3

MATRICES

Objectives

After studying the material of this chapter, you should be able to :

- **Understand the definitions of matrix and types of matrices.**
- **Understand the conditions for the equality of two matrices.**
- **Understand the operations on matrices.**
- **Understand the Transpose of a Matrix and Symmetric & Skew-symmetric matrices.**
- **Understand the definition of Invertible Matrix.**
- **Understand to the find inverse of a Matrix by elementary transformations.**

God made integers, all else is the work of man

—L. Kronecker

Chapter at Glance

INTRODUCTION

In Mathematics, Matrices are most important tools and the knowledge of its properties is essential in various fields. Matrices are not only being used for solving the systems of simultaneous linear equations. Now a days, it is also widely applied in various branches of Sciences, Engineering, Economics, Industrial management; etc. Physical operations like rotation and reflection through a plane can also be represented mathematically by matrices.

In this chapter we will study the following concepts :

- Introduction to Matrices
- Types of Matrices
- Operations on Matrices
- Transpose of a Matrix
- Inverse of a Matrix by Elementary Operations.

3.1. MATRICES

(I) Let us express that *"Anil has 8 books."*

We express this as [8] with the idea that the number written inside [ ] is the number of books Anil has.

(II) Let us express that *"Anil has 8 books and 6 pencils."*

We express this as [8   6] with the idea that first number inside [ ] is the number of books and the second number is the number of pencils Anil has.

(III) Suppose there are three boys Anil, Akram and Manjit. Anil has 9 books and 4 pencils, Akram has 4 books and 3 pencils while Manjit has 6 books and 5 pencils.

Let us express the above statement in the form of a matrix.

First of all, we express the above information in the tabular form as below :

	Books	*Pencils*
Anil	9	4
Akram	4	3
Manjit	6	5

Now, we write this as below :

$$\begin{array}{cc} \begin{bmatrix} 9 & 4 \\ 4 & 3 \\ 6 & 5 \end{bmatrix} & \begin{array}{l} \rightarrow \; \textit{1st row} \\ \rightarrow \; \textit{2nd row} \\ \rightarrow \; \textit{3rd row} \end{array} \\ \begin{array}{cc} \downarrow & \downarrow \\ \textit{1st} & \textit{2nd} \\ \textit{column} & \textit{column} \end{array} & \end{array}$$

We gather the following information from the above display :

(*i*) The entries in the first, second and third rows represent the objects (books and pencils) possessed by Anil, Akram and Manjit respectively.

(*ii*) The entries in the first and second columns represent the number of books and pencils respectively.

Thus the entry in the third row and second column represents the number of pencils possessed by Manjit.

Similar is the case with other items of the above display.

The arrangement or display of the above kind is called a **matrix.**

Formally, we have the following definition :

Definition

A system of mn-numbers (real or complex) arranged in the form of an ordered set of m horizontal lines (called **rows**) *and n vertical lines* (called **columns**) *is said to be an m × n* ***matrix*** (to be read as *m* by *n* matrix).

Thus the **order** of the matrix is $m \times n$.

Or

Definition

An ordered rectangular array of numbers (or functions) is called a matrix. The numbers (or functions) are called **elements (entries)** of the matrix.

We write $m \times n$ matrix as :

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2j} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{i1} & a_{i2} & a_{i3} & \cdots & \boxed{a_{ij}} & \cdots & a_{in} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix},$$

where $a_{ij} \in \mathbf{K}$, when $\mathbf{K}$ is the number system.

We denote the general element by a_{ij} and the matrix by $[a_{ij}]$ or by $[a_{ij}]_{m \times n}$ or by $[a_{ij}]_{(m \times n)}$

The suffixes i and j in a_{ij} denote ith row and jth column respectively to which the element a_{ij} belongs.

Notations :

Matrices are generally denoted by capital letters of English alphabet *viz.*, A, B, C,....

Elements are generally denoted by corresponding small letters of English alphabet *viz.* a_{ij}, b_{ij}, c_{ij}.....

The following notations are used for enclosing the elements which constitute a matrix :

[ ], ( ), ‖ ‖, { }.

In this chapter, we shall use the first two.

Examples :

(1) *Consider* $A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \\ 7 & 9 & 11 \end{bmatrix}$.

Clearly A is 3 × 3 matrix.

Here $a_{11} = 1$, *i.e.* the element, which occurs in the first row and first column $= 1$.

$a_{12} = 3$, *i.e.* the element, which occurs in the first row and second column $= 3$.

$a_{13} = 5$, *i.e.* the element, which occurs in the first row and third column $= 5$.

Similarly, we have :

$$a_{21} = 2, a_{22} = 4, a_{23} = 6 \quad \text{and} \quad a_{31} = 7, a_{32} = 9, a_{33} = 11.$$

(2) *Consider the following information regarding the number of men and women workers in three factories I, II and III :*

	Men Workers	*Women Workers*
I	*25*	*10*
II	*30*	*11*
III	*24*	*7*

Represent the above information in the form of a 3 × 2 matrix.

What does the entry in the third row and second column represent ?

The given information is represented in the form of a 3 × 2 matrix as below :

$$\begin{bmatrix} 25 & 10 \\ 30 & 11 \\ 24 & \boxed{7} \end{bmatrix}.$$

The entry in the third row and second column represents the number of women workers in factory III. (Here $a_{32} = 7$)

(3) *Consider the linear equations :*

$$2x + 3y - 5z = 4$$
$$x + y + 2z = 5.$$

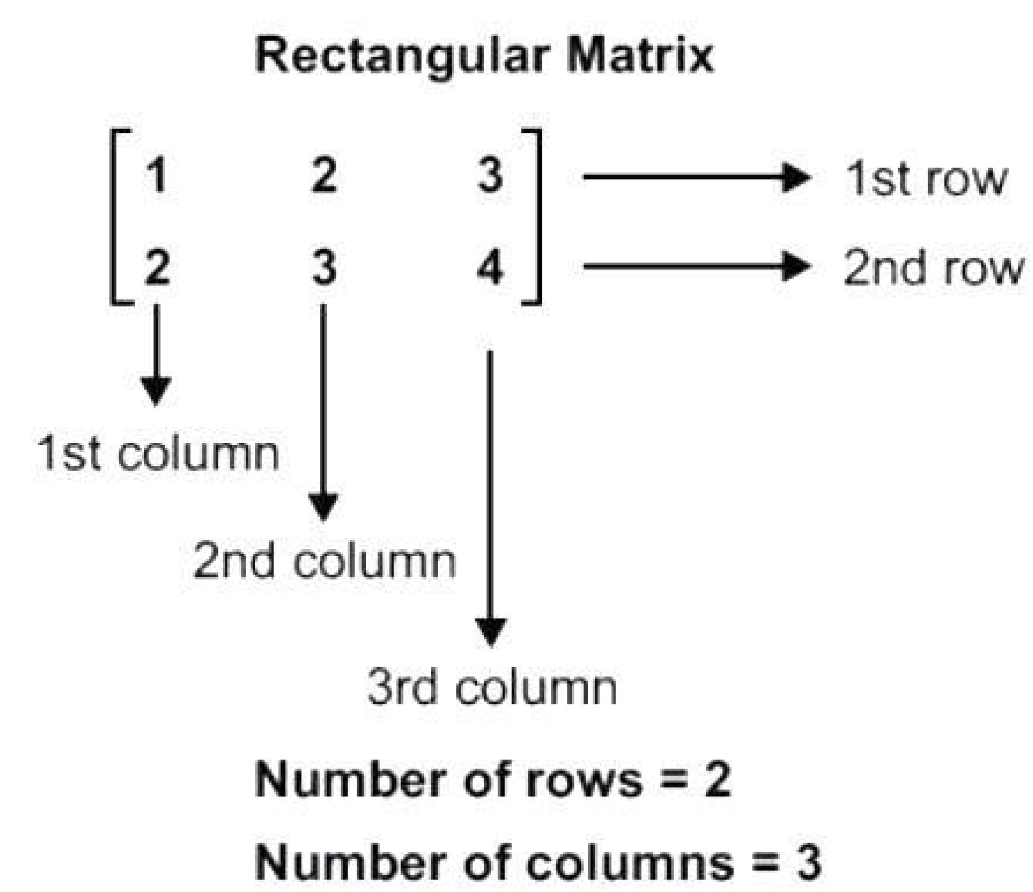
The coefficients of x, y and z in the above equations can be expressed in the form of the following matrix :

$$\begin{bmatrix} 2 & 3 & -5 \\ 1 & 1 & 2 \end{bmatrix}.$$

3.2. TYPES OF MATRICES

(*a*) (*i*) **Rectangular Matrix.** *Any $m \times n$ matrix ($m \neq n$) is called a rectangular matrix.*

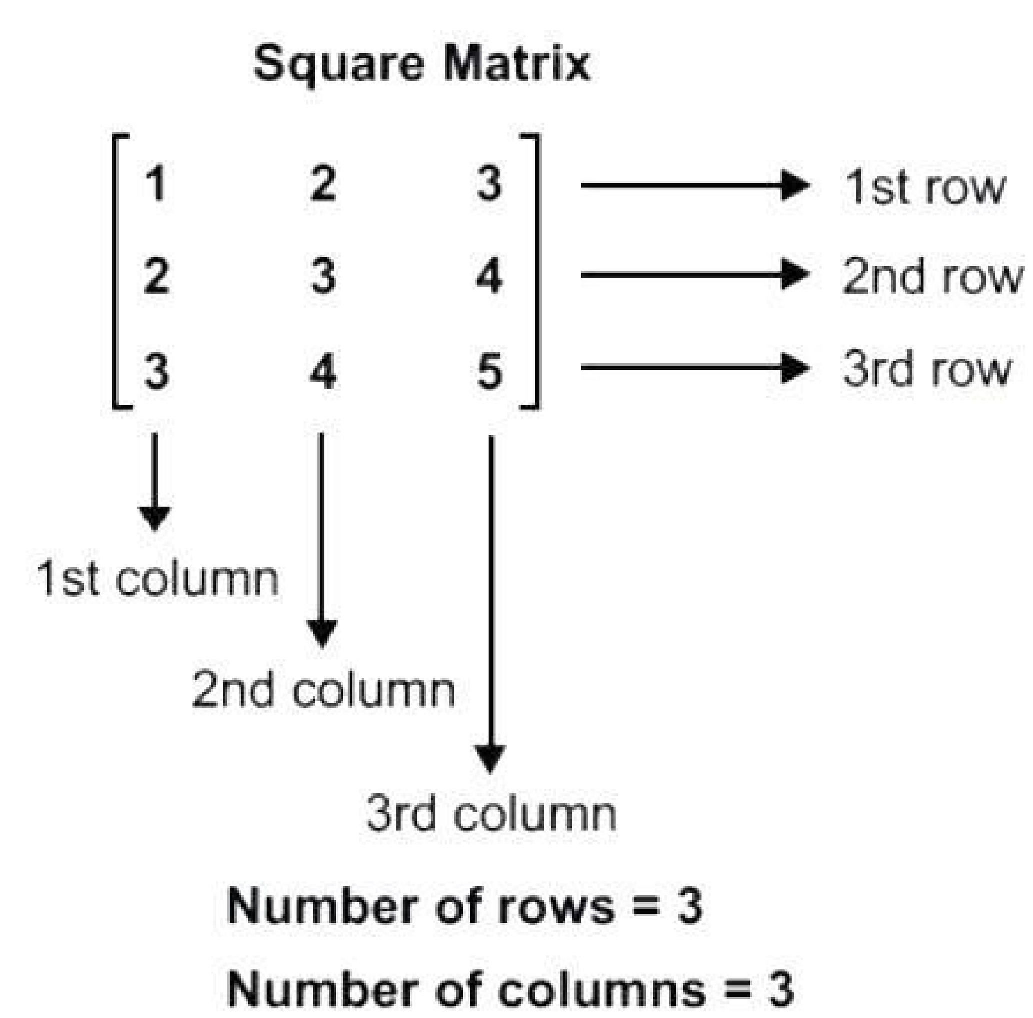
For Example : $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix}$ is a rectangular matrix of order 2 × 3.



(*ii*) **Square Matrix.** *Any $n \times n$ matrix is called a square matrix of order n (or n-rowed matrix).* ***(Jammu B. 2013)***

In this case, the number of rows = the number of columns.

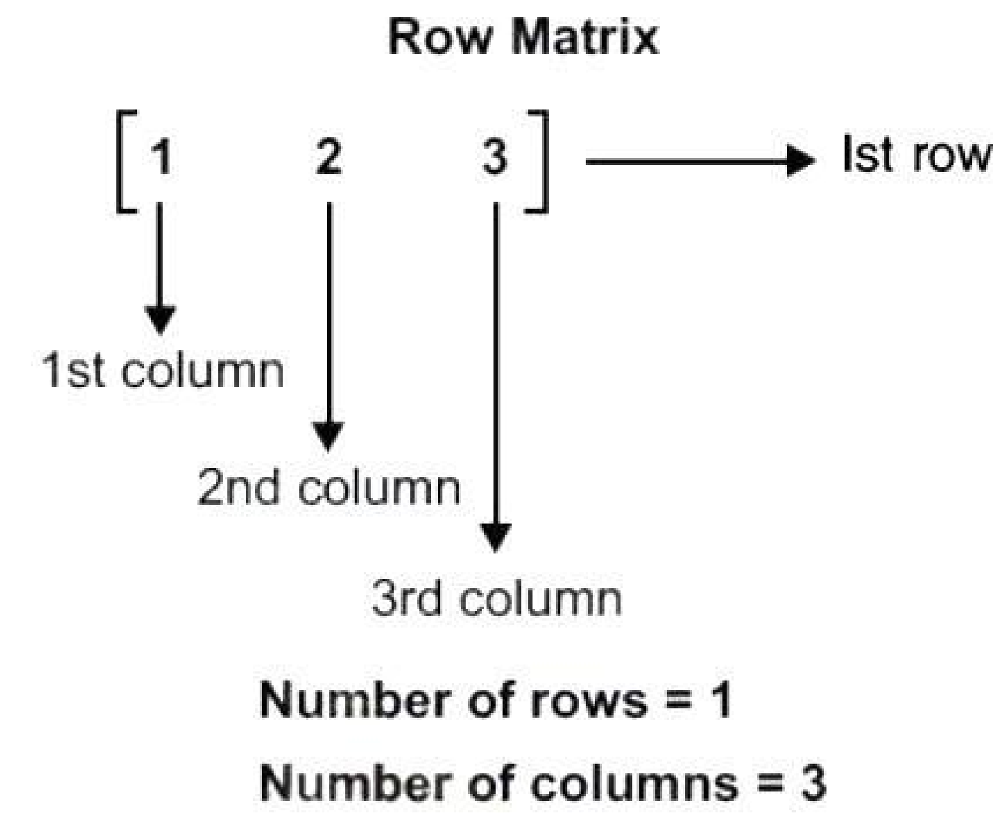
For Example : $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix}$ is a square matrix of order 3.



(*iii*) **Row Matrix.** *Any* $1 \times n$ *matrix is called a* **row matrix** (or **row vector**).

A row matrix has only one row.

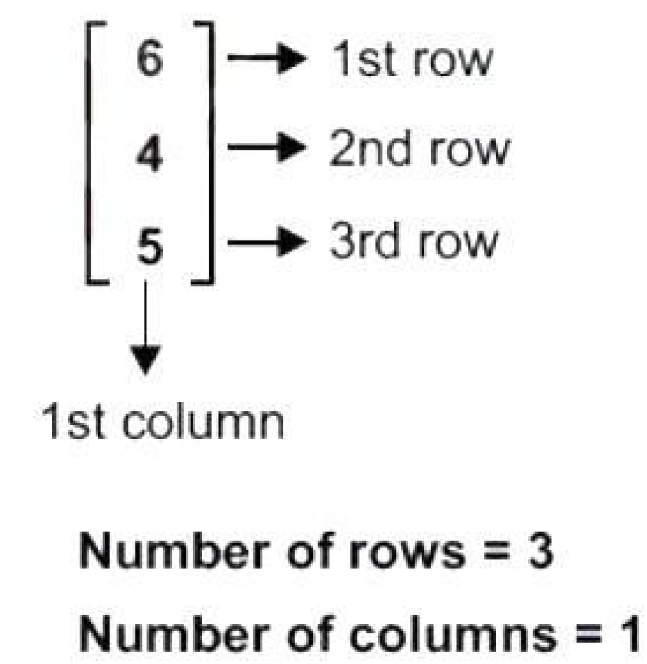
For Example : $[1 \quad 2 \quad 3]$ is a 1×3 row matrix.



(*iv*) **Column Matrix.** *Any* $m \times 1$ *matrix is called a* **column matrix** (*or* **column vector**).

A column matrix has only one column.

For Example : $\begin{bmatrix} 6 \\ 4 \\ 5 \end{bmatrix}$ is a 3×1 column matrix.



(***b***) (*i*) **Diagonal elements of a matrix.**

An element a_{ij} of the square matrix $A = [a_{ij}]$ is said to be a diagonal element if $i = j$.

Thus, $a_{11}, a_{22}, \ldots\ldots, a_{ii} \ldots\ldots\ldots$ are diagonal elements.

(*ii*) **Principal Diagonal.** *The line on which the diagonal elements lie is said to be the principal diagonal.*

This is also known as **leading diagonal** or simply **diagonal.**

Thus, $a_{11}, a_{22}, \ldots., a_{ii}, \ldots..$ form the principal diagonal.

For Example : If $A = \begin{bmatrix} 2 & -3 & 1 \\ 2 & 1 & 1 \\ -1 & -2 & 0 \end{bmatrix}$,

then the diagonal elements are 2, 1, 0 and principal diagonal is the line on which these elements lie.

(*iii*) **Diagonal Matrix.** *A square matrix $A = [a_{ij}]$ is said to be diagonal matrix if $a_{ij} = 0$ when $i \neq j$.*

Thus, it is a square matrix in which all the elements except the diagonal elements are zero.

For Example : $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 7 \end{bmatrix}$ is diagonal matrix of order 3.

This is also denoted by the symbol diag. [1, 4, 7].

(*iv*) **Scalar Matrix.** *A diagonal matrix is said to be scalar matrix if all its diagonal entries are equal.*

(Karnataka B. 2014)

Thus, $A = [a_{ij}]_{m \times n}$ is said to be a scalar matrix

if $\begin{cases} a_{ij} = 0 & \text{when} \quad i \neq j \\ a_{ij} = \alpha & \text{when} \quad i = j \end{cases}$, where α is a number.

For Examples : $[3]$, $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

are scalar matrices of order 1, 2 and 3 respectively.

(***c***) (*i*) **Identity Matrix.** *A diagonal matrix is said to be an identity matrix if each of its diagonal elements is unity.*

This is also known as **unit matrix.**

Thus, $A = [a_{ij}]_{n \times n}$ is called an identity matrix

if (I) $a_{ij} = 0$ when $i \neq j$ (II) $a_{ij} = 1$ when $i = j$.

The identity matrix of order n is usually denoted by $\mathbf{I}_n$ or simply by **I.**

For Examples : $\mathbf{I_1} = [1]$, $\mathbf{I_2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\mathbf{I_3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$; etc.

Definition

The square matrix $[a_{ij}]$ is an identity matrix

if $a_{ij} = \begin{cases} 1 & \text{when} \quad i = j \\ 0 & \text{when} \quad i \neq j. \end{cases}$

(*ii*) **Zero Matrix.** *A matrix is said to be a zero matrix if each of its elements is zero.* ***(Jammu B. 2013)***

This is also known as **null matrix.**

The zero matrix of the type $m \times n$ is denoted by $\mathbf{O}_{m,\,n}$ or simply by **O**.

For Examples : $O_{2,\,3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $O_{3,\,3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$; etc.

(*d*) Triangular Matrices.

(*i*) *A square matrix* $A = [a_{ij}]$ *is said to be* **upper triangular matrix** if $a_{ij} = 0$ *for* $i > j$.

For Example : $\begin{bmatrix} 2 & 4 & 6 \\ 0 & 3 & 5 \\ 0 & 0 & 9 \end{bmatrix}$ is an upper triangular matrix.

(*ii*) *A square matrix* $A = [a_{ij}]$ *is said to be* **lower triangular matrix** if $a_{ij} = 0$ *for* $i < j$.

For Example : $\begin{bmatrix} 5 & 0 & 0 \\ 6 & 2 & 0 \\ 3 & 4 & 1 \end{bmatrix}$ is a lower triangular matrix.

(*e*) Matrix Polynomial.

Let $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + ... + a_{n-1}x + a_n$ *be a polynomial and A be a square matrix of order n. Then* $f(A) = a_0A^n + a_1A^{n-1} + a_2A^{n-2} + ... + a_{n-1}A + a_n I$ *is called a matrix polynomial.*

For Example : If $f(x) = x^2 - 5x + 3$, then

$f(A) = A^2 - 5A + 3I$, where A is a square matrix.

3.3. EQUALITY OF MATRICES

Two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are said to be *equal* if and only if :

(*i*) *they are of the same order*

(*i.e.* they have the same number of rows and the same number of columns)

(*ii*) *the elements in the corresponding positions are equal (i.e.* $a_{ij} = b_{ij}$ for all i, j).

For Examples : (I) $[a \quad b] = [3 \quad 4]$ if and only if $a = 3$ and $b = 4$

(II) $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ if and only if $a = 1$, $b = 2$, $c = 3$ and $d = 4$.

Frequently Asked Questions

FAQs

Example 1. Write the element a_{23} of a 3 × 3 matrix A = $[a_{ij}]$ whose elements a_{ij} are given by :

$$a_{ij} = \frac{|i-j|}{2}.$$ *(C.B.S.E. 2015)*

Solution. We have : $a_{ij} = \frac{|i-j|}{2}$.

$\therefore$ $a_{23} = \frac{|2-3|}{2} = \frac{|-1|}{2} = \frac{1}{2}$.

Example 2. If matrix A = $[a_{ij}]_{3\times2}$ and $a_{ij} = (3i - 2j)^2$, then find matrix A. *(P.B. 2018)*

Solution.

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$

$$= \begin{bmatrix} (3-2)^2 & (3-4)^2 \\ (6-2)^2 & (6-4)^2 \\ (9-2)^2 & (9-4)^2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ 16 & 4 \\ 49 & 25 \end{bmatrix}.$$

Example 3. If $\begin{bmatrix} x+y & 1 \\ 2y & 5 \end{bmatrix} = \begin{bmatrix} 7 & 1 \\ 4 & 5 \end{bmatrix}$, find 'x'. *(C.B.S.E. 2010 C)*

Solution. We have : $\begin{bmatrix} x+y & 1 \\ 2y & 5 \end{bmatrix} = \begin{bmatrix} 7 & 1 \\ 4 & 5 \end{bmatrix}$.

Comparing, $x + y = 7$ ....(1)

and $2y = 4$ ....(2)

From (2), $y = 2$.

Putting in (1), $x + 2 = 7 \Rightarrow x = 7 - 2$.

Hence, $x = 5$.

Example 4. If $\begin{bmatrix} x-y & z \\ 2x-y & w \end{bmatrix} = \begin{bmatrix} -1 & 4 \\ 0 & 5 \end{bmatrix}$, find the value of x + y. *(A.I.C.B.S.E. 2014)*

Solution. We have : $\begin{bmatrix} x-y & z \\ 2x-y & w \end{bmatrix} = \begin{bmatrix} -1 & 4 \\ 0 & 5 \end{bmatrix}$.

Comparing the corresponding elements of first column, we get :

$x - y = -1$ ...(1) $2x - y = 0$ ...(2)

Subtracting (1) from (2), $x = 1$.

Putting in (1), $1 - y = -1 \Rightarrow y = 2$.

Hence, $x + y = 1 + 2 = 3$.

Example 5. If A = $\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$, then for what value of 'α' is A an identity matrix ? *(C.B.S.E. 2010)*

Solution. Here $A = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$.

Now $A = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ when :

$\cos\alpha = 1$ and $\sin\alpha = 0$.

Hence, $\alpha = 0$.

Example 6. Find the values of a, b, c and d from the following equation :

$$\begin{bmatrix} 2a+b & a-2b \\ 5c-d & 4c+3d \end{bmatrix} = \begin{bmatrix} 4 & -3 \\ 11 & 24 \end{bmatrix}.$$

(N.C.E.R.T.; H.P.B. 2010 S)

Solution. We have :

$$\begin{bmatrix} 2a+b & a-2b \\ 5c-d & 4c+3d \end{bmatrix} = \begin{bmatrix} 4 & -3 \\ 11 & 24 \end{bmatrix}.$$

Comparing the corresponding elements of two given matrices, we get :

$2a + b = 4$ ...(1) $a - 2b = -3$ ...(2)

$5c - d = 11$ ...(3) $4c + 3d = 24$ ...(4)

Solving (1) and (2) :

From (1), $b = 4 - 2a$ ...(5)

Putting in (2), $a - 2(4 - 2a) = -3$

$\Rightarrow$ $a - 8 + 4a = -3 \Rightarrow 5a = 5 \Rightarrow a = 1$.

Putting in (5), $b = 4 - 2(1) = 4 - 2 = 2$.

Solving (3) and (4) :

From (3), $d = 5c - 11$ ....(6)

Putting in (4), $4c + 3(5c - 11) = 24$

$\Rightarrow$ $4c + 15c - 33 = 24 \Rightarrow 19c = 57 \Rightarrow c = 3$.

Putting in (6), $d = 5(3) - 11 = 15 - 11 = 4$.

Hence, $a = 1, b = 2, c = 3$ and $d = 4$.

EXERCISE 3 (a)

Fast Track Answer Type Questions

FTATQ

1. Consider the following information regarding the number of men and women workers in three factories I, II and III :

	Men Workers	Women Workers
I	30	25
II	25	31
III	27	26.

Represent the information in the form of a 3 × 2 matrix. What does the entry in the third row and second column represent ? *(N.C.E.R.T.)*

2. (*a*) If a matrix has 8 elements, what are the possible orders it can have ? *(N.C.E.R.T.)*

(*b*) If a matrix has 24 elements, what are possible orders it can have ? What, if it has 13 elements ? *(N.C.E.R.T.; H.P.B. 2010 S)*

3. Name the square matrix $A = [a_{ij}]$ in which $a_{ij} = 0, i \neq j$. *(Uttarakhand B. 2015)*

4. If $[5\ 6\ 7]A = [13\ 23]$, what is the order of the matrix A ? *(Assam B. 2015)*

5. (*a*) Write the element a_{12} of the matrix $A = [a_{ij}]_{2\times 2}$, whose elements a_{ij} are given by :

$$a_{ij} = e^{2ix} \sin jx.$$ *(A.I.C.B.S.E. 2015)*

(*b*) For a 2×2 matrix, $A = [a_{ij}]$, whose elements are given by :

$a_{ij} = \frac{i}{j}$, write the value of a_{12}. *(C.B.S.E. 2011)*

6. (*a*) (*i*) Find the value of 'x', if :

$$\begin{bmatrix} 3x + y & -y \\ 2y - x & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -5 & 3 \end{bmatrix}$$

(A.I.C.B.S.E. 2009)

(*ii*) $\begin{bmatrix} x+2y & 5 \\ -y & 3 \end{bmatrix} = \begin{bmatrix} 7 & 5 \\ -2 & 3 \end{bmatrix}$ *(Kashmir B. 2013)*

(*b*) Find the value of 'y', if :

$$\begin{bmatrix} y + 2x & 5 \\ -x & 3 \end{bmatrix} = \begin{bmatrix} 7 & 5 \\ -2 & 3 \end{bmatrix}$$ *(Jammu B. 2012)*

(*c*) Find the values of 'x' and 'y' when :

$$\begin{bmatrix} x+2y & 3y \\ 4x & 2 \end{bmatrix} = \begin{bmatrix} 0 & -3 \\ 8 & 2 \end{bmatrix}$$

(H.B. 2012, 09)

(*d*) Find the value of 'a' if :

$$\begin{bmatrix} a-b & 2a+c \\ 2a-b & 3c+d \end{bmatrix} = \begin{bmatrix} -1 & 5 \\ 0 & 13 \end{bmatrix}.$$ *(C.B.S.E. 2013)*

(*e*) Write the value of $x - y + z$ from the following equation :

$$\begin{bmatrix} x + y + z \\ x + z \\ y + z \end{bmatrix} = \begin{bmatrix} 9 \\ 5 \\ 7 \end{bmatrix}.$$

(Kashmir B. 2016; C.B.S.E. (F) 2011)

7. If A is a square matrix of order m, and if there exists another square matrix B of the same order m, such that AB = BA = I, then B is called the .... **(Fill in the blank)** *(Kashmir B. 2012)*

Very Short Answer Type Questions

VSATQ

8. Construct a 2×2 matrix $A = [a_{ij}]$ whose elements are given by :

(*i*) $a_{ij} = \frac{i}{j}$ *(N.C.E.R.T.; Karnataka B. 2017; H.P.B. 2013 S, 10 S)*

(*ii*) $a_{ij} = \frac{2i - j}{3}$ *(Bihar B. 2014)*

(*iii*) $a_{ij} = \frac{(i + j)^2}{2}$ *(N.C.E.R.T. ; Jammu B. 2015 W; H.P.B. 2013 S)*

(*iv*) $a_{ij} = \frac{(i + 2j)^2}{2}$ *(H.P.B. 2016)*

(*v*) $a_{ij} = \frac{1}{3}|(2i + 3j|$. *(Nagaland B. 2018)*

9. (*a*) Construct a 2×3 matrix whose elements in the ith row and jth column are given by :

(*i*) $a_{ij} = \frac{(i + j)^2}{2}$ *(H.P.B. 2010)*

(*ii*) $a_{ij} = \begin{cases} i - j, & \text{if } i \geq j \\ i + j, & \text{if } i < j. \end{cases}$ *(J & K. B. 2010)*

(*b*) Construct a 3×2 matrix whose elements in the ith row and jth column are given by :

(*i*) $a_{ij} = \frac{i + 4j}{2}$ (*ii*) $a_{ij} = \frac{(i + 2j)^2}{2}$

(*iii*) $a_{ij} = \frac{1}{2}|i - 3j|$. *(N.C.E.R.T.; H.P.B. 2016, 10)*

10. (*a*) Construct a 3×3 matrix whose elements a_{ij} are given by :

(*i*) $a_{ij} = i + j$ (*ii*) $a_{ij} = i \times j$

(*iii*) $a_{ij} = (i + j)^2$.

(*b*) Construct a 3×4 matrix whose elements are given by :

(*i*) $a_{ij} = 2i - j$ *(N.C.E.R.T.; H.P.B. 2010)*

(*ii*) $a_{ij} = \frac{1}{2}|-3i + j|$. *(H.P.B. 2016, 10 S)*

11. Find the values of 'x' and 'y' from the following matrix equation :

$$\begin{bmatrix} 2x+1 & 2y \\ 0 & y^2 - 5y \end{bmatrix} = \begin{bmatrix} x+3 & y^2 + 2 \\ 0 & -6 \end{bmatrix}.$$

12. Find the values of x, y and z from the following matrix equations :

(*i*) $\begin{bmatrix} 4 & 3 \\ x & 5 \end{bmatrix} = \begin{bmatrix} y & z \\ 1 & 5 \end{bmatrix}$ *(N.C.E.R.T.)*

(*ii*) $\begin{bmatrix} x+y & 2 \\ 5+z & xy \end{bmatrix} = \begin{bmatrix} 6 & 2 \\ 5 & 8 \end{bmatrix}$ *(N.C.E.R.T.)*

(*iii*) $\begin{bmatrix} x + y + z \\ x + z \\ y + z \end{bmatrix} = \begin{bmatrix} 9 \\ 5 \\ 7 \end{bmatrix}$. *(N.C.E.R.T.; H.P.B. 2011)*

13. Find values of a, b, c and d from the equation :

$$\begin{bmatrix} a-b & 2a+c \\ 2a-b & 3c+d \end{bmatrix} = \begin{bmatrix} -1 & 5 \\ 0 & 13 \end{bmatrix}.$$ *(N.C.E.R.T.)*

14. If $\begin{bmatrix} x+3 & z+4 & 2y-7 \\ 4x+6 & a-1 & 0 \\ b-3 & 3b & z+2c \end{bmatrix}$

$$= \begin{bmatrix} 0 & 6 & 3y-2 \\ 2x & -3 & 2c+2 \\ 2b+4 & -21 & 0 \end{bmatrix},$$

obtain the values of a, b, c; x, y and z.

(N.C.E.R.T.)

Answers

1. $A = \begin{bmatrix} 30 & 25 \\ 25 & 31 \\ 27 & 26 \end{bmatrix}$;

Number of women workers in factory III.

2. (a) $1 \times 8, 8 \times 1, 2 \times 4, 4 \times 2$

(b) $1 \times 24, 24 \times 1, 2 \times 12, 12 \times 2, 3 \times 8, 8 \times 3, 4 \times 6,$
6×4 ; $1 \times 13, 13 \times 1$.

3. Diagonal matrix.

4. 3×2 matrix.

5. (a) $e^{2x} \sin 2x$ (b) $\frac{1}{2}$.

6. (a) (i) $x = 1$ (ii) $x = 3$ (b) $y = 3$

(c) $x = 2, y = -1$ (d) $a = 1$ (e) 1.

7. Inverse of A.

8. (i) $\begin{bmatrix} 1 & 1/2 \\ 2 & 1 \end{bmatrix}$ (ii) $\begin{bmatrix} 1/3 & 0 \\ 1 & 2/3 \end{bmatrix}$

(iii) $\begin{bmatrix} 2 & 9/2 \\ 9/2 & 8 \end{bmatrix}$ (iv) $\begin{bmatrix} 9/2 & 25/2 \\ 8 & 18 \end{bmatrix}$

(v) $\begin{bmatrix} 1/3 & 4/3 \\ 1/3 & 2/3 \end{bmatrix}$.

9. (a) (i) $\begin{bmatrix} 2 & 9/2 & 8 \\ 9/2 & 8 & 25/2 \end{bmatrix}$ (ii) $\begin{bmatrix} 0 & 3 & 4 \\ 1 & 0 & 5 \end{bmatrix}$

(b) (i) $\begin{bmatrix} 5/2 & 9/2 \\ 3 & 5 \\ 7/2 & 11/2 \end{bmatrix}$ (ii) $\begin{bmatrix} 9/2 & 25/2 \\ 8 & 18 \\ 25/2 & 49/2 \end{bmatrix}$

(iii) $\begin{bmatrix} 1 & 5/2 \\ 1/2 & 2 \\ 0 & 3/2 \end{bmatrix}$.

10. (a) (i) $\begin{bmatrix} 2 & 3 & 4 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$

(iii) $\begin{bmatrix} 4 & 9 & 16 \\ 9 & 16 & 25 \\ 16 & 25 & 36 \end{bmatrix}$

(b) (i) $\begin{bmatrix} 1 & 0 & -1 & -2 \\ 3 & 2 & 1 & 0 \\ 5 & 4 & 3 & 2 \end{bmatrix}$

(ii) $\begin{bmatrix} 1 & 1/2 & 0 & 1/2 \\ 5/2 & 2 & 3/2 & 1 \\ 4 & 7/2 & 3 & 5/2 \end{bmatrix}$.

11. Given matrices can't be equal for any value of y

12. (i) $x = 1, y = 4, z = 3$

(ii) $x = 4, y = 2, z = 0$ ***OR*** $x = 2, y = 4, z = 0$

(iii) $x = 2, y = 4, z = 3$.

13. $a = 1, b = 2, c = 3, d = 4$.

14. $a = -2, b = -7, c = -1; x = -3, y = -5, z = 2$.

Hints to Selected Questions

6. (e) Comparing, $x + y + z = 9$ ...(1)

$x + z = 5$ ...(2)

and $y + z = 7$ ...(3)

Solving, $x = 2, y = 4$ and $z = 3$.

$\therefore$ $x - y + z = 2 - 4 + 3 = 1$.

8. (v) $a_{11} = \frac{1}{2}|2-3| = \frac{1}{2}(1) = \frac{1}{2}$,

$a_{12} = \frac{1}{2}|2-6| = \frac{1}{2}(4) = 2$

and $a_{21} = \frac{1}{2}|4-3| = \frac{1}{2}(1) = \frac{1}{2}$,

$a_{22} = \frac{1}{2}|4-6| = \frac{1}{2}(2) = 1$.

Hence, $A = \begin{bmatrix} 1/2 & 2 \\ 1/2 & 1 \end{bmatrix}$.

9. (a) (ii) $a_{11} = 1 - 1 = 0; a_{12} = 1 + 2 = 3; a_{13} = 1 + 3 = 4$

and $a_{21} = 2 - 1 = 1; a_{22} = 2 - 2 = 0; a_{23} = 2 + 3 = 5$.

Hence, $A = \begin{bmatrix} 0 & 3 & 4 \\ 1 & 0 & 5 \end{bmatrix}$.

13. Comparing, $a - b = -1, 2a + c = 5, 2a - b = 0$ and $3c + d = 13$. Solve.

14. Comparing, $x + 3 = 0, z + 4 = 6, 2y - 7 = 3y - 2$; etc.

Solving, $x = -3, y = -5, z = 2$; etc.

3.4. OPERATIONS ON MATRICES

Now we shall define the following three operations on matrices :

(A) Addition of matrices (B) Multiplication of matrix by a scalar (C) Multiplication of matrices.

Let us deal with one by one.

(A) Operation I. Addition of Matrices.

(*i*) Suppose Ajay has two factories at places A and B. Each factory produces clothing for boys and girls in three price styles, labelled 1, 2 and 3. The quantities (in standard units) produced at each factory are given in the matrices as shown below :

$$\begin{array}{c} \\ \text{Boys} \rightarrow \\ \text{Girls} \rightarrow \end{array}\begin{array}{c} \textit{Factory at A} \\ \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 71 & 69 & 73 \\ 50 & 63 & 45 \end{bmatrix} \end{array} \qquad \begin{array}{c} \textit{Factory at B} \\ \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 40 & 33 & 37 \\ 31 & 25 & 35 \end{bmatrix} \end{array}\begin{array}{c} \\ \leftarrow \text{Boys} \\ \leftarrow \text{Girls} \end{array}$$

In order to find the total production of clothing in each style, we have :

	Style 1	*Style 2*	*Style 3*
Boys →	(71 + 40)	(69 + 33)	(73 + 37)
Girls →	(50 + 31)	(63 + 25)	(45 + 35).

This can be represented in the following matrix form :

$$\begin{bmatrix} 71+40 & 69+33 & 73+37 \\ 50+31 & 63+25 & 45+35 \end{bmatrix}.$$

The above matrix is the *sum* of first two matrices. It is clearly observed that the sum of two matrices is a matrix, which is obtained by adding the corresponding elements of the given matrices. Moreover, the sum is defined if the matrices are of the same order. Formally, we have the following definition :

Definition

Let $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{m \times n}$ be two matrices. Then their sum (denoted by $A + B$) is defined to be the matrix $[c_{ij}]_{m \times n}$, where :
$c_{ij} = a_{ij} + b_{ij}$ for $1 \le i \le m,\ 1 \le j \le n$.

For Example :

Let $A = \begin{bmatrix} 1 & 2 & -3 \\ 4 & -5 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 7 & -8 & 9 \\ 2 & 8 & -4 \end{bmatrix}$, then $A + B = \begin{bmatrix} 1+7 & 2-8 & -3+9 \\ 4+2 & -5+8 & 6-4 \end{bmatrix} = \begin{bmatrix} 8 & -6 & 6 \\ 6 & 3 & 2 \end{bmatrix}$.

KEY POINT

Addition is defined only for those matrices, which are of the same type.

If two matrices are of the same order, they are said to be **conformable for addition.**

If two matrices are not of the same order, we cannot find their sum.

For Example : If $A = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 4 & 5 \\ 1 & 2 & 3 \end{bmatrix}$, then $A + B$ is not defined. [$\because$ *A and B are of different orders*]

(*ii*) **Properties of Matrix addition.**

(I) Matrix addition is Commutative. *If A, B are any two matrices of order m × n, then A + B = B + A.*

Let $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{m \times n}$

LHS $= A + B = [a_{ij}]_{m \times n} + [b_{ij}]_{m \times n} = [a_{ij} + b_{ij}]_{m \times n}$ [*Def.*]

$= [b_{ij} + a_{ij}]_{m \times n}$ [$\because$ *Addition in number system is commutative*]

$= [b_{ij}]_{m \times n} + [a_{ij}]_{m \times n} = B + A =$ **RHS.**

(II) Matrix addition is Associative. *If A, B, C are any three matrices of order m × n, then :*

$$(A + B) + C = A + (B + C).$$

Let $A = [a_{ij}]_{m \times n}$, $B = [b_{ij}]_{m \times n}$ and $C = [c_{ij}]_{m \times n}$.

LHS $= (A + B) + C = ([a_{ij}]_{m \times n} + [b_{ij}]_{m \times n}) + [c_{ij}]_{m \times n} = [a_{ij} + b_{ij}]_{m \times n} + [c_{ij}]_{m \times n}$ [*Def.*]

$= [(a_{ij} + b_{ij}) + c_{ij}]_{m \times n}$ [*Def.*]

$= [a_{ij} + (b_{ij} + c_{ij})]_{m \times n}$ [$\because$ *Addition in number system is associative*]

$= [a_{ij}]_{m \times n} + ([b_{ij}]_{m \times n} + [c_{ij}]_{m \times n}) = A + (B + C) =$ **RHS.**

(III) Existence of additive Identity. *If O be the null matrix of order m × n, then :*

$A + O = O + A = A$, *where A is any matrix of the same order m × n.*

Let $A = [a_{ij}]_{m \times n}$.

$\therefore$ $A + O = [a_{ij}]_{m \times n} + [0]_{m \times n} = [a_{ij} + 0]_{m \times n} = [a_{ij}]_{m \times n} = A$

and $O + A = [0]_{m \times n} + [a_{ij}]_{m \times n} = [0 + a_{ij}]_{m \times n} = [a_{ij}]_{m \times n} = A.$

Hence, $A + O = O + A = A.$

KEY POINT

The zero matrix O is the ***additive identity,*** *which is unique.*

(IV) Existence of additive Inverse. *Let A be a matrix of the order m × n. There exists a matrix X of the same order m × n such that A + X = O = X + A.*

Let $A = [a_{ij}]_{m \times n}$ and $X = [x_{ij}]_{m \times n}$.

$\therefore$ $A + X = [a_{ij}]_{m \times n} + [x_{ij}]_{m \times n} = [a_{ij} + x_{ij}]_{m \times n}$ [*Def.*]

If $A + X = O$, then $a_{ij} + x_{ij} = 0 \ \forall i, j$

$\Rightarrow$ $x_{ij} = -a_{ij}$ $\Rightarrow$ x_{ij} exists and is unique.

Thus $X = [-a_{ij}]_{m \times n}$.

Similarly $X + A = O$ gives : $X = [-a_{ij}]_{m \times n}$.

Thus if $A = [a_{ij}]_{m \times n}$ is given, there exists unique matrix $[-a_{ij}]_{m \times n}$ s.t.

$[a_{ij}]_{m \times n} + [-a_{ij}]_{m \times n} = O.$

The matrix $[-a_{ij}]_{m \times n}$ is the **negative** or **inverse** of A and is written as $-A$.

Hence, $A + (-A) = O = (-A) + A.$

(*iii*) **Difference of Matrices.**

Definition

Let A and B be two matrices of the same order m × n. Then the difference of A and B, denoted by A – B, is the sum of A and the negative of B.
Thus A – B = A + (– B).

Note. The difference $A - B$ is obtained by subtracting from each element of A the corresponding element of B.

(*iv*) **Cancellation Laws.** *If A, B, C are the matrices of order m × n, then :*

$A + B = A + C \Rightarrow B = C$ [*Left Cancellation*]

and $B + A = C + A \Rightarrow B = C.$ [*Right Cancellation*]

Now $A + B = A + C \Rightarrow -A + (A + B) = -A + (A + C)$ [*Adding – A to both sides*]

$\Rightarrow (-A + A) + B = (-A + A) + C$ [*Associative Property*]

$\Rightarrow O + B = O + C$ [$\because -A + A = O$]

$\Rightarrow B = C.$

Similarly $B + A = C + A \Rightarrow B = C.$

Frequently Asked Questions

FAQs

Example 1. If $\begin{bmatrix} 9 & -1 & 4 \\ -2 & 1 & 3 \end{bmatrix} = A + \begin{bmatrix} 1 & 2 & -1 \\ 0 & 4 & 9 \end{bmatrix}$, **then find the matrix A.** *(C.B.S.E. 2013)*

Solution. Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$.

Then $\begin{bmatrix} 9 & -1 & 4 \\ -2 & 1 & 3 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} + \begin{bmatrix} 1 & 2 & -1 \\ 0 & 4 & 9 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 9 & -1 & 4 \\ -2 & 1 & 3 \end{bmatrix} = \begin{bmatrix} a_{11}+1 & a_{12}+2 & a_{13}-1 \\ a_{21}+0 & a_{22}+4 & a_{23}+9 \end{bmatrix}$.

Comparing, $9 = a_{11}+1$, $-1 = a_{12}+2$, $4 = a_{13}-1$,

$-2 = a_{21}$, $1 = a_{22}+4$, $3 = a_{23}+9$

$\Rightarrow a_{11} = 8$, $a_{12} = -3$, $a_{13} = 5$,

$a_{21} = -2$, $a_{22} = -3$, $a_{23} = -6$.

Hence, $A = \begin{bmatrix} 8 & -3 & 5 \\ -2 & -3 & -6 \end{bmatrix}$.

Example 2. If $A = \begin{bmatrix} 2 & 2 \\ -3 & 1 \\ 4 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 6 & 2 \\ 1 & 3 \\ 0 & 4 \end{bmatrix}$, **find the matrix C such that A + B + C is a zero matrix.**

Solution. Let $C = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \\ c_{31} & c_{32} \end{bmatrix}$.

Then A + B + C = O

$\Rightarrow \begin{bmatrix} 2 & 2 \\ -3 & 1 \\ 4 & 0 \end{bmatrix} + \begin{bmatrix} 6 & 2 \\ 1 & 3 \\ 0 & 4 \end{bmatrix} + \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \\ c_{31} & c_{32} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 2+6 & 2+2 \\ -3+1 & 1+3 \\ 4+0 & 0+4 \end{bmatrix} + \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \\ c_{31} & c_{32} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 2+6+c_{11} & 2+2+c_{12} \\ -3+1+c_{21} & 1+3+c_{22} \\ 4+0+c_{31} & 0+4+c_{32} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 8+c_{11} & 4+c_{12} \\ -2+c_{21} & 4+c_{22} \\ 4+c_{31} & 4+c_{32} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$.

Comparing :

$8 + c_{11} = 0 \Rightarrow c_{11} = -8$,

$4 + c_{12} = 0 \Rightarrow c_{12} = -4$,

$-2 + c_{21} = 0 \Rightarrow c_{21} = 2$,

$4 + c_{22} = 0 \Rightarrow c_{22} = -4$,

$4 + c_{31} = 0 \Rightarrow c_{31} = -4$

and $4 + c_{32} = 0 \Rightarrow c_{32} = -4$.

Hence, $C = \begin{bmatrix} -8 & -4 \\ 2 & -4 \\ -4 & -4 \end{bmatrix}$.

Example 3. Mahesh has three factories at A, B and C places respectively. Each factory produces clothing for boys and girls in three price styles labelled 1, 2, 3. The quantities produced by each factory are given in the matrices shown :

Factory at 'A' S Factory at 'B' Factory at 'C'

$$\begin{array}{l} \\ \textbf{Boys} \rightarrow \\ \textbf{Girls} \rightarrow \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 75 & 70 & 72 \\ 60 & 65 & 40 \end{bmatrix} \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 80 & 84 & 72 \\ 65 & 69 & 54 \end{bmatrix} \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 25 & 35 & 40 \\ 30 & 22 & 36 \end{bmatrix} \end{array}.$$

Find the total production of clothing in each style for boys and girls.

Solution. The required total production of clothing requires the addition of three matrices, *i.e.*

$$\begin{bmatrix} 75 & 70 & 72 \\ 60 & 65 & 40 \end{bmatrix} + \begin{bmatrix} 80 & 84 & 72 \\ 65 & 69 & 54 \end{bmatrix} + \begin{bmatrix} 25 & 35 & 40 \\ 30 & 22 & 36 \end{bmatrix}$$

$$= \begin{bmatrix} 75+80 & 70+84 & 72+72 \\ 60+65 & 65+69 & 40+54 \end{bmatrix} + \begin{bmatrix} 25 & 35 & 40 \\ 30 & 22 & 36 \end{bmatrix}$$

$$= \begin{bmatrix} 75+80+25 & 70+84+35 & 72+72+40 \\ 60+65+30 & 65+69+22 & 40+54+36 \end{bmatrix}$$

$$= \begin{bmatrix} 180 & 189 & 184 \\ 155 & 156 & 130 \end{bmatrix}.$$

Hence, the total production of clothing in each style for boys and girls is given by :

$$\begin{array}{l} \\ \text{Boys} \rightarrow \\ \text{Girls} \rightarrow \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 180 & 189 & 184 \\ 155 & 156 & 130 \end{bmatrix} \end{array}.$$

EXERCISE 3 (b)

Fast Track Answer Type Questions

FTATQ

1. If A is of order $m \times n$ and B is also of order $m \times n$; then (A + B) is a matrix of order.......... **(Fill in the Blank)**

(Jammu B. 2017)

2. (*a*) Compute the following :

(*i*) $\begin{bmatrix} 2 & -1 \\ 4 & 2 \end{bmatrix} + \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$ *(H.B. 2012)*

(*ii*) $\begin{bmatrix} 0 & 1 & 5 \\ -3 & 2 & 1 \end{bmatrix} + \begin{bmatrix} 6 & 2 & -3 \\ -1 & 4 & 2 \end{bmatrix}$.

(*b*) (*i*) $\begin{bmatrix} a & b \\ -b & a \end{bmatrix} + \begin{bmatrix} a & b \\ b & a \end{bmatrix}$ *(N.C.E.R.T.)*

(*ii*) $\begin{bmatrix} \cos^2 x & \sin^2 x \\ \sin^2 x & \cos^2 x \end{bmatrix} + \begin{bmatrix} \sin^2 x & \cos^2 x \\ \cos^2 x & \sin^2 x \end{bmatrix}$ *(N.C.E.R.T.)*

(*iii*) $\begin{bmatrix} \sin(\theta+\phi) & \cos(\theta+\phi) \\ \sin(\theta-\phi) & \cos(\theta-\phi) \end{bmatrix} + \begin{bmatrix} \sin(\theta-\phi) & \cos(\theta-\phi) \\ \sin(\theta+\phi) & \cos(\theta+\phi) \end{bmatrix}$.

3. Compute the following :

$\begin{bmatrix} -1 & 4 & -6 \\ 8 & 5 & 16 \\ 2 & 8 & 5 \end{bmatrix} + \begin{bmatrix} 12 & 7 & 6 \\ 8 & 0 & 5 \\ 3 & 2 & 4 \end{bmatrix}$. *(N.C.E.R.T.)*

4. If $A = \begin{bmatrix} 2 & -1 \\ 4 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 3 \\ -2 & 1 \end{bmatrix}$,

$C = \begin{bmatrix} -2 & -3 \\ -1 & -2 \end{bmatrix}$, find the value of A + B + C.

5. Does the sum $\begin{bmatrix} 5 & 3 & 2 \\ 2 & 5 & 3 \\ 5 & 2 & 3 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ make sense ?

If so, find the sum and if not, give the reason.

Very Short Answer Type Questions

VSATQ

6. (*i*) If $A = \begin{bmatrix} 1 & -3 & 2 \\ 2 & 0 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 2 & -1 & -1 \\ 1 & 0 & -1 \end{bmatrix}$, find the matrix C such that A + B + C is a zero matrix.

(*ii*) Given $X = \begin{bmatrix} 2 & 0 & 2 \\ 1 & 0 & -1 \end{bmatrix}$ and $Y = \begin{bmatrix} 3 & -1 & 0 \\ -2 & 0 & -1 \end{bmatrix}$, find Z such that

X + Y + Z = O. *(Meghalaya B. 2013)*

Short Answer Type Questions

SATQ

7. Verify associative law of matrix addition for the matrices :

$$A = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}, B = \begin{bmatrix} 3 & 7 \\ 4 & 8 \end{bmatrix} \text{ and } C = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}.$$

8. If $A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 2 \\ 1 & -3 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 5 & 6 \\ -1 & 0 & 1 \\ 2 & 1 & 2 \end{bmatrix}$

and $C = \begin{bmatrix} -1 & -2 & 1 \\ -1 & 2 & 3 \\ -1 & -2 & 2 \end{bmatrix}$, verify that :

A + (B + C) = (A + B) + C.

9. If A and B are two $m \times n$ matrices and O is the null matrix of the type $m \times n$, then show that :

A + B = O ⇒ A = – B and B = – A.

10. A retail wool store stocks three kinds of wool : ordinary, standard and deluxe. These wools come in three colours : yellow, green and red. The store's weekly sales are given in the matrix form as :

$$\begin{array}{cc} & \begin{array}{ccc} \textit{Yellow} & \textit{Green} & \textit{Red} \end{array} \\ P = & \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{bmatrix} \begin{array}{l} \to \text{Ordinary} \\ \to \text{Standard} \\ \to \text{Deluxe,} \end{array} \end{array}$$

where for instance p_{31} denotes the number of kgs. of deluxe yellow wool that was sold during the week.

Suppose the store's inventory at the beginning of the week is given by :

$$\begin{array}{cc} & \begin{array}{ccc} \textit{Yellow} & \textit{Green} & \textit{Red} \end{array} \\ Q = & \begin{bmatrix} q_{11} & q_{12} & q_{13} \\ q_{21} & q_{22} & q_{23} \\ q_{31} & q_{32} & q_{33} \end{bmatrix} \begin{array}{l} \to \text{Ordinary} \\ \to \text{Standard} \\ \to \text{Deluxe.} \end{array} \end{array}$$

Find the inventory at the end of the week.

Answers

1. $m \times n$

2. (a) (i) $\begin{bmatrix} 4 & 2 \\ 5 & 4 \end{bmatrix}$ (ii) $\begin{bmatrix} 6 & 3 & 2 \\ -4 & 6 & 3 \end{bmatrix}$.

(b) (i) $\begin{bmatrix} 2a & 2b \\ 0 & 2a \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

(iii) $\begin{bmatrix} 2\sin\theta\cos\phi & 2\cos\theta\cos\phi \\ 2\sin\theta\cos\phi & 2\cos\theta\cos\phi \end{bmatrix}$.

3. $\begin{bmatrix} 11 & 11 & 0 \\ 16 & 5 & 21 \\ 5 & 10 & 9 \end{bmatrix}$. 4. $\begin{bmatrix} 4 & -1 \\ 1 & 1 \end{bmatrix}$. 5. No.

6. (i) $\begin{bmatrix} -3 & 4 & -1 \\ -3 & 0 & -1 \end{bmatrix}$ (ii) $\begin{bmatrix} -5 & 1 & -2 \\ 1 & 0 & 2 \end{bmatrix}$.

10. $\begin{bmatrix} q_{11}-p_{11} & q_{12}-p_{12} & q_{13}-p_{13} \\ q_{21}-p_{21} & q_{22}-p_{22} & q_{23}-p_{23} \\ q_{31}-p_{31} & q_{32}-p_{32} & q_{33}-p_{33} \end{bmatrix}$.

Hints to Selected Questions

7. Verify : A + (B + C) = (A + B) + C.

9. A + B = O

$\Rightarrow [a_{ij}]_{m \times n} + [b_{ij}]_{m \times n} = [0]_{m \times n}$

$\Rightarrow [a_{ij} + b_{ij}]_{m \times n} = [0]_{m \times n}$

$\Rightarrow a_{ij} + b_{ij} = 0 \Rightarrow a_{ij} = -b_{ij}$.

Now $A = [a_{ij}]_{m \times n} = [-b_{ij}]_{m \times n}$

$= -[b_{ij}]_{m \times n} = -B$; etc.

10. Inventory at the end of the week

$= Q - P$

$$= \begin{bmatrix} q_{11}-p_{11} & q_{12}-p_{12} & q_{13}-p_{13} \\ q_{21}-p_{21} & q_{22}-p_{22} & q_{23}-p_{23} \\ q_{31}-p_{31} & q_{32}-p_{32} & q_{33}-p_{33} \end{bmatrix}.$$

(B) Operation II. (*i*) (*a*) Multiplication of a Matrix by a Scalar.

Definition

*Let A be any m × n matrix and k be any scalar (i.e., an element of number system **K**). The m × n matrix obtained by multiplying each element of the matrix A by k is said to be the scalar multiple of A by k and is denoted by kA or Ak.*

In Symbols : If $A = [a_{ij}]_{m \times n}$, then $kA = Ak = [ka_{ij}]_{m \times n}$.

For Example : $\frac{1}{2}\begin{bmatrix} 1 & -1 & 2 \\ 2 & 4 & 6 \end{bmatrix} = \begin{bmatrix} 1/2 & -1/2 & 1 \\ 1 & 2 & 3 \end{bmatrix}$.

(*b*) **Negative of a matrix.** The negative of a matrix A is denoted by – A.

We define : – A = (– 1) A.

For Example : If $A = \begin{bmatrix} 3 & 4 \\ -4 & x \end{bmatrix}$, then $-A = (-1)A = (-1)\begin{bmatrix} 3 & 4 \\ -4 & x \end{bmatrix} = \begin{bmatrix} -3 & -4 \\ 4 & -x \end{bmatrix}$.

(*c*) **Difference of Matrices.** Let $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{m \times n}$.

Then (A – B) is defined as $D = [d_{ij}]_{m \times n}$, where $d_{ij} = a_{ij} - b_{ij}$.

Thus D = A – B = A + (– 1) B, which is the sum of matrix A and the matrix – B.

(*ii*) **Properties of Multiplication of a Matrix by a Scalar.**

(I) Scalar multiplication is distributive over matrix addition.

If A and B are two matrices of the same order m × n, then k (A + B) = kA + kB, where k is any scalar.

Let $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{m \times n}$.

LHS $= k(A + B) = k([a_{ij}]_{m \times n} + [b_{ij}]_{m \times n})$

$= [k(a_{ij} + b_{ij})]_{m \times n}$ [*Def. of addition of two matrices*]

$= [ka_{ij} + kb_{ij}]_{m \times n}$ [*Def. of scalar multiplication*]

$= [ka_{ij}]_{m \times n} + [kb_{ij}]_{m \times n}$ [*Distributive Law*]

$= k[a_{ij}]_{m \times n} + k[b_{ij}]_{m \times n} = kA + kB =$ **RHS.**

(II) *If A is any matrix of the type m × n, then $(k_1 + k_2)A = k_1A + k_2A$, where k_1, k_2 are scalars.*

Let $A = [a_{ij}]_{m \times n}$.

$$\text{LHS} = (k_1 + k_2)\,A = (k_1 + k_2)\,[a_{ij}]_{m \times n}$$
$$= [(k_1 + k_2)\,a_{ij}]_{m \times n} = [k_1 a_{ij} + k_2 a_{ij}]_{m \times n} = [k_1 a_{ij}]_{m \times n} + [k_2 a_{ij}]_{m \times n}$$
$$= k_1\,[a_{ij}]_{m \times n} + k_2\,[a_{ij}]_{m \times n} = k_1 A + k_2 A = \textbf{RHS.}$$

(III) *If k_1 and k_2 are two scalars and A is any $m \times n$ matrix, then $k_1\,(k_2 A) = (k_1 k_2)\,A$.*

Let $A = [a_{ij}]_{\cdot m \times n}$

$$\text{LHS} = k_1\,(k_2 A) = k_1\,(k_2\,[a_{ij}]_{m \times n}) = k_1\,[k_2 a_{ij}]_{m \times n} = [k_1\,(k_2 a_{ij})]_{m \times n} = [(k_1 k_2)\,a_{ij}]_{m \times n}$$

[*Associative Law for real numbers under multiplication*]

$$= (k_1 k_2)\,[a_{ij}]_{m \times n} = (k_1 k_2)\,A = \textbf{RHS.}$$

(IV) *If A is any $m \times n$ matrix, then $(-k)\,A = -(kA) = k(-A)$, where k is any scalar.*

Let $A = [a_{ij}]_{\cdot m \times n}$

Then $(-k)\,A = -k\,[a_{ij}]_{m \times n} = [(-k)\,a_{ij}]_{m \times n} = [-(k a_{ij})]_{m \times n} = -(kA)$ ...(1)

Also $(-k)\,A = (-k)\,[a_{ij}]_{m \times n} = [(-k)\,a_{ij}]_{m \times n} = [k\,(-a_{ij})]_{m \times n} = k\,[-a_{ij}]_{m \times n} = k\,(-A)$ ...(2)

(1) and (2) $\Rightarrow$ $(-k)\,A = -(kA) = k\,(-A)$.

(V) *If A is any $m \times n$ matrix, then :*

(1) $IA = A$ (2) $(-1)\,A = -A$.

Let $A = [a_{ij}]_{\cdot m \times n}$

(1) $1A = 1\,[a_{ij}]_{m \times n} = [1 a_{ij}]_{m \times n} = [a_{ij}]_{\cdot m \times n}$

Hence, IA = A.

(2) $(-1)\,A = (-1)\,[a_{ij}]_{m \times n} = [(-1)\,a_{ij}]_{m \times n} = [-a_{ij}]_{m \times n} = -A$.

Hence, $(-1)\,A = -A$.

Frequently Asked Questions

FAQs

Example 1. Find the values of x, y, z and t, if :

$$2\begin{bmatrix} \mathbf{x} & \mathbf{z} \\ \mathbf{y} & \mathbf{t} \end{bmatrix} + 3\begin{bmatrix} \mathbf{1} & \mathbf{-1} \\ \mathbf{0} & \mathbf{2} \end{bmatrix} = 3\begin{bmatrix} \mathbf{3} & \mathbf{5} \\ \mathbf{4} & \mathbf{6} \end{bmatrix}.$$

(Meghalaya B. 2016)

Solution. We have : $2\begin{bmatrix} x & z \\ y & t \end{bmatrix} + 3\begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} = 3\begin{bmatrix} 3 & 5 \\ 4 & 6 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} 2x & 2z \\ 2y & 2t \end{bmatrix} + \begin{bmatrix} 3 & -3 \\ 0 & 6 \end{bmatrix} = \begin{bmatrix} 9 & 15 \\ 12 & 18 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2x+3 & 2z-3 \\ 2y & 2t+6 \end{bmatrix} = \begin{bmatrix} 9 & 15 \\ 12 & 18 \end{bmatrix}$$

$\Rightarrow$ $2x + 3 = 9$ ...(1) $2z - 3 = 15$ ...(2)

$2y = 12$ ...(3) $2t + 6 = 18$ ...(4)

From (1), $2x = 9 - 3 \Rightarrow 2x = 6 \Rightarrow x = 3$.

From (3), $2y = 12 \Rightarrow y = 6$.

From (2), $2z - 3 = 18 \Rightarrow 2z = 21 \Rightarrow z = \dfrac{21}{2}$.

From (4), $2t + 6 = 18 \Rightarrow 2t = 12 \Rightarrow t = 6$.

Hence, $x = 3$, $y = 6$, $z = \dfrac{21}{2}$ and $t = 6$.

Example 2. If $\mathbf{x}\begin{bmatrix} \mathbf{2} \\ \mathbf{3} \end{bmatrix} + \mathbf{y}\begin{bmatrix} \mathbf{-1} \\ \mathbf{1} \end{bmatrix} = \begin{bmatrix} \mathbf{10} \\ \mathbf{5} \end{bmatrix}$, **write the values of 'x' and 'y'.**

(Kerala B. 2018; Jammu B. 2017; Kashmir B. 2016; C.B.S.E. (F) 2012)

Solution. We have : $x\begin{bmatrix} 2 \\ 3 \end{bmatrix} + y\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 10 \\ 5 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} 2x \\ 3x \end{bmatrix} + \begin{bmatrix} -y \\ y \end{bmatrix} = \begin{bmatrix} 10 \\ 5 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2x - y \\ 3x + y \end{bmatrix} = \begin{bmatrix} 10 \\ 5 \end{bmatrix}.$$

Comparing, $2x - y = 10$ ...(1)

and $3x + y = 5$ ...(2)

Adding (1) and (2), $5x = 15$

$\Rightarrow$ $x = 3$.

Putting in (1), $2(3) - y = 10 \Rightarrow y = -4$

Hence, $x = 3$ and $y = -4$.

Example 3. If $2\begin{bmatrix} \mathbf{3} & \mathbf{4} \\ \mathbf{5} & \mathbf{x} \end{bmatrix} + \begin{bmatrix} \mathbf{1} & \mathbf{y} \\ \mathbf{0} & \mathbf{1} \end{bmatrix} = \begin{bmatrix} \mathbf{7} & \mathbf{0} \\ \mathbf{10} & \mathbf{5} \end{bmatrix}$, **find (x – y).**

(C.B.S.E. 2014)

Solution. We have : $2\begin{bmatrix} 3 & 4 \\ 5 & x \end{bmatrix} + \begin{bmatrix} 1 & y \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 0 \\ 10 & 5 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} 6 & 8 \\ 10 & 2x \end{bmatrix} + \begin{bmatrix} 1 & y \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 0 \\ 10 & 5 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 6+1 & 8+y \\ 10+0 & 2x+1 \end{bmatrix} = \begin{bmatrix} 7 & 0 \\ 10 & 5 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 7 & 8+y \\ 10 & 2x+1 \end{bmatrix} = \begin{bmatrix} 7 & 0 \\ 10 & 5 \end{bmatrix}.$$

Comparing, $8 + y = 0$, $2x + 1 = 5$

$\Rightarrow$ $y = -8$, $2x = 4$ *i.e.* $x = 2$.

Hence, $x - y = 2 - (-8) = 2 + 8 = 10$.

Example 4. Find the value of (x + y) from the following equation :

$$2\begin{bmatrix} x & 5 \\ 7 & y-3 \end{bmatrix} + \begin{bmatrix} 3 & -4 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 6 \\ 15 & 14 \end{bmatrix}.$$

(Kashmir B. 2017; A.I.C.B.S.E. 2012)

Solution. We have :

$$2\begin{bmatrix} x & 5 \\ 7 & y-3 \end{bmatrix} + \begin{bmatrix} 3 & -4 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 6 \\ 15 & 14 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} 2x & 10 \\ 14 & 2y-6 \end{bmatrix} + \begin{bmatrix} 3 & -4 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 6 \\ 15 & 14 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} 2x+3 & 10-4 \\ 14+1 & 2y-6+2 \end{bmatrix} = \begin{bmatrix} 7 & 6 \\ 15 & 14 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} 2x+3 & 6 \\ 15 & 2y-4 \end{bmatrix} = \begin{bmatrix} 7 & 6 \\ 15 & 14 \end{bmatrix}.$$

Comparing, $2x + 3 = 7$ and $2y - 4 = 14$

$\Rightarrow$ $2x = 4 \Rightarrow x = 2$ and $2y = 18 \Rightarrow y = 9$.

Hence, $x + y = 2 + 9 = 11$.

Example 5. If $A = \begin{bmatrix} 4 & 2 & 7 \\ 2 & 1 & 4 \end{bmatrix}$ **and** $B = \begin{bmatrix} 2 & 1 & 5 \\ 3 & 4 & 6 \end{bmatrix}$, **find 2A – B.** *(H.B. 2014)*

Solution. $2A - B = 2A + (-B) = 2A + (-1)B$

$$= 2\begin{bmatrix} 4 & 2 & 7 \\ 2 & 1 & 4 \end{bmatrix} + (-1)\begin{bmatrix} 2 & 1 & 5 \\ 3 & 4 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 4 & 14 \\ 4 & 2 & 8 \end{bmatrix} + \begin{bmatrix} -2 & -1 & -5 \\ -3 & -4 & -6 \end{bmatrix}$$

$$= \begin{bmatrix} 8-2 & 4-1 & 14-5 \\ 4-3 & 2-4 & 8-6 \end{bmatrix} = \begin{bmatrix} 6 & 3 & 9 \\ 1 & -2 & 2 \end{bmatrix}.$$

Example 6. If $A = \begin{bmatrix} 8 & 0 \\ 4 & -2 \\ 3 & 6 \end{bmatrix}$ **and** $B = \begin{bmatrix} 2 & -2 \\ 4 & 2 \\ -5 & 1 \end{bmatrix}$, **then find the matrix 'X', of order 3 × 2, such that :**

2A + 3X = 5B. *(N.C.E.R.T.)*

Solution. We have : $2A + 3X = 5B$

$\Rightarrow \quad 2A + 3X - 2A = 5B - 2A$

$\Rightarrow \quad 2A - 2A + 3X = 5B - 2A$

$\Rightarrow \quad (2A - 2A) + 3X = 5B - 2A$

$\Rightarrow \quad O + 3X = 5B - 2A$

[$\because$ *– 2A is the inverse of 2A*]

$\Rightarrow \quad 3X = 5B - 2A.$

[$\because$ *O is the additive identity*]

Hence, $X = \frac{1}{3}(5B - 2A)$

$$= \frac{1}{3}\left(5\begin{bmatrix} 2 & -2 \\ 4 & 2 \\ -5 & 1 \end{bmatrix} - 2\begin{bmatrix} 8 & 0 \\ 4 & -2 \\ 3 & 6 \end{bmatrix}\right)$$

$$= \frac{1}{3}\left(\begin{bmatrix} 10 & -10 \\ 20 & 10 \\ -25 & 5 \end{bmatrix} + \begin{bmatrix} -16 & 0 \\ -8 & 4 \\ -6 & -12 \end{bmatrix}\right)$$

$$= \frac{1}{3}\begin{bmatrix} 10-16 & -10+0 \\ 20-8 & 10+4 \\ -25-6 & 5-12 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} -6 & -10 \\ 12 & 14 \\ -31 & -7 \end{bmatrix} = \begin{bmatrix} -2 & -10/3 \\ 4 & 14/3 \\ -31/3 & -7/3 \end{bmatrix}.$$

Example 7. If A = diag. [3, – 5, 7]* and B = diag. [– 1, 2, 4], then find (2A + 3B).

Solution. We have :

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & 7 \end{bmatrix} \text{ and } B = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}.$$

$$\therefore \quad 2A + 3B = 2\begin{bmatrix} 3 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & 7 \end{bmatrix} + 3\begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 0 & 0 \\ 0 & -10 & 0 \\ 0 & 0 & 14 \end{bmatrix} + \begin{bmatrix} -3 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 12 \end{bmatrix}$$

$$= \begin{bmatrix} 6-3 & 0+0 & 0+0 \\ 0+0 & -10+6 & 0+0 \\ 0+0 & 0+0 & 14+12 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 26 \end{bmatrix} = \text{diag. } [3, -4, 26].$$

Example 8. Two farmers Ram Kishan and Gurcharan Singh cultivate only three varieties of rice namely Basmati, Permal and Naura. The sale (in ₹) of these varieties of rice by both the farmers in the month of September and October are given by the following matrices A and B :

September sales (in ₹)

$$A = \begin{array}{c} \begin{array}{ccc} \textbf{Basmati} & \textbf{Permal} & \textbf{Naura} \end{array} \\ \begin{bmatrix} 10,000 & 20,000 & 30,000 \\ 50,000 & 30,000 & 10,000 \end{bmatrix} \end{array} \begin{array}{l} \rightarrow \textbf{Ram Kishan} \\ \rightarrow \textbf{Gurucharan Singh} \end{array}$$

October sales (in ₹)

$$B = \begin{array}{c} \begin{array}{ccc} \textbf{Basmati} & \textbf{Permal} & \textbf{Naura} \end{array} \\ \begin{bmatrix} 5,000 & 10,000 & 6,000 \\ 20,000 & 10,000 & 10,000 \end{bmatrix} \end{array} \begin{array}{l} \rightarrow \textbf{Ram Kishan} \\ \rightarrow \textbf{Gurucharan Singh.} \end{array}$$

Find :

(i) What were combined sales in September and October for each farmer in each variety ?

---

* $A = \text{diag } [a_1, a_2, \ldots\ldots\ldots\ldots, a_n] = \begin{bmatrix} a_1 & 0 & 0 \ldots\ldots\ldots 0 \\ 0 & a_2 & 0 \ldots\ldots\ldots 0 \\ & & \\ 0 & 0 & 0 \ldots\ldots\ldots a_n \end{bmatrix}$, where A is $n \times n$ matrix.

(ii) What was the change in sales from September to October ?

(iii) If both farmers receive 2% profit on gross sales, compute the profit for each farmer and for each variety in October. *(N.C.E.R.T.)*

Solution. (*i*) Combined sales in September and October :

$$\begin{array}{cccc} & Basmati & Permal & Naura \\ A+B= & \left[\begin{matrix} 10000+5000 & 20000+10000 & 30000+6000 \\ 50000+20000 & 30000+10000 & 10000+10000 \end{matrix}\right] \end{array}$$

$$= \begin{bmatrix} 15000 & 30000 & 36000 \\ 70000 & 40000 & 20000 \end{bmatrix} \begin{matrix} \to \text{Ram Kishan} \\ \to \text{Gurucharan Singh.} \end{matrix}$$

(*ii*) Change in sales from September to October :

$$\begin{array}{cccc} & Basmati & Permal & Naura \\ A-B= & \left[\begin{matrix} 10000-5000 & 20000-10000 & 30000-6000 \\ 50000-20000 & 30000-10000 & 10000-10000 \end{matrix}\right] \end{array}$$

$$= \begin{bmatrix} 5000 & 10000 & 24000 \\ 30000 & 20000 & 0 \end{bmatrix} \begin{matrix} \to \text{Ram Kishan} \\ \to \text{Gurucharan Singh.} \end{matrix}$$

(*iii*) 2% of $B = \frac{2}{100} \times B = 0 \cdot 02\ B$

$$\begin{array}{ccc} & & Basmati \quad Permal \quad Naura \\ = & (0 \cdot 02) & \left[\begin{matrix} 5000 & 10000 & 6000 \\ 20000 & 10000 & 10000 \end{matrix}\right] \begin{matrix} \to \text{Ram Kishan} \\ \to \text{Gurucharan Singh} \end{matrix} \end{array}$$

$$= \begin{bmatrix} 100 & 200 & 120 \\ 400 & 200 & 200 \end{bmatrix} \begin{matrix} \to \text{Ram Kishan} \\ \to \text{Gurucharan Singh} \end{matrix}$$

Hence, Ram Kishan receives ₹ 100, ₹ 200 and ₹ 120 as profit and Gurucharan Singh receives ₹ 400, ₹ 200 and ₹ 200 in each variety of rice in the month of October.

EXERCISE 3 (c)

Fast Track Answer Type Questions

FTATQ

1. (*a*) If $A = \begin{bmatrix} 2 & 7 \\ 9 & 8 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$, then find :

(A – B). *(Jharkhand B.2013)*

(*b*) If $A = \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}$, find (*i*) 2A (*ii*) – 3A.

2. Find 'x' and 'y' if $2\begin{bmatrix} x & 5 \\ 3 & y \end{bmatrix} = \begin{bmatrix} 4 & 10 \\ 6 & 6 \end{bmatrix}$.

(Uttarakhand B. 2013)

3. (*i*) If $A = \begin{bmatrix} 4 & 2 & 3 \\ 1 & 5 & 7 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 3 & 7 \\ 0 & 4 & 1 \end{bmatrix}$, find 2A + B

(H.B. 2014)

(*ii*) If $A = \begin{bmatrix} 2 & -1 \\ 4 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$, find 2A + 3B

(H. B. 2012)

(*iii*) If $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & -1 & 3 \\ -1 & 0 & 2 \end{bmatrix}$, then find 2A – B.

(N.C.E.R.T.)

4. (*i*) Find the value of $(y - x)$ from the following equation :

$$2\begin{pmatrix} x & 5 \\ 7 & y-3 \end{pmatrix} + \begin{pmatrix} 3 & -4 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 7 & 6 \\ 15 & 14 \end{pmatrix}.$$

(A.I.C.B.S.E. 2012)

(*ii*) Find the value of $(x + y)$ from the following equation :

$$2\begin{pmatrix} 1 & 3 \\ 0 & x \end{pmatrix} + \begin{pmatrix} y & 0 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 5 & 6 \\ 1 & 8 \end{pmatrix}.$$ *(A.I.C.B.S.E. 2012)*

5. If $A = \begin{bmatrix} 2/3 & 1 & 5/3 \\ 1/3 & 2/3 & 4/3 \\ 7/3 & 2 & 2/3 \end{bmatrix}$ and $B = \begin{bmatrix} 2/5 & 3/5 & 1 \\ 1/5 & 2/5 & 4/5 \\ 7/5 & 6/5 & 2/5 \end{bmatrix}$, then compute 3A – 5B. *(N.C.E.R.T.)*

Very Short Answer Type Questions

VSATQ

6. Find a matrix 'X' such that :

2A – B + X = O, where $A = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & 1 \\ 0 & 3 \end{bmatrix}$.

(Meghalaya B. 2018)

7. Find matrices 'X' and 'Y' if :

(*i*) $X + Y = \begin{bmatrix} 5 & 2 \\ 0 & 9 \end{bmatrix}$ and $X - Y = \begin{bmatrix} 3 & 6 \\ 0 & -1 \end{bmatrix}$

(N.C.E.R.T. ; H.P.B. 2013 S ; H.B. 2012)

(*ii*) $X+Y=\begin{bmatrix}7 & 0\\2 & 5\end{bmatrix}$ and $X-Y=\begin{bmatrix}3 & 0\\0 & 3\end{bmatrix}$

(Jammu B. 2017)

(*iii*) $2X+3Y=\begin{bmatrix}2 & 3\\4 & 0\end{bmatrix}$ and $3X+2Y=\begin{bmatrix}2 & -2\\-1 & 5\end{bmatrix}$.

(N.C.E.R.T. ; H.P.B. 2013 S)

8. Find 'X', if $Y=\begin{bmatrix}3 & 2\\1 & 4\end{bmatrix}$ and $2X+Y=\begin{bmatrix}1 & 0\\-3 & 2\end{bmatrix}$.

(N.C.E.R.T., Jammu B. 2018; Assam B. 2018)

9. If $x\begin{bmatrix}2\\3\end{bmatrix}+y\begin{bmatrix}-1\\1\end{bmatrix}=\begin{bmatrix}10\\5\end{bmatrix}$, then find the values of 'x' and 'y'. **(N.C.E.R.T.; Kashmir B. 2011)**

10. Find 'x' and 'y', if (*i*) $2\begin{bmatrix}1 & 3\\0 & x\end{bmatrix}+\begin{bmatrix}y & 0\\1 & 2\end{bmatrix}=\begin{bmatrix}5 & 6\\1 & 8\end{bmatrix}$

(N.C.E.R.T.; Kashmir B. 2016)

(*ii*) $2\begin{bmatrix}x & 5\\7 & y-3\end{bmatrix}+\begin{bmatrix}3 & -4\\1 & 2\end{bmatrix}=\begin{bmatrix}7 & 6\\15 & 14\end{bmatrix}$.

(N.C.E.R.T.)

11. Solve the equation for x, y, z and t :

(*i*) $2\begin{bmatrix}x & z\\y & t\end{bmatrix}+3\begin{bmatrix}1 & -1\\0 & 2\end{bmatrix}=3\begin{bmatrix}3 & 5\\4 & 6\end{bmatrix}$

(N.C.E.R.T.; Jammu B. 2015)

(*ii*) $2\begin{bmatrix}x & y\\z & t\end{bmatrix}+3\begin{bmatrix}1 & -1\\0 & 2\end{bmatrix}=3\begin{bmatrix}3 & 5\\4 & 6\end{bmatrix}$.

(H.B. 2012)

12. Given $3\begin{bmatrix}x & y\\z & w\end{bmatrix}=\begin{bmatrix}x & 6\\-1 & 2w\end{bmatrix}+\begin{bmatrix}4 & x+y\\z+w & 2\end{bmatrix}$, find the values of x, y, z and w. **(N.C.E.R.T.)**

Short Answer Type Questions

SATQ

13. If $A=\begin{bmatrix}2 & -1\\4 & 2\end{bmatrix}$, $B=\begin{bmatrix}4 & 3\\-2 & 1\end{bmatrix}$ and

$C=\begin{bmatrix}-2 & -3\\-1 & 2\end{bmatrix}$, find each of the following :

(*i*) 2B + 3C (*ii*) – 2A + (B + C)

(*iii*) (2A – 3B) – C (*iv*) A + (2B – C)

(*v*) A + (B + C) (*vi*) (A + B) + C.

14. If $A=\begin{bmatrix}1 & 2 & 3\\-1 & 0 & 2\\1 & -8 & -1\end{bmatrix}$,

$B=\begin{bmatrix}4 & 5 & 6\\-1 & 0 & 1\\2 & 1 & 2\end{bmatrix}$, $C=\begin{bmatrix}-1 & -2 & 1\\-1 & 2 & 3\\-1 & -2 & 2\end{bmatrix}$,

find (*i*) 2B – 3C (*ii*) A – 2B + 3C.

15. If A = diag. [2, –5, 9], B = diag. [–3, 7, 14] and C = diag. [4, – 6, 3], find :

(*i*) A + 2B (*ii*) 2A + B – 5C.

Answers

1. (*a*) $\begin{bmatrix}1 & 5\\9 & 5\end{bmatrix}$ (*b*) (*i*) $\begin{bmatrix}4 & 6\\-2 & 8\end{bmatrix}$ (*ii*) $\begin{bmatrix}-6 & -9\\3 & -12\end{bmatrix}$.

2. $x = 2, y = 3$.

3. (*i*) $\begin{bmatrix}9 & 7 & 13\\2 & 14 & 15\end{bmatrix}$ (*ii*) $\begin{bmatrix}10 & 7\\11 & 10\end{bmatrix}$ (*iii*) $\begin{bmatrix}-1 & 5 & 3\\5 & 6 & 0\end{bmatrix}$.

4. (*i*) 7 (*ii*) 6.

5. $\begin{bmatrix}0 & 0 & 0\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}$. **6.** $\begin{bmatrix}-8 & -1\\0 & -1\end{bmatrix}$.

7. (*i*) $X=\begin{bmatrix}4 & 4\\0 & 4\end{bmatrix}$, $Y=\begin{bmatrix}1 & -2\\0 & 5\end{bmatrix}$

(*ii*) $X=\begin{bmatrix}5 & 0\\1 & 4\end{bmatrix}$, $Y=\begin{bmatrix}2 & 0\\1 & 1\end{bmatrix}$

(*iii*) $X=\begin{bmatrix}2/5 & -12/5\\-11/5 & 3\end{bmatrix}$, $Y=\begin{bmatrix}2/5 & 13/5\\14/5 & -2\end{bmatrix}$.

8. $X=\begin{bmatrix}-1 & -1\\-2 & -1\end{bmatrix}$. **9.** $x = 3, y = -4$.

10. (*i*) $x = 3, y = 3$ (*ii*) $x = 2, y = 9$.

11. (*i*) $x = 3, y = 6, z = 9, t = 6$

(*ii*) $x = 3, y = 9, z = 6, t = 6$.

12. $x = 2, y = 4, z = \frac{1}{2}, w = 2$.

13. (*i*) $\begin{bmatrix}2 & -3\\-7 & 8\end{bmatrix}$ (*ii*) $\begin{bmatrix}-2 & 2\\-11 & -1\end{bmatrix}$ (*iii*) $\begin{bmatrix}-6 & -8\\15 & -1\end{bmatrix}$

(*iv*) $\begin{bmatrix}12 & 8\\1 & 2\end{bmatrix}$ (*v*) – (*vi*) $\begin{bmatrix}4 & -1\\1 & 5\end{bmatrix}$.

14. (*i*) $\begin{bmatrix}11 & 16 & 9\\1 & -6 & -7\\7 & 8 & -2\end{bmatrix}$ (*ii*) $\begin{bmatrix}-10 & -14 & -6\\-2 & 6 & 9\\-6 & -16 & 1\end{bmatrix}$.

15. (*i*) diag. [– 4, 9, 37] (*ii*) diag. [– 19, 27, 17].

Hints to Selected Questions

6. X = B – 2A.

7. (*i*) Add and Subtract.

8. $2X + Y = \begin{bmatrix} 1 & 0 \\ -3 & 2 \end{bmatrix}$

$\Rightarrow \quad 2X + \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -3 & 2 \end{bmatrix}$;etc.

11. (*i*) $\begin{bmatrix} 2x & 2z \\ 2y & 2t \end{bmatrix} + \begin{bmatrix} 3 & -3 \\ 0 & 6 \end{bmatrix} = \begin{bmatrix} 9 & 15 \\ 12 & 18 \end{bmatrix}$

$\Rightarrow \quad \begin{bmatrix} 2x+3 & 2z-3 \\ 2y & 2t+6 \end{bmatrix} = \begin{bmatrix} 9 & 15 \\ 12 & 18 \end{bmatrix}$;etc.

15. (*i*) A + 2B

$= \begin{bmatrix} 2 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & 9 \end{bmatrix} + 2 \begin{bmatrix} -3 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 14 \end{bmatrix}$;etc.

(C) Operation III. Multiplication of two Matrices.

(*i*) Anil and Akram are two friends. Anil wants to buy 4 pencils and 4 note-books, whereas Akram wants to buy 7 pencils and 6 note-books. They go to a shop and know the rates as below :

Pencil → ₹ 5 each. Note-book → ₹ 10 each.

Let us find out the money does each need to spend ? Clearly Anil needs ₹ (4 × 5 + 4 × 10) *i.e.* ₹ 60, whereas Akram needs ₹ (7 × 5 + 6 × 10) *i.e.* ₹ 95. We write the above information in terms of matrix representation as below :

Requirements $\begin{bmatrix} 4 & 4 \\ 7 & 6 \end{bmatrix}$ *Prices (in ₹)* $\begin{bmatrix} 5 \\ 10 \end{bmatrix}$ *Money needed (in ₹)* $\begin{bmatrix} 4 \times 5 + 4 \times 10 \\ 7 \times 5 + 6 \times 10 \end{bmatrix} = \begin{bmatrix} 60 \\ 95 \end{bmatrix}$.

Suppose there is another shop in the market quoting the following rates :

Pencil → ₹ 6 each. Note-book → ₹ 12 each.

Now Anil needs ₹ (4 × 6 + 4 × 12) *i.e.* ₹ 72 and Akram needs ₹ (7 × 6 + 6 × 12) *i.e.* ₹ 114.

We write the above information in terms of matrix representation as below :

Requirements $\begin{bmatrix} 4 & 4 \\ 7 & 6 \end{bmatrix}$ *Prices (in ₹)* $\begin{bmatrix} 6 \\ 12 \end{bmatrix}$ *Money needed (in ₹)* $\begin{bmatrix} 4 \times 6 + 4 \times 12 \\ 7 \times 6 + 6 \times 12 \end{bmatrix} = \begin{bmatrix} 72 \\ 114 \end{bmatrix}$.

Combining the above two informations, we have :

Requirements $\begin{bmatrix} 4 & 4 \\ 7 & 6 \end{bmatrix}$ *Prices (in ₹) at two shops* $\begin{bmatrix} 5 & 6 \\ 10 & 12 \end{bmatrix}$ *Money needed (in ₹) at two shops* $\begin{bmatrix} 4 \times 5 + 4 \times 10 & 4 \times 6 + 4 \times 12 \\ 7 \times 5 + 6 \times 10 & 7 \times 6 + 6 \times 12 \end{bmatrix} = \begin{bmatrix} 60 & 72 \\ 95 & 114 \end{bmatrix}$.

The above example illustrates the multiplication of matrices. It is observed that for multiplication of two matrices A and B,

No. of columns of A = No. of rows of B.

Formally we have the following definition :

Definition

Let A = $[a_{ij}]$ be m × n matrix and B = $[a_{jk}]$ be n × p matrix such that the number of columns in A is equal to the number of rows in B. Then the matrix C = $[c_{ik}]$, which is of order m × p such that :

$c_{ik} = \sum_{j=1}^{n} a_{ij} b_{jk}$, *where i = 1, 2, 3,............, m ; k = 1, 2, ..., p, is called the product of the matrices A and B and is written as :*

$$C = AB.$$

Here C is of the order $m \times p$.

In AB, A is called the **pre-factor** and B the **post-factor.**

KEY POINT

The product AB of two matrices A and B is defined if and only if the number of columns of A = number of rows of B.

Two such matrices are said to be **conformable for multiplication.**

If the product AB is defined, then BA may not be defined.

Thus the multiplication of matrices is not necessarily commutative.

For Example : If A is 4×5 and B is 5×6, then AB is defined. [$\because$ *No. of columns of A = No. of rows of B = 5*]

but BA is not defined. [$\because$ *No. of columns of B $\neq$ No. of rows of A as $6 \neq 4$*]

ALGORITHMIC APPROACH TO MULTIPLICATION

Let C = AB.

To obtain C :

(i) Multiply the elements of first row of A with the corresponding elements of the first column of B and add.

This gives us the first element of the first row of C.

(ii) Multiply the elements of first row of A with the corresponding elements of the second column of B and add.

This gives us the second element of the first row of C.

Similarly we can get the remaining elements of the first row of C.

Proceeding as above, we get the elements of the remaining rows of C.

For Example :

If $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ and $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$ be two 2×2 matrices over a number system **K**, then the product A.B is defined to be the 2×2 matrix :

$$AB = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix}.$$

Thus (i, j) – entry in A.B is $a_{i1}b_{1j} + a_{i2}b_{2j} = \sum_{r=1}^{2} a_{ir}b_{rj}$, which is the sum of the products of corresponding entries in the ith row of A and jth column of B.

Thus if $A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$ and $B = \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1p} \\ b_{21} & b_{22} & \dots & b_{2p} \\ \dots & \dots & \dots & \dots \\ b_{n1} & b_{n2} & \dots & b_{np} \end{bmatrix}$

be two matrices of the type $m \times n$ and $n \times p$ respectively over a number system **K**, then the product A.B is defined to be the $m \times p$ matrix :

$$\begin{bmatrix} c_{11} & c_{12} & \dots & c_{1p} \\ c_{21} & c_{22} & \dots & c_{2p} \\ \dots & \dots & \dots & \dots \\ c_{m1} & c_{m2} & \dots & c_{mp} \end{bmatrix}, \text{ where } c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \ \dots + a_{in}b_{nj} = \sum_{r=1}^{n} a_{ir}b_{rj} \text{ for } 1 \leq i \leq m,\ 1 \leq j \leq p.$$

(ii) Properties of Matrix Multiplication.

(I) Matrix multiplication is Associative.

If A, B, C are three matrices such that A.B and B.C are defined, then (A.B).C and A.(B.C) are both defined and (A.B).C = A.(B.C).

Let A be of the order $m \times n$.

Since A.B and B.C are both defined,

$\therefore$ B must be of the order $n \times p$ and C of the order $p \times q$.

By *def.*, A.B is of the order $m \times p$ and B.C of the order $n \times q$.

Consequently (A.B).C is defined of the order $m \times q$ and A.(B.C) is defined of the order $m \times q$.

Let $A = [a_{ij}]$, $B = [b_{ij}]$ and $C = [c_{ij}]$.

Let $A.B = [d_{ij}]$ and $B.C = [e_{ij}]$.

Then $d_{ij} = \sum_{r=1}^{n} a_{ir}\, b_{rj}$ and $e_{ij} = \sum_{s=1}^{p} b_{is}\, c_{sj}$.

For $1 \le i \le m,\ 1 \le j \le q$, (i, j) – entry of (A.B).C $= \sum_{s=1}^{p} d_{is}\, c_{sj} = \sum_{s=1}^{p}\left(\sum_{r=1}^{n} a_{ir}\, b_{rs}\right) c_{sj} = \sum_{s=1}^{p}\sum_{r=1}^{n} (a_{ir}\, b_{rs})\, c_{sj}$

and (i, j) – entry of A. (B.C) $= \sum_{r=1}^{n} a_{ir}\, e_{rj} = \sum_{r=1}^{n} a_{ir}\left(\sum_{s=1}^{p} b_{rs}\, c_{sj}\right) = \sum_{r=1}^{n}\sum_{s=1}^{p} a_{ir}\,(b_{rs}\, c_{sj})$.

Thus both the matrices (A.B).C and A.(B.C) have identical (i, j) –entries and are of the same order $m \times q$.

[$\because$ $(a_{ir}\, b_{rs})\, c_{sj} = a_{ir}\,(b_{rs}\, c_{sj})$ *for all r, s*]

Hence, (A.B).C = A.(B.C).

(II) Matrix Multiplication is distributive over addition.

(*i*) *If B and C are matrices of the same order and A is a matrix such that A.(B + C) is defined, then :*

$$A.\,(B + C) = A.B + A.C.$$

(*ii*) *If A and B are matrices of the same order and C is a matrix such that (A + B).C is defined, then :*

$$(A + B).C = A.C + B.C.$$

(*i*) Let $A = [a_{ij}]$ be any $m \times n$ matrix.

Let $B = [b_{ij}]$ and $C = [c_{ij}]$ be $n \times p$ matrices.

Then $B + C = [d_{ij}]$, where $d_{ij} = b_{ij} + c_{ij}$.

For $1 \le i \le m,\ 1 \le j \le p$, then (i, j) – entry of A.(B + C)

$$= \sum_{r=1}^{n} a_{ir} d_{rj} = \sum_{r=1}^{n} a_{ir}(b_{rj} + c_{rj}) = \sum_{r=1}^{n} (a_{ir}b_{rj} + a_{ir}c_{rj}) = \sum_{r=1}^{n} a_{ir}b_{rj} + \sum_{r=1}^{n} a_{ir}c_{rj}$$

= (i, j) – entry of A.B + (i, j) – entry of A.C = (i, j) – entry of (A.B + A.C).

Hence A. (B + C) = A.B + A.C.

(*ii*) This is left for the readers to try.

(III) Multiplication by an Identity Matrix.

If A is an $m \times n$ *matrix and* I_m *and* I_n *are identity matrices of orders m and n respectively, then* $I_m A = A = A\, I_n$.

Let $A = [a_{ij}]_{m \times n}$

Then $I_m A$ and AI_n are of same order $m \times n$ such that :

$$(I_m A)_{ij} = \sum_{r=1}^{m} (I_m)_{ir} (A)_{rj}$$

$$\Rightarrow \quad (I_m A)_{ij} = \sum_{r=1}^{m} (I_m)_{ir}\, a_{rj}$$

$$= (I_m)_{i1} a_{1j} + (I_m)_{i2} a_{2j} + \dots\dots\dots + (I_m)_{im} a_{mj}$$

$$= a_{ij}\ \forall\ i, j. \qquad \left[\because (I_m)_{ir} = \begin{cases} 0 & if\ r \ne i \\ 1 & if\ r = i \end{cases}\right]$$

3.5. POSITIVE INTEGRAL POWERS OF MATRICES

If A is a square matrix, then AA is defined.

We write it as $AA = A^2$.

Now $A^2A = (AA)\,A = A\,(AA) = AA^2 = A^3$. [*Associative Law*]

Thus $A^3 = AAA = AA^2 = A^2A$.

In general, AA ...to m factors = A^m.

Now $A^m A^n$ = (AA ... to m factors) (AA ... to n factors)

= AA ... to $(m + n)$ factors = A^{m+n}.

Again, $(A^m)^n = A^m A^m$ ..... to n factors = (AA ... to m factors) (AA ... to m factors) ... to n factors

= AA ... to mn factors = $(A)^{mn}$.

Cor. If I be a unit matrix of any order, then $I^2 = I^3 = \dots = I^n = I$.

Frequently Asked Questions

FAQs

Example 1. Write the order of the product matrix :

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} [2 \quad 3 \quad 4].$$ *(C.B.S.E. (F) 2011)*

Solution. Here A is 3 × 1 matrix and B is 1 × 3 matrix. Hence, AB is defined and is of order 3 × 3.

Example 2. If $(5 \quad x \quad 1)\begin{pmatrix} 4 \\ 2 \\ 7 \end{pmatrix} = (35)$, **find x.**

(Assam B. 2016)

Solution. We have : $(5 \quad x \quad 1)\begin{pmatrix} 4 \\ 2 \\ 7 \end{pmatrix} = (35)$

$\Rightarrow (20 + 2x + 7) = (35)$
$\Rightarrow 20 + 2x + 7 = 35 \quad \Rightarrow \quad 2x = 8.$
Hence, $x = 4$.

Example 3. If matrix $A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ **and** $A^2 = kA$, **then write the value of 'k'.**
(Meghalaya B. 2014; A.I.C.B.S.E. 2013)

Solution. We have : $A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$.

$$\therefore \quad A^2 = AA = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+1 & -1-1 \\ -1-1 & 1+1 \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}.$$

By the question, $A^2 = kA$

$$\Rightarrow \quad \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} = k\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\Rightarrow \quad 2\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = k\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$

Comparing, $2 = k$.
Hence, $k = 2$.

Example 4. If $\begin{pmatrix} 2 & 3 \\ 5 & 7 \end{pmatrix}\begin{pmatrix} 1 & -3 \\ -2 & 4 \end{pmatrix} = \begin{pmatrix} -4 & 6 \\ -9 & x \end{pmatrix}$, **write the value of 'x'.** *(C.B.S.E. 2012)*

Solution. We have :

$$\begin{pmatrix} 2 & 3 \\ 5 & 7 \end{pmatrix}\begin{pmatrix} 1 & -3 \\ -2 & 4 \end{pmatrix} = \begin{pmatrix} -4 & 6 \\ -9 & x \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 2-6 & -6+12 \\ 5-14 & -15+28 \end{pmatrix} = \begin{pmatrix} -4 & 6 \\ -9 & x \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} -4 & 6 \\ -9 & 13 \end{pmatrix} = \begin{pmatrix} -4 & 6 \\ -9 & x \end{pmatrix}.$$

Hence, $x = 13$. [*By Comparison*]

Example 5. If A is a square matrix such that $A^2 = I$, then find the simplified value of $(A - I)^3 + (A + I)^3 - 7A$.
(C.B.S.E. 2016)

Solution. $(A - I)^3 + (A + I)^3 - 7A$

$$= (A^3 - I^3 - 3A^2I + 3AI^2) + (A^3 + I^3 + 3A^2I + 3AI^2) - 7A$$
$$= 2A^3 + 6AI^2 - 7A$$
$$= 2AA^2 + 6AI - 7A \quad [\because I^2 = I]$$
$$= 2AI + 6AI - 7A \quad [\because A^2 = I]$$
$$= 2A + 6A - 7A = A.$$

Example 6. If A is a square matrix such that $A^2 = A$, then write the value of $7A - (I + A)^3$, where I is an identity matrix. *(Type : W. Bengal B. 2016; A.I.C.B.S.E. 2014)*

Solution.
$$(I + A)^2 = (I + A)(I + A)$$
$$= II + IA + AI + AA$$
$$= I + A + A + A^2$$
$$= I + 2A + A \quad [\because A^2 = A]$$
$$= I + 3A \quad ...(1)$$

$$\therefore \quad (I + A)^3 = (I + A)^2 (I + A)$$
$$= (I + 3A)(I + A) \quad [\text{Using } (1)]$$
$$= II + IA + 3AI + 3AA$$
$$= I + A + 3A + 3A^2$$
$$= I + A + 3A + 3A \quad [\because A^2 = A]$$
$$= I + 7A \quad ...(2)$$

Hence, $7A - (I + A)^3 = 7A - (I + 7A) \quad [\text{Using } (2)]$
$= -I.$

Example 7. Find matrix A such that

$$\begin{bmatrix} 2 & -1 \\ 1 & 0 \\ -3 & 4 \end{bmatrix} A = \begin{bmatrix} -1 & -8 \\ 1 & -2 \\ 9 & 22 \end{bmatrix}.$$ *(A.I.C.B.S.E. 2017)*

Solution. Let $P = \begin{bmatrix} 2 & -1 \\ 1 & 0 \\ -3 & 4 \end{bmatrix}$, which is of order 3 × 2

and $Q = \begin{bmatrix} -1 & -8 \\ 1 & -2 \\ 9 & 22 \end{bmatrix}$, which is of order 3 × 2.

Let A be $p \times q$.
Then PA = Q $\Rightarrow$ $2 = p$ and $q = 2$.
Thus A is 2×2 matrix.

Let $$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Then $$\begin{bmatrix} 2 & -1 \\ 1 & 0 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} -1 & -8 \\ 1 & -2 \\ 9 & 22 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2a-c & 2b-d \\ a & b \\ -3a+4c & -3b+4d \end{bmatrix} = \begin{bmatrix} -1 & -8 \\ 1 & -2 \\ 9 & 22 \end{bmatrix}.$$

Comparing, $2a - c = -1$ ...(1) $\quad 2b - d = -8$ ...(2)
$a = 1$ ...(3) $\quad b = -2$ ...(4)
$-3a + 4c = 9$ ...(5) $\quad -3b + 4d = 22$ ...(6)
From (1) and (3), $a = 1$ and $c = 3$.
From (2) and (4), $b = -2$ and $d = 4$.
(5) and (6) are also verified.

Hence, $$A = \begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix}.$$

Example 8. Solve the following equation for x :

$$[x \quad 1]\begin{bmatrix} 1 & 0 \\ -2 & 0 \end{bmatrix} = O.$$ ***(C.B.S.E. 2014)***

Solution. We have : $[x \quad 1]\begin{bmatrix} 1 & 0 \\ -2 & 0 \end{bmatrix} = O$

$\Rightarrow$ $[x-2 \quad 0+0] = O$
$\Rightarrow$ $[x-2 \quad 0] = [0 \quad 0]$.
Comparing, $x - 2 = 0$. Hence, $x = 2$.

Example 9. Compute the indicated product :

$$\begin{bmatrix} 2 & 3 & 4 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \end{bmatrix}\begin{bmatrix} 1 & -3 & 5 \\ 0 & 2 & 4 \\ 3 & 0 & 5 \end{bmatrix}.$$ ***(N.C.E.R.T.)***

Solution. Let $$A = \begin{bmatrix} 2 & 3 & 4 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \end{bmatrix}$$

and $$B = \begin{bmatrix} 1 & -3 & 5 \\ 0 & 2 & 4 \\ 3 & 0 & 5 \end{bmatrix}.$$

Number of columns of A = Number of rows of B = 3.
Thus AB is defined and is a 3×3 matrix.

$$\text{And AB} = \begin{bmatrix} 2\times1+3\times0+4\times3 & 2\times(-3)+3\times2+4\times0 & 2\times5+3\times4+4\times5 \\ 3\times1+4\times0+5\times3 & 3\times(-3)+4\times2+5\times0 & 3\times5+4\times4+5\times5 \\ 4\times1+5\times0+6\times3 & 4\times(-3)+5\times2+6\times0 & 4\times5+5\times4+6\times5 \end{bmatrix}$$

$$= \begin{bmatrix} 2+0+12 & -6+6+0 & 10+12+20 \\ 3+0+15 & -9+8+0 & 15+16+25 \\ 4+0+18 & -12+10+0 & 20+20+30 \end{bmatrix}$$

$$= \begin{bmatrix} 14 & 0 & 42 \\ 18 & -1 & 56 \\ 22 & -2 & 70 \end{bmatrix}.$$

Example 10. If $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, find 'k' so that $A^2 = 5A + kI$.

(C.B.S.E. Sample Paper 2019)

Solution. We have : $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$.

$$\therefore \quad A^2 = AA = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 9-1 & 3+2 \\ -3-2 & -1+4 \end{bmatrix} = \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix}.$$

Also, $$5A = 5\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 15 & 5 \\ -5 & 10 \end{bmatrix}$$

and $$kI = k\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}.$$

Now, $A^2 = 5A + kI$

$$\Rightarrow \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix} = \begin{bmatrix} 15 & 5 \\ -5 & 10 \end{bmatrix} + \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix} - \begin{bmatrix} 15 & 5 \\ -5 & 10 \end{bmatrix} = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 8-15 & 5-5 \\ -5+5 & 3-10 \end{bmatrix} = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -7 & 0 \\ 0 & -7 \end{bmatrix} = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}.$$

Hence, $k = -7$.

Example 11. If $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$ and $A^3 - 6A^2 + 7A + kI_3 = O$, find k. **(*A.I.C.B.S.E. 2016*)**

Solution. We have : $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$.

$$\therefore A^2 = AA = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1+0+4 & 0+0+0 & 2+0+6 \\ 0+0+2 & 0+4+0 & 0+2+3 \\ 2+0+6 & 0+0+0 & 4+0+9 \end{bmatrix} = \begin{bmatrix} 5 & 0 & 8 \\ 2 & 4 & 5 \\ 8 & 0 & 13 \end{bmatrix}.$$

$$\text{And } A^3 = AA^2 = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}\begin{bmatrix} 5 & 0 & 8 \\ 2 & 4 & 5 \\ 8 & 0 & 13 \end{bmatrix}$$

$$= \begin{bmatrix} 5+0+16 & 0+0+0 & 8+0+26 \\ 0+4+8 & 0+8+0 & 0+10+13 \\ 10+0+24 & 0+0+0 & 16+0+39 \end{bmatrix} = \begin{bmatrix} 21 & 0 & 34 \\ 12 & 8 & 23 \\ 34 & 0 & 55 \end{bmatrix}.$$

Now $A^3 - 6A^2 + 7A + kI_3 = O$

$$\Rightarrow \begin{bmatrix} 21 & 0 & 34 \\ 12 & 8 & 23 \\ 34 & 0 & 55 \end{bmatrix} - 6\begin{bmatrix} 5 & 0 & 8 \\ 2 & 4 & 5 \\ 8 & 0 & 13 \end{bmatrix} + 7\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$$

$$+ k\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 21-30+7+k & 0-0+0+0 & 34-48+14+0 \\ 12-12+0+0 & 8-24+14+k & 23-30+7+0 \\ 34-48+14+0 & 0-0+0+0 & 55-78+21+k \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -2+k & 0 & 0 \\ 0 & k-2 & 0 \\ 0 & 0 & -2+k \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Comparing, $-2 + k = 0$.
Hence, $k = 2$.

Example 12. If $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$, show that :

$A^3 - 6A^2 + 7A + 2I = O$. *(N.C.E.R.T; Jammu B. 2015)*

Solution. Here $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$.

$\therefore \quad A^2 = AA$

$$= \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1\times1+0\times0+2\times2 & 1\times0+0\times2+2\times0 & 1\times2+0\times1+2\times3 \\ 0\times1+2\times0+1\times2 & 0\times0+2\times2+1\times0 & 0\times2+2\times1+1\times3 \\ 2\times1+0\times0+3\times2 & 2\times0+0\times2+3\times0 & 2\times2+0\times1+3\times3 \end{bmatrix}$$

$$= \begin{bmatrix} 1+0+4 & 0+0+0 & 2+0+6 \\ 0+0+2 & 0+4+0 & 0+2+3 \\ 2+0+6 & 0+0+0 & 4+0+9 \end{bmatrix} = \begin{bmatrix} 5 & 0 & 8 \\ 2 & 4 & 5 \\ 8 & 0 & 13 \end{bmatrix}$$

and $\quad A^3 = A^2A$

$$= \begin{bmatrix} 5 & 0 & 8 \\ 2 & 4 & 5 \\ 8 & 0 & 13 \end{bmatrix}\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 5\times1+0\times0+8\times2 & 5\times0+0\times2+8\times0 & 5\times2+0\times1+8\times3 \\ 2\times1+4\times0+5\times2 & 2\times0+4\times2+5\times0 & 2\times2+4\times1+5\times3 \\ 8\times1+0\times0+13\times2 & 8\times0+0\times2+13\times0 & 8\times2+0\times1+13\times3 \end{bmatrix}$$

$$= \begin{bmatrix} 5+0+16 & 0+0+0 & 10+0+24 \\ 2+0+10 & 0+8+0 & 4+4+15 \\ 8+0+26 & 0+0+0 & 16+0+39 \end{bmatrix} = \begin{bmatrix} 21 & 0 & 34 \\ 12 & 8 & 23 \\ 34 & 0 & 55 \end{bmatrix}.$$

$\therefore \ A^3 - 6A^2 + 7A + 2I$

$$= \begin{bmatrix} 21 & 0 & 34 \\ 12 & 8 & 23 \\ 34 & 0 & 55 \end{bmatrix} - 6\begin{bmatrix} 5 & 0 & 8 \\ 2 & 4 & 5 \\ 8 & 0 & 13 \end{bmatrix} + 7\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix} + 2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 21 & 0 & 34 \\ 12 & 8 & 23 \\ 34 & 0 & 55 \end{bmatrix} + \begin{bmatrix} -30 & 0 & -48 \\ -12 & -24 & -30 \\ -48 & 0 & -78 \end{bmatrix} + \begin{bmatrix} 7 & 0 & 14 \\ 0 & 14 & 7 \\ 14 & 0 & 21 \end{bmatrix} + \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 21-30+7+2 & 0+0+0+0 & 34-48+14+0 \\ 12-12+0+0 & 8-24+14+2 & 23-30+7+0 \\ 34-48+14+0 & 0+0+0+0 & 55-78+21+2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O.$$

which verifies the result.

Example 13. Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ -1 & 1 & 2 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & 2 & -1 \\ 1 & 3 & 4 \\ 0 & -2 & -3 \end{bmatrix}.$$

Find AB and BA. Is AB = BA ?

Solution. Here A is 3×3 matrix and B is also 3×3 matrix.

Since no. of columns in A = no. of rows in B = 3,

$\therefore$ AB is defined.

$$\therefore \ AB = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ -1 & 1 & 2 \end{bmatrix}\begin{bmatrix} 0 & 2 & -1 \\ 1 & 3 & 4 \\ 0 & -2 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 1\times0+2\times1+3\times0 & 1\times2+2\times3+3\times(-2) & 1\times(-1)+2\times4+3\times(-3) \\ 2\times0+3\times1+4\times0 & 2\times2+3\times3+4\times(-2) & 2\times(-1)+3\times4+4\times(-3) \\ -1\times0+1\times1+2\times0 & -1\times2+1\times3+2\times(-2) & -1\times(-1)+1\times4+2\times(-3) \end{bmatrix}$$

$$= \begin{bmatrix} 0+2+0 & 2+6-6 & -1+8-9 \\ 0+3+0 & 4+9-8 & -2+12-12 \\ -0+1+0 & -2+3-4 & 1+4-6 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 2 & -2 \\ 3 & 5 & -2 \\ 1 & -3 & -1 \end{bmatrix}.$$

Again since no. of columns of B = no. of rows in A = 3,

$\therefore$ BA is defined.

$$\therefore \ BA = \begin{bmatrix} 0 & 2 & -1 \\ 1 & 3 & 4 \\ 0 & -2 & -3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ -1 & 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 0\times1+2\times2+(-1)\times(-1) & 0\times2+2\times3+(-1)\times1 & 0\times3+2\times4+(-1)\times2 \\ 1\times1+3\times2+4\times(-1) & 1\times2+3\times3+4\times1 & 1\times3+3\times4+4\times2 \\ 0\times1+(-2)\times2+(-3)\times(-1) & 0\times2+(-2)\times3+(-3)\times1 & 0\times3+(-2)\times4+(-3)\times2 \end{bmatrix}$$

$$= \begin{bmatrix} 0+4+1 & 0+6-1 & 0+8-2 \\ 1+6-4 & 2+9+4 & 3+12+8 \\ 0-4+3 & 0-6-3 & 0-8-6 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 5 & 6 \\ 3 & 15 & 23 \\ -1 & -9 & -14 \end{bmatrix}.$$

Hence, AB $\neq$ BA.

Example 14. If the product of two matrices is a zero matrix, it is not necessary that one of the matrices is a zero matrix.

Or

Give an example of two matrices A and B such that AB = O when neither A= O nor B = O.

(N.C.E.R.T.)

Solution. Take $A = \begin{bmatrix} 0 & -1 \\ 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 5 \\ 0 & 0 \end{bmatrix}$.

Here $\quad A \neq O$ and $B \neq O$,

But $\quad AB = \begin{bmatrix} 0 & -1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 3 & 5 \\ 0 & 0 \end{bmatrix}$

$$= \begin{bmatrix} 0-0 & 0-0 \\ 0+0 & 0+0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O.$$

Example 15. Let $A = \begin{bmatrix} 3 & 4 \\ -4 & -3 \end{bmatrix}$, find f(A), where $f(x) = x^2 - 5x + 7$.

Solution. We have : $f(x) = x^2 - 5x + 7$.

$$\therefore \qquad f(A) = A^2 - 5A + 7I.$$

Now $\qquad A^2 = AA = \begin{bmatrix} 3 & 4 \\ -4 & -3 \end{bmatrix} \begin{bmatrix} 3 & 4 \\ -4 & -3 \end{bmatrix}$

$$= \begin{bmatrix} 3\times3+4\times(-4) & 3\times4+4\times(-3) \\ (-4)\times3+(-3)\times(-4) & (-4)\times4+(-3)\times(-3) \end{bmatrix}$$

$$= \begin{bmatrix} 9-16 & 12-12 \\ -12+12 & -16+9 \end{bmatrix} = \begin{bmatrix} -7 & 0 \\ 0 & -7 \end{bmatrix}.$$

$\therefore \ f(A) = A^2 - 5A + 7I$

$$= \begin{bmatrix} -7 & 0 \\ 0 & -7 \end{bmatrix} - 5\begin{bmatrix} 3 & 4 \\ -4 & -3 \end{bmatrix} + 7\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -7 & 0 \\ 0 & -7 \end{bmatrix} + \begin{bmatrix} -15 & -20 \\ 20 & 15 \end{bmatrix} + \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} -7-15 & 0-20 \\ 0+20 & -7+15 \end{bmatrix} + \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} -22 & -20 \\ 20 & 8 \end{bmatrix} + \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} -22+7 & -20+0 \\ 20+0 & 8+7 \end{bmatrix} = \begin{bmatrix} -15 & -20 \\ 20 & 15 \end{bmatrix}.$$

PRINCIPLE OF MATHEMATICAL INDUCTION

Denote the given problem by P (n), where $\in$ **N.**

GUIDE-LINES

Step I. Verify that P (n) is true for $n = 1$.

Step II. Assume P (n) to be true for $n = m$, where $1 \leq m < n$.

Step III. Verify the truth of P (n) when $n = m + 1$.

Then P (n) is true for all $n \in$ **N.**

Example 16. Prove the following by the principle of Mathematical Induction :

If $A = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$, then $A^n = \begin{bmatrix} 1+2n & -4n \\ n & 1-2n \end{bmatrix}$, where n $\in$ N.

(N.C.E.R.T. ; Assam B. 2018 ; H.B. 2014 ; Kashmir B. 2012 ; P.B. 2012)

Solution. Step I. When $n = 1$.

$$A^1 = A = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1+2.1 & -4.1 \\ 1 & 1-2.1 \end{bmatrix}.$$

Thus the result is true when $n = 1$.

Step II. Let us assume that the result is true for any natural number m, where $1 \leq m < n$

$$i.e. \ A^m = \begin{bmatrix} 1+2m & -4m \\ m & 1-2m \end{bmatrix} \qquad ...(1)$$

Step III.

$$A^{m+1} = A^m A$$

$$= \begin{bmatrix} 1+2m & -4m \\ m & 1-2m \end{bmatrix} \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix} \quad [Using\ (1)]$$

$$= \begin{bmatrix} 3+6m-4m & -4-8m+4m \\ 3m+1-2m & -4m-1+2m \end{bmatrix}$$

$$= \begin{bmatrix} 3+2m & -4-4m \\ 1+m & -1-2m \end{bmatrix}$$

$$= \begin{bmatrix} 1+2(m+1) & -4(m+1) \\ m+1 & 1-2(m+1) \end{bmatrix}.$$

Thus the result is true when $n = m + 1$.

Hence, by Mathematical Induction, the required result is true for all $n \in \mathbf{N}$.

Example 17. Three schools A, B and C organised a mela for collecting funds for helping the rehabilitation of flood victims. They sold hand made fans, mats and plates from recycled meterial at a cost of ₹ 25, ₹ 100 and ₹ 50 each. The number of articles sold are given below :

School / Articles	A	B	C
Hand fans	**40**	**25**	**35**
Mats	**50**	**40**	**50**
Plates	**20**	**30**	**40**

Find the funds collected by each school separately by selling the above articles. Also, find the total fund collected for the purpose. *(C.B.S.E. 2015)*

Solution. Quantity matrix, $A = \begin{bmatrix} 40 & 25 & 35 \\ 50 & 40 & 50 \\ 20 & 30 & 40 \end{bmatrix}$.

Cost matrix, $B = \begin{bmatrix} 25 \\ 100 \\ 50 \end{bmatrix}$.

Total fund = Quantity matrix × Cost matrix = A × B

$$= \begin{bmatrix} 40 & 25 & 35 \\ 50 & 40 & 50 \\ 20 & 30 & 40 \end{bmatrix} \begin{bmatrix} 25 \\ 100 \\ 50 \end{bmatrix}$$

$$= \begin{bmatrix} 40 \times 25 + 25 \times 100 + 35 \times 50 \\ 50 \times 25 + 40 \times 100 + 50 \times 50 \\ 20 \times 25 + 30 \times 100 + 40 \times 50 \end{bmatrix}$$

$$= \begin{bmatrix} 1000 + 2500 + 1750 \\ 1250 + 4000 + 2500 \\ 500 + 3000 + 2000 \end{bmatrix} = \begin{bmatrix} 5250 \\ 7750 \\ 5500 \end{bmatrix}.$$

∴ Fund collected by school A = ₹ 5,250

Fund collected by school B = ₹ 7,750

Fund collected by school C = ₹ 5,500

and total fund collected = 5250 + 7750 + 5500

= ₹ 18,500.

Values : We should show sympathy and humanity towards flood victims.

Example 18. In a legislative assembly election, a political group hired a public relations firm to promote its candidate in three ways ; telephone, house calls and letters. The cost per contact (in paise) is given in matrix A as :

$$\begin{array}{c} \textbf{Cost per contact} \\ A = \begin{bmatrix} 40 \\ 100 \\ 50 \end{bmatrix} \end{array} \begin{array}{l} \\ \textbf{Telephone} \\ \textbf{House calls} \\ \textbf{Letter} \end{array}$$

The number of contacts of each type made in two cities X and Y is given in matrix B as :

$$\begin{array}{c} \begin{array}{ccc} \textbf{Telephone} & \textbf{House calls} & \textbf{Letter} \end{array} \\ B = \begin{bmatrix} 100 & 500 & 5000 \\ 3000 & 1000 & 10000 \end{bmatrix} \end{array} \begin{array}{l} \to X \\ \to Y. \end{array}$$

Find the total amount spent by the group in two cities X and Y. *(N.C.E.R.T.)*

Solution. Here $BA = \begin{bmatrix} 100 & 500 & 5000 \\ 3000 & 1000 & 10000 \end{bmatrix} \begin{bmatrix} 40 \\ 100 \\ 50 \end{bmatrix}$

$$= \begin{bmatrix} 100 \times 40 + 500 \times 100 + 5000 \times 50 \\ 3000 \times 40 + 1000 \times 100 + 10000 \times 50 \end{bmatrix}$$

$$= \begin{bmatrix} 4000 + 50000 + 250000 \\ 120000 + 100000 + 500000 \end{bmatrix} = \begin{bmatrix} 304{,}000 \\ 720{,}000 \end{bmatrix} \begin{array}{l} \to X \\ \to Y \end{array}$$

Hence, the total amount spent in two cities is ₹ 3,040 and ₹ 7,200 respectively.

EXERCISE 3 (d)

Fast Track Answer Type Questions

FTATQ

1. (*a*) (*i*) If A is a $m \times n$ matrix such that AB and BA are defined, then B is of order....... **(Fill in the Blank)** (*Jammu B. 2017*)

(*ii*) If A is of order $m \times n$ and B is of order $n \times p$, then AB is a matrix of order as $m \times p$.

(True/False) (*Kashmir B. 2017*)

(*b*) If A is a matrix of order 3×4 and B is a matrix of order 4×3, find the order of the matrix (AB). (*C.B.S.E. 2010 C*)

2. Compute the indicated products :

(*a*) (*i*) $\begin{bmatrix} a & b \\ -b & a \end{bmatrix}\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ (*N.C.E.R.T.*)

(*ii*) $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\begin{bmatrix} 2 & 3 & 4 \end{bmatrix}$ (*N.C.E.R.T.*)

(*iii*) $\begin{bmatrix} 1 & -2 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix}$ (*N.C.E.R.T.*)

(*iv*) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$ (*Meghalaya B. 2015*)

(*v*) $\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ (*Uttarakhand B. 2015*)

(*b*) (*i*) $\begin{bmatrix} 6 & 9 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 2 & 6 & 0 \\ 7 & 9 & 8 \end{bmatrix}$ (*ii*) $\begin{bmatrix} 2 & 3 & 4 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \end{bmatrix}\begin{bmatrix} 1 & -3 & 5 \\ 0 & 2 & 4 \\ 3 & 0 & 5 \end{bmatrix}$

(*iii*) $\begin{bmatrix} 3 & -1 & 3 \\ -1 & 0 & 2 \end{bmatrix}\begin{bmatrix} 2 & -3 \\ 1 & 0 \\ 3 & 1 \end{bmatrix}$. (*N.C.E.R.T.*)

(*c*) If $A = \begin{bmatrix} 4 \\ 2 \\ 3 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & 2 \end{bmatrix}$, the find the value of AB.

(*Jharkhand B. 2016*)

(*d*) If $A = \begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$, compute AB.

(*Meghalaya B. 2017*)

3. (*a*) If $P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 2 & 1 \\ 2 & 3 & 0 \end{bmatrix}$, $Q = \begin{bmatrix} 1 & 2 \\ 3 & 0 \\ 4 & 1 \end{bmatrix}$, find PQ.

(*J.&K.B. 2010*)

(*b*) If $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ 2 & 5 \end{pmatrix} = \begin{pmatrix} 7 & 11 \\ k & 23 \end{pmatrix}$, then write the value of 'k'. (*C.B.S.E. 2010*)

(*c*) (*i*) If matrix $A = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}$ and $A^2 = \lambda A$, then write the value of 'λ'. (*A.I.C.B.S.E. 2013*)

(*ii*) If matrix $A = \begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix}$ and $A^2 = \lambda A$, then write the value of 'λ'. (*A.I.C.B.S.E. 2013*)

(*d*) If A is a square matrix such that $A^2 = A$, then write the value of $(I + A)^2 - 3A$. (*Tripura B. 2016 C.B.S.E. (F) 2012*)

(*e*) If $A = \begin{bmatrix} 1 & 0 \\ -1 & 7 \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, then find K so that $A^2 = 8A + KI$. (*Mizoram B. 2017*)

4. (*a*) Find the values of 'a' and 'b' for which :

$$\begin{bmatrix} a & b \\ -a & 2b \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}.$$

(*Bihar B. 2014; Kerala B. 2017; Rajasthan B. 2017*)

(*b*) If $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$, then prove that :

$$A^2 = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ -\sin 2\theta & \cos 2\theta \end{bmatrix}.$$

(*Kerala B. 2017; Rajasthan B. 2013*)

Very Short Answer Type Questions

VSATQ

5. (*i*) If $A = \begin{bmatrix} 2 & -3 & 1 \\ -2 & 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 5 \\ 3 & 1 \\ 4 & 2 \end{bmatrix}$, then find AB. (*H.B. 2016*)

(*ii*) If $A = \begin{bmatrix} 1 & -2 & 3 \\ -4 & 2 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 2 & 1 \end{bmatrix}$, then find AB and BA. (*Jammu B. 2017*) (*iii*) If $A = \begin{bmatrix} 5 & -1 \\ 6 & 7 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}$, find AB.

(Jammu B. 2017)

6. Evaluate the following :

(*i*) $\begin{bmatrix} 4 \\ 5 \end{bmatrix} [7 \;\; 9] + \begin{bmatrix} 4 & 0 \\ 0 & -5 \end{bmatrix}$

(*ii*) $[x \;\; y \;\; z] \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

(*iii*) $\begin{bmatrix} 1 & -1 \\ 0 & 2 \\ 2 & 3 \end{bmatrix} \left(\begin{bmatrix} 1 & 0 & 2 \\ 2 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 2 \end{bmatrix} \right)$.

7. (*a*) Solve the matrix equations :

$[\, 2x \quad 3 \,] \begin{bmatrix} 1 & 2 \\ -3 & 0 \end{bmatrix} \begin{bmatrix} x \\ 3 \end{bmatrix} = O$ *(H.B. 2012)*

(*b*) Find the value of '*x*' such that :

(*i*) $[1 \;\; 2 \;\; 1] \begin{bmatrix} 1 & 2 & 0 \\ 2 & 0 & 1 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ x \end{bmatrix} = O$ *(N.C.E.R.T.)*

(*ii*) $[x \;\; -5 \;\; -1] \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ 4 \\ 1 \end{bmatrix} = O$. *(N.C.E.R.T)*

(*c*) Find the values of '*a*' and '*b*' for which the following hold :

$$\begin{bmatrix} 3 & 2 \\ 7 & a \end{bmatrix} \begin{bmatrix} 5 & -2 \\ -7 & b \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

8. Let $A = \begin{bmatrix} 2 & 4 \\ 1 & -3 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -1 & 5 \\ 0 & 2 & 6 \end{bmatrix}$

(*a*) Find AB.

(*b*) Is BA defined ? Justify your answer.

(Kerala B. 2013)

9. If $A = \begin{bmatrix} 1 & -2 & 3 \\ -4 & 2 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 2 & 1 \end{bmatrix}$, then find AB, BA. Show that $AB \neq BA$. *(N.C.E.R.T.; Jammu B. 2017, 16)*

10. Prove that AB = BA when :

$$A = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \text{ and } B = \begin{bmatrix} \cos\phi & \sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}.$$

11. Show that :

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 2 & 3 & 4 \end{bmatrix} \neq \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}.$$

(N.C.E.R.T.)

12. Show with the help of an example that AB = O, whereas BA ≠ O, where O is a zero matrix and A, B are both non-zero matrices.

13. Give an example of matrices A, B and C such that AB = AC but B ≠ C, A ≠ O.

14. If $A = \begin{bmatrix} 5 & 2 \\ -1 & 2 \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, show that :

$(A - 3I)(A - 4I) = O$.

15. If $A = \begin{bmatrix} 3 & -5 \\ -4 & 2 \end{bmatrix}$, show that $A^2 - 5A$ is a scalar matrix.

Short Answer Type Questions

SATQ

16. Consider the matrices :

$A = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

If $AB = \begin{bmatrix} 2 & 9 \\ 5 & 6 \end{bmatrix}$, find the values of *a*, *b*, *c* and *d*.

(Kerala B. 2014)

17. (*i*) If $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 4 \\ -1 & 1 \end{bmatrix}$, does $(A + B)^2 = A^2 + 2AB + B^2$ hold ?

(*ii*) If $A = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$, verify that :

$(A + B)^2 \neq A^2 + 2AB + B^2$. *(J. & K.B. 2010)*

18. (*i*) If $A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 0 & 3 \\ 3 & -1 & 2 \end{bmatrix}, B = \begin{bmatrix} 1 & 3 \\ 0 & 2 \\ -1 & 4 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 2 & 3 & -4 \\ 2 & 0 & -2 & 1 \end{bmatrix}$, find A(BC), (AB)C and show that (AB)C = A (BC). *(N.C.E.R.T.)*

(*ii*) Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$.

Calculate AC, BC and (A + B) C.
Also verify that : (A + B) C = AC + BC.

(Karnataka B. 2014)

(*iii*) Let $A = \begin{bmatrix} 0 & 6 & 7 \\ -6 & 0 & 8 \\ 7 & -8 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix}$, $C = \begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix}$.

Calculate AC, BC and (A + B) C.
Also show that (A + B) C = AC + BC. ***(N.C.E.R.T.)***

19. Find the matrix 'X' so that :

$$X\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} -7 & -8 & -9 \\ 2 & 4 & 6 \end{bmatrix}.$$ ***(N.C.E.R.T.)***

20. (*i*) If $A = \begin{bmatrix} 1 & -2 \\ -3 & 4 \end{bmatrix}$,

find $-A^2 + 5A$. ***(P.B. 2010)***

(*ii*) If $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$, show that $A^2 - 5A + 7I = O$.

Use this result to find A^4. ***(N.C.E.R.T.)***

(*iii*) If $A = \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}$, then find :

(I) $A^2 - 5A + 6I$
(II) $A^2 - 3A + 2I$. ***(N.C.E.R.T.; A.I.C.B.S.E 2010)***

21. (*i*) If $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$, show that $A^2 - 5A + 7I = O$.

(N.C.E.R.T.)

(*ii*) If $M = \begin{bmatrix} 7 & 5 \\ 2 & 3 \end{bmatrix}$, then verify the equation :

$M^2 - 10M + 11I_2 = O$. ***(Kerala B. 2015)***

22. (*i*) If $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 4 & 2 & 1 \end{bmatrix}$, then show that :

$A^2 - 23A - 40I \neq O$. ***(N.C.E.R.T.)***

(*ii*) If $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$, prove that :

$A^3 - 6A^2 + 7A + 2I = O$. ***(Karnataka B. 2017)***

23. (*i*) If $A = \begin{bmatrix} -1 & 2 \\ 3 & 1 \end{bmatrix}$, find $f(A)$,

where $f(x) = x^2 - 2x + 3$. ***(Mizoram B. 2015)***

(*ii*) If $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$, then find $f(A)$,

where $f(x) = x^2 - 5x + 7$. ***(H.B. 2018)***

24. (*a*) (*i*) If $A = \begin{bmatrix} 1 & 0 \\ -1 & 7 \end{bmatrix}$, find '$k$' so that $A^2 - 8A = kI$.

(W. Bengal B. 2018; P.B. 2010)

(*ii*) If $A = \begin{bmatrix} 3 & -2 \\ 4 & -2 \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, then find '$k$' so that $A^2 = kA - 2I$. ***(N.C.E.R.T.)***

(*iii*) If $A = \begin{bmatrix} 0 & 3 \\ -7 & 5 \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$,

then find 'k' so that $kA^2 = 5A - 21I$. ***(H.B. 2015)***

(*b*) If $A = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$, then find a and b for which

$A^2 + aA + bI = O$. ***(H.B. 2016)***

25. Let $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, show that :

$(aI + bA)^n = a^n I + na^{n-1} bA$,

where I is the identity matrix of order 2 and $n \in \mathbf{N}$.

(N.C.E.R.T. ; H.B. 2014; Kashmir B. 2012 ; Jammu B. 2012)

26. A matrix X has $a + b$ rows and $a + 2$ columns while the matrix Y has $b + 1$ rows and $a + 3$ columns. Both matrices XY and YX exist. Find 'a' and 'b'. Can you say XY and YX are of the same type ? Are they equal ?

27. Let $A = \begin{bmatrix} -1 & -4 \\ 1 & 3 \end{bmatrix}$, prove by Mathematical Induction that :

$$A^n = \begin{bmatrix} 1-2n & -4n \\ n & 1+2n \end{bmatrix}, \text{ where } n \in \mathbf{N}.$$

(P.B. 2012)

28. (*i*) If $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$, prove that :

$$A^n = \begin{bmatrix} \cos n\theta & \sin n\theta \\ -\sin n\theta & \cos n\theta \end{bmatrix}, n \in \mathbf{N}.$$

(N.C.E.R.T. ; H.B. 2015, 12 ; Kashmir B. 2012)

(*ii*) Let $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$. Show by Mathematical Induction that :

$$A^n = \begin{bmatrix} \cos n\theta & -\sin n\theta \\ \sin n\theta & \cos n\theta \end{bmatrix}$$ for every positive integer n.

29. Three schools X, Y and Z organised a fete (mela) for the following funds for flood victims in which they sold hand-hold fans, mats and toys made from recycled material, the sale price of each being ₹ 25, ₹ 100 and ₹ 50 respectively. The following table shows the number of articles of each type sold :

Articles \ School	A	B	C
Hand fans	30	40	35
Mats	12	15	20
Toys	70	55	75

Using materials, find the funds collected by each school by selling the above articles and the total funds collected. **(A.I.C.B.S.E. 2015)**

30. A trust fund has ₹ 30,000 that must be invested in two different types of bonds. The first bond pays 5% and second 7% interest per year, which will be given to an NGO Cancer Aid Society. Using matrix multiplication, determine how to divide ₹ 30,000 among the two types of bonds if the trust fund must obtain an annual total interest of :

(*a*) ₹ 1,800 (*b*) ₹ 2,000. **(N.C.E.R.T.)**

31. There are two families A and B. There are 4 men, 6 women and 2 children in family A ; 2 men, 2 women and 4 children in family B. The recommended daily allowance for calories is Man : 2400, Woman : 1900, Child : 1800 and for proteins is : Man : 55 gm, Woman : 45 gm and Child : 33 gm. Represent the above information by matrices. Using matrix multiplication, calculate the total requirement of calories and proteins for each of the two families.

Long Answer Type Questions

LATQ

32. Let $A = \begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix}$ and $f(x) = x^2 - 4x + 7$. Show that $f(A) = O_{2 \times 2}$. Use this result to find A^5.

33. Let $A = \begin{bmatrix} 0 & -\tan\frac{\alpha}{2} \\ \tan\frac{\alpha}{2} & 0 \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, show that :

$$(I + A) = (I - A)\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}.$$

(N.C.E.R.T.; Jammu B. 2015; Assam B. 2013 ; Rajasthan B. 2013 ; H.B. 2012)

Answers

1. (*a*) (*i*) $n \times m$ (*ii*) True (*b*) 3×3.

2. (*a*) (*i*) $\begin{bmatrix} a^2+b^2 & 0 \\ 0 & a^2+b^2 \end{bmatrix}$ (*ii*) $\begin{bmatrix} 2 & 3 & 4 \\ 4 & 6 & 8 \\ 6 & 9 & 12 \end{bmatrix}$

(*iii*) $\begin{bmatrix} -3 & -4 & 1 \\ 8 & 13 & 9 \end{bmatrix}$ (*iv*) $\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$ (*v*) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

(*b*) (*i*) $\begin{bmatrix} 75 & 117 & 72 \\ 25 & 39 & 24 \end{bmatrix}$ (*ii*) $\begin{bmatrix} 14 & 0 & 42 \\ 18 & -1 & 56 \\ 22 & -2 & 70 \end{bmatrix}$

(*iii*) $\begin{bmatrix} 14 & -6 \\ 4 & 5 \end{bmatrix}$. (*c*) $\begin{bmatrix} 4 & 0 & 8 \\ 2 & 0 & 4 \\ 3 & 0 & 6 \end{bmatrix}$

(*d*) [30].

3. (*a*) $\begin{bmatrix} 3 & 0 \\ 10 & 1 \\ 11 & 4 \end{bmatrix}$ (*b*) $k = 17$

(*c*) (*i*) $\lambda = 4$ (*ii*) $\lambda = 6$ (*d*) I (*e*) $K = -7$.

4. (*a*) $a = 1, b = -3$.

5. (*i*) $\begin{bmatrix} -1 & 9 \\ 21 & 1 \end{bmatrix}$ (*ii*) $\begin{bmatrix} 0 & -4 \\ 10 & 3 \end{bmatrix}$, $\begin{bmatrix} -10 & 2 & 21 \\ -16 & 2 & 37 \\ -2 & -2 & 11 \end{bmatrix}$

(*iii*) $\begin{bmatrix} 7 & 1 \\ 33 & 34 \end{bmatrix}$.

6. (*i*) $\begin{bmatrix} 32 & 36 \\ 35 & 40 \end{bmatrix}$

(*ii*) $[ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy]$

(*iii*) $\begin{bmatrix} 0 & -1 & 1 \\ 2 & 0 & -2 \\ 5 & -2 & -3 \end{bmatrix}$.

7. (*a*) $x = 0, -3/2$

(*b*) (*i*) $x = -1$ (*ii*) $x = \pm 4\sqrt{3}$

(*c*) $a = 5, b = 3$.

8. (*a*) $\begin{bmatrix} 2 & 6 & 34 \\ 1 & -7 & -13 \end{bmatrix}$

(*b*) BA is not defined because number of columns of B ≠ number. of rows of A.

9. $AB = \begin{bmatrix} 0 & -4 \\ 10 & 3 \end{bmatrix}, BA = \begin{bmatrix} -10 & 2 & 21 \\ -16 & 2 & 37 \\ -2 & -2 & 11 \end{bmatrix}$.

12. Take $A = \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}$.

13. $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, C = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.

16. $a = 16, b = 39, c = 7$ and $d = 15$.

17. (*i*) No.

18. (*i*) $(AB)\,C = A(BC) = \begin{bmatrix} 4 & 4 & 4 & -7 \\ 35 & -2 & -39 & 22 \\ 31 & 2 & -27 & 11 \end{bmatrix}$

(*ii*) $AC = \begin{bmatrix} 5 & 7 \\ 4 & 5 \end{bmatrix}, BC = \begin{bmatrix} 2 & 2 \\ 7 & 10 \end{bmatrix}$

and $(A + B)\,C = \begin{bmatrix} 7 & 9 \\ 11 & 15 \end{bmatrix}$

(*iii*) $AC = \begin{bmatrix} 9 \\ 12 \\ 30 \end{bmatrix}, BC = \begin{bmatrix} 1 \\ 8 \\ -2 \end{bmatrix}, AC + BC = \begin{bmatrix} 10 \\ 20 \\ 28 \end{bmatrix}$.

19. $X = \begin{bmatrix} 1 & -2 \\ 2 & 0 \end{bmatrix}$.

20. (*i*) $\begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$ (*ii*) $\begin{bmatrix} 39 & 55 \\ -55 & -16 \end{bmatrix}$

(*iii*) (I) $\begin{bmatrix} 1 & -1 & -3 \\ -1 & -1 & -10 \\ -5 & 4 & 4 \end{bmatrix}$ (II) $\begin{bmatrix} 1 & -1 & -1 \\ 3 & -3 & -4 \\ -3 & 2 & 0 \end{bmatrix}$.

23. (*i*) $\begin{bmatrix} 12 & -4 \\ -6 & 8 \end{bmatrix}$ (*ii*) O.

24. (*a*) (*i*) $k = -7$ (*ii*) $k = 1$ (*iii*) $k = 1$ (*b*) $a = -4, b = 1$.

26. $a = 2, b = 3$; No. ; No.

29. X : ₹ 6,500 ; Y : ₹ 2,800 ; Z ; ₹ 11,000

Total fund = ₹ 20,300.

30. (*a*) ₹ 15,000, ₹ 15,000 (*b*) ₹ 5,000, ₹ 25,000.

31. A : 24600 calories, 556 gm proteins

B : 15800 calories : 332 gm proteins.

32. $\begin{bmatrix} -118 & -93 \\ 31 & -118 \end{bmatrix}$.

Hints to Selected Questions

15. $$A^2 = \begin{bmatrix} 3 & -5 \\ -4 & 2 \end{bmatrix}\begin{bmatrix} 3 & -5 \\ -4 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 9+20 & -15-10 \\ -12-8 & 20+4 \end{bmatrix}\begin{bmatrix} 29 & -25 \\ -20 & 24 \end{bmatrix}.$$

$$\therefore\ A^2 - 5A = \begin{bmatrix} 29 & -25 \\ -20 & 24 \end{bmatrix} - 5\begin{bmatrix} 3 & -5 \\ -4 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 29-15 & -25+25 \\ -20+20 & 24-10 \end{bmatrix} = \begin{bmatrix} 14 & 0 \\ 0 & 14 \end{bmatrix},$$

which is a scalar matrix.

19. Let $X = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$.

Then $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} -7 & -8 & -9 \\ 2 & 4 & 6 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} a_{11}+4a_{12} & 2a_{11}+5a_{12} & 3a_{11}+6a_{12} \\ a_{21}+4a_{22} & 2a_{21}+5a_{22} & 3a_{21}+6a_{22} \end{bmatrix} = \begin{bmatrix} -7 & -8 & -9 \\ 2 & 4 & 6 \end{bmatrix}.$$

Compare and solve.

23. $f(A) = A^2 - 2A + 3I$.

26. XY is defined $\Rightarrow a + 2 = b + 1 \Rightarrow a - b = -1$ ...(1)

YX is defined $\Rightarrow a + 3 = a + b \Rightarrow b = 3$.

Putting in (1), $a - 3 = -1 \Rightarrow a = 2$.

30. (*a*) $$[x \quad 30{,}000 - x]\begin{bmatrix} \frac{5}{100} \\ \frac{7}{100} \end{bmatrix} = [1800]$$

$\Rightarrow\ x = 15000$.

31. $F = \begin{bmatrix} 4 & 6 & 2 \\ 2 & 2 & 4 \end{bmatrix}, R = \begin{bmatrix} 2400 & 55 \\ 1900 & 45 \\ 1800 & 33 \end{bmatrix}$.

33. Use $\cos\alpha = \dfrac{1 - \tan^2 \alpha/2}{1 + \tan^2 \alpha/2}$

and $\sin\alpha = \dfrac{2\tan\alpha/2}{1 + \tan^2 \alpha/2}$.

3.6. TRANSPOSE OF A MATRIX

(*a*) The matrix obtained from any given matrix A by interchanging its rows and columns is called its transpose and is denoted by A′ or A^t or A^T. ***(Kashmir B. 2015)***

Definition

If A = $[a_{ij}]$ is an m × n matrix, then the transpose of A, denoted by A′ or A^t or A^T, is defined to be n × m matrix obtained from A by writing the rows of A as columns and columns of A as rows in the same order.

Thus A′ = $[b_{ij}]$, where $b_{ij} = a_{ji}$ for $1 \le j \le m$, $1 \le i \le n$.

For Examples : (I) If $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 2 & 5 \end{bmatrix}$, then $A' = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 2 \\ 4 & 1 & 5 \end{bmatrix}$.

(II) If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, then $A' = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$.

(*b*) **Theorem.** *If A′ and B′ be the transpose of A and B respectively, then :*

(I) *(A′)′ = A* (II) *(A + B)′ = A′ + B′, A and B being of the same order*

(III) *(kA)′ = kA′, k being any scalar*

(IV) **Reversal Law.** *(AB)′ = B′A′, A and B being conformable to multiplication.*

Proof. (I) Let A be an $m \times n$ matrix.

Then A′ will be an $n \times m$ matrix and thus (A′)′ will be an $m \times n$ matrix.

Thus A and (A′)′ are of the same type ...(1)

Now (*i, j*)th element of (A′)′ = (*j, i*)th element of A′ = (*i, j*)th element of A ...(2)

(1) and (2) ⇒ (A′)′ and A are of the same order and their corresponding elements are equal.

Hence, (A′)′ = A.

(II) Let = $[a_{ij}]_{m \times n}$ and B = $[b_{ij}]_{m \times n.}$ Then (A + B) is a matrix of the order $m \times n$

⇒ (A + B)′ is a matrix of the order $n \times m$.

Also as A′ and B′ are both of the order $n \times m$ ⇒ (A′ + B′) is a matrix of the order $n \times m$.

Thus (A + B)′ and (A′ + B′) are both of the same order ...(3)

Now (*i, j*)th element of (A + B)′ = (*j, i*)th element of (A + B) = $a_{ji} + b_{ji}$

= (*j, i*)th element of A + (*j, i*)th element of B

= (*i, j*)th element of A′ + (*i, j*)th element of B′

= (*i, j*)th element of (A′ + B′) ...(4)

(3) and (4) ⇒ (A + B)′ and A′ + B′ are both of the same order and their corresponding elements are equal.

Hence, (A + B)′ = A′ + B′.

(III) Let A = $[a_{ij}]_{\cdot m \times n}$

If *k* is any scalar, then *k*A is a matrix of the same order $m \times n$ and thus (*k*A)′ is a matrix of the order $n \times m$.

Again A′ is a matrix of the order $n \times m$, ∴ *k*A′ is a matrix of the order $n \times m$.

Thus (*k*A)′ and *k*A′ are both of the same order ...(5)

Now (*j, i*)th element of (*k*A)′ = (*i, j*)th element of *k*A = *k* . (*i, j*)th element of A

= *k* . (*j, i*)th elemenet of A′ = (*j, i*)th element of *k*A′ ...(6)

(5) and (6) $\Rightarrow$ $(kA)'$ and kA' are both of the same order and their corresponding elements are equal.

Hence, $(kA)' = kA'$.

(IV) Let $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{n \times p}$

Thus the product AB is defined.

$\therefore$ AB is a matrix of the order $m \times p$ $\Rightarrow$ $(AB)'$ is a matrix of the order $p \times m$.

Now B' is a matrix of the order $p \times n$ and A' is a matrix of the order $n \times m$.

$\therefore$ $B'A'$ is defined and is a matrix of the order $p \times m$.

Thus $(AB)'$ and $B'A'$ are matrices of the same order ...(7)

Now (i, j)th element of $(AB)'$ = (j, i)th element of AB

= Sum of the products of the jth row of A and ith column of B

$$= [a_{j1} \;\; a_{j2} \;\; \ldots \;\; a_{jn}] \begin{bmatrix} b_{1i} \\ b_{2i} \\ \ldots \\ b_{ni} \end{bmatrix} = a_{j1}\, b_{1i} + a_{j2}\, b_{2i} + \ldots.. + a_{jn}\, b_{ni}$$

$= b_{1i}\, a_{j1} + b_{2i}\, a_{j2} + \ldots. + b_{ni}\, a_{jn}$

= Sum of the products of the ith column of B and jth row of A

= Sum of the products of the ith row of B' and jth column of A'

= (i, j)th element of $B'A'$ ...(8)

(7) and (8) $\Rightarrow$ $(AB)'$ and $B'A'$ are both of the same order and their corresponding elements are equal.

Hence, $(AB)' = B'A'$.

Generalisation. $(A_1A_2 \ldots\ldots\ldots A_n)' = A'_n A'_{n-1} \ldots\ldots\ldots\ldots A_2' A_1'$.

3.7. SYMMETRIC AND SKEW-SYMMETRIC MATRICES

(a) Symmetric Matrix.

Definition

A square matrix $A = [a_{ij}]$ is said to be symmetric if (i, j)th element is the same as its (j, i)th element. **(*Kashmir B. 2012*)**

Thus matrix A is symmetric if $A' = A$ *i.e.* if $a_{ji} = a_{ij}$ for all i and j.

For Examples : $\begin{bmatrix} 2 & 4 \\ 4 & 5 \end{bmatrix}, \begin{bmatrix} a_1 & a_2 & a_3 \\ a_2 & b_2 & c_2 \\ a_3 & c_2 & c_3 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$

are all symmetric matrices.

Theorem. *The necessary and sufficient condition for the matrix A to be symmetric is that $A' = A$.*

Proof. The condition is necessary.

Let $A = [a_{ij}]$ be n-rowed symmetric matrix.

Then, by *def.*, $a_{ij} = a_{ji}$ ...(1)

Also A' will be n-rowed square matrix ...(2)

Now (i, j)th element of A' = (j, i)th element of $A = a_{ji} = a_{ij}$ [*Using (1)*]

= (i, j)th element of A ...(3)

(2) and (3) $\Rightarrow$ $A' = A$.

The condition is sufficient.

Here $A' = A$ [*Given*]

$\Rightarrow$ A is a square matrix.

Also (i, j)th element of A $=$ (i, j)th element of A′ $[\because A' = A]$

$=$ (j, i)th element of A.

Hence, A is a symmetric matrix.

(b) Skew-Symmetric Matrix.

Definition

A square matrix A = $[a_{ij}]$ is said to be skew-symmetric if (i, j)th element is negative of its (j, i)th element. **(Kashmir B. 2012)**

Thus matrix A is skew-symmetric if $A' = -A$ *i.e.* if $a_{ji} = -a_{ij}$ for all i and j.

For Example : $\begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}$ is a skew-symmetric matrix.

Theorem. *The necessary and sufficient condition for the matrix A to be skew-symmetric is that $A' = -A$.*

Proof. The condition is necessary.

Let $A = [a_{ij}]$ be n-rowed skew-symmetric matrix.

Then, by *def.*, $a_{ij} = -a_{ji}$ ...(1)

Also A′ will be n-rowed square matrix ...(2)

Now (i, j)th element of A′ $= (j, i)$th element of A

$= a_{ji} = -a_{ij}$ [*Using (1)*]

$= -(i, j)$th element of A ...(3)

(2) and (3) $\Rightarrow$ $A' = -A$.

The condition is sufficient.

Here $A' = -A$ [*Given*]

$\Rightarrow$ A is a square matrix.

Also (i, j)th element of A $= -(i, j)$th element of A′ $[\because A' = -A]$

$= -(j, i)$th element of A.

Hence, A is a skew-symmetric.

KEY POINT

The symmetry of a matrix is that it is symmetric along the main diagonal, whereas the matrix is skew-symmetric if all the elements on the main diagonal are zero and the elements symmetric about the main diagonal are equal in magnitude but of opposite signs.

3.7.1 MORE THEOREMS

Theorem I. *For any square matrix A with real entries, $A + A'$ is a symmetric matrix and $A - A'$ is a skew-symmetric matrix.*

Proof. (*i*) Let $B = A + A'$.

Then $B' = (A + A')'$

$= A' + (A')'$ $[\because (A + B)' = A' + B']$

$= A' + A$ $[\because (A')' = A]$

$= A + A'$ $[\because A + B = B + A]$

$= B$.

By *def.*, $B = A + A'$ is symmetric matrix.

(*ii*) Let $\quad C = A - A'$.

Then $\quad C' = (A - A')'$

$= A' - (A')'$ $\quad$ $[\because (A-B)' = A' - B']$

$= A' - A$ $\quad$ $[\because (A')' = A]$

$= -(A - A')$

$= -C$.

By *def.*, $C = A - A'$ is skew-symmetric matrix.

Theorem II. *Any square matrix can be expressed as that sum of a symmetric and skew-symmetric matrix.*

(Kashmir B. 2012)

Proof. Let A be a square matrix.

Then $\quad A = \frac{1}{2}(A + A') + \frac{1}{2}(A - A')$.

But $A + A'$ is symmetric and $(A - A')$ is skew symmetric $\quad$ [*Th. I.*]

$\Rightarrow \frac{1}{2}(A + A')$ is symmetric and $\frac{1}{2}(A - A')$ is skew-symmetric.

Hence, any square matrix can be expressed as the sum of symmetric matrix and skew-symmetric matrix.

3.8. MORE MATRICES

(*i*) Conjugate Matrix. *The matrix obtained from the matrix A by replacing the elements by their conjugates is said to be conjugate matrix and is denoted by* $\overline{A}$.

(*ii*) Hermitian Matrix. *A matrix A is said to be hermitian if the transpose of the conjugate matrix is the same as A*

i.e. $\quad (\overline{A})' = A$ or $A^{\Theta} = A$.

(*iii*) Skew-Hermitian Matrix. *A matrix A is said to be skew-hermitian if the transpose of the conjugate matrix is negative of A*

i.e. $\quad (\overline{A})' = -A$ *i.e.* $A^{\Theta} = -A$.

(*iv*) Orthogonal Matrix. *A square matrix A is said to be orthogonal* if $A'A = AA' = I$.

(*v*) Unitary Matrix. *A matrix said to be unitary* if $A^{\Theta}A = I$.

(*vi*) Nilpotent Matrix. *A square matrix A is said to be nilpotent if there exists a positive integer 'm' such that* $A^m = O$. The least positive integer '*m*' satisfying $A^m = O$ is said to be the **index** of the nilpotent matrix.

(*vii*) Idempotent Matrix. *A square matrix A is said to be idempotent matrix if* $A^2 = A$.

(*viii*) Involutory Matrix. *A square matrix A is said to be involutory matrix if* $A^2 = I$.

(*ix*) Trace of a Matrix. *Let* $A = [a_{ij}]_{n \times n}$ *be a square matrix. Then the trace of A, denoted by tr (A), is the sum of all main diagonal elements.*

Thus $tr\,(A) = a_{11} + a_{22} + \ldots\ldots\ldots. + a_{nn}$.

Frequently Asked Questions

FAQs

Example 1. Matrix $A = \begin{bmatrix} 0 & 2b & -2 \\ 3 & 1 & 3 \\ 3a & 3 & -1 \end{bmatrix}$ **is given to be symmetric, find values of a and b.** ***(C.B.S.E. 2016)***

Solution. Since A is symmetric, $\quad$ [*Given*]

$\therefore \quad A' = A$

$$\Rightarrow \begin{bmatrix} 0 & 3 & 3a \\ 2b & 1 & 3 \\ -2 & 3 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 2b & -2 \\ 3 & 1 & 3 \\ 3a & 3 & -1 \end{bmatrix}.$$

Comparing, $3 = 2b$, $3a = -2$.

Hence, $a = -\frac{2}{3}$ and $b = \frac{3}{2}$.

Example 2. If the matrix $\begin{bmatrix} 0 & a & -3 \\ 2 & 0 & -1 \\ b & 1 & 0 \end{bmatrix}$ is skew-symmetric, find the values of 'a' and 'b'. (*C.B.S.E. 2018*)

Solution. Let $A = \begin{bmatrix} 0 & a & -3 \\ 2 & 0 & -1 \\ b & 1 & 0 \end{bmatrix}$.

Since A is skew-symmetric,

$\therefore \quad A' = -A$

$$\Rightarrow \begin{bmatrix} 0 & a & -3 \\ 2 & 0 & -1 \\ b & 1 & 0 \end{bmatrix}' = -\begin{bmatrix} 0 & a & -3 \\ 2 & 0 & -1 \\ b & 1 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0 & 2 & b \\ a & 0 & 1 \\ -3 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -a & 3 \\ -2 & 0 & 1 \\ -b & -1 & 0 \end{bmatrix}.$$

Comparing, $\quad 2 = -a$ and $b = 3$.

Hence, $\quad a = -2$ and $b = 3$.

Example 3. If $A = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}$, find '$\alpha$' satisfying $0 < \alpha < \frac{\pi}{2}$ when $A + A^T = \sqrt{2}\, I_2$, where A^T is transpose of A. (*A.I.C.B.S.E. 2016*)

Solution. We have : $A = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}$.

$\therefore \quad A^T = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$.

Now $\quad A + A^T = \sqrt{2}\, I_2$

$$\Rightarrow \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} + \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} = \sqrt{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 2\cos\alpha & 0 \\ 0 & 2\cos\alpha \end{pmatrix} = \begin{pmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{2} \end{pmatrix}.$$

Comparing, $\quad 2\cos\alpha = \sqrt{2} \Rightarrow \cos\alpha = \frac{1}{\sqrt{2}}$.

Hence, $\quad \alpha = \frac{\pi}{4}. \quad \left[\because 0 < \alpha < \frac{\pi}{2}\right]$

Example 4. If $\begin{pmatrix} a+b & 2 \\ 5 & b \end{pmatrix} = \begin{pmatrix} 6 & 5 \\ 2 & 2 \end{pmatrix}'$, then find 'a'. (*C.B.S.E. 2010 C*)

Solution. We have : $\begin{pmatrix} a+b & 2 \\ 5 & b \end{pmatrix} = \begin{pmatrix} 6 & 5 \\ 2 & 2 \end{pmatrix}'$

$$\Rightarrow \begin{pmatrix} a+b & 2 \\ 5 & b \end{pmatrix} = \begin{pmatrix} 6 & 2 \\ 5 & 2 \end{pmatrix}.$$

Comparing, $\quad a + b = 6 \quad$ ...(1)

and $\quad b = 2 \quad$ ...(2)

Putting the value of b from (2) in (1), $a + 2 = 6$.

Hence, $\quad a = 4$.

Example 5. If $A = \begin{bmatrix} 3 & \sqrt{3} & 2 \\ 4 & 2 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -1 & 2 \\ 1 & 2 & 4 \end{bmatrix}$, then verify that :

(i) $(A')' = A$

(ii) $(A + B)' = A' + B'$

(iii) $(kB)' = kB'$, where k is any constant. (*N.C.E.R.T.*)

Solution. (*i*) We have : $A = \begin{bmatrix} 3 & \sqrt{3} & 2 \\ 4 & 2 & 0 \end{bmatrix}$.

$$\therefore \quad A' = \begin{bmatrix} 3 & 4 \\ \sqrt{3} & 2 \\ 2 & 0 \end{bmatrix}.$$

$$\therefore \quad (A')' = \begin{bmatrix} 3 & \sqrt{3} & 2 \\ 4 & 2 & 0 \end{bmatrix} = A.$$

Hence, $(A')' = A$.

(*ii*) We have : $A = \begin{bmatrix} 3 & \sqrt{3} & 2 \\ 4 & 2 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -1 & 2 \\ 1 & 2 & 4 \end{bmatrix}$.

$$\therefore \quad A + B = \begin{bmatrix} 3+2 & \sqrt{3}-1 & 2+2 \\ 4+1 & 2+2 & 0+4 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & \sqrt{3}-1 & 4 \\ 5 & 4 & 4 \end{bmatrix}.$$

$$\therefore \quad (A + B)' = \begin{bmatrix} 5 & 5 \\ \sqrt{3}-1 & 4 \\ 4 & 4 \end{bmatrix} \quad \text{...(1)}$$

Also $\quad A' = \begin{bmatrix} 3 & 4 \\ \sqrt{3} & 2 \\ 2 & 0 \end{bmatrix}$ and $B' = \begin{bmatrix} 2 & 1 \\ -1 & 2 \\ 2 & 4 \end{bmatrix}$.

$$\therefore \quad A' + B' = \begin{bmatrix} 3+2 & 4+1 \\ \sqrt{3}-1 & 2+2 \\ 2+2 & 0+4 \end{bmatrix} = \begin{bmatrix} 5 & 5 \\ \sqrt{3}-1 & 4 \\ 4 & 4 \end{bmatrix} \quad ...(2)$$

From (1) and (2), $(A + B)' = A' + B'$.

(*iii*) $$kB = k\begin{bmatrix} 2 & -1 & 2 \\ 1 & 2 & 4 \end{bmatrix} = \begin{bmatrix} 2k & -k & 2k \\ k & 2k & 4k \end{bmatrix}.$$

$$\therefore \quad (kB)' = \begin{bmatrix} 2k & k \\ -k & 2k \\ 2k & 4k \end{bmatrix} = k\begin{bmatrix} 2 & 1 \\ -1 & 2 \\ 2 & 4 \end{bmatrix} = kB'.$$

Hence, $(kB)' = kB'$.

Example 6. Show that A + A′ is symmetric when $A = \begin{bmatrix} 2 & 4 \\ 5 & 6 \end{bmatrix}$. *(H.B. 2014)*

Solution. We have : $A = \begin{bmatrix} 2 & 4 \\ 5 & 6 \end{bmatrix}$.

$$\therefore \quad A' = \begin{bmatrix} 2 & 5 \\ 4 & 6 \end{bmatrix}.$$

$$\therefore \quad A + A' = \begin{bmatrix} 2 & 4 \\ 5 & 6 \end{bmatrix} + \begin{bmatrix} 2 & 5 \\ 4 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 2+2 & 4+5 \\ 5+4 & 6+6 \end{bmatrix} = \begin{bmatrix} 4 & 9 \\ 9 & 12 \end{bmatrix} \quad ...(1)$$

Now $$(A + A')' = \begin{bmatrix} 4 & 9 \\ 9 & 12 \end{bmatrix} = A + A' \quad [\text{Using } (1)]$$

Hence, $A + A'$ is symmetric matrix.

Example 7. Show that A – A′ is skew-symmtric when $A = \begin{bmatrix} 1 & 4 \\ 3 & 7 \end{bmatrix}$. *(H. B. 2014)*

Solution. We have : $A = \begin{bmatrix} 1 & 4 \\ 3 & 7 \end{bmatrix}$.

$$\therefore \quad A' = \begin{bmatrix} 1 & 3 \\ 4 & 7 \end{bmatrix}.$$

$$\therefore \quad A - A' = \begin{bmatrix} 1 & 4 \\ 3 & 7 \end{bmatrix} - \begin{bmatrix} 1 & 3 \\ 4 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 4 \\ 3 & 7 \end{bmatrix} + \begin{bmatrix} -1 & -3 \\ -4 & -7 \end{bmatrix}$$

$$= \begin{bmatrix} 1-1 & 4-3 \\ 3-4 & 7-7 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad ...(1)$$

Now $$(A - A')' = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = -\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$= -(A - A'). \quad [\text{Using } (1)]$$

Hence, $A - A'$ is skew-symmetric matrix.

Example 8. If $A' = \begin{bmatrix} 3 & 4 \\ -1 & 2 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix}$, then verify that : (A + B)′ = A′ + B′. *(N.C.E.R.T.)*

Solution. We have : $A' = \begin{bmatrix} 3 & 4 \\ -1 & 2 \\ 0 & 1 \end{bmatrix}$

and $$B = \begin{bmatrix} -1 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix}.$$

$$\therefore \quad A = \begin{bmatrix} 3 & -1 & 0 \\ 4 & 2 & 1 \end{bmatrix} \text{ and } B' = \begin{bmatrix} -1 & 1 \\ 2 & 2 \\ 1 & 3 \end{bmatrix}.$$

(*i*) $$A + B = \begin{bmatrix} 3 & -1 & 0 \\ 4 & 2 & 1 \end{bmatrix} + \begin{bmatrix} -1 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 3-1 & -1+2 & 0+1 \\ 4+1 & 2+2 & 1+3 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 5 & 4 & 4 \end{bmatrix}.$$

$$\therefore \quad (A + B)' = \begin{bmatrix} 2 & 5 \\ 1 & 4 \\ 1 & 4 \end{bmatrix} \quad ...(1)$$

And $$A' + B' = \begin{bmatrix} 3 & 4 \\ -1 & 2 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} -1 & 1 \\ 2 & 2 \\ 1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 3-1 & 4+1 \\ -1+2 & 2+2 \\ 0+1 & 1+3 \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ 1 & 4 \\ 1 & 4 \end{bmatrix} \quad ...(2)$$

From (1) and (2), $(A + B)' = A' + B'$, which verifies the result.

Example 9. If A = $\begin{bmatrix} -2 \\ 4 \\ 5 \end{bmatrix}$, B = [1 3 −6], then verify that (AB)' = B'A'.

(N.C.E.R.T.; H.P.B. 2018, 12 ; P.B. 2014; Kashmir B. 2012)

Solution. We have :

$$A = \begin{bmatrix} -2 \\ 4 \\ 5 \end{bmatrix} \quad \text{and } B = [1 \ \ 3 \ \ -6].$$

$$\therefore \qquad A' = [-2 \ 4 \ 5] \text{ and } B' = \begin{bmatrix} 1 \\ 3 \\ -6 \end{bmatrix}.$$

Also $\qquad AB = \begin{bmatrix} -2 \\ 4 \\ 5 \end{bmatrix} [1 \ \ 3 \ \ -6]$

$$= \begin{bmatrix} -2 & -6 & 12 \\ 4 & 12 & -24 \\ 5 & 15 & -30 \end{bmatrix}.$$

$$\therefore \qquad (AB)' = \begin{bmatrix} -2 & 4 & 5 \\ -6 & 12 & 15 \\ 12 & -24 & -30 \end{bmatrix} \qquad ...(1)$$

And $\qquad B'A' = \begin{bmatrix} 1 \\ 3 \\ -6 \end{bmatrix} [-2 \ \ 4 \ \ 5]$

$$= \begin{bmatrix} -2 & 4 & 5 \\ -6 & 12 & 15 \\ 12 & -24 & -30 \end{bmatrix} \qquad ...(2)$$

From (1) and (2), $(AB)' = B'A'$,

which verifies the result.

Example 10. If A and B are symmetric matrices of the same order, then show that AB is symmetric iff AB = BA i.e. A and B commute.

(N.C.E.R.T.; Karnataka B. 2017)

Solution. AB is symmetric

$\Rightarrow \quad (AB)' = AB$ [*Def.*]

$\Rightarrow \quad B'A' = AB$ [*Reversal Law*]

$\Rightarrow \quad BA = AB.$

[$\because$ *A is symmetric,* $\therefore$ *A' = A. Similarly B' = B*]

Hence, A and B commute.

Conversely :

$(AB)' = B'A'$

$= BA$ [$\because$ *A and B are symmetric*]

$= AB.$ [$\because$ *A and B commute*]

Hence, AB is symmetric.

Example 11. Let A be a square symmetric matrix. Show that :

(a) (i) $\frac{1}{2}(A + A')$ is a symmetric matrix

(ii) $\frac{1}{2}(A - A')$ is a skew-symmetric matrix.

(b) Also prove that any square matrix can be $\boxed{\text{uniquely}}$ expressed as the sum of a symmetric matrix and a skew-symmetric matrix.

Solution. Since A is a square symmetric matrix, [*Given*]

$\therefore A' = A.$

(*a*) (*i*) $\left[\frac{1}{2}(A + A')\right]' = \frac{1}{2}(A + A')'$

$$= \frac{1}{2}(A' + (A')') = \frac{1}{2}(A' + A) = \frac{1}{2}(A + A').$$

Hence, $\frac{1}{2}(A + A')$ is a symmetric matrix.

(*ii*) $\left[\frac{1}{2}(A - A')\right]' = \frac{1}{2}(A - A')'$

$$= \frac{1}{2}(A' - (A')') = \frac{1}{2}(A' - A) = -\frac{1}{2}(A - A').$$

Hence, $\frac{1}{2}(A - A')$ is a skew-symmetric matrix.

(*b*) Given a square matrix A.

Take $P = \frac{1}{2}(A + A')$ and $Q = \frac{1}{2}(A - A')$.

Clearly $A = \frac{1}{2}(2A) = \frac{1}{2}(A + A' + A - A')$

$$= \frac{1}{2}(A + A') + \frac{1}{2}(A - A') = P + Q.$$

Thus A has been expressed as the sum of P and Q matrices.

But P is symmetric and Q is skew-symmetric matrix.

[*See Part* (*a*) (*i*) *and* (*ii*)]

Hence, any square matrix can be expressed as the sum of symmetric matrix and skew-symmetric matrix.

Uniqueness :

If possible, let A have two representations *viz.*

$A = P + Q$ ...(1) and $A = R + S$ ...(2),

where P and R are symmetric

i.e. $P' = P$ and $R' = R$ ...(3)

and Q and S are skew-symmetric

i.e. $Q' = -Q$ and $S' = -S$ ...(4)

Taking transposes on both sides of (1) and (2), we get :

$A' = P' + Q'$ and $A' = R' + S'$

$\Rightarrow A' = P - Q$ ...(5) and $A' = R - S$ ...(6)

[*Using (3) & (4)*]

Adding (1) and (5), $A + A' = 2P$ ...(7)

Adding (2) and (6), $A + A' = 2R$ ...(8)

Subtracting (5) from (1), $A - A' = 2Q$ ...(9)

Subtracting (6) from (2), $A - A' = 2S$ ...(10)

From (7) and (8), $2P = 2R \Leftrightarrow R = P$.

From (9) and (10), $2Q = 2S \Leftrightarrow S = Q$.

Thus the second representation

$A = R + S$ becomes $A = P + Q$,

which establishes the uniqueness.

Hence, any square matrix can be **uniquely** expressed as the sum of a symmetric matrix and skew-symmetric matrix.

Example 12. Express $\begin{bmatrix} 3 & 4 \\ -1 & 5 \end{bmatrix}$ as the sum of symmetric and a skew symmetric matrix.

(*Jammu B. 2013*)

Solution. Let $A = \begin{bmatrix} 3 & 4 \\ -1 & 5 \end{bmatrix}$.

$$\therefore \quad A' = \begin{bmatrix} 3 & -1 \\ 4 & 5 \end{bmatrix}.$$

$$\therefore \quad A + A' = \begin{bmatrix} 3 & 4 \\ -1 & 5 \end{bmatrix} + \begin{bmatrix} 3 & -1 \\ 4 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 3+3 & 4-1 \\ -1+4 & 5+5 \end{bmatrix} = \begin{bmatrix} 6 & 3 \\ 3 & 10 \end{bmatrix}$$

$$\Rightarrow \frac{1}{2}(A + A') = \begin{bmatrix} 3 & 3/2 \\ 3/2 & 5 \end{bmatrix},$$

which is symmetric.

$$\text{And} \quad A - A' = \begin{bmatrix} 3 & 4 \\ -1 & 5 \end{bmatrix} - \begin{bmatrix} 3 & -1 \\ 4 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 3-3 & 4+1 \\ -1-4 & 5-5 \end{bmatrix} = \begin{bmatrix} 0 & 5 \\ -5 & 0 \end{bmatrix}$$

$$\Rightarrow \frac{1}{2}(A - A') = \begin{bmatrix} 0 & 5/2 \\ -5/2 & 0 \end{bmatrix},$$

which is skew-symmetric.

$$\text{Hence,} \quad A = \frac{1}{2}(A + A') + \frac{1}{2}(A - A')$$

$$= \begin{bmatrix} 3 & 3/2 \\ 3/2 & 5 \end{bmatrix} + \begin{bmatrix} 0 & 5/2 \\ -5/2 & 0 \end{bmatrix}.$$

Example 13. Express the matrix A as the sum of a symmetric and a skew-symmetric matrix, where :

$$\mathbf{A} = \begin{bmatrix} \mathbf{3} & \mathbf{-2} & \mathbf{-4} \\ \mathbf{3} & \mathbf{-2} & \mathbf{-5} \\ \mathbf{-1} & \mathbf{1} & \mathbf{2} \end{bmatrix}.$$

(*Type :W. Bengal B. 2016; A.I.C.B.S.E. 2010*)

Solution. We have :

$$A = \begin{bmatrix} 3 & -2 & -4 \\ 3 & -2 & -5 \\ -1 & 1 & 2 \end{bmatrix}.$$

$$\therefore \quad A' = \begin{bmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{bmatrix}.$$

$$\therefore \quad A + A' = \begin{bmatrix} 3 & -2 & -4 \\ 3 & -2 & -5 \\ -1 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 3+3 & -2+3 & -4-1 \\ 3-2 & -2-2 & -5+1 \\ -1-4 & 1-5 & 2+2 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 1 & -5 \\ 1 & -4 & -4 \\ -5 & -4 & 4 \end{bmatrix}$$

$$\Rightarrow \frac{1}{2}(A + A') = \begin{bmatrix} 3 & 1/2 & -5/2 \\ 1/2 & -2 & -2 \\ -5/2 & -2 & 2 \end{bmatrix},$$

which is symmetric.

$$\text{And} \quad A - A' = \begin{bmatrix} 3 & -2 & -4 \\ 3 & -2 & -5 \\ -1 & 1 & 2 \end{bmatrix} - \begin{bmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 3-3 & -2-3 & -4+1 \\ 3+2 & -2+2 & -5-1 \\ -1+4 & 1+5 & 2-2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -5 & -3 \\ 5 & 0 & -6 \\ 3 & 6 & 0 \end{bmatrix}$$

$$\Rightarrow \frac{1}{2}(A - A') = \begin{bmatrix} 0 & -5/2 & -3/2 \\ 5/2 & 0 & -3 \\ 3/2 & 3 & 0 \end{bmatrix},$$

which is skew-symmetric.

Hence, $A = \frac{1}{2}(A + A') + \frac{1}{2}(A - A')$

$$= \begin{bmatrix} 3 & 1/2 & -5/2 \\ 1/2 & -2 & -2 \\ -5/2 & -2 & 2 \end{bmatrix} + \begin{bmatrix} 0 & -5/2 & -3/2 \\ 5/2 & 0 & -3 \\ 3/2 & 3 & 0 \end{bmatrix}.$$

EXERCISE 3 (e)

Fast Track Answer Type Questions

FTATQ

1. (*a*) For any square matrix A, A – A′ is ..... matrix.

(Fill in the Blank)

(*b*) If A and B are matrices of same order, then (AB)′ = B′A′. **(True/False)**

(Jammu B. 2018 ; Kashmir B. 2017)

(*c*) If A is a matrix of order 3 × 4 and B is a matrix of order 4 × 5, what is the order of the matrix $(AB)^T$?

(Assam B. 2017)

2. Find the transpose of each of the following matrices :

(*i*) $[1 \quad -1 \quad i]$ (*ii*) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

(*iii*) $\begin{bmatrix} 1 & 3 & 2 \\ 0 & 2 & 1 \\ 9 & 5 & 3 \end{bmatrix}$.

3. For what value of 'x', is the matrix :

$$A = \begin{bmatrix} 0 & 1 & -2 \\ -1 & 0 & 3 \\ x & -3 & 0 \end{bmatrix}$$

a skew-symmetric matrix. ***(A.I.C.B.S.E. 2013)***

4. (*a*) If $A = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 2 & 3 & 4 \end{bmatrix}$, verify that :

$$\frac{3}{4}A' = \left(\frac{3}{4}A\right)'.$$

(Uttarakhand B. 2013 ; Rajasthan B. 2013)

(*b*) If matrix $A = (1 \quad 2 \quad 3)$, write AA′, where A′ is the transpose of matrix A. ***(C.B.S.E. 2009)***

(*c*) If $A = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$ and $A + A^T = I$, write down the general value of α. ***(Assam B. 2017)***

5. Consider a 2 × 2 matrix $A = [a_{ij}]$, where $a_{ij} = \frac{(i+2j)^2}{2}$. Find A + A′. ***(Kerala B. 2014)***

Very Short Answer Type Questions

VSATQ

6. Show the $A = \frac{1}{3}\begin{bmatrix} -1 & 2 & -2 \\ -2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$ is proper orthogonal matrix. ***(W. Bengal B. 2017)***

7. (*a*) If $A = \begin{bmatrix} -1 & 2 & 3 \\ 5 & 7 & 9 \\ -2 & 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} -4 & 1 & -5 \\ 1 & 2 & 0 \\ 1 & 3 & 1 \end{bmatrix}$, then verify that :

(*i*) (A + B)′ = A′ + B′ (*ii*) (A – B)′ = A′ – B′.

(N.C.E.R.T.; H.P.B. 2014)

(*b*) If $A = \begin{bmatrix} -2 & 3 \\ 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 0 \\ 1 & 2 \end{bmatrix}$,

then find (A + 2B)′. ***(N.C.E.R.T.)***

(*c*) If $A' = \begin{bmatrix} 3 & 4 \\ -1 & 2 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix}$, then verify that (*i*) (A + B)′ = A′ + B′ (*ii*) (A – B)′ = A′ – B′.

(N.C.E.R.T)

8. If $X' = \begin{bmatrix} 3 & 4 \\ -1 & 2 \\ 0 & 1 \end{bmatrix}$ and $Y = \begin{bmatrix} -1 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix}$, then find X′ – Y′. ***(Meghalaya B. 2014)***

9. (*i*) If $A = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$, $B = [1 \quad 5 \quad 7]$, verify that (AB)′ = B′A′.

(N.C.E.R.T. ; H.P.B. 2012)

(*ii*) If $A = \begin{bmatrix} 1 \\ -4 \\ 3 \end{bmatrix}$, $B = [-1 \quad 2 \quad 1]$, verify that : (AB)′ = B′A′.

(N.C.E.R.T.; H.P.B. 2012 ; A.I.C.B.S.E. 2010)

(*iii*) If $A = \begin{bmatrix} 3 \\ -1 \\ 5 \end{bmatrix}$, $B = [-6 \quad 7 \quad 10]$, verify that :

$(AB)' = B'A'$. **(P.B. 2016)**

(*iv*) If $A = \begin{bmatrix} 2 \\ -4 \\ 1 \end{bmatrix}$, $B = [6 \quad 3 \quad -1]$, verify that :

$(AB)' = B'A'$. **(P.B. 2017)**

(*v*) If $A = \begin{bmatrix} 3 \\ -4 \\ 5 \end{bmatrix}$, $B = [6 \quad 1 \quad -1]$, verify that :

$(AB)' = B'A'$. **(P.B. 2017)**

Short Answer Type Questions

11. Show that $A + A^T$ is symmetric matrix, where A^T denotes the transpose of A :

(*i*) $A = \begin{bmatrix} 1 & 5 \\ 6 & 7 \end{bmatrix}$ **(H.P.B. 2018; H.B. 2017; Jammu B. 2017; H.P.B. 2015, 13)**

(*ii*) $\begin{bmatrix} 1 & 5 \\ -6 & 7 \end{bmatrix}$ **(H.B. 2016)**

(*iii*) $A = \begin{bmatrix} 2 & 3 & -1 \\ 4 & 5 & 2 \\ 0 & 6 & 1 \end{bmatrix}$. **(H.B. 2011)**

12. Show that $A - A^T$ is skew-symmetric matrix, where A^T denotes the transpose of A :

(*i*) $A = \begin{bmatrix} 1 & 5 \\ 6 & 7 \end{bmatrix}$ **(H.P.B. 2018; Jammu B. 2017)**

(*ii*) $\begin{bmatrix} 6 & -8 \\ 7 & -9 \end{bmatrix}$ **(H.B. 2016)**

(*iii*) $A = \begin{bmatrix} 1 & 5 \\ 8 & 7 \end{bmatrix}$ **(Jammu B. 2013)**

(*iv*) $A = \begin{bmatrix} 4 & -1 & 2 \\ 3 & 0 & 5 \\ 6 & 1 & 3 \end{bmatrix}$. **(H.B. 2011)**

13. If $A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 2 \\ 1 & -3 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 5 & 6 \\ -1 & 0 & 1 \\ 2 & 1 & 2 \end{bmatrix}$,

$C = \begin{bmatrix} -1 & -2 & 1 \\ -1 & 2 & 3 \\ -1 & -2 & 2 \end{bmatrix}$, find $(A + B + C)'$.

Is $(A + B + C)' = A' + B' + C'$?

10. (*a*) Show that the matrix, $A = \begin{bmatrix} 1 & -1 & 5 \\ -1 & 2 & 1 \\ 5 & 1 & 3 \end{bmatrix}$ is a symmetric matrix. **(N.C.E.R.T. ; H.P.B. 2013)**

(*b*) Show that the matrix, $A = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}$ is a skew-symmetric matrix. **(N.C.E.R.T.)**

(*c*) Show that the matrix A is skew-symmetric, where :

$$A = \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}.$$ **(H.B. 2010)**

SATQ

14. (*i*) If $A = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$, then verify that :

$A'A = I$. **(N.C.E.R.T.; H.P.B. 2015; Nagaland B. 2015)**

(*ii*) If $A = \begin{bmatrix} \sin\alpha & \cos\alpha \\ -\cos\alpha & \sin\alpha \end{bmatrix}$, then prove that $A'A = I$.

(N.C.E.R.T. ; H.P.B. 2015)

Show that (AB)′ = B′A′ (15 – 16) :

15. $A = \begin{bmatrix} -1 & 3 & 0 \\ -7 & 2 & 8 \end{bmatrix}$, $B = \begin{bmatrix} -5 & 0 \\ 0 & 3 \\ 1 & -8 \end{bmatrix}$.

16. $A = \begin{bmatrix} 3 & 4 \\ 4 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 5 & 3 \\ 2 & 1 \end{bmatrix}$. **(P.B. 2010)**

17. If $A = \begin{bmatrix} 5 & -1 \\ 6 & 7 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 3 \\ -1 & 4 \end{bmatrix}$,

verify the following :

(*i*) $(A')' = A$

(*ii*) $(A + B)' = A' + B'$

(*iii*) $(3A)' = 3A'$

(*iv*) $(AB)' = B'A'$.

18. If A be a square matrix, prove that :

AA' and $A'A$ are both symmetric matrices.

19. Verify that :

(*a*) $A + A'$ is a Symmetric Matrix

(*b*) $A - A'$ is a Skew-symmetric Matrix when :

(*i*) $A = \begin{bmatrix} 1 & 5 \\ 6 & 7 \end{bmatrix}$ (*ii*) $A = \begin{bmatrix} 2 & 5 \\ 4 & 1 \end{bmatrix}$

(*iii*) $A = \begin{bmatrix} 6 & 2 \\ 4 & 5 \end{bmatrix}$,

where A' is the transpose of A. **(P.B. 2012)**

20. (*a*) For the matrix $A = \begin{bmatrix} 1 & 5 \\ 6 & 7 \end{bmatrix}$, verify that :

(*i*) $(A + A')$ is a symmetric matrix

(*ii*) $(A - A')$ is a skew-symmetric matrix.

(N.C.E.R.T. ; Jammu B. 2013)

(*b*) If $A = \begin{bmatrix} 3 & 1 & -1 \\ 0 & 1 & 2 \end{bmatrix}$, then show that '$AA'$' is a symmetric matrix. ***(H.B. 2015)***

21. Find $\frac{1}{2}(A + A')$ and $\frac{1}{2}(A - A')$ when :

$$A = \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}.$$

(N.C.E.R.T.; Jammu B. 2015; Meghalaya B. 2015)

22. Express (*i*) $\begin{bmatrix} 3 & 5 \\ 1 & -1 \end{bmatrix}$

(Kerala B. 2016; Jammu B. 2013)

(*ii*) $\begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$

as the sum of symmetric and skew-symmetric matrices.

(N.C.E.R.T.; Mizoram B. 2016)

23. Show that all the diagonal elements of a skew-symmetric matrix are zero. ***(C.B.S.E. 2017)***

24. Show that the matrix $B'AB$ is symmetric or skew-symmetric according as A is symmetric or skew-symmetric.

(N.C.E.R.T.)

25. Let A and B be symmetric matrices of the same order. Then show that :

(*i*) $A + B$ is a symmetric matrix

(*ii*) $AB - BA$ is skew-symmetric matrix

(N.C.E.R.T. ; H.B. 2017; Jammu B. 2012 ; H.P.B. 2009 S)

(*iii*) $AB + BA$ is a symmetric matrix.

Long Answer Type Questions

LATQ

26. Express the following as the sum of symmetric and skew-symmetric matrices :

(*i*) $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$ ***(N.C.E.R.T.)***

(*ii*) $\begin{bmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{bmatrix}$

(N.C.E.R.T.; Assam B. 2018; Jammu B. 2016; P.B. 2010)

(*iii*) $\begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$

(N.C.E.R.T.; Kerala B. 2018 Kashmir B. 2016, 12, 11; Jammu B. 2015 W; P.B. 2010)

(*iv*) $\begin{bmatrix} 1 & 3 & 1 \\ 1 & 3 & 2 \\ 5 & -4 & 5 \end{bmatrix}$ ***(N.C.E.R.T.)***

(*v*) $\begin{bmatrix} 2 & 5 & -1 \\ 3 & 1 & 5 \\ 7 & 6 & 9 \end{bmatrix}$ ***(P.B. 2018)***

(*vi*) $\begin{bmatrix} 2 & -4 & 5 \\ 1 & 8 & -2 \\ 7 & 3 & 9 \end{bmatrix}$ ***(P.B. 2017)***

(*vii*) $\begin{bmatrix} 1 & 2 & -3 \\ 7 & 0 & 5 \\ -4 & 8 & 9 \end{bmatrix}$ ***(P.B. 2016)***

(*viii*) $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 5 & 6 & 7 \end{bmatrix}$ ***(Tripura B. 2016)***

(*ix*) $\begin{bmatrix} 3 & 2 & 3 \\ 4 & 5 & 3 \\ 2 & 4 & 5 \end{bmatrix}$. ***(W. Bengal B. 2018)***

Answers

1. (*a*) skew-symmetric (*b*) True (*c*) 5×3.

2. (*i*) $\begin{bmatrix} 1 \\ -1 \\ i \end{bmatrix}$ (*ii*) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ (*iii*) $\begin{bmatrix} 1 & 0 & 9 \\ 3 & 2 & 5 \\ 2 & 1 & 3 \end{bmatrix}$.

3. $x = 2$.

4. (*b*) (14) (*c*) $\alpha = \frac{\pi}{3}$

5. $\begin{bmatrix} 9 & 41/2 \\ 41/2 & 36 \end{bmatrix}$.

7. (*b*) $\begin{bmatrix} -4 & 3 \\ 3 & 6 \end{bmatrix}$.

8. $\begin{bmatrix} 4 & 3 \\ -3 & 0 \\ -1 & -2 \end{bmatrix}$.

13. $\begin{bmatrix} 4 & -3 & 2 \\ 5 & 2 & -4 \\ 10 & 6 & 3 \end{bmatrix}$; Yes.

21. (*i*) O (*ii*) $\begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}$.

22. (*i*) $\begin{bmatrix} 3 & 3 \\ 3 & -1 \end{bmatrix} + \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$

(*ii*) $\begin{bmatrix} 3 & -3/2 \\ -3/2 & -1 \end{bmatrix} + \begin{bmatrix} 0 & -5/2 \\ 5/2 & 0 \end{bmatrix}$.

26. (*i*) $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

(*ii*) $\begin{bmatrix} 3 & 1/2 & -5/2 \\ 1/2 & -2 & -2 \\ -5/2 & -2 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 5/2 & 3/2 \\ -5/2 & 0 & 3 \\ -3/2 & -3 & 0 \end{bmatrix}$

(*iii*) $\begin{bmatrix} 2 & -3/2 & -3/2 \\ -3/2 & 3 & 1 \\ -3/2 & 1 & -3 \end{bmatrix} + \begin{bmatrix} 0 & -1/2 & -5/2 \\ 1/2 & 0 & 3 \\ 5/2 & -3 & 0 \end{bmatrix}$

(*iv*) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & -1 \\ 3 & -1 & 5 \end{bmatrix} + \begin{bmatrix} 0 & 1 & -2 \\ -1 & 0 & 3 \\ 2 & -3 & 0 \end{bmatrix}$

(*v*) $\begin{bmatrix} 2 & 4 & 3 \\ 4 & 1 & 11/2 \\ 3 & 11/2 & 9 \end{bmatrix} + \begin{bmatrix} 0 & 1 & -4 \\ -1 & 0 & -1/2 \\ 4 & 1/2 & 0 \end{bmatrix}$

(*vi*) $\begin{bmatrix} 2 & -3/2 & 6 \\ -3/2 & 8 & 1/2 \\ 6 & 1/2 & 9 \end{bmatrix} + \begin{bmatrix} 0 & -5/2 & -1 \\ 5/2 & 0 & -5/2 \\ 1 & 5/2 & 0 \end{bmatrix}$

(*vii*) $\begin{bmatrix} 1 & 9/2 & -7/2 \\ 9/2 & 0 & 13/2 \\ -7/2 & 13/2 & 9 \end{bmatrix} + \begin{bmatrix} 0 & -5/2 & 1/2 \\ 5/2 & 0 & -3/2 \\ -1/2 & 3/2 & 0 \end{bmatrix}$

(*viii*) $\begin{bmatrix} 1 & 5/2 & 4 \\ 5/2 & 4 & 11/2 \\ 4 & 11/2 & 7 \end{bmatrix} + \begin{bmatrix} 0 & -1/2 & -1 \\ 1/2 & 0 & -1/2 \\ 1 & 1/2 & 0 \end{bmatrix}$.

(*ix*) $\begin{bmatrix} 3 & 5 & 5/2 \\ 3 & 5 & 7/2 \\ 5/2 & 7/2 & 5 \end{bmatrix} + \begin{bmatrix} 0 & -1 & -1/2 \\ 0 & 0 & -1/2 \\ -1/2 & 1/2 & 0 \end{bmatrix}$.

Hints to Selected Questions

3. Use $A = -A'$.

12. (*i*) $A - A^T = \begin{bmatrix} 1 & 5 \\ 6 & 7 \end{bmatrix} - \begin{bmatrix} 1 & 6 \\ 5 & 7 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

Hence, $(A - A^T)^T$

$= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = -\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = -(A - A^T)$.

18. $(AA')' = (A')'\,A' = AA'$

$(A'A)' = A'\,(A')' = A'A$.

23. (*a*) $a_{ii} = -a_{ii} \Rightarrow 2a_{ii} = 0 \Rightarrow a_{ii} = 0$.

24. Since A is symmetric, $\therefore\ A' = A$.

Now $(B'AB)' = B'A'\,(B')'$

$= B'A'\,B = B'AB$.

$\therefore$ $B'AB$ is symmetric.

25. (*ii*) Let $C = AB - BA$.

$\therefore\ C' = (AB - BA)' = (AB)' - (BA)' = B'A' - A'B'$

$= BA - AB = -C$.

3.9. ELEMENTARY OPERATIONS TRANSFORMATIONS)

There are six operations (transformations) of a matrix – three for rows and three for columns, which are known as **elementary operations** or **elementary transformations.**

(I) *Interchange of any two rows (columns).*

Symbolically : (*a*) Interchange of *i*th and *j*th rows is denoted by $R_i \leftrightarrow R_j$.

(*b*) Interchange of *i*th and *j*th columns is denoted by $C_i \leftrightarrow C_j$.

For Example : Let $A = \begin{bmatrix} 5 & 6 & 7 \\ -1 & \sqrt{3} & 1 \\ 2 & 4 & 8 \end{bmatrix}$.

Applying $R_1 \leftrightarrow R_2$, $A = \begin{bmatrix} -1 & \sqrt{3} & 1 \\ 5 & 6 & 7 \\ 2 & 4 & 8 \end{bmatrix}$.

(II) *Multiplication of the elements of any row (column) by a non-zero number.*

Symbolically : (*a*) Multiplication of each element of *i*th row by non-zero '*k*' is denoted by $R_i \to kR_i$.

(*b*) Multiplication of each element of *i*th column by non-zero '*k*' is denoted by $C_i \to kC_i$.

For Example : Let $B = \begin{bmatrix} 1 & 2 & 3 \\ -1 & \sqrt{3} & 1 \end{bmatrix}$.

Applying $C_3 \to \frac{1}{5} C_3$, $B = \begin{bmatrix} 1 & 2 & 3/5 \\ -1 & \sqrt{3} & 1/5 \end{bmatrix}$.

(III) *Addition to the elements of any row (column) the corresponding elements of any other row (column) multiplied by any non-zero number.*

Symbolically : (*a*) Addition to the elements of *i*th row, the corresponding elements of *j*th row multiplied by non-zero '*k*' is denoted by $R_i \to R_i + k\,R_j$.

(*b*) Addition to the elements of *i*th column, the corresponding elements of *j*th column multiplied by non-zero '*k*' is denoted by $C_i \to C_i + kC_j$.

For Example : Let $C = \begin{bmatrix} 2 & 3 \\ 3 & -2 \end{bmatrix}$.

Applying $R_2 \to R_2 - 2R_1$, $C = \begin{bmatrix} 2 & 3 \\ -1 & -8 \end{bmatrix}$.

3.9.1. EQUIVALENT MATRICES

Definition

Two matrices are said to be equivalent if one can be obtained from the other by a sequence of elementary operations.

If A and B are equivalent, then we write : A ~ B.

For Example : $\begin{bmatrix} 5 & 6 & 7 \\ -1 & \sqrt{3} & 1 \\ 2 & 4 & 8 \end{bmatrix} \sim \begin{bmatrix} -1 & \sqrt{3} & 1 \\ 5 & 6 & 7 \\ 2 & 4 & 8 \end{bmatrix}$

because second can be obtained from first by $R_1 \leftrightarrow R_2$.

3.10. INVERTIBLE MATRIX AND INVERSE OF A MATRIX

Definition

Any n–rowed square matrix A is said to be invertible if there exists an n–rowed matrix B such that

$$AB = BA = I_n,$$

where I_n is the unit matrix of order n.

B is called the **inverse** of A or **reciprocal** of A.

Inverse of A is generally denoted by A^{-1}.

For Example : *Let* $A = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$ *and* $B = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$.

Here $AB = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 4-3 & 6-6 \\ -2+2 & -3+4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I.$

And $$BA = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 4-3 & -6+6 \\ 2-2 & -3+4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I.$$

Thus B is the inverse of A and A is the inverse of B.

In other words, $B = A^{-1}$ and $A = B^{-1}$.

Note. A rectangular matrix does not possess inverse matrix.

[$\because$ *For products AB and BA to be defined and to be equal, it is necessary that matrices A and B are to be square matrices of the same order*]

KEY POINT

If A^{-1} is the inverse of A, then A is the inverse of A^{-1}.

Theorem. *Inverse of every square matrix, if it exists, is unique.*

Proof. Let A be any n–rowed square invertible matrix.

Let B and C be two inverses of A.

Then $AB = BA = I_n$ ...(1) [$\because$ *B is the inverse of A*]

and $AC = CA = I_n$ ...(2) [$\because$ *C is the inverse of A*]

From (1), we have :

$$AB = I_n.$$

Pre-multiplying by C, we have :

$$C(AB) = CI_n = C \quad ...(3)$$

From (2), we have :

$$CA = I_n$$

Post-multiplying by B, we have :

$$(CA)B = I_nB = B \quad ...(4)$$

Since $C(AB) = (CA)B$,

$\therefore$ from (3) and (4), we have : C = B.

Hence, the inverse of A is unique.

3.11. INVERSE OF A MATRIX BY ELEMENTARY OPERATIONS

We can find the inverse of a matrix, if it exists, by using either row operations (transformations) or column operations (transformations).

Let A be a square matrix of order n.

Then $A = I_nA$ (or $A = AI_n$)

Let a sequence of elementary row (column) operations reduce A on the left (right) side to I_n and the same elementary row (column) operation reduces I_n on the left (right) side to a matrix B.

Then $I_n = BA$

$\Rightarrow$ $I_nA^{-1} = (BA)A^{-1}$ [*Multiplying by A^{-1}*]

$\Rightarrow$ $A^{-1} = B(AA^{-1})$ [*Associative Law*]

$\Rightarrow$ $A^{-1} = B(I) = B.$

KEY POINT

By doing one or more elementary operations if we obtain all 0's in one or more rows (columns) of left side matrix of $I_n = BA$, then $\mathbf{A^{-1}}$ **does not exist.**

ILLUSTRATIVE EXAMPLES

Example 1. By using elementary transformations, find the inverse of the matrix A = $\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}$.

(N.C.E.R.T.; H.P.B. 2018, 15, 12)

Solution. (By Elementary Row Transformations)

We know that $A = I_2A$

i.e. $$\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} A$$

$$\Rightarrow \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} A$$

[*Applying* $R_2 \to R_2 - 2R_1$]

$$\Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & -3 \\ -2 & 1 \end{bmatrix} A.$$

[*Applying* $R_1 \to R_1 - 3R_2$]

Hence, $$A^{-1} = \begin{bmatrix} 7 & -3 \\ -2 & 1 \end{bmatrix}.$$

(By Elementary Coloumn Transformations)

We know that $A = AI_2$

i.e. $$\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix} = A\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} = A\begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}$$

[*Applying* $C_2 \to C_2 - 3C_1$]

$$\Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = A\begin{bmatrix} 7 & -3 \\ -2 & 1 \end{bmatrix}.$$

[*Applying* $C_1 \to C_1 - 2C_2$]

Hence, $$A^{-1} = \begin{bmatrix} 7 & -3 \\ -2 & 1 \end{bmatrix}.$$

Example 2. By using Elementary Row Transformations, find P^{-1}, if it exists, when :

$$P = \begin{bmatrix} 10 & -2 \\ -5 & 1 \end{bmatrix}.$$ *(N.C.E.R.T.)*

Solution. We know that $P = I_2P$

i.e. $$\begin{bmatrix} 10 & -2 \\ -5 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} P$$

$$\Rightarrow \begin{bmatrix} 1 & -1/5 \\ -5 & 1 \end{bmatrix} = \begin{bmatrix} 1/10 & 0 \\ 0 & 1 \end{bmatrix} P \quad \left[\text{Applying } R_1 \to \frac{1}{10} R_1\right]$$

$$\Rightarrow \begin{bmatrix} 1 & -1/5 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1/10 & 0 \\ 1/2 & 1 \end{bmatrix}.$$

[*Applying* $R_2 \to R_2 + 5R_1$]

Since second row of **LHS** matrix has all zeros,

$\therefore$ P^{-1} does not exist.

Example 3. Obtain the inverse of the following matrix by using elementary row transformations :

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}.$$ *(N.C.E.R.T.; P.B. 2010)*

Solution. We know that $A = I_3A$

i.e. $$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

[*Applying* $R_1 \leftrightarrow R_2$]

$$\Rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & -5 & -8 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & -3 & 1 \end{bmatrix} A$$

[*Applying* $R_3 \to R_3 - 3R_1$]

$$\Rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & -5 & -8 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & -3 & 1 \end{bmatrix} A$$

[*Applying* $R_1 \to R_1 - 2R_2$]

$$\Rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & 0 \\ 5 & -3 & 1 \end{bmatrix} A$$

[*Applying* $R_3 \to R_3 + 5R_2$]

$$\Rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & 0 \\ 5/2 & -3/2 & 1/2 \end{bmatrix} A$$

$\left[\text{Applying } R_3 \to \frac{1}{2} R_3\right]$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/2 & -1/2 & 1/2 \\ 1 & 0 & 0 \\ 5/2 & -3/2 & 1/2 \end{bmatrix} A.$$

[*Applying* $R_1 \to R_1 + R_3$]

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/2 & -1/2 & 1/2 \\ -4 & 3 & -1 \\ 5/2 & -3/2 & 1/2 \end{bmatrix} A.$$

[*Applying* $R_2 \to R_2 - 2R_3$]

Hence, $$A^{-1} = \begin{bmatrix} 1/2 & -1/2 & 1/2 \\ -4 & 3 & -1 \\ 5/2 & -3/2 & 1/2 \end{bmatrix}.$$

Example 4. If $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$**, find the inverse of A, using elementary row transformations and hence solve the following equation XA = [1 0 1].**

(C.B.S.E. Sample Paper 2018)

Solution. (*i*) We know that $A = I_3 A$

i.e. $$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

[*Applying* $R_1 \to R_1 - R_2$]

$$\Rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 2 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

[*Applying* $R_1 \leftrightarrow R_2$]

$$\Rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 2 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

[*Applying* $R_2 \to R_2 - R_1$]

$$\Rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -2 & 0 \\ -1 & 2 & 1 \end{bmatrix} A$$

[*Applying* $R_3 \to R_3 - R_2$]

$$\Rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 2 & 1 \\ 1 & -2 & 0 \end{bmatrix} A$$

[*Applying* $R_2 \leftrightarrow R_3$]

$$\Rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 2 & 1 \\ 2 & -4 & -1 \end{bmatrix} A$$

[*Applying* $R_3 \to R_3 - R_2$]

$$\Rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 2 & 1 \\ -2 & 4 & 1 \end{bmatrix} A$$

[*Applying* $R_3 \to (-1) R_3$]

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -3 & -1 \\ -1 & 2 & 1 \\ -2 & 4 & 1 \end{bmatrix}.$$

[*Applying* $R_1 \to R_1 - R_3$]

Hence, $$A^{-1} = \begin{bmatrix} 2 & -3 & -1 \\ -1 & 2 & 1 \\ -2 & 4 & 1 \end{bmatrix}.$$

(*ii*) Here, $XA = [1 \quad 0 \quad 1]$

$$\Rightarrow \quad X = [1 \quad 0 \quad 1]A^{-1}$$

$$= [1 \quad 0 \quad 1]_{1\times 3} \begin{bmatrix} 2 & -3 & -1 \\ -1 & 2 & 1 \\ -2 & 4 & 1 \end{bmatrix}_{3\times 3}$$

$$= \begin{bmatrix} 2-0-2 \\ -3+0+4 \\ -1+0+1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

Hence, $x = 0$, $y = 1$ and $z = 0$.

EXERCISE 3 (f)

Very Short Answer Type Questions

VSATQ

Find the inverse of the following, if it exists, by using elementary row (column) transformations :

1. (*i*) $\begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix}$ ***(N.C.E.R.T., Kashmir B. 2013)***

(*ii*) $\begin{bmatrix} 2 & 1 \\ 7 & 4 \end{bmatrix}$. ***(N.C.E.R.T.; H.P.B. 2016)***

2. (*i*) $\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$

(N.C.E.R.T.; H.P.B. 2016, 13 ; H.B. 2013; C.B.S.E. 2010)

(*ii*) $\begin{bmatrix} 2 & -6 \\ 1 & -2 \end{bmatrix}$

(N.C.E.R.T.; H.P.B. 2017, 10; H.B. 2017; P.B. 2013)

(*iii*) $\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$.

(N.C.E.R.T. ; H.P.B. 2018, 15; Jammu B. 2012)

3. (*i*) $\begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}$ *(N.C.E.R.T.; H.P.B. 2013; P.B. 2012)*

(*ii*) $\begin{bmatrix} 3 & -1 \\ -4 & 2 \end{bmatrix}$

(N.C.E.R.T.; H.P.B. 2017, 10; H.B. 2017; P.B. 2012)

(*iii*) $\begin{bmatrix} 3 & 10 \\ 2 & 7 \end{bmatrix}$. *(N.C.E.R.T.; Jammu B. 2014)*

4. (*i*) $\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$ *(Kerala B. 2018; H.P.B. 2015; P.B. 2012)*

(*ii*) $\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$ *(Karnataka B. 2014)*

(*iii*) $\begin{bmatrix} 4 & 5 \\ 3 & 4 \end{bmatrix}$

(N.C.E.R.T.; H.P.B. 2016, 13; P.B. 2012)

(*iv*) $\begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$. *(N.C.E.R.T.; H.P.B. 2017)*

5. $\begin{bmatrix} 6 & -3 \\ -2 & 1 \end{bmatrix}$. *(N.C.E.R.T.; H.P.B. 2010)*

6. (*i*) $\begin{bmatrix} -5 & 4 \\ -6 & 5 \end{bmatrix}$ *(P.B. 2011)*

(*ii*) $\begin{bmatrix} -4 & 3 \\ -5 & 4 \end{bmatrix}$. *(P.B. 2011)*

Short Answer Type Questions

SATQ

7. Find the inverse of the following, if it exists, using elementary row (column) transformations :

(*i*) $\begin{bmatrix} 2 & -1 & 3 \\ -5 & 3 & 1 \\ -3 & 2 & 3 \end{bmatrix}$ *(C.B.S.E. Sample Paper 2019)*

(*ii*) $\begin{bmatrix} 1 & 3 & -2 \\ -3 & 0 & -5 \\ 2 & 5 & 0 \end{bmatrix}$

(Type : Nagaland B. 2016; P.B. 2014, 10)

(*iii*) $\begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$

(A.I.C.B.S.E. 2015; Uttarakhand B. 2015; Assam B. 2015; C.B.S.E. (F) 2011; P.B. 2010)

(*iv*) $\begin{bmatrix} 2 & -3 & 3 \\ 2 & 2 & 3 \\ 3 & -2 & 2 \end{bmatrix}$. *(N.C.E.R.T.)*

(*v*) $\begin{bmatrix} 2 & 4 & 1 \\ 1 & 2 & 3 \\ 1 & -3 & 0 \end{bmatrix}$ *(P.B. 2018)*

(*vi*) $\begin{bmatrix} 2 & 3 & 1 \\ 5 & -3 & 1 \\ 1 & 1 & 3 \end{bmatrix}$ *(P.B. 2015)*

(*vii*) $\begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$ *(A.I.C.B.S.E. 2010)*

(*viii*) $\begin{bmatrix} 1 & 3 & -2 \\ -3 & 0 & -1 \\ 2 & 1 & 0 \end{bmatrix}$ *(Bihar 2014; C.B.S.E. 2011)*

(*ix*) $\begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 3 \\ 2 & -1 & 2 \end{bmatrix}$ *(P.B. 2017)*

(*x*) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ -2 & -4 & -5 \end{bmatrix}$. *(C.B.S.E. 2018)*

Long Answer Type Questions

LATQ

8. Find the inverse of the matrix A by elementary operations and verify that $A^{-1}A = I$ when :

(*i*) $A = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$ *(H.B. 2010)*

(*ii*) $A = \begin{bmatrix} 1 & 2 & 5 \\ 2 & 3 & 1 \\ -1 & 1 & 1 \end{bmatrix}$.

(H.B. 2010)

Answers

1. (i) $\begin{bmatrix} -7 & 3 \\ 5 & -2 \end{bmatrix}$ (ii) $\begin{bmatrix} 4 & -1 \\ -7 & 2 \end{bmatrix}$.

2. (i) $\begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$ (ii) $\begin{bmatrix} -1 & 3 \\ \frac{-1}{2} & 1 \end{bmatrix}$

(iii) $\frac{1}{5}\begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix}$.

3. (i) $\begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix}$ (ii) $\frac{1}{2}\begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}$

(iii) $\begin{bmatrix} 7 & -10 \\ -2 & 3 \end{bmatrix}$.

4. (i) $\begin{bmatrix} 1/5 & 2/5 \\ 2/5 & -1/5 \end{bmatrix}$ (ii) $\begin{bmatrix} 1/5 & 2/5 \\ -2/5 & 1/5 \end{bmatrix}$

(iii) $\begin{bmatrix} 4 & -5 \\ -3 & 4 \end{bmatrix}$ (iv) $\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$.

5. Does not exist.

6. (i) $\begin{bmatrix} -5 & 4 \\ -6 & 5 \end{bmatrix}$ (ii) Itself.

7. (i) $\begin{bmatrix} -7 & -9 & 10 \\ -12 & -15 & 17 \\ 1 & 1 & -1 \end{bmatrix}$

(ii) $\begin{bmatrix} 1 & -2/5 & -3/5 \\ -2/5 & 4/25 & 11/25 \\ -3/5 & 1/25 & 9/25 \end{bmatrix}$

(iii) $\begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$ (iv) $\begin{bmatrix} -2/5 & 0 & 3/5 \\ -1/5 & 1/5 & 0 \\ 2/5 & 1/5 & -2/5 \end{bmatrix}$.

(v) $\frac{1}{14}\begin{bmatrix} 9 & 3 & 14 \\ 3 & 1 & 0 \\ -2 & 4 & 0 \end{bmatrix}$ (vi) $\frac{1}{54}\begin{bmatrix} 10 & 8 & -6 \\ 14 & -5 & -3 \\ -8 & -1 & 21 \end{bmatrix}$

(vii) $\begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}$ (viii) $\begin{bmatrix} 1 & -2 & -3 \\ -2 & 4 & 7 \\ -3 & 5 & 9 \end{bmatrix}$

(ix) $\begin{bmatrix} 11 & -5 & 2 \\ 2 & 4 & -7 \\ -10 & 7 & 8 \end{bmatrix}$ (x) $\begin{bmatrix} 2 & -2 & -1 \\ -4 & 1 & -1 \\ 2 & 0 & 1 \end{bmatrix}$.

NCERT-FILE

Questions from NCERT Book

(For each unsolved question, refer : "Solution of Modern's abc of Mathematics")

Exercise 3.1

1. In the matrix $A = \begin{bmatrix} 2 & 5 & 19 & -7 \\ 35 & -2 & 5/2 & 12 \\ \sqrt{3} & 1 & -5 & 17 \end{bmatrix}$, write :

(i) The order of the matrix
(ii) The number of elements
(iii) Write the elements $a_{13}, a_{21}, a_{33}, a_{24}, a_{23}$.

Solution : We have : $A = \begin{bmatrix} 2 & 5 & 19 & -7 \\ 35 & -2 & 5/2 & 12 \\ \sqrt{3} & 1 & -5 & 17 \end{bmatrix}$.

(i) The order of A is 3×4.

[$\because$ *A has 3 rows and 4 columns*]

(ii) Number of elements $= 3 \times 4 = 12$.

(iii) $a_{13} = 19$, $a_{21} = 35$,

$a_{33} = -5$, $a_{24} = 12$ and $a_{23} = \frac{5}{2}$.

2. If a matrix has 24 elements, what are the possible orders it can have ? What, if it has 13 elements ?

[Solution : Refer Q. 2 (*b*), Ex. 3(*a*)]

3. If a matrix has 18 elements, what are the possible orders it can have ? What,if it has 5 elements ?

Solution : We know that a matrix of order $m \times n$ has mn elements.

(i) In order to find all possible orders of a matrix having 18 elements, we find all ordered pairs, the product of whose elements is 18.

Thus all possible ordered pairs are :

(1, 18), (18, 1), (2, 9), (9, 2), (3, 6), (6, 3).

Hence, the possible orders are :

1×18, 18×1, 2×9, 9×2, 3×6, 6×3.

(ii) Possible orders in case of 5 elements are :

1×5, 5×1.

4. Construct a 2×2 matrix, $A = [a_{ij}]$, whose elements are given by :

(*i*) $a_{ij} = \frac{(i+j)^2}{2}$ (*ii*) $a_{ij} = \frac{i}{j}$

[Solution : Refer Q. 8; Ex. 3(*a*)]

(*iii*) $a_{ij} = \frac{(i+2j)^2}{2}$

Solution : Since $a_{ij} = \frac{(i+2j)^2}{2}$;

where $1 \le i \le 2$, $1 \le j \le 2$, [*Given*]

$$\therefore \quad a_{11} = \frac{(1+2(1))^2}{2} = \frac{(1+2)^2}{2} = \frac{9}{2};$$

$$a_{12} = \frac{(1+2(2))^2}{2} = \frac{(1+4)^2}{2} = \frac{25}{2};$$

$$a_{21} = \frac{(2+2(1))^2}{2} = \frac{(2+2)^2}{2} = \frac{16}{2} = 8$$

and $a_{22} = \frac{(2+2(2))^2}{2} = \frac{(2+4)^2}{2} = \frac{36}{2} = 18.$

Hence, the required matrix $A = \begin{bmatrix} 9/2 & 25/2 \\ 8 & 18 \end{bmatrix}$.

5. Construct a 3×4 matrix, whose elements are given by :

(*i*) $a_{ij} = \frac{1}{2}|-3i+j|$ (*ii*) $a_{ij} = 2i - j$

Solution : (*i*) Since $a_{ij} = \frac{1}{2}|-3i+j|$;

where $1 \le i \le 3$, $1 \le j \le 4$. [*Given*]

$$\therefore \; a_{11} = \frac{1}{2}|-3(1)+1| = \frac{1}{2}|-3+1| = \frac{1}{2}|-2| = \frac{1}{2}(2) = 1;$$

$$a_{12} = \frac{1}{2}|-3(1)+2| = \frac{1}{2}|-3+2| = \frac{1}{2}|-1| = \frac{1}{2}(1) = \frac{1}{2};$$

$$a_{13} = \frac{1}{2}|-3(1)+3| = \frac{1}{2}|-3+3| = \frac{1}{2}|0| = \frac{1}{2}(0) = 0;$$

$$a_{14} = \frac{1}{2}|-3(1)+4| = \frac{1}{2}|-3+4| = \frac{1}{2}|1| = \frac{1}{2}(1) = \frac{1}{2};$$

$$a_{21} = \frac{1}{2}|-3(2)+1| = \frac{1}{2}|-6+1| = \frac{1}{2}|-5| = \frac{1}{2}(5) = \frac{5}{2};$$

$$a_{22} = \frac{1}{2}|-3(2)+2| = \frac{1}{2}|-6+2| = \frac{1}{2}|-4| = \frac{1}{2}(4) = 2;$$

$$a_{23} = \frac{1}{2}|-3(2)+3| = \frac{1}{2}|-6+3| = \frac{1}{2}|-3| = \frac{1}{2}(3) = \frac{3}{2};$$

$$a_{24} = \frac{1}{2}|-3(2)+4| = \frac{1}{2}|-6+4| = \frac{1}{2}|-2| = \frac{1}{2}(2) = 1;$$

$$a_{31} = \frac{1}{2}|-3(3)+1| = \frac{1}{2}|-9+1| = \frac{1}{2}|-8| = \frac{1}{2}(8) = 4;$$

$$a_{32} = \frac{1}{2}|-3(3)+2| = \frac{1}{2}|-9+2| = \frac{1}{2}|-7| = \frac{1}{2}(7) = \frac{7}{2};$$

$$a_{33} = \frac{1}{2}|-3(3)+3| = \frac{1}{2}|-9+3| = \frac{1}{2}|-6| = \frac{1}{2}(6) = 3$$

and $a_{34} = \frac{1}{2}|-3(3)+4| = \frac{1}{2}|-9+4|$

$$= \frac{1}{2}|-5| = \frac{1}{2}(5) = \frac{5}{2}.$$

Hence, the required matrix $= \begin{bmatrix} 1 & 1/2 & 0 & 1/2 \\ 5/2 & 2 & 3/2 & 1 \\ 4 & 7/2 & 3 & 5/2 \end{bmatrix}$.

(*ii*) **[Solution :** Refer Q. 10 (*b*) (*i*) ; Ex. 3(*a*)]

6. Find the values of x, y and z from the following equations :

(*i*) $\begin{bmatrix} 4 & 3 \\ x & 5 \end{bmatrix} = \begin{bmatrix} y & z \\ 1 & 5 \end{bmatrix}$ (*ii*) $\begin{bmatrix} x+y & 2 \\ 5+z & xy \end{bmatrix} = \begin{bmatrix} 6 & 2 \\ 5 & 8 \end{bmatrix}$

(*iii*) $\begin{bmatrix} x+y+z \\ x+z \\ y+z \end{bmatrix} = \begin{bmatrix} 9 \\ 5 \\ 7 \end{bmatrix}$.

[Solution : Refer Q. 12 (*i*) – (*ii*) & Q. 6(*e*); Ex. 3(*a*)]

7. Find the values of a, b, c and d from the equation :

$$\begin{bmatrix} a-b & 2a+c \\ 2a-b & 3c+d \end{bmatrix} = \begin{bmatrix} -1 & 5 \\ 0 & 13 \end{bmatrix}.$$

[Solution : Refer Q. 6(*d*); Ex. 3(*a*)]

8. $A = [a_{ij}]_{m \times n}$ is a square matrix, if

(A) $m < n$ (B) $m > n$

(C) $m = n$ (D) None of these.

[Ans. (C)]

9. Which of the given values of x and y make the following pair of matrices equal $\begin{bmatrix} 3x+2 & 5 \\ y+1 & 2-3x \end{bmatrix}, \begin{bmatrix} 0 & y-2 \\ 8 & 4 \end{bmatrix}$:

(A) $x = \frac{-1}{3}$, $y = 7$ (B) Not possible to find

(C) $y = 7$, $x = \frac{-2}{3}$ (D) $x = \frac{-1}{3}$, $y = \frac{-2}{3}$.

[Ans. (B)]

10. The number of all possible matrices of order 3×3 with each entry 0 or 1 is :

(A) 27 (B) 18

(C) 81 (D) 512. **[Ans.** (D)]

Exercise 3.2

1. Let $A = \begin{bmatrix} 2 & 4 \\ 3 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 3 \\ -2 & 5 \end{bmatrix}$, $C = \begin{bmatrix} -2 & 5 \\ 3 & 4 \end{bmatrix}$.

Find each of the following :

(*i*) A + B (*ii*) A – B
(*iii*) 3A – C (*iv*) AB
(*v*) BA.

Solution. (*i*) $A + B = \begin{bmatrix} 2 & 4 \\ 3 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 3 \\ -2 & 5 \end{bmatrix}$

$$= \begin{bmatrix} 2+1 & 4+3 \\ 3-2 & 2+5 \end{bmatrix} = \begin{bmatrix} 3 & 7 \\ 1 & 7 \end{bmatrix}.$$

(*ii*) $A - B = A + (-1)\, B = \begin{bmatrix} 2 & 4 \\ 3 & 2 \end{bmatrix} + (-1) \begin{bmatrix} 1 & 3 \\ -2 & 5 \end{bmatrix}$

$$= \begin{bmatrix} 2 & 4 \\ 3 & 2 \end{bmatrix} + \begin{bmatrix} -1 & -3 \\ 2 & -5 \end{bmatrix}$$

$$= \begin{bmatrix} 2-1 & 4-3 \\ 3+2 & 2-5 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 5 & -3 \end{bmatrix}.$$

(*iii*) $3A - C = 3 \begin{bmatrix} 2 & 4 \\ 3 & 2 \end{bmatrix} - \begin{bmatrix} -2 & 5 \\ 3 & 4 \end{bmatrix}$

$$= \begin{bmatrix} 6 & 12 \\ 9 & 6 \end{bmatrix} + \begin{bmatrix} 2 & -5 \\ -3 & -4 \end{bmatrix}$$

$$= \begin{bmatrix} 6+2 & 12-5 \\ 9-3 & 6-4 \end{bmatrix} = \begin{bmatrix} 8 & 7 \\ 6 & 2 \end{bmatrix}.$$

(*iv*) Number of columns of A = 2 = Number of rows of B.

Thus AB is defined and $AB = \begin{bmatrix} 2 & 4 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ -2 & 5 \end{bmatrix}$

$$= \begin{bmatrix} 2 \times 1 + 4 \times (-2) & 2 \times 3 + 4 \times 5 \\ 3 \times 1 + 2 \times (-2) & 3 \times 3 + 2 \times 5 \end{bmatrix}$$

$$= \begin{bmatrix} 2-8 & 6+20 \\ 3-4 & 9+10 \end{bmatrix} = \begin{bmatrix} -6 & 26 \\ -1 & 19 \end{bmatrix}.$$

(*v*) Number of columns of B = 2 = Number of rows of A.

Thus BA is defined and $BA = \begin{bmatrix} 1 & 3 \\ -2 & 5 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ 3 & 2 \end{bmatrix}$

$$= \begin{bmatrix} 1 \times 2 + 3 \times 3 & 1 \times 4 + 3 \times 2 \\ (-2) \times 2 + 5 \times 3 & (-2) \times 4 + 5 \times 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2+9 & 4+6 \\ -4+15 & -8+10 \end{bmatrix} = \begin{bmatrix} 11 & 10 \\ 11 & 2 \end{bmatrix}.$$

2. Compute the following :

(*i*) $\begin{bmatrix} a & b \\ -b & a \end{bmatrix} + \begin{bmatrix} a & b \\ b & a \end{bmatrix}$

(*ii*) $\begin{bmatrix} a^2+b^2 & b^2+c^2 \\ a^2+c^2 & a^2+b^2 \end{bmatrix} + \begin{bmatrix} 2ab & 2bc \\ -2ac & -2ab \end{bmatrix}$

(*iii*) $\begin{bmatrix} -1 & 4 & -6 \\ 8 & 5 & 16 \\ 2 & 8 & 5 \end{bmatrix} + \begin{bmatrix} 12 & 7 & 6 \\ 8 & 0 & 5 \\ 3 & 2 & 4 \end{bmatrix}$

(*iv*) $\begin{bmatrix} \cos^2 x & \sin^2 x \\ \sin^2 x & \cos^2 x \end{bmatrix} + \begin{bmatrix} \sin^2 x & \cos^2 x \\ \cos^2 x & \sin^2 x \end{bmatrix}$.

[**Solution :** Refer Q. 2(*i*), Q. 3, Q. 2 (*b*) (*ii*); Ex. 3(*b*)]

Solution : (*ii*) $\begin{bmatrix} a^2+b^2 & b^2+c^2 \\ a^2+c^2 & a^2+b^2 \end{bmatrix} + \begin{bmatrix} 2ab & 2bc \\ -2ac & -2ab \end{bmatrix}$

$$= \begin{bmatrix} a^2+b^2+2ab & b^2+c^2+2bc \\ a^2+c^2-2ac & a^2+b^2-2ab \end{bmatrix}$$

$$= \begin{bmatrix} (a+b)^2 & (b+c)^2 \\ (a-c)^2 & (a-b)^2 \end{bmatrix}.$$

3. Compute the indicated products :

(*i*) $\begin{bmatrix} a & b \\ -b & a \end{bmatrix} \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ (*ii*) $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} [2 \;\; 3 \;\; 4]$

(*iii*) $\begin{bmatrix} 1 & -2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix}$ (*iv*) $\begin{bmatrix} 2 & 3 & 4 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1 & -3 & 5 \\ 0 & 2 & 4 \\ 3 & 0 & 5 \end{bmatrix}$

(*v*) $\begin{bmatrix} 2 & 1 \\ 3 & 2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ -1 & 2 & 1 \end{bmatrix}$ (*vi*) $\begin{bmatrix} 3 & -1 & 3 \\ -1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 2 & -3 \\ 1 & 0 \\ 3 & 1 \end{bmatrix}$.

[**Solution :** Refer Q. 2 ; Ex. 3(*d*)]

4. If $A = \begin{bmatrix} 1 & 2 & -3 \\ 5 & 0 & 2 \\ 1 & -1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 3 & -1 & 2 \\ 4 & 2 & 5 \\ 2 & 0 & 3 \end{bmatrix}$ and

$C = \begin{bmatrix} 4 & 1 & 2 \\ 0 & 3 & 2 \\ 1 & -2 & 3 \end{bmatrix}$, then compute (A + B) and (B – C). Also, verify that A + (B – C) = (A + B) – C.

Solution : (*i*) A + B

$$= \begin{bmatrix} 1 & 2 & -3 \\ 5 & 0 & 2 \\ 1 & -1 & 1 \end{bmatrix} + \begin{bmatrix} 3 & -1 & 2 \\ 4 & 2 & 5 \\ 2 & 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1+3 & 2-1 & -3+2 \\ 5+4 & 0+2 & 2+5 \\ 1+2 & -1+0 & 1+3 \end{bmatrix} = \begin{bmatrix} 4 & 1 & -1 \\ 9 & 2 & 7 \\ 3 & -1 & 4 \end{bmatrix}.$$

(*ii*) B – C = B + (– 1)

$$C = \begin{bmatrix} 3 & -1 & 2 \\ 4 & 2 & 5 \\ 2 & 0 & 3 \end{bmatrix} + (-1) \begin{bmatrix} 4 & 1 & 2 \\ 0 & 3 & 2 \\ 1 & -2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -1 & 2 \\ 4 & 2 & 5 \\ 2 & 0 & 3 \end{bmatrix} + \begin{bmatrix} -4 & -1 & -2 \\ -0 & -3 & -2 \\ -1 & 2 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 3-4 & -1-1 & 2-2 \\ 4-0 & 2-3 & 5-2 \\ 2-1 & 0+2 & 3-3 \end{bmatrix} = \begin{bmatrix} -1 & -2 & 0 \\ 4 & -1 & 3 \\ 1 & 2 & 0 \end{bmatrix}.$$

(*iii*) A + (B – C)

$$= \begin{bmatrix} 1 & 2 & -3 \\ 5 & 0 & 2 \\ 1 & -1 & 1 \end{bmatrix} + \begin{bmatrix} -1 & -2 & 0 \\ 4 & -1 & 3 \\ 1 & 2 & 0 \end{bmatrix}$$

[*Using part (ii)*]

$$= \begin{bmatrix} 1-1 & 2-2 & -3+0 \\ 5+4 & 0-1 & 2+3 \\ 1+1 & -1+2 & 1+0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -3 \\ 9 & -1 & 5 \\ 2 & 1 & 1 \end{bmatrix} \quad ...(1)$$

$$(A + B) - C = \begin{bmatrix} 4 & 1 & -1 \\ 9 & 2 & 7 \\ 3 & -1 & 4 \end{bmatrix} - \begin{bmatrix} 4 & 1 & 2 \\ 0 & 3 & 2 \\ 1 & -2 & 3 \end{bmatrix}$$

[*Using part (i)*]

$$= \begin{bmatrix} 4 & 1 & -1 \\ 9 & 2 & 7 \\ 3 & -1 & 4 \end{bmatrix} + \begin{bmatrix} -4 & -1 & -2 \\ -0 & -3 & -2 \\ -1 & 2 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 4-4 & 1-1 & -1-2 \\ 9-0 & 2-3 & 7-2 \\ 3-1 & -1+2 & 4-3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -3 \\ 9 & -1 & 5 \\ 2 & 1 & 1 \end{bmatrix} \quad ....(2)$$

From (1) and (2), A + (B – C) = (A + B) – C.
Hence, the verification.

5. If $A = \begin{bmatrix} \frac{2}{3} & 1 & \frac{5}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{4}{3} \\ \frac{7}{3} & 2 & \frac{2}{3} \end{bmatrix}$ and $B = \begin{bmatrix} \frac{2}{3} & \frac{3}{5} & 1 \\ \frac{1}{5} & \frac{2}{5} & \frac{4}{5} \\ \frac{7}{5} & \frac{6}{5} & \frac{2}{5} \end{bmatrix}$, then compute 3A – 5B.

[**Solution :** Refer Q. 5; Ex. 3(*c*)]

6. Simplify :

$$\cos\theta \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} + \sin\theta \begin{bmatrix} \sin\theta & -\cos\theta \\ \cos\theta & \sin\theta \end{bmatrix}.$$

Solution : $\cos\theta \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

$$+ \sin\theta \begin{bmatrix} \sin\theta & -\cos\theta \\ \cos\theta & \sin\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\theta & \sin\theta\cos\theta \\ -\sin\theta\cos\theta & \cos^2\theta \end{bmatrix} + \begin{bmatrix} \sin^2\theta & -\sin\theta\cos\theta \\ \sin\theta\cos\theta & \sin^2\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\theta + \sin^2\theta & \sin\theta\cos\theta - \sin\theta\cos\theta \\ -\sin\theta\cos\theta + \sin\theta\cos\theta & \cos^2\theta + \sin^2\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$= I_2$.

7. Find X and Y, if :

(*i*) $X + Y = \begin{bmatrix} 7 & 0 \\ 2 & 5 \end{bmatrix}$ and $X - Y = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$.

Solution : We have : $X + Y = \begin{bmatrix} 7 & 0 \\ 2 & 5 \end{bmatrix}$ ...(1)

and $X - Y = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$ ....(2)

Adding (1) and (2),

$$(X + Y) + (X - Y) = \begin{bmatrix} 7 & 0 \\ 2 & 5 \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

$$\Rightarrow \quad 2X = \begin{bmatrix} 7+3 & 0+0 \\ 2+0 & 5+3 \end{bmatrix}$$

$$\Rightarrow \quad 2X = \begin{bmatrix} 10 & 0 \\ 2 & 8 \end{bmatrix}.$$

Hence, $X = \frac{1}{2}\begin{bmatrix} 10 & 0 \\ 2 & 8 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 1 & 4 \end{bmatrix}$.

Subtracting (2) from (1),

$$(X + Y) - (X - Y) = \begin{bmatrix} 7 & 0 \\ 2 & 5 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & 0 \\ 2 & 5 \end{bmatrix} + \begin{bmatrix} -3 & -0 \\ -0 & -3 \end{bmatrix}$$

$$\Rightarrow \quad 2Y = \begin{bmatrix} 7-3 & 0-0 \\ 2-0 & 5-3 \end{bmatrix}$$

$$\Rightarrow \quad 2Y = \begin{bmatrix} 4 & 0 \\ 2 & 2 \end{bmatrix}.$$

Hence, $Y = \frac{1}{2}\begin{bmatrix} 4 & 0 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$.

(*ii*) $2X + 3Y = \begin{bmatrix} 2 & 3 \\ 4 & 0 \end{bmatrix}$ and $3X + 2Y = \begin{bmatrix} 2 & -2 \\ -1 & 5 \end{bmatrix}$.

[Solution : Refer Q. 7(*iii*); Ex. 3(*c*)]

8. Find X, if $Y = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}$ and $2X + Y = \begin{bmatrix} 1 & 0 \\ -3 & 2 \end{bmatrix}$.

[Solution : Refer Q. 8; Ex. 3(*c*)]

9. Find x and y, if $2\begin{bmatrix} 1 & 3 \\ 0 & x \end{bmatrix} + \begin{bmatrix} y & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 6 \\ 1 & 8 \end{bmatrix}$.

[Solution : Refer Q. 4 (*ii*) ; Ex. 3(*c*)]

10. Solve the equation for x, y, z and t, if $2\begin{bmatrix} x & z \\ y & t \end{bmatrix} + 3\begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} = 3\begin{bmatrix} 3 & 5 \\ 4 & 6 \end{bmatrix}$.

[Solution. Refer Q. 11 (*i*); Ex. 3(*c*)]

11. If $x\begin{bmatrix} 2 \\ 3 \end{bmatrix} + y\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 10 \\ 5 \end{bmatrix}$, find the values of x and y.

[Solution. Refer Q. 9; Ex. 3(*c*)]

12. Given $3\begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} x & 6 \\ -1 & 2w \end{bmatrix} + \begin{bmatrix} 4 & x+y \\ z+w & 3 \end{bmatrix}$, find the values of x, y, z and w.

[Solution. Refer Q. 12; Ex. 3(*c*)]

13. If $F(x) = \begin{bmatrix} \cos x & -\sin x & 0 \\ \sin x & \cos x & 0 \\ 0 & 0 & 1 \end{bmatrix}$, show that :

$F(x)\,F(y) = F(x + y)$. ***(P.B. 2016)***

Solution : Here $F(x)\,F(y) = \begin{bmatrix} \cos x & -\sin x & 0 \\ \sin x & \cos x & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \cos y & -\sin y & 0 \\ \sin y & \cos y & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} \cos x \cos y - \sin x \sin y & -\sin y \cos x - \sin x \cos y & 0 \\ \sin x \cos y + \cos x \sin y & -\sin x \sin y + \cos x \cos y & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos(x+y) & -\sin(x+y) & 0 \\ \sin(x+y) & \cos(x+y) & 0 \\ 0 & 0 & 1 \end{bmatrix} = F(x+y),$$

which is true.

14. Show that

(*i*) $\begin{bmatrix} 5 & -1 \\ 6 & 7 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix} \neq \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 5 & -1 \\ 6 & 7 \end{bmatrix}$

(*ii*) $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 2 & 3 & 4 \end{bmatrix} \neq \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 2 & 3 & 4 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$.

[Solution : Refer Q. 11; Ex. 3(*d*)]

15. Find $A^2 - 5A + 6I$, if $A = \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}$.

[Solution : Refer Q. 20 (*iii*) (I); Ex. 3(*d*)]

16. If $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$, prove that $A^3 - 6A^2 + 7A + 2I = O$.

[Solution : Refer Ex. 12; Page 3/23]

17. If $A = \begin{bmatrix} 3 & -2 \\ 4 & -2 \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, find k so that $A^2 = kA - 2I$. **[Solution :** Refer Q. 24 (*ii*); Ex. 3(*d*)]

18. If $A = \begin{bmatrix} 0 & -\tan\frac{\alpha}{2} \\ \tan\frac{\alpha}{2} & 0 \end{bmatrix}$ and I is the identity matrix of order 2, show that $I + A = (1 - A)\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$.

[Solution : Refer Q. 33; Ex. 3(*d*)]

19. A trust fund has ₹ 30,000 that must be invested in two different types of bonds. The first bond pays 5% interest per year, and the second bond pays 7% interest per year. Using matrix multiplication, determine how to divide ₹ 30,000 among the two types of bonds.If the trust fund must obtain an annual total interest :

(*i*) ₹ 1800 (*ii*) ₹ 2000. **[Solution :** Refer Q. 30; Ex. 3(*d*)]

20. The bookshop of a particular school has 10 dozen chemistry books, 8 dozen physics books, 10 dozen economics books. Their selling prices are ₹ 80, ₹ 60 and ₹ 40 each respectively. Find the total amount the bookshop will receive from selling all the books using matrix algebra.

Solution : Inventory,

$$A = \begin{array}{c} \quad\textit{Chem.} \quad \textit{Phy.} \quad \textit{Eco.} \\ [10 \times 12 \quad 8 \times 12 \quad 10 \times 12] \end{array}$$

$$= \begin{bmatrix} 120 & 96 & 120 \end{bmatrix}$$

Selling Price matrix, $B = \begin{array}{c} \begin{bmatrix} 80 \\ 60 \\ 40 \end{bmatrix} \end{array} \begin{array}{l} \to \textit{Chem.} \\ \to \textit{Phy.} \\ \to \textit{Eco.} \end{array}$

$\therefore$ Reqd. amount = AB

$$= \begin{bmatrix} 120 & 96 & 120 \end{bmatrix} \begin{bmatrix} 80 \\ 60 \\ 40 \end{bmatrix}$$

$$= \begin{bmatrix} 120 \times 80 + 96 \times 60 + 120 \times 40 \end{bmatrix}$$

$$= \begin{bmatrix} 9600 + 5760 + 4800 \end{bmatrix} = \begin{bmatrix} 20160 \end{bmatrix}.$$

Hence, the book-shop will receive ₹ 20,160 by selling all the books.

Assume X, Y, Z, W and P are matrices of order $2 \times n$, $3 \times k$, $2 \times p$, $n \times 3$ and $p \times k$, respectively. **Choose the correct answer in Exercises 21 and 22.**

21. The restriction on n, k and p so that PY + WY will be defined are :

(A) $k = 3, p = n$ (B) k is arbitrary, $p = 2$

(C) p is arbitrary, $k = 3$ (D) $k = 2, p = 3$. [**Ans.** (A)]

22. If $n = p$, then the order of the matrix 7X – 5Z is :

(A) $p \times 2$ (B) $2 \times n$

(C) $n \times 3$ (D) $p \times n$. [**Ans.** (B)]

Exercise 3.3

1. Find the transpose of each of the following matrices :

(*i*) $\begin{bmatrix} 5 \\ \frac{1}{2} \\ -1 \end{bmatrix}$ (*ii*) $\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$

(*iii*) $\begin{bmatrix} -1 & 5 & 6 \\ \sqrt{3} & 5 & 6 \\ 2 & 3 & -1 \end{bmatrix}$.

Solution. (*i*) We have : $A = \begin{bmatrix} 5 \\ 1/2 \\ -1 \end{bmatrix}$.

$\therefore$ $A' = \begin{bmatrix} 5 & \frac{1}{2} & -1 \end{bmatrix}$.

(*ii*) We have : $A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$.

$\therefore$ $A' = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$.

(*iii*) We have : $A = \begin{bmatrix} -1 & 5 & 6 \\ \sqrt{3} & 5 & 6 \\ 2 & 3 & -1 \end{bmatrix}$.

$\therefore$ $A' = \begin{bmatrix} -1 & \sqrt{3} & 2 \\ 5 & 5 & 3 \\ 6 & 6 & -1 \end{bmatrix}$.

2. If $A = \begin{bmatrix} -1 & 2 & 3 \\ 5 & 7 & 9 \\ -2 & 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} -4 & 1 & -5 \\ 1 & 2 & 0 \\ 1 & 3 & 1 \end{bmatrix}$, then verify that :

(*i*) $(A + B)' = A' + B'$, (*ii*) $(A - B)' = A' - B'$.

[**Solution :** Refer Q. 7 (*a*) ; Ex. 3(*e*)]

3. If $A' = \begin{bmatrix} 3 & 4 \\ -1 & 2 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix}$, then verify that :

(*i*) $(A + B)' = A' + B'$ (*ii*) $(A - B)' = A' - B'$.

[**Solution :** Refer Q. 7(*c*); Ex. 3(*e*)]

4. If $A' = \begin{bmatrix} -2 & 3 \\ 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 0 \\ 1 & 2 \end{bmatrix}$, then find $(A + 2B)'$. [**Solution :** Refer Q. 7(*b*); Ex. 3(*e*)]

5. For the matrices A and B, verify that $(AB)' = B'A'$, where :

(*i*) $A = \begin{bmatrix} 1 \\ -4 \\ 3 \end{bmatrix}$, $B = [-1 \ \ 2 \ \ 1]$ (*ii*) $A = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$, $B = [1 \ \ 5 \ \ 7]$.

[**Solution :** Refer Q. 9 (*ii*); Ex. 3(*e*)]

6. If (*i*) $A = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$, then verify that $A'A = I$.

(*ii*) If $A = \begin{bmatrix} \sin\alpha & \cos\alpha \\ -\cos\alpha & \sin\alpha \end{bmatrix}$, then verify that $A'A = I$.

[**Solution :** Refer Q. 14; Ex. 3(*e*)]

7. (*i*) Show that the matrix $A = \begin{bmatrix} 1 & -1 & 5 \\ -1 & 2 & 1 \\ 5 & 1 & 3 \end{bmatrix}$ is a symmetric matrix.

(*ii*) Show that the matrix $A = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}$ is a skew symmetric matrix.

[**Solution :** Refer Q. 10 (*b*); Ex. 3(*e*)]

8. For the matrix $A = \begin{bmatrix} 1 & 5 \\ 6 & 7 \end{bmatrix}$, verify that

(*i*) $(A + A')$ is a symmetric matrix

(*ii*) $(A' - A')$ is a skew symmetric matrix.

[Solution : Refer Q. 20 (*a*) ; Ex. 3(*e*)]

9. Find $\frac{1}{2}(A + A')$ and $\frac{1}{2}(A - A')$, when

$A = \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}$. **[Solution :** Refer Q. 21; Ex. 3(*e*)]

10. Express the following matrices as the sum of a symmetric and a skew symmetric matrix :

(*i*) $\begin{bmatrix} 3 & 5 \\ 1 & -1 \end{bmatrix}$ (*ii*) $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$

(*iii*) $\begin{bmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{bmatrix}$ (*iv*) $\begin{bmatrix} 1 & 5 \\ -1 & 2 \end{bmatrix}$.

[Solution : Refer Q. 22 and 26; Ex. 3(*e*)]

Choose the correct answer in the Exercises 11 and 12.

11. If A, B are symmetric matrices of same order, then AB – BA is a :

(A) Skew symmetric matrix

(B) Symmetric matrix

(C) Zero matrix

(D) Identity matrix. **[Ans.** (A)]

12. If $A = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$, then $A + A' = I$, if the value of α is :

(A) $\frac{\pi}{6}$ (B) $\frac{\pi}{3}$

(C) π (D) $\frac{3\pi}{2}$. **[Ans.** (B)]

Exercise 3.4

Using elementary transformations, find the inverse of each of the matrices, if it exists in Exercises 1 to 17 :

1. $\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$ **2.** $\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ **3.** $\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}$

4. $\begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix}$ **5.** $\begin{bmatrix} 2 & 1 \\ 7 & 4 \end{bmatrix}$ **6.** $\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$

7. $\begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}$ **8.** $\begin{bmatrix} 4 & 5 \\ 3 & 4 \end{bmatrix}$ **9.** $\begin{bmatrix} 3 & 10 \\ 2 & 7 \end{bmatrix}$

10. $\begin{bmatrix} 3 & -1 \\ -4 & 2 \end{bmatrix}$ **11.** $\begin{bmatrix} 2 & -6 \\ 1 & -2 \end{bmatrix}$ **12.** $\begin{bmatrix} 6 & -3 \\ -2 & 1 \end{bmatrix}$

13. $\begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$ **14.** $\begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$ **15.** $\begin{bmatrix} 2 & -3 & 3 \\ 2 & 2 & 3 \\ 3 & -2 & 2 \end{bmatrix}$

16. $\begin{bmatrix} 1 & 3 & -2 \\ -3 & 0 & -5 \\ 2 & 5 & 0 \end{bmatrix}$ **17.** $\begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$.

[Solution : Refer Ex. 3(*f*)]

18. Matrices A and B will be inverse of each other only if :

(A) AB = BA (B) AB = BA = O

(C) AB = O, BA = I (D) AB = BA = I. **[Ans.** (D)]

Miscellaneous Exercise on Chapter 3

1. Let $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, show that $(aI + bA)^n = a^n I + na^{n-1} bA$, where I is the identity matrix of order 2 and $n \in \mathbf{N}$.

[Solution : Refer Q. 25; Ex. 3(*d*)]

2. If $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$, prove that :

$$A^n = \begin{bmatrix} 3^{n-1} & 3^{n-1} & 3^{n-1} \\ 3^{n-1} & 3^{n-1} & 3^{n-1} \\ 3^{n-1} & 3^{n-1} & 3^{n-1} \end{bmatrix}, n \in \mathbf{N}.$$

Solution. Step I. When $n = 1$,

$$A^1 = \begin{bmatrix} 3^0 & 3^0 & 3^0 \\ 3^0 & 3^0 & 3^0 \\ 3^0 & 3^0 & 3^0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = A.$$

Thus the result is true when $n = 1$.

Step II. Let us assume that the result is true for any positive integer m, where $1 \le m < n$

$$i.e.\ A^m = \begin{bmatrix} 3^{m-1} & 3^{m-1} & 3^{m-1} \\ 3^{m-1} & 3^{m-1} & 3^{m-1} \\ 3^{m-1} & 3^{m-1} & 3^{m-1} \end{bmatrix} \quad ...(1)$$

Step III. $A^{m+1} = A^m A$

$$= \begin{bmatrix} 3^{m-1} & 3^{m-1} & 3^{m-1} \\ 3^{m-1} & 3^{m-1} & 3^{m-1} \\ 3^{m-1} & 3^{m-1} & 3^{m-1} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3^{m-1}+3^{m-1}+3^{m-1} & 3^{m-1}+3^{m-1}+3^{m-1} & 3^{m-1}+3^{m-1}+3^{m-1} \\ 3^{m-1}+3^{m-1}+3^{m-1} & 3^{m-1}+3^{m-1}+3^{m-1} & 3^{m-1}+3^{m-1}+3^{m-1} \\ 3^{m-1}+3^{m-1}+3^{m-1} & 3^{m-1}+3^{m-1}+3^{m-1} & 3^{m-1}+3^{m-1}+3^{m-1} \end{bmatrix}$$

$$= \begin{bmatrix} 3.3^{m-1} & 3.3^{m-1} & 3.3^{m-1} \\ 3.3^{m-1} & 3.3^{m-1} & 3.3^{m-1} \\ 3.3^{m-1} & 3.3^{m-1} & 3.3^{m-1} \end{bmatrix} = \begin{bmatrix} 3^m & 3^m & 3^m \\ 3^m & 3^m & 3^m \\ 3^m & 3^m & 3^m \end{bmatrix}$$

$$= \begin{bmatrix} 3^{(m+1)-1} & 3^{(m+1)-1} & 3^{(m+1)-1} \\ 3^{(m+1)-1} & 3^{(m+1)-1} & 3^{(m+1)-1} \\ 3^{(m+1)-1} & 3^{(m+1)-1} & 3^{(m+1)-1} \end{bmatrix}.$$

Thus the result is true when $n = m + 1$.

Hence, by Mathematical Induction, the required result is true for all $n \in \mathbf{N}$.

3. If $A = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$ then prove that :

$A^n = \begin{bmatrix} 1+2n & -4n \\ n & 1-2n \end{bmatrix}$, where n is any positive integer.

[**Ans.** Refer Ex. 16; Page 3/24]

4. If A and B are symmetric matrices, prove that AB – BA is a skew symmetric matrix.

[**Solution :** Refer Q. 25 (*ii*); Ex. 3(*e*)]

5. Show that the matrix $B'AB$ is symmetric or skew-symmetric according as A is symmetric or skew symmetric.

[**Solution :** Refer Q. 24; Ex. 3(*e*)]

6. Find the values of x, y, z if the matrix $A = \begin{bmatrix} 0 & 2y & z \\ x & y & -z \\ x & -y & z \end{bmatrix}$ satisfy the equation $A'A = I$.

[**Solution :** Refer Q. 9; Rev. Ex.]

7. For what values of x : $[1 \ \ 2 \ \ 1] \begin{bmatrix} 1 & 2 & 0 \\ 2 & 0 & 1 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ x \end{bmatrix} = 0$?

[**Solution :** Refer Q. 7(*b*) (*i*) ; Ex. 3(*d*)]

8. If $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$, show that $A^2 - 5A + 7I = O$.

[**Solution :** Refer Q. 20 (*ii*); Ex. 3(*d*)]

9. Find x, if $[x \ -5 \ -1] \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ 4 \\ 1 \end{bmatrix} = O$.

[**Solution :** Refer Q. 7(*b*) (*ii*) ; Ex. 3(*d*)]

10. A manufacturer produces three products x, y, z which he sells in two markets. Annual sales are indicated below :

Market		**Products**	
I	10,000	2,000	18,000
II	6,000	20,000	8,000

(*a*) If unit sale prices of x, y and z are ₹ 2.50, ₹ 1.50 and ₹ 1.00, respectively, find the total revenue in each market with the help of matrix algebra.

(*b*) If the unit costs of the above three commodities are ₹ 2.00, ₹ 1.00 and 50 paise respectively. Find the gross profit.

[**Solution :** Refer Q. 4; Rev. Ex.]

11. Find the matrix X so that :

$$X \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} -7 & -8 & -9 \\ 2 & 4 & 6 \end{bmatrix}.$$

[**Solution :** Refer Q. 19; Ex. 3(*d*)]

12. If A and B are square matrices of the same order such that AB = BA, then prove by induction that $AB^n = B^nA$. Further, prove that $(AB)^n = A^n B^n$ for all $n \in N$.

Solution. (*i*) Let P (n) : $AB^n = B^nA$.

Step I. When $n = 1$, $AB^1 = AB = BA$ [$\because$ *AB = BA (given)*]

$= B^1A$.

Thus P (1) is true.

Step II. Let us assume that P (n) is true for $n = m$, where $1 \le m < n$

i.e. $AB^m = B^mA$ ...(1)

Step III. Now $AB^{m+1} = A(BB^m)$

$= (AB)\ B^m$ [*Associative Law*]

$= (BA)\ B^m$ [$\because$ *AB = BA, (given)*]

$= B\ (AB^m)$ [*Associative Law*]

$= B\ (B^m A)$ [*Using (1)*]

$= (BB^m)\ A.$ [*Associative Law*]

$= B^{m+1}\ A.$

Thus P (n) is true for $n = m + 1$.

Hence, by Mathematical Induction, P (n) is true for all $n \in \mathbf{N}$.

(*ii*) Let P (n) : $(AB)^n = A^nB^n$.

Step I. When $n = 1$, $(AB)^1 = A^1B^1$ *i.e.* AB = AB.

Thus P (1) is true.

Step II. Let us assume that P (n) is true for $n = m$, where $1 \leq m < n$

i.e. $(AB)^m = A^mB^m$ ....(1)

Step III. Now $(AB)^{m+1} = (AB)^m (AB)$

$$\begin{aligned} &= (A^mB^m)(AB) && [\textit{Using (1)}] \\ &= A^mB^m(BA) && [\because AB = BA, \textit{(given)}] \\ &= A^m(B^mB)A && [\textit{Associative Law}] \\ &= A^m(B^{m+1}A) \\ &= A^m(AB^{m+1}) && [\textit{Commutative Law}] \\ &= (A^mA)B^{m+1} && [\textit{Associative Law}] \\ &= A^{m+1}B^{m+1}. \end{aligned}$$

Thus P (n) is true for $n = m + 1$.

Hence, by Mathematical Induction, P (n) in true for all $n \in \mathbf{N}$.

Choose the correct answer in the following questions :

13. If $A = \begin{bmatrix} \alpha & \beta \\ \gamma & -\alpha \end{bmatrix}$ is such that $A^2 = I$, then :

(A) $1 + \alpha^2 + \beta\gamma = 0$ (B) $1 - \alpha^2 + \beta\gamma = 0$

(C) $1 - \alpha^2 - \beta\gamma = 0$ (D) $1 + \alpha^2 - \beta\gamma = 0$.

[Ans. (C)]

14. If the matrix A is both symmetric and skew symmetric, then :

(A) A is a diagonal matrix (B) A is a zero matrix

(C) A is a square matrix (D) None of these.

[Ans. (B)]

15. If A is square matrix such that $A^2 = A$, then $(1 + A)^3 - 7A$ is equal to :

(A) A (B) I – A

(C) I (D) 3A. **[Ans.** (C)]

Questions From NCERT Exemplar

Example 1. Construct a matrix $A = [a_{ij}]_{2\times 2}$ whose elements a_{ij} are given by $a_{ij} = e^{2ix} \sin jx$.

Solution. Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$.

Now
$$a_{11} = e^{2x} \sin x \,;\, a_{12} = e^{2x} \sin 2x \,;$$
$$a_{21} = e^{4x} \sin x \,;\, a_{22} = e^{4x} \sin 2x.$$

Hence, $$A = \begin{bmatrix} e^{2x} \sin x & e^{2x} \sin 2x \\ e^{4x} \sin x & e^{4x} \sin 2x \end{bmatrix}.$$

Example 2. Show that a matrix, which is both symmetric and skew symmetric, is a zero matrix.

Solution. Let $A = [a_{ij}]$.

Since A is symmetric matrix, therefore, $A' = A$

i.e., $a_{ji} = a_{ij}$ for all i and j ...(1)

Since A is skew-symmteric metrix, therefore, $A' = -A$

i.e., $a_{ij} = -a_{ji}$ ...(2)

From (1) and (2), $a_{ij} = -a_{ij}$ for all i and j

$\Rightarrow 2a_{ij} = 0 \Rightarrow a_{ij} = 0$ for all i and j.

Hence, A is a zero matrix.

Example 3. If $[2x \quad 3] \begin{bmatrix} 1 & 2 \\ -3 & 0 \end{bmatrix} \begin{bmatrix} x \\ 8 \end{bmatrix} = O$, find the value of 'x'.

Solution. We have :

$$[2x \quad 3] \begin{bmatrix} 1 & 2 \\ -3 & 0 \end{bmatrix} \begin{bmatrix} x \\ 8 \end{bmatrix} = O$$

$$\Rightarrow \quad [2x - 9 \quad 4x] \begin{bmatrix} x \\ 8 \end{bmatrix} = O$$

$$\Rightarrow \quad [2x^2 - 9x + 32x] = [0]$$
$$\Rightarrow \quad [2x^2 + 23x] = [0]$$
$$\Rightarrow \quad 2x^2 + 23x = 0$$
$$\Rightarrow \quad x(2x + 23) = 0.$$

Hence, $x = 0, -\frac{23}{2}$.

Example 4. If A is 3 × 3 invertible matrix, then show that for any scalar 'k' (non-zero), kA is invertible and $(kA)^{-1} = \frac{1}{k} A^{-1}$.

Solution. We have :

$$(kA)\left(\frac{1}{k} A^{-1}\right) = \left(k.\frac{1}{k}\right)(A.\, A^{-1}) = 1.\,(I) = I.$$

$\therefore$ (kA) is the inverse of $\left(\frac{1}{k} A^{-1}\right)$.

Hence, $$(kA)^{-1} = \frac{1}{k} A^{-1}.$$

Example 5. Let $A = \begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix}$. Then show that $A^2 - 4A + 7I = O$. Using this result calculate A^5 also.

Solution. We have :

$$A = \begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix}.$$

$$\therefore \quad A^2 = AA = \begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 4-3 & 6+6 \\ -2-2 & -3+4 \end{bmatrix} = \begin{bmatrix} 1 & 12 \\ -4 & 1 \end{bmatrix},$$

$$-4A = -4\begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} -8 & -12 \\ 4 & -8 \end{bmatrix}$$

and $7I = 7\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}.$

$$\therefore \quad A^2 - 4A + 7I = \begin{bmatrix} 1 & 12 \\ -4 & 1 \end{bmatrix} + \begin{bmatrix} -8 & -12 \\ 4 & -8 \end{bmatrix} + \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} 1-8+7 & 12-12+0 \\ -4+4+0 & 1-8+7 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$= O.$$

Now $A^2 - 4A + 7I = O \Rightarrow A^2 = 4A - 7I$ ...(1)

Thus $A^3 = A.A^2 = A(4A - 7I)$ [*Using (1)*]

$= 4A^2 - 7AI$

$= 4(4A - 7I) - 7A$ [*Using (1)*]

$= 16A - 28I - 7A = 9A - 28I$ ...(2)

Now $A^5 = A^3A^2$

$= (9A - 28I)(4A - 7I)$ [*Using (1) & (2)*]

$= 36A^2 - 63AI - 112\,IA + 196\,II$

$= 36(4A - 7I) - 63A - 112A + 196I$ [*Using (1)*]

$= 36(4A - 7I) - 175A + 196I$

$= -31A - 56I$

$$= -31\begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix} - 56\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -62 & -93 \\ 31 & -62 \end{bmatrix} + \begin{bmatrix} -56 & 0 \\ 0 & -56 \end{bmatrix}$$

$$= \begin{bmatrix} -62-56 & -93+0 \\ 31+0 & -62-56 \end{bmatrix}$$

$$= \begin{bmatrix} -118 & -93 \\ 31 & -118 \end{bmatrix}.$$

Exercise

1. If a matrix has 28 elements, what are the possible orders it can have ? What if it has 13 elements ?

2. Construct $a_{2\times 2}$ matrix, where $a_{ij} = |-2i + 3j|$.

3. If X and Y are 2×2 matrices, then solve the following matrix equations of X and Y :

$$2X + 3Y = \begin{bmatrix} 2 & 3 \\ 4 & 0 \end{bmatrix},\ 3x + 2Y = \begin{bmatrix} -2 & 2 \\ 1 & -5 \end{bmatrix}.$$

4. If A is a square matrix such that $A^2 = A$, show that $(I + A)^3 = 7A + I$.

5. The matrix $\begin{bmatrix} 0 & 0 & 5 \\ 0 & 5 & 0 \\ 5 & 0 & 0 \end{bmatrix}$ is a scalar matrix.

State true or false. If false, then what type of matrix is this ?

6. Find non-zero values of 'x', satisfying the matrix equation :

$$x\begin{bmatrix} 2x & 2 \\ 3 & x \end{bmatrix} + 2\begin{bmatrix} 8 & 5x \\ 4 & 4x \end{bmatrix} = 2\begin{bmatrix} x^2+8 & 24 \\ 10 & 6x \end{bmatrix}.$$

7. Express the matrix $\begin{bmatrix} 2 & 4 & -6 \\ 7 & 3 & 5 \\ 1 & -2 & 4 \end{bmatrix}$ as a sum of symmetric and a skew-symmetric matrix.

Answers

1. $1 \times 28, 2 \times 14, 4 \times 7, 7 \times 4, 14 \times 2, 28 \times 1$;
$1 \times 13, 13 \times 1$.

2. $\begin{bmatrix} 1 & 4 \\ 1 & 2 \end{bmatrix}.$

3. $X = \begin{bmatrix} -2 & 0 \\ -1 & -3 \end{bmatrix}, Y = \begin{bmatrix} 2 & 1 \\ 2 & 2 \end{bmatrix}.$

5. False ; Square matrix.

6. $x = 4$.

7. Given Matrix $= \begin{bmatrix} 2 & \frac{11}{2} & -\frac{5}{2} \\ \frac{11}{2} & 3 & \frac{3}{2} \\ -\frac{5}{2} & \frac{3}{2} & 4 \end{bmatrix} + \begin{bmatrix} 0 & -\frac{3}{2} & -\frac{7}{2} \\ \frac{3}{2} & 0 & \frac{7}{2} \\ \frac{7}{2} & -\frac{7}{2} & 0 \end{bmatrix}.$

Revision Exercise

1. Let $A = \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 5 & 2 \\ 7 & 4 \end{bmatrix}$, $C = \begin{bmatrix} 2 & 5 \\ 3 & 8 \end{bmatrix}$. **Find a matrix D such that CD – AB = O.**

(N.C.E.R.T. ; C.B.S.E. 2017; Kashmir B. 2011)

Solution. Since A, B and C are all square matrices of order 2,

$\therefore$ D must also be a square matrix of order 2.

Let $D = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$

By the question, CD – AB = O

$$\Rightarrow \begin{bmatrix} 2 & 5 \\ 3 & 8 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 5 & 2 \\ 7 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2a+5c & 2b+5d \\ 3a+8c & 3b+8d \end{bmatrix} - \begin{bmatrix} 10-7 & 4-4 \\ 15+28 & 6+16 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2a+5c-3 & 2b+5d \\ 3a+8c-43 & 3b+8d-22 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Comparing, $2a + 5c - 3 = 0$ ...(1)

$3a + 8c - 43 = 0$ ...(2)

$2b + 5d = 0$ ...(3)

and $3b + 8d - 22 = 0$ ...(4)

Solving (1) and (2), $a = -191$, $c = 77$.

Solving (3) and (4), $b = -110$, $d = 44$.

Hence, $D = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} -191 & -110 \\ 77 & 44 \end{bmatrix}$.

2. If A and B be square matrices of the same order such that AB = BA, prove that :

(*i*) $(A + B)(A - B) = A^2 - B^2$

(*ii*) $(A - B)^2 = A^2 - 2AB + B^2$

(*iii*) $(A + B)^3 = A^3 + 3A^2B + 3AB^2 + B^3$.

3. If $A_\alpha = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$, then show that :

$A_\alpha . A_\beta = A_{\alpha+\beta}$ and $(A_\alpha)^n = A_{n\alpha}$, where $n \in \mathbf{N}$.

4. A manufacturer produces three products x, y, z which he sells in two markets. Annual sales are indicated below :

Market	*Products*		
I	10,000	2,000	18,000
II	6,000	20,000	8,000

(*a*) If unit sale prices of x, y and z are ₹ 2·50, ₹ 1·50 and ₹ 1·00 respectively. Find the total revenue in each market with the help of matrix algebra.

(*b*) If the unit costs of the above three commodities are ₹ 2·00, ₹ 1·00 and ₹ 0·50 respectively, find the gross profit. **(*N.C.E.R.T.*)**

5. Show that :

$$\left[\begin{pmatrix} 1 & \omega & \omega^2 \\ \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \end{pmatrix} + \begin{pmatrix} \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \\ \omega & \omega^2 & 1 \end{pmatrix}\right] \begin{bmatrix} 1 \\ \omega \\ \omega^2 \end{bmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

where ω is a cube root of unity.

6. If $A(x) = \dfrac{1}{\sqrt{1-x^2}} \begin{bmatrix} 1 & -x \\ -x & 1 \end{bmatrix}$, prove that :

$A(x)\,A(y) = A\left(\dfrac{x+y}{1+xy}\right)$, where $|x| < 1$.

Hence, deduce that $(A(x))^{-1} = A(-x)$.

7. Let $A = \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}$, where $a \neq 1$. Show that :

$$A^n = \begin{bmatrix} a^n & \dfrac{b(a^n-1)}{a-1} \\ 0 & 1 \end{bmatrix} \text{ for all } n \in \mathbf{N}.$$

(*H.B. 2012*)

8. Let A be a square matrix and k be a scalar. Prove that :

(*i*) If A is symmetric, then kA is symmetric.

(*ii*) If A is skew–symmetric, then kA is skew–symmetric.

9. Find the values of x, y, z if the matrix $A = \begin{bmatrix} 0 & 2y & z \\ x & y & -z \\ x & -y & z \end{bmatrix}$ satisfies the equation $A'A = I_3$.

(*N.C.E.R.T.*)

10. If $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -3 \\ 2 & 4 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$, then prove that :

(*i*) $AB \neq BA$ (*ii*) $A(BC) = (AB)C$

(*iii*) $A(B + C) = AB + AC$ (*iv*) $BB' = 10C$

(*v*) $(AB)' = B'A'$ (*vi*) $A^2 - 2A + I = O$.

(*Jharkhand B. 2016*)

Answers

4. (*a*) Total revenue in market I = ₹ 46,000

Total revenue in market II = ₹ 53,000

(*b*) ₹ 15,000, ₹ 17,000.

9. $x = \pm \dfrac{1}{\sqrt{2}}, y = \pm \dfrac{1}{\sqrt{6}}, z = \pm \dfrac{1}{\sqrt{3}}$.

Hints to Selected Questions

6. $A(x)\,A(-x) = A\left(\dfrac{x-x}{1-x^2}\right) = A(0)$

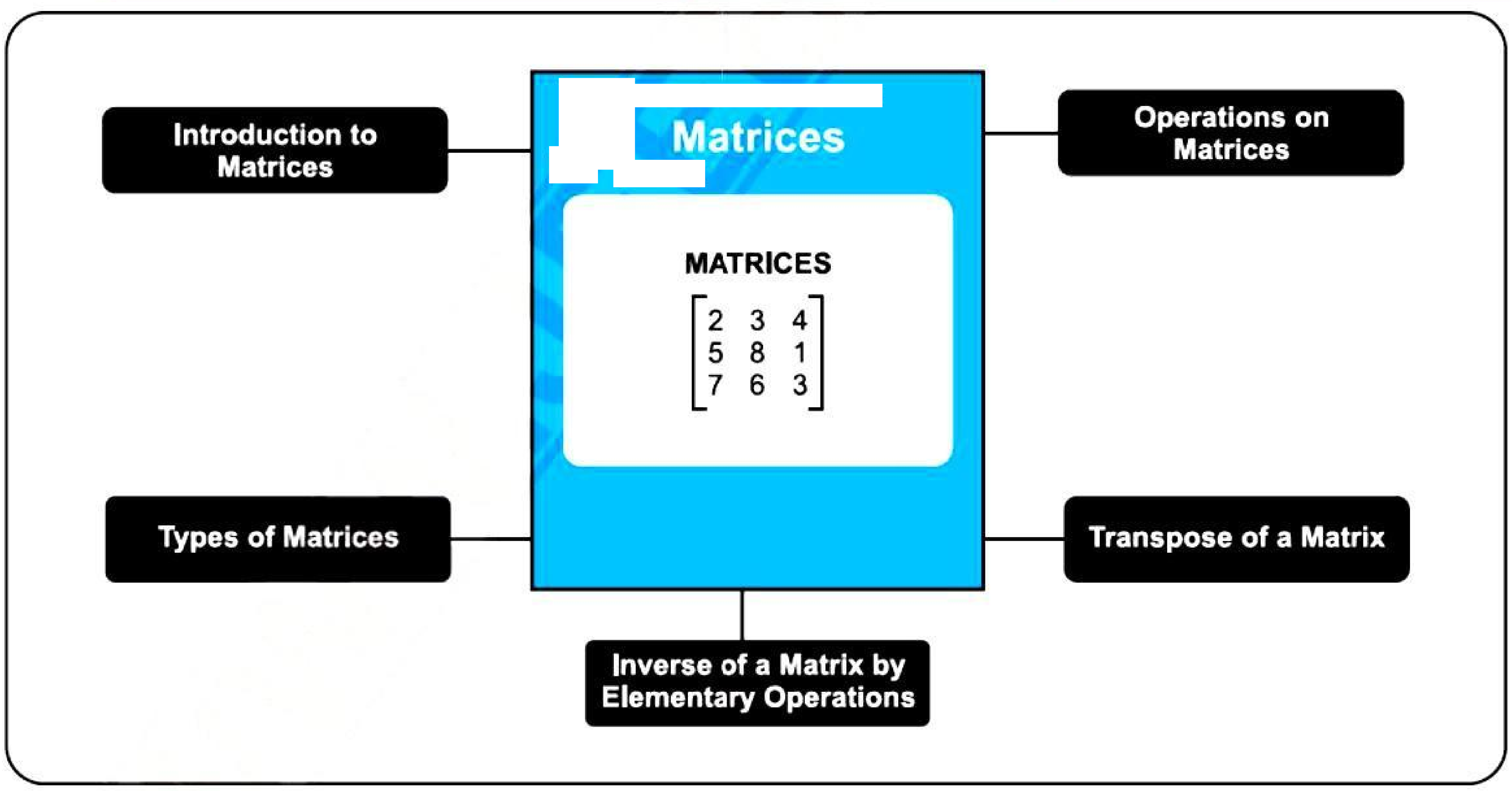
$$= \frac{1}{\sqrt{1-0}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I.$$

Hence, $(A(x))^{-1} = A(-x)$.

CHECK YOUR UNDERSTANDING

1. A matrix is an ordered rectangular array of numbers or functions. **(True/False)**
   Ans. True.
2. A diagonal matrix is said to be a .... if its diagonal elements are equal (other than unity). **(Fill in the blank)**
   Ans. Scalar matrix.
3. Construct a 2×2 matrix whose elements a_{ij} are given by $a_{ij} = i + j$.
   Ans. $\begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}$.
4. Compute $\begin{bmatrix} p & q \\ q & p \end{bmatrix} + \begin{bmatrix} p & q \\ -q & p \end{bmatrix}$. **Ans.** $\begin{bmatrix} 2p & 2q \\ 0 & 2p \end{bmatrix}$.
5. If $A = \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}$, find 4A. **Ans.** $\begin{bmatrix} 8 & 12 \\ -4 & 16 \end{bmatrix}$.
6. What is the order of the product matrix ?
   $\begin{bmatrix} a \\ b \\ c \end{bmatrix} [1 \quad 2 \quad 3]$. **Ans.** 3×3.
7. Compute $\begin{bmatrix} a & 0 \\ a & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. **Ans.** $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.
8. Find the transpose of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$. **Ans.** $\begin{bmatrix} a & c \\ b & d \end{bmatrix}$.
9. If $A = \begin{bmatrix} 1 & 5 \\ 6 & 7 \end{bmatrix}$, then find $A + A'$. **Ans.** $\begin{bmatrix} 2 & 11 \\ 11 & 14 \end{bmatrix}$.
10. If A and B are two symmetric matrices of the same order, then A + B is also symmetric. **(True/False)** **(*Kashmir B. 2015*)**
    Ans. True.

SUMMARY — MATRICES



DEFINITIONS AND IMPORTANT RESULTS

1. MATRIX

Def. A system of mn–numbers (real or complex) arranged in the form of an ordered set of m horizontal lines (called *rows*) and n vertical lines (called columns) is called an $m \times n$ matrix.

2. TYPES OF MATRICES

(i) Rectangular Matrix. Any $m \times n$ matrix $(m \neq n)$ is called a rectangular matrix.

(ii) Square Matrix. Any $n \times n$ matrix is called a square matrix of order n.

(iii) Row Matrix. Any $1 \times n$ matrix is called a row matrix.

(iv) Column Matrix. Any $m \times 1$ matrix is called a column matrix.

(v) Diagonal Matrix. A square matrix $A = [a_{ij}]$ is said to be a diagonal matrix if $a_{ij} = 0$ when $i \neq j$.

(vi) Scalar Matrix. A diagonal matrix is said to be a scalar matrix if all its diagonal entries are equal.

(vii) Identity Matrix. A diagonal matrix is said to be an identity matrix if each of its diagonal elements is unity.

(viii) Zero Matrix. A matrix is said to be a zero matrix if each of its elements is zero.

(ix) Triangular Matrices.

(I) A square matrix $A = [a_{ij}]$ is said to be **upper triangular matrix** if $a_{ij} = 0$ for $i > j$.

(II) A square matrix $A = [a_{ij}]$ is said to be **lower triangular matrix** if $a_{ij} = 0$ for $i < j$.

3. EQUALITY OF MATRICES

Two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are said to be equal iff (*i*) they are of the same order (*ii*) their corresponding elements are equal.

4. OPERATIONS ON MATRICES

(i) Addition of Matrices.

Let $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{m \times n}$ be two matrices. Then the sum $A + B = C = [c_{ij}]_{m \times n}$, where $c_{ij} = a_{ij} + b_{ij}$ for

$$1 \le i \le m,\ 1 \le j \le n.$$

KEY POINT

Addition is defined only for matrices, which are of the same order.

(ii) Multiplication of a matrix by a scalar.

Let A be any $m \times n$ matrix and k be any scalar. Then $m \times n$ matrix obtained by multiplying each element by k is said to be scalar multiple of A by k and is denoted by kA or Ak.

(iii) Multiplication of Matrices.

Let $A = [a_{ij}]$ be $m \times n$ matrix and $B = [b_{jk}]$ be $n \times p$ matrix such that the number of columns of A equals the number of rows of B. Then matrix $C = [c_{ik}]$, which is of the order $m \times p$ such that :

$$c_{ik} = \sum_{j=1}^{n} a_{ij} b_{jk};\ \text{where } i = 1, 2, ..., m;\ p = 1, 2, 3, ..., k$$

is called the product of the matrices A and B and is written as C = AB.

KEY POINT

Product AB is defined iff number of columns of A = number of rows of B.

5. TRANSPOSE OF A MATRIX

(i) Def. If $A = [a_{ij}]_{m \times n}$, then the transpose of A, denoted by A' (or A^t or A^T) is defined by $n \times m$ matrix obtained from A by writing the rows of A as columns and columns of A as rows in the same order.

(ii) Properties : (*i*) $(A')' = A$

(*ii*) $(A + B)' = A' + B'$, A and B being of same type

(*iii*) $(kA)' = kA'$, k being any scalar

(*iv*) $(AB)^{-1} = B^{-1} A^{-1}$.

6. SYMMETRIC AND SKEW–SYMMETRIC MATRICES

(i) Symmetric Matrix. Def. A square matrix $A = [a_{ij}]$ is said to be symmetric if (i, j)th element is the same as its (j, i)th element.

KEY POINT

A is symmetric if $A' = A$.

(ii) Skew–Symmetric Matrix. A square matrix $A = [a_{ij}]$ is said to be skew–symmetric if (i, j)th element is negative of its (j, i)th element.

KEY POINT

A is skew–symmetric if $A' = -A$.

MULTIPLE CHOICE QUESTIONS

For Board Examinations

1. If A, B are symmetric matrices of same order, then AB – BA is a :
   (A) skew-symmetric matrix
   (B) symmetric matrix
   (C) zero matrix
   (D) identity matrix. ***(H.P.B. 2018, 17, 16)***

2. If $A = \begin{bmatrix} 4 & 2 & 3 \\ 1 & 5 & 7 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 3 & 7 \\ 0 & 4 & 1 \end{bmatrix}$, then 2A + B is :
   (A) $\begin{bmatrix} 9 & 7 & 13 \\ 2 & 14 & 15 \end{bmatrix}$ (B) $\begin{bmatrix} 9 & 13 & 7 \\ 2 & 14 & 13 \end{bmatrix}$
   (C) $\begin{bmatrix} 7 & 9 & 13 \\ 2 & 14 & 15 \end{bmatrix}$ (D) None of these.
   (H.B. 2018)

3. If the matrix A is both symmetric and skew-symmetric, then A is :
   (A) a unit matrix (B) a zero matrix
   (C) a scalar matrix (D) a diagonal matrix.
   (Mizoram B. 2018)

4. If order of matrix A is 2 × 3 and order of matrix B is 3 × 5, then order of matrix $B'A'$ is :
   (A) 5 × 2 (B) 2 × 5
   (C) 5 × 3 (D) 3 × 2. ***(P.B. 2017)***

5. If A and B are invertible matrices of the same order, then $(AB)^{-1}$ is equal to :
   (A) BA (B) $B^{-1}A$
   (C) BA^{-1} (D) $B^{-1}A^{-1}$. ***(H.P.B. 2017)***

6. If the matrices $\begin{bmatrix} 3x+7 & 5 \\ y+1 & 2-3x \end{bmatrix} = \begin{bmatrix} 5 & y-2 \\ 8 & 4 \end{bmatrix}$, then the values of x and y are :

(A) $x = -\frac{1}{3}, y = 7$ (B) $x = -\frac{1}{3}, y = -\frac{2}{3}$

(C) $x = -\frac{2}{3}, y = 7$ (D) $x = 5, y = -\frac{2}{3}$.

(H.B. 2017)

7. (*i*) If $x\begin{bmatrix} 2 \\ 3 \end{bmatrix} + y\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 10 \\ 5 \end{bmatrix}$, the values of x and y are :

(A) $x = 2, y = -3$ (B) $x = 10, y = 0$

(C) $x = 3, y = -4$ (D) $x = 5, y = -1$.

(H.B. 2017)

(*ii*) If $2\begin{bmatrix} 1 & 3 \\ 0 & x \end{bmatrix} + \begin{bmatrix} y & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 6 \\ 1 & 5 \end{bmatrix}$, then the values of x and y are :

(A) $x = 2, y = 3$ (B) $x = 3, y = 3$

(C) $x = 4, y = 2$ (D) None of these.

(H.B. 2017)

8. Let $A = \begin{bmatrix} 0 & 2 \\ 0 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 3 \\ 0 & 0 \end{bmatrix}$, then AB equals :

(A) $\begin{bmatrix} 0 & 6 \\ 0 & 0 \end{bmatrix}$ (B) $\begin{bmatrix} 0 & 4 \\ 0 & 0 \end{bmatrix}$

(C) $\begin{bmatrix} 0 & 6 \\ 0 & 4 \end{bmatrix}$ (D) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

(Nagaland B. 2017)

9. If A is a square matrix such that :

$|A| \neq 0$ and $A^2 - A + 2I = O$, then A^{-1} equals :

(A) $I - A$ (B) $\frac{1}{2}(I - A)$

(C) $\frac{1}{2}(I + A)$ (D) $I + A$. ***(Mizoram B. 2017)***

10. The value of 'k' such that the metrix $\begin{pmatrix} 1 & k \\ -k & 1 \end{pmatrix}$ is symmetric is :

(A) 0 (B) 1

(C) – 1 (D) 2. ***(Kerala B. 2017)***

11. If AB = C, where B and C are matrices of order 3×5, then the order of matrix A is :

(A) 3×5 (B) 3×3

(C) 5×5 (D) 5×3. ***(P.B. 2016)***

12. If A and B are invertible matrices, then :

(A) $(AB)^{-1} = B^{-1} A^{-1}$ (B) $(AB)^{-1} = A^{-1} B^{-1}$

(C) $(AB)^{-1} = (BA)^{-1}$ (D) None of these.

(H.P.B. 2016)

13. (*i*) If $A = [a_{ij}]$ is a matrix of order 2×3 and $a_{ij} = \frac{i+2j}{2}$, then the matrix is :

(A) $\begin{bmatrix} 3/2 & 3 \\ 2 & 3 \\ 5/2 & 7/2 \end{bmatrix}$ (B) $\begin{bmatrix} 3/2 & 5/2 & 7/2 \\ 2 & 3 & 4 \end{bmatrix}$

(C) $\begin{bmatrix} 3/2 & 2 \\ 5/2 & 3 \\ 7/2 & 4 \end{bmatrix}$ (D) $\begin{bmatrix} 3/2 & 2 & 5/2 \\ 3 & 3 & 7/2 \end{bmatrix}$.

(H.B. 2016)

14. If $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, then BA =

(A) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (B) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

(C) $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ (D) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

(Kerala B. 2016)

RCQ Pocket

(Single Correct Answer Type)

(JEE-Main and Advanced)

15. The number of 3×3 non-singular matrices, with four entries as 1 and all other entries as 0, is :

(A) less than 4 (B) 5

(C) 6 (D) at least 7. ***(A.I.E.E.E. 2010)***

16. The number of 3×3 matrices A whose entries are either 0 or 1 and for which the system :

$A\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ has exactly two distinct solutions, is :

(A) 0 (B) $2^9 - 1$

(C) 168 (D) 2. ***(I.I.T. 2010)***

17. If $\omega \neq 1$ is the complex cube root of unity and matrix $N = \begin{bmatrix} \omega & 0 \\ 0 & \omega \end{bmatrix}$, then H^{70} is equal to :

(A) O (B) – H

(C) H^2 (D) H. ***(A.I.E.E.E. 2011 S)***

18. Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$. If u_1 and u_2 are column matrices such that $Au_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $Au_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, then $u_1 + u_2$ is equal to :

(A) $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ (B) $\begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}$

(C) $\begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix}$ (D) $\begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$. *(A.I.E.E.E. 2012)*

19. If P is a 3 × 3 matrix such that $P^7 = 2P + I$, where P^T is the transpose of A and I is the 3 × 3 identity matrix, then there exists a column matrix $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ such that

(A) $PX = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ (B) PX = X

(C) PX = 2X (D) PX = – X. *(I.I.T. 2012)*

20. If A is an 3 × 3 non-singular matrix such that $AA' = A'A$ and $B = A^{-1}A'$, then BB' equals :

(A) I (B) B^{-1}

(C) $(B^{-1})'$ (D) I + B.

(J.E.E (Main) 2014)

21. If $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ a & 2 & b \end{bmatrix}$ is a matrix satisfying the equation $AA^T = 9I$, where I is a 3 × 3 identity matrix, then the ordered pair (a, b) is equal to :

(A) (2, –1) (B) (–2, 1)

(C) (2, 1) (D) (–2, –1).

(J.E.E. (Main) 2015)

22. If $A = \begin{bmatrix} 5a & -b \\ 3 & 2 \end{bmatrix}$ and A. adj A = A A^T, then $5a + b$ is equal to :

(A) 5 (B) 4

(C) 13 (D) – 1. *(J.E.E. (Main) 2016)*

23. Let $P = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 16 & 4 & 1 \end{bmatrix}$ and I be the identity matrix of order 3.

If $Q = [q_{ij}]$ is a matrix such that :

$P^{50} - Q = I$, then $\dfrac{q_{31} + q_{32}}{q_{21}}$ equals :

(A) 52 (B) 103

(C) 201 (D) 205.

(J.E.E. (Advanced) 2016)

24. How many 3 × 3 matrices M with entries from {0, 1, 2} are there, for which the sum of the diagonal entries of M^TM is ?

(A) 162 (B) 135

(C) 126 (D) 198.

(J.E.E. (Advanced) 2017)

Answers

1. (A)	**2.** (A)	**3.** (B)	**4.** (A)	**5.** (D)	**6.** (C)	**7.** (C)	**8.** (D)	**9.** (B)	**10.** (A)
11. (B)	**12.** (A)	**13.** (B)	**14.** (C)	**15.** (D)	**16.** (A)	**17.** (D)	**18.** (D)	**19.** (D)	**20.** (A)
21. (D)	**22.** (A)	**23.** (B)	**24.** (D).						

Hints/Solutions

15. (D) First row with exactly one zero
total number of cases = 6.
First row with 2 zeroes, we get more cases.
$\therefore$ Total no. of non-singular matrices = 7.

16. (A) There are three possible cases.
Either unique solution or no solution
or infinite number of solutions.
Hence, two distinct solutions are not possible.

17. (D) $H^2 = \begin{bmatrix} \omega & 0 \\ 0 & \omega \end{bmatrix}\begin{bmatrix} \omega & 0 \\ 0 & \omega \end{bmatrix} = \begin{bmatrix} \omega^2 & 0 \\ 0 & \omega^2 \end{bmatrix}$.

Let $H^k = \begin{bmatrix} \omega^k & 0 \\ 0 & \omega^k \end{bmatrix}$, where $1 \le k < n$.

Then $H^{k+1} = H^k H^1 = \begin{bmatrix} \omega^k & 0 \\ 0 & \omega^k \end{bmatrix}\begin{bmatrix} \omega & 0 \\ 0 & \omega \end{bmatrix}$

$= \begin{bmatrix} \omega^{k+1} & 0 \\ 0 & \omega^{k+1} \end{bmatrix}$.

By Mathematical Induction, $H^{70} = \begin{bmatrix} \omega^{70} & 0 \\ 0 & \omega^{70} \end{bmatrix}$

$= \begin{bmatrix} \omega & 0 \\ 0 & \omega \end{bmatrix}$ $\quad [\because \omega^{70} = \omega^{69} . \omega^1 = 1 . \omega = \omega]$

= H.

18. (D) $\quad A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$.

Now $Au_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \Rightarrow u_1 = A^{-1}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ ....(1)

But $|A| = 1$

and adj $A = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}' = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix}$.

$\therefore A^{-1} = \frac{1}{1}\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix}$.

From (1), $u_1 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$= \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$.

And $\quad Au_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \Rightarrow u_2 = A^{-1}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$\Rightarrow \quad u_2 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}$.

$\therefore \quad u_1 + u_2 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$.

19. (D) We have : $P^T = 2P + I$

$\Rightarrow \quad P^T - 2P = I$ ....(1)

Taking transposes, $(P^T - 2P)^T = I^T$

$\Rightarrow \quad (P^T)^T - 2P^T = I$

$\Rightarrow \quad P - 2P^T = I$ ....(2)

Multiplying (1) by 2, $\quad 2P^T - 4P = 2I$ ...(3)

Adding (2) and (3), $-3P = 3I \Rightarrow P = -I \Rightarrow P + I = O$.

Thus $(P + I)X = O \Rightarrow PX = -X$.

20. (A) $B = A^{-1}A' \Rightarrow AB = A'$

$ABB' = A'B' \Rightarrow ABB' = (BA)' = (A^{-1}A'A)'$
$= (A^{-1}AA')' = (A')' = A$

$BB' = I$.

21. (D) We have : $AA^T = 9I$

$\Rightarrow \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ a & 2 & b \end{bmatrix}\begin{bmatrix} 1 & 2 & a \\ 2 & 1 & 2 \\ 2 & -2 & b \end{bmatrix} = 9\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 1+4+4 & 2+2-4 & a+4+2b \\ 2+2-4 & 4+1+4 & 2a+2-2b \\ a+4+2b & 2a+2-2b & a^2+4+b^2 \end{bmatrix}$

$= \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} 9 & 0 & a+4+2b \\ 0 & 9 & 2a+2-2b \\ a+4+2b & 2a+2-2b & a^2+4+b^2 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}.$$

Comparing, $a + 4 + 2b = 0 \Rightarrow a + 2b = -4$ ...(1)
$2a + 2 - 2b = 0 \Rightarrow 2a - 2b = -2$ ...(2)
and $a^2 + 4 + b^2 = 9 \Rightarrow a^2 + b^2 = 5$ ...(3)
Adding (1) and (2), $3a = -6 \Rightarrow a = -2$.
Putting in (1), $-2 + 2b = -4 \Rightarrow 2b = -2$
$\Rightarrow b = -1$.
Hence, $(a, b) = (-2, -1)$.
[*$a = -2$, $b = -1$ also satisfy (3)*]

22. (A)

We have : $AA^T = \begin{bmatrix} 5a & -b \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 5a & 3 \\ -b & 2 \end{bmatrix}$

$$= \begin{bmatrix} 25a^2+b^2 & 15a-2b \\ 15a-2b & 13 \end{bmatrix} \quad ...(1)$$

Now adj $A = \begin{bmatrix} 2 & -3 \\ b & 5a \end{bmatrix}' = \begin{bmatrix} 2 & b \\ -3 & 5a \end{bmatrix}$.

$\therefore$ A adj A $= \begin{bmatrix} 5a & -b \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 2 & b \\ -3 & 5a \end{bmatrix}$

$$= \begin{bmatrix} 10a+3b & 0 \\ 0 & 3b+10a \end{bmatrix} \quad ...(2)$$

From (1) and (2), $\begin{bmatrix} 25a^2+b^2 & 15a-2b \\ 15a-2b & 13 \end{bmatrix}$

$$= \begin{bmatrix} 10a+3b & 0 \\ 0 & 10a+3b \end{bmatrix}.$$

Comparing, $10a + 3b = 13$ and $15a - 2b = 0$.

Solving, $a = \frac{2}{5}, b = 3$.

Hence, $5a + b = 2 + 3 = 5$.

23. (B) $P = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 16 & 4 & 1 \end{bmatrix}$.

$$\therefore P^2 = PP = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 16 & 4 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 16 & 4 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 8 & 1 & 0 \\ 16(1+2) & 8 & 1 \end{bmatrix}.$$

Similarly $P^3 = \begin{bmatrix} 1 & 0 & 0 \\ 12 & 1 & 0 \\ 16(1+2+3) & 12 & 1 \end{bmatrix}$.

By Induction, $P^{51} = \begin{bmatrix} 1 & 0 & 0 \\ 200 & 1 & 0 \\ \frac{16.50.51}{2} & 200 & 1 \end{bmatrix}$.

Since $P^{50} - Q - I = I$,

we have : $q_{31} = \frac{16.50.51}{2}$

and $q_{32} = 200$, $q_{21} = 200$.

Hence, $\frac{q_{31}+q_{32}}{q_{21}} = \frac{16.50.51}{2.200} + 1 = 102 + 1 = 103$.

24. (D) Let $M = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$.

$$\therefore M^TM = \begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix}\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

$$= \begin{bmatrix} a^2+d^2+g^2 & ab+de+gh & ac+df+gi \\ ab+de+gh & b^2+e^2+h^2 & bc+ef+hi \\ ca+fd+ig & bc+ef+hi & c^2+f^2+i^2 \end{bmatrix}.$$

$\therefore tr(M^T M) = (a^2 + d^2 + g^2) + (b^2 + e^2 + h^2) + (c^2 + f^2 + i^2)$
$= a^2 + b^2 + c^2 + d^2 + e^2 + f^2 + g^2 + h^2 + i^2 = 5$,
when entries are from {0, 1, 2}.

Two cases arise :

Case I. When five entries are 1 and other four are zero.

$\therefore$ ${}^9C_5 \times 1 = 126$.

Case II. When one entry is 2, one entry is 1 and others are 0.

$\therefore$ ${}^9C_2 \times 21! = 72$.

Hence, total = 126 + 72 = 198.

CHAPTER TEST 3

Time Allowed : 1 Hour **Max. Marks : 34**

Notes : **1.** *All questions are compulsory.*

2. *Marks have been indicated against each question.*

1. Write the element a_{12} of the matrix $A = [a_{ij}]_{2\times 2}$, whose elements are given by $a_{ij} = e^{2ix} \sin jx$. **(1)**

2. If the matrix $A = \begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix}$ and $A^2 = \lambda A$, then write the value of 'λ'. **(1)**

3. For what value of 'x', is the matrix :

$$A = \begin{bmatrix} 0 & 1 & -2 \\ -1 & 0 & 3 \\ x & -3 & 0 \end{bmatrix}$$

a skew-symmetric matrix. **(2)**

4. If $2\begin{bmatrix} 3 & 4 \\ 5 & x \end{bmatrix} + \begin{bmatrix} 1 & y \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 0 \\ 10 & 5 \end{bmatrix}$, find $(x - y)$. **(2)**

5. If $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$, show that $A^2 - 5A + 7I = O$. **(4)**

6. Show that $A + A^T$ is symmetric when $A = \begin{bmatrix} 2 & 4 \\ 5 & 6 \end{bmatrix}$. **(4)**

7. By using elementary transformations, find the inverse of :

$$A = \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}.$$ **(4)**

8. If $\begin{bmatrix} x+3 & z+4 & 2y-7 \\ 4x+6 & a-1 & 0 \\ b-3 & 3b & z+2c \end{bmatrix} = \begin{bmatrix} 0 & 6 & 3y-2 \\ 2x & -3 & 2c+2 \\ 2b+4 & -21 & 0 \end{bmatrix}$,

obtain the values of a, b, c ; x, y and z. **(4)**

9. Let $A = \begin{bmatrix} -1 & -4 \\ 1 & 3 \end{bmatrix}$, by Mathematical Induction prove that :

$A^n = \begin{bmatrix} 1-2n & -4n \\ n & 1+2n \end{bmatrix}$, where $n \in \mathbf{N}$. **(6)**

10. Express the matrix A as the sum of a symmetric and a skew-symmetric matrix, where :

$$A = \begin{bmatrix} 3 & -2 & -4 \\ 3 & -2 & -5 \\ -1 & 1 & 2 \end{bmatrix}.$$ **(6)**

Answers

1. $e^{2x} \sin 2x$. **2.** $\lambda = 6$. **3.** $x = 2$. **4.** 10.

7. $\begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix}$. **8.** $a = -2, b = -7, c = -1$; $x = -3, y = -5, z = 2$.

10. $\begin{bmatrix} 3 & 1/2 & -5/2 \\ 1/2 & -2 & -2 \\ -5/2 & -2 & 2 \end{bmatrix} + \begin{bmatrix} 0 & -5/2 & -3/2 \\ 5/2 & 0 & -3 \\ 3/2 & 3 & 0 \end{bmatrix}$.