$$= 0$$

(a) Mixed Properties of Modulus and Arguments

(i) $|z_1 + z_2| = |z_1| + |z_2| \Leftrightarrow \arg(z_1) = \arg(z_2)$

(ii) $|z_1 + z_2| = |z_1| - |z_2| \Leftrightarrow \arg(z_1) - \arg(z_2) = \pi$

Proof (i) Let $\arg(z_1) = \theta$ and $\arg(z_2) = \phi$

$$\therefore \quad |z_1 + z_2| = |z_1| + |z_2|$$

On squaring both sides, we get

$$|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2|z_1||z_2|$$

$$\Rightarrow \quad |z_1|^2 + |z_2|^2 + 2|z_1||z_2| \cos(\theta - \phi)$$
$$= |z_1|^2 + |z_2|^2 + 2|z_1||z_2|$$

$$\Rightarrow \quad \cos(\theta - \phi) = 1$$

$$\therefore \quad \theta - \phi = 0 \text{ or } \theta = \phi$$

$$\therefore \quad \arg(z_1) = \arg(z_2)$$

(ii) $\because$ $|z_1 + z_2| = |z_1| - |z_2|$

On squaring both sides, we get

$$|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 - 2|z_1||z_2|$$

$$\Rightarrow \quad |z_1|^2 + |z_2|^2 + 2|z_1||z_2| \cos(\theta - \phi)$$
$$= |z_1|^2 + |z_2|^2 - 2|z_1||z_2|$$

$$\Rightarrow \quad \cos(\theta - \phi) = -1$$

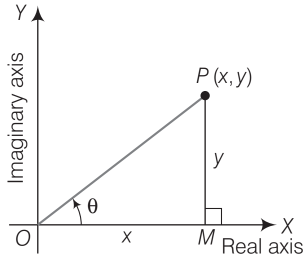
$$\therefore \quad \theta - \phi = \pi \text{ or } \arg(z_1) - \arg(z_2) = \pi$$

Remark

1. $|z_1 - z_2| = |z_1| + |z_2| \Leftrightarrow \arg(z_1) = \arg(z_2)$
2. $|z_1 - z_2| = |z_1| - |z_2| \Leftrightarrow \arg(z_1) - \arg(z_2) = \pi$
3. $|z_1 - z_2| = |z_1 + z_2| \Leftrightarrow \arg(z_1) - \arg(z_2) = \pm \frac{\pi}{2}$, $\bar{z}_1 z_2$ and $\frac{z_1}{z_2}$ are purely imaginary.

(b) Trigonometric or Polar or Modulus Argument Form of a Complex Number

Let $z = x + iy$, where $x, y \in R$ and $i = \sqrt{-1}$, z is represented by $P(x, y)$ in the argand plane.



By geometrical representation,

$$OP = \sqrt{(x^2 + y^2)} = |z|$$

$$\angle POM = \theta = \arg(z)$$

In ΔOPM, $\quad x = OP \cos(\angle POM) = |z| \cos(\arg z)$

and $\quad y = OP \sin(\angle POM) = |z| \sin(\arg z)$

$\because \quad z = x + iy$

$\therefore \quad z = |z|(\cos(\arg z) + i \sin(\arg z))$

or $\quad z = r(\cos\theta + i \sin\theta)$

$\quad \bar{z} = r(\cos\theta - i\sin\theta)$

where, $r = |z|$ and $\theta =$ principal value of $\arg(z)$.

Remark

1. $\cos\theta + i\sin\theta$ is also written as $CiS\ \theta$.
2. **Remember**

$$1 = \cos 0 + i \sin 0 \quad \Rightarrow \quad -1 = \cos\pi + i\sin\pi$$
$$i = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} \quad \Rightarrow \quad -i = \cos\frac{\pi}{2} - i\sin\frac{\pi}{2}$$

Example 38. ***Write the polar form of*** $-\frac{1}{2} - \frac{i\sqrt{3}}{2}$ (where, $i = \sqrt{-1}$).

Sol. Let $z = -\frac{1}{2} - \frac{i\sqrt{3}}{2}$. Since, $\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$ lies in III quadrant.

$\therefore$ Principal value of $\arg(z) = -\pi + \tan^{-1}\left|\frac{-\sqrt{3}/2}{-1/2}\right|$

$$= -\pi + \tan^{-1}\sqrt{3} = -\pi + \frac{\pi}{3} = -\frac{2\pi}{3}$$

and $|z| = \sqrt{\left(-\frac{1}{2}\right)^2 + \left(-\frac{\sqrt{3}}{2}\right)^2} = \sqrt{\left(\frac{1}{4} + \frac{3}{4}\right)} = \sqrt{1} = 1$

$\therefore$ Polar form of $z = |z|[\cos(\arg z) + i\sin(\arg z)]$

i.e. $-\frac{1}{2} - \frac{i\sqrt{3}}{2} = \left[\cos\left(-\frac{2\pi}{3}\right) + i\sin\left(-\frac{2\pi}{3}\right)\right]$

(c) Euler's Form

If $\theta \in R$ and $i = \sqrt{-1}$, then $e^{i\theta} = \cos\theta + i\sin\theta$ is known as Euler's identity.

Now, $\quad e^{-i\theta} = \cos\theta - i\sin\theta$

Let $\quad z = e^{i\theta}$

$\therefore \quad |z| = 1$ and $\arg(z) = \theta$

Also, $\quad e^{i\theta} + e^{-i\theta} = 2\cos\theta$ and $e^{i\theta} - e^{-i\theta} \; 2i\sin\theta$

and if $\theta, \phi \in R$ and $i = \sqrt{-1}$, then

(i) $e^{i\theta} + e^{i\phi} = e^{i\left(\frac{\theta+\phi}{2}\right)} \cdot 2\cos\left(\frac{\theta-\phi}{2}\right)$

$\therefore \quad |e^{i\theta} + e^{i\phi}| = 2\left|\cos\left(\frac{\theta-\phi}{2}\right)\right|$

and $\quad \arg(e^{i\theta} + e^{i\phi}) = \left(\frac{\theta+\phi}{2}\right)$

(ii) $e^{i\theta} - e^{i\phi} = e^{i\left(\frac{\theta+\phi}{2}\right)} \cdot 2i\sin\left(\frac{\theta-\phi}{2}\right)$

$\therefore \quad |e^{i\theta} - e^{i\phi}| = 2\left|\sin\left(\frac{\theta-\phi}{2}\right)\right|$

and $\arg(e^{i\theta} - e^{i\phi}) = \frac{\theta+\phi}{2} + \frac{\pi}{2} \qquad [\because i = e^{i\pi/2}]$

Remark

1. $e^{i\theta} + 1 = e^{i\theta/2} \cdot 2\cos(\theta/2)$ (Remember)
2. $e^{i\theta} - 1 = e^{i\theta/2} \cdot 2i\sin(\theta/2)$ (Remember)
3. $\frac{e^{i\theta} - 1}{e^{i\theta} + 1} = i\tan(\theta/2)$ (Remember)
4. If $z = re^{i\theta}$; $|z| = r$, then $\arg(z) = \theta$, $\bar{z} = re^{-i\theta}$
5. If $|z - z_0| = 1$, then $z - z_0 = e^{i\theta}$

Example 39. ***Given that*** $|z-1| = 1$, ***where*** z ***is a point on the argand plane, show that*** $\frac{z-2}{z} = i\tan(\arg z)$, where $i = \sqrt{-1}$.

Sol. Given, $|z-1| = 1$

$\therefore \quad z - 1 = e^{i\theta} \Rightarrow z = e^{i\theta} + 1 = e^{i\theta/2} \cdot 2\cos(\theta/2)$

$\therefore \quad \arg(z) = \theta/2 \qquad \ldots$(i)

$$\text{LHS} = \frac{z-2}{z} = \frac{1 + e^{i\theta} - 2}{1 + e^{i\theta}} = \frac{e^{i\theta} - 1}{e^{i\theta} + 1} = i\tan(\theta/2)$$

$= i\tan(\arg z) = \text{RHS}$ [from Eq. (i)]

Example 40. Let z be a non-real complex number lying on $|z| = 1$, prove that $z = \dfrac{1 + i\tan\left(\frac{\arg(z)}{2}\right)}{1 - i\tan\left(\frac{\arg(z)}{2}\right)}$ (where, $i = \sqrt{-1}$).

Sol. Given, $\quad |z| = 1$

$\therefore \quad z = e^{i\theta} \qquad \ldots$(i)

$\Rightarrow \quad \arg(z) = \theta \qquad \ldots$(ii)

$$\text{RHS} = \frac{1 + i\tan\left(\frac{\arg(z)}{2}\right)}{1 - i\tan\left(\frac{\arg(z)}{2}\right)} = \frac{1 + i\tan(\theta/2)}{1 - i\tan(\theta/2)} \quad \text{[from Eq. (ii)]}$$

$$= \frac{\cos\theta/2 + i\sin\theta/2}{\cos\theta/2 - i\sin\theta/2} = \frac{e^{i\theta/2}}{e^{-i\theta/2}}$$

$= e^{i\theta} = z = \text{LHS}$ [from Eq. (i)]

Example 41. ***Prove that*** $\tan\left(i\ln\left(\frac{a-ib}{a+ib}\right)\right) = \frac{2ab}{a^2-b^2}$ (where $a, b \in R^+$ and $i = \sqrt{-1}$).

Sol. $\because \quad \left|\frac{a-ib}{a+ib}\right| = \frac{|a-ib|}{|a+ib|} = 1 \qquad [\because |\bar{z}| = |z|]$

Let $\quad \frac{a-ib}{a+ib} = e^{i\theta} \qquad \ldots$(i)

By componendo and dividendo , we get

$$\frac{(a-ib)-(a+ib)}{(a-ib)+(a+ib)} = \frac{e^{i\theta}-1}{e^{i\theta}+1} - \frac{b}{a}i = i\tan(\theta/2)$$

or $\quad \tan\left(\frac{\theta}{2}\right) = -\frac{b}{a} \qquad \ldots$(ii)

$\therefore \quad \text{LHS} = \tan\left(i\ln\left(\frac{a-ib}{a+ib}\right)\right)$

$= \tan(i\ln(e^{i\theta}))$ [from Eq. (i)]

$= \tan(i \cdot i\theta) = -\tan\theta$

$= -\frac{2\tan\theta/2}{1-\tan^2\theta/2}$

$= -\frac{2(-b/a)}{1-(-b/a)^2}$ [from Eq. (ii)]

$= \frac{2ab}{a^2-b^2} = \text{RHS}$

Applications of Euler's Form

If $x, y, \theta \in R$ and $i = \sqrt{-1}$, then

let $\quad z = x + iy$ [cartesian form]

$= |z|(\cos\theta + i\sin\theta)$ [polar form]

$= |z|e^{i\theta}$ [Euler's form]

(i) Product of Two Complex Numbers

Let two complex numbers be

$$z_1 = |z_1|e^{i\theta_1} \text{ and } z_2 = |z_2|e^{i\theta_2},$$

where $\theta_1, \theta_2 \in R$ and $i = \sqrt{-1}$

$\therefore \quad z_1 \cdot z_2 = |z_1| e^{i\theta_1} \cdot |z_2| e^{i\theta_2} = |z_1||z_2| e^{i(\theta_1 + \theta_2)}$

$$= |z_1||z_2|(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2))$$

Thus, $\quad |z_1 z_2| = |z_1||z_2|$

and $\quad \arg(z_1 z_2) = \theta_1 + \theta_2 = \arg(z_1) + \arg(z_2)$

(ii) Division of Two Complex Numbers

Let two complex numbers be

$$z_1 = |z_1| e^{i\theta_1} \quad \text{and} \quad z_2 = |z_2| e^{i\theta_2},$$

where $\theta_1, \theta_2 \in R$ and $i = \sqrt{-1}$

$$\therefore \quad \frac{z_1}{z_2} = \frac{|z_1| e^{i\theta_1}}{|z_2| e^{i\theta_2}} = \frac{|z_1|}{|z_2|} e^{i(\theta_1 - \theta_2)}$$

$$= \frac{|z_1|}{|z_2|}(\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2))$$

Thus, $\quad \left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}, (z_2 \neq 0)$

and $\arg\left(\frac{z_1}{z_2}\right) = \theta_1 - \theta_2 = \arg(z_1) - \arg(z_2)$

(iii) Logarithm of a Complex Number

$\log_e(z) = \log_e(|z| e^{i\theta}) = \log_e|z| + \log_e(e^{i\theta})$

$$= \log_e|z| + i\theta = \log_e|z| + i\arg(z)$$

So, the general value of $\log_e(z)$

$$= \log_e(z) + 2n\pi i \, (-\pi < \arg z < \pi).$$

Example 42. If m and x are two real numbers and $i = \sqrt{-1}$, prove that $e^{2mi\cot^{-1}x}\left(\frac{xi+1}{xi-1}\right)^m = 1$.

Sol. Let $\cot^{-1}x = \theta$, then $\cot\theta = x$

$$\therefore \text{LHS} = e^{2mi\cot^{-1}x}\left(\frac{xi+1}{xi-1}\right)^m = e^{2mi\theta}\left(\frac{i\cot\theta+1}{i\cot\theta-1}\right)^m$$

$$= e^{2mi\theta}\left(\frac{i(\cot\theta - i)}{i(\cot\theta + i)}\right)^m = e^{2mi\theta}\left(\frac{\cos\theta - i\sin\theta}{\cos\theta + i\sin\theta}\right)^m$$

$$= e^{2mi\theta}\cdot\left(\frac{e^{-i\theta}}{e^{i\theta}}\right)^m = e^{2mi\theta}\cdot(e^{-2i\theta})^m$$

$$= e^{2mi\theta}\cdot e^{-2mi\theta} = e^0 = 1 = \text{RHS}$$

Example 43. If z and w are two non-zero complex numbers such that $|z| = |w|$ and $\arg(z) + \arg(w) = \pi$, prove that $z = -\overline{w}$.

Sol. Let $\arg(w) = \theta$, then $\arg(z) = \pi - \theta$

$$\therefore \quad z = |z|(\cos(\arg z) + i\sin(\arg z))$$

$$= |z|(\cos(\pi - \theta) + i\sin(\pi - \theta))$$

$$= |z|(-\cos\theta + i\sin\theta) = -|z|(\cos\theta - i\sin\theta)$$

$$= -|w|(\cos(\arg w) - i\sin(\arg w))$$

$$= -|w|(\cos(-\arg w) + i\sin(-\arg w))$$

$$= -|\overline{w}|(\cos(\arg\overline{w}) + i\sin(\arg\overline{w})) = -\overline{w}$$

Example 44. Express $(1+i)^{-i}$, (where, $i = \sqrt{-1}$) in the form $A + iB$.

Sol. Let $A + iB = (1+i)^{-i}$

On taking logarithm both sides, we get

$\log_e(A + iB) = -i\log_e(1+i)$

$$= -i\log_e\left(\sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)\right)$$

$$= -i\log_e\left(\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\right)$$

$$= -i\log_e(\sqrt{2}\, e^{i\pi/4}) = -i(\log_e\sqrt{2} + \log_e e^{i\pi/4})$$

$$= -i\left(\frac{1}{2}\log_e 2 + \frac{i\pi}{4}\right) = -\frac{i}{2}\log_e 2 + \frac{\pi}{4}$$

$$\therefore A + iB = e^{-\frac{i}{2}\log_e 2 + \frac{\pi}{4}} = e^{\pi/4}\cdot e^{i\log_e 2^{-1/2}}$$

$$= e^{\pi/4}\cdot(\cos(\log_e 2^{-1/2}) + i\sin(\log_e 2^{-1/2}))$$

$$= e^{\pi/4}\cdot\cos\left(\log_e\left(\frac{1}{\sqrt{2}}\right)\right) + i\, e^{\pi/4}\sin\left(\log_e\left(\frac{1}{\sqrt{2}}\right)\right)$$

Example 45. If $\sin(\log_e i^i) = a + ib$, where $i = \sqrt{-1}$, find a and b, hence and find $\cos(\log_e i^i)$.

Sol. $a + ib = \sin(\log_e i^i) = \sin(i\log_e i)$

$$= \sin(i(\log_e|i| + i\arg i))$$

$$= \sin(i(\log_e 1 + (i\pi/2)))$$

$$= \sin(i(0 + (i\pi/2))) = \sin(-\pi/2) = -1$$

$\therefore \quad a = -1, b = 0$

Now, $\quad \cos(\log_e i^i) = \sqrt{1 - \sin^2(\log_e i^i)}$

$$= \sqrt{1 - (-1)^2} = \sqrt{(1-1)} = 0$$

Aliter

$\because \quad i^i = (e^{i\pi/2})^i = e^{-\pi/2}$

$\therefore \quad \sin(\log_e i^i) = \sin(\log_e e^{-\pi/2}) = \sin\left(-\frac{\pi}{2}\log_e e\right)$

$$= \sin(-\pi/2) = -1 = a + ib \qquad \text{[given]}$$

$\therefore \quad a = -1, b = 0$

and $\quad \cos(\log_e i^i) = \cos(\log_e e^{-\pi/2})$

$$= \cos\left(-\frac{\pi}{2}\log_e e\right) = \cos\left(-\frac{\pi}{2}\right) = 0$$

Example 46. ***Find the general value of*** $\log_2(5i)$, ***where*** $i = \sqrt{-1}$.

Sol. $\log_2 5i = \frac{\log_e 5i}{\log_e 2} = \frac{1}{\log_e 2}\{\log_e|5i| + i\arg(5i) + 2n\pi i\}$

$$= \frac{1}{\log_e 2}\left\{\log_e 5 + \frac{i\pi}{2} + 2n\pi i\right\}, n \in I$$

Exercise for Session 2

1 If $\frac{1-ix}{1+ix}=a-ib$ and $a^2+b^2=1$, where $a,b\in R$ and $i=\sqrt{-1}$, then x is equal to

(a) $\frac{2a}{(1+a)^2+b^2}$ (b) $\frac{2b}{(1+a)^2+b^2}$ (c) $\frac{2a}{(1+b)^2+a^2}$ (d) $\frac{2b}{(1+b)^2+a^2}$

2 The least positive integer n for which $\left(\frac{1+i}{1-i}\right)^n=\frac{2}{\pi}\left(\sec^{-1}\frac{1}{x}+\sin^{-1}x\right)$ (where, $x\neq 0$, $-1\leq x\leq 1$ and $i=\sqrt{-1}$), is

(a) 2 (b) 4 (c) 6 (d) 8

3 If $z=(3+4i)^6+(3-4i)^6$, where $i=\sqrt{-1}$, then $\mathrm{Im}(z)$ equals to

(a) -6 (b) 0 (c) 6 (d) None of these

4 If $(x+iy)^{1/3}=a+ib$, where $i=\sqrt{-1}$, then $\left(\frac{x}{a}+\frac{y}{b}\right)$ is equal to

(a) $4a^2b^2$ (b) $4(a^2-b^2)$ (c) $4a^2-b^2$ (d) a^2+b^2

5 If $\frac{3}{2+\cos\theta+i\sin\theta}=a+ib$, where $i=\sqrt{-1}$ and $a^2+b^2=\lambda a-3$, the value of λ is

(a) 3 (b) 4 (c) 5 (d) 6

6 If $\frac{z-1}{z+1}$ is purely imaginary, then $|z|$ is equal to

(a) $\frac{1}{2}$ (b) 1 (c) $\sqrt{2}$ (d) 2

7 The complex numbers $\sin x+i\cos 2x$ and $\cos x-i\sin 2x$, where $i=\sqrt{-1}$, are conjugate to each other, for

(a) $x=n\pi, n\in I$ (b) $x=0$ (c) $x=\left(n+\frac{1}{2}\right), n\in I$ (d) no value of x

8 If α and β are two different complex numbers with $|\beta|=1$, then $\left|\frac{\beta-\alpha}{1-\bar{\alpha}\beta}\right|$ is equal to

(a) 0 (b) $\frac{1}{2}$ (c) 1 (d) 2

9 If $x=3+4i$ (where, $i=\sqrt{-1}$), the value of $x^4-12x^3+70x^2-204x+225$, is

(a) -45 (b) 0 (c) 35 (d) 15

10 If $|z_1-1|\leq 1$, $|z_2-2|\leq 2$, $|z_3-3|\leq 3$, the greatest value of $|z_1+z_2+z_3|$ is

(a) 6 (b) 12 (c) 17 (d) 23

11 The principal value of $\arg(z)$, where $z=\left(1+\cos\frac{8\pi}{5}\right)+i\sin\frac{8\pi}{5}$ (where, $i=\sqrt{-1}$) is given by

(a) $-\frac{\pi}{5}$ (b) $-\frac{4\pi}{5}$ (c) $\frac{\pi}{5}$ (d) $\frac{4\pi}{5}$

12 If $|z_1|=2$, $|z_2|=3$, $|z_3|=4$ and $|z_1+z_2+z_3|=5$, then $|4z_2z_3+9z_3z_1+16z_1z_2|$ is

(a) 24 (b) 60 (c) 120 (d) 240

13 If $|z-i|\leq 5$ and $z_1=5+3i$ (where, $i=\sqrt{-1}$), the greatest and least values of $|iz+z_1|$ are

(a) 7 and 3 (b) 9 and 1 (c) 10 and 0 (d) None of these

14 If z_1, z_2 and z_3, z_4 are two pairs of conjugate complex numbers, then $\arg\left(\frac{z_1}{z_4}\right)+\arg\left(\frac{z_2}{z_3}\right)$ equals to

(a) 0 (b) $\frac{\pi}{2}$ (c) π (d) $\frac{3\pi}{2}$

Answers

Exercise for Session 2

1. (b)	**2.** (b)	**3.** (b)	**4.** (b)	**5.** (b)	**6.** (b)
7. (d)	***8.*** (c)	**9.** (b)	**10.** (b)	**11.** (a)	**12.** (c)
13. (c)	**14.** (a)				