JEE Type Solved Examples :
Single Option Correct Type Questions

- This section contains **10 multiple choice examples.** Each example has four choices (a), (b), (c) and (d) out of which **ONLY ONE** is correct.

Ex. 1 *If $b-c$, $2b-\lambda$, $b-a$ are in HP, then $a-\frac{\lambda}{2}$, $b-\frac{\lambda}{2}$, $c-\frac{\lambda}{2}$ are is*

(a) AP (b) GP
(c) HP (d) None of these

Sol. (b) $(2b-\lambda)=\frac{2(b-c)(b-a)}{(b-c)+(b-a)}$

$$\Rightarrow \quad (2b-\lambda)=(2b-(a+c))=2[b^2-(a+c)b+ac]$$

$$\Rightarrow \quad 2b^2-2b\lambda+\lambda(a+c)-2ac=0$$

$$\Rightarrow \quad b^2-b\lambda+\frac{\lambda}{2}(a+c)-ac=0$$

$$\Rightarrow \left(b-\frac{\lambda}{2}\right)^2-\frac{\lambda^2}{4}+\frac{\lambda}{2}(a+c)-ac=0$$

$$\Rightarrow \quad \left(b-\frac{\lambda}{2}\right)^2=\frac{\lambda^2}{4}-\frac{\lambda}{2}(a+c)+ac$$

$$\Rightarrow \quad \left(b-\frac{\lambda}{2}\right)^2=\left(a-\frac{\lambda}{2}\right)\left(c-\frac{\lambda}{2}\right)$$

Hence, $a-\frac{\lambda}{2}, b-\frac{\lambda}{2}, c-\frac{\lambda}{2}$ are in GP.

Ex. 2 *Let $a_1, a_2, a_3, \ldots, a_{10}$ are in GP with $a_{51}=25$ and $\sum_{i=1}^{101} a_i=125$, then the value of $\sum_{i=1}^{101}\left(\frac{1}{a_i}\right)$ equals*

(a)5 (b) $\frac{1}{5}$ (c) $\frac{1}{25}$ (d) $\frac{1}{125}$

Sol. (b) Let 1st term be a and common ratio be r, then

$$\sum_{i=1}^{101}\frac{1}{a_i}=125$$

$$\Rightarrow \quad (a_1+a_1r+a_1r^2+\ldots+a_1r^{100})=125$$

$$\Rightarrow \quad \frac{a_1(1-r^{101})}{(1-r)}=125 \qquad [\text{let } 0<r<1] \ldots\text{(i)}$$

$$\therefore \quad \sum_{i=1}^{101}\frac{1}{a_i}=\frac{1}{a_i}+\frac{1}{a_ir}+\frac{1}{a_ir^2}+\ldots+\frac{1}{a_ir^{100}}=\frac{\frac{1}{a_i}\left[\left(\frac{1}{r}\right)^{101}-1\right]}{\left(\frac{1}{r}-1\right)}$$

$$\left[\text{here } \frac{1}{r}>1\right]$$

$$=\frac{(1-r^{101})}{a_1r^{100}(1-r)}=\frac{1}{a_1r^{100}}\times\frac{125}{a_1} \qquad [\text{from Eq. (i)}]$$

$$=\frac{125}{(a_1r^{50})^2}=\frac{125}{(a_{51})^2}=\frac{125}{(25)^2}=\frac{1}{5}$$

Ex. 3 *If $x=111\ldots1$ (20 digits), $y=333\ldots3$ (10 digits) and $z=222\ldots2$ (10 digits), then $\frac{x-y^2}{z}$ equals*

(a) $\frac{1}{2}$ (b) 1 (c) 2 (d) 4

Sol. (b) $\because$ $x=\frac{1}{9}(999\ldots9)=\frac{1}{9}(10^{20}-1)$,

$$y=\frac{1}{3}(999\ldots9)=\frac{1}{3}(10^{10}-1)$$

and $$z=\frac{2}{9}(999\ldots9)=\frac{2}{9}(10^{10}-1)\,]$$

$$\therefore \quad \frac{x-y^2}{z}=\frac{\frac{1}{9}(10^{20}-1)-\frac{1}{9}(10^{10}-1)^2}{\frac{2}{9}(10^{10}-1)}$$

$$=\frac{10^{10}+1-(10^{10}-1)}{2}=1$$

Ex. 4 *Consider the sequence 1, 2, 2, 3, 3, 3, ..., where n occurs n times. The number that occurs as 2011th terms is*

(a) 61 (b) 62
(c) 63 (d) 64

Sol. (c) The last 4 occurs as $1+2+3+4=10$th term. The last n occurs as $\left(\frac{n(n+1)}{2}\right)^{\text{th}}$ term, the last 62 occurs as $\left(\frac{62\times63}{2}\right)^{\text{th}}=1953$rd term and the last 63 occurs as $\left(\frac{63\times64}{2}\right)^{\text{th}}=2016$th term.

$\therefore$ 63 occurs from 1954th term to 2016th term.

Hence, $(2011)^{\text{th}}$ term is 63.

Ex. 5 *Let $S=\sum_{r=1}^{117}\frac{1}{2[\sqrt{r}]+1}$, when $[\cdot]$ denotes the greatest integer function and if $S=\frac{p}{q}$, when p and q are co-primes, the value of $p+q$ is*

(a) 20 (b) 76 (c) 19 (d) 69

Sol. (b) $\because \quad S = \sum_{r=1}^{117} \frac{1}{2[\sqrt{r}]+1}$

$$= \frac{3}{2\cdot 1+1} + \frac{5}{2\cdot 2+1} + \frac{7}{2\cdot 3+1} + \ldots + \frac{19}{2\cdot 9+1} + \frac{18}{2\cdot 10+1}$$

$$= 9 + \frac{18}{21} = 9 + \frac{6}{7} = \frac{69}{7}$$

$\therefore \quad p = 69$ and $q = 7 \Rightarrow p + q = 69 + 7 = 76$

● **Ex. 6** *If a, b, c are non-zero real numbers, then the minimum value of the expression* $\frac{(a^8 + 4a^4 + 1)(b^4 + 3b^2 + 1)(c^2 + 2c + 2)}{a^4 b^2}$ *equals*

(a) 12 (b) 24 (c) 30 (d) 60

Sol. (c) Let $P = \frac{(a^8 + 4a^4 + 1)(b^4 + 3b^2 + 1)(c^2 + 2c + 2)}{a^4 b^2}$

$$= \left(a^4 + 4 + \frac{1}{a^4}\right)\left(b^2 + 3 + \frac{1}{b^2}\right)\{(c+1)^2 + 1)\}$$

$\because \quad = a^4 + 4 + \frac{1}{a^4} \geq 6, b^2 + 3 + \frac{1}{b^2} \geq 5$ and $(c+1)^2 + 1 \geq 1$

$\left[\because x + \frac{1}{x} \geq 2 \text{ for } x > 0\right]$

$\therefore \quad P \geq 6\cdot 5\cdot 1 = 30 \Rightarrow P \geq 30$

Hence, the required minimum value is 30.

● **Ex. 7** *If the sum of m consecutive odd integers is* m^4*, then the first integer is*

(a) $m^3 + m + 1$ (b) $m^3 + m - 1$
(c) $m^3 - m - 1$ (d) $m^3 - m + 1$

Sol. (d) Let $2a+1, 2a+3, 2a+5, \ldots$ be the AP, then

$$m^4 = (2a+1) + (2a+3) + (2a+5) + \ldots \text{ upto } m \text{ terms}$$

$$= \frac{m}{2}\{2(2a+1) + (m-1)\cdot 2\} = m(2a+1+m-1)$$

$\Rightarrow \quad m^3 = (2a+1) + m - 1$

$\therefore \quad 2a + 1 = m^3 - m + 1$

● **Ex. 8** *The value of* $\sum_{r=1}^{\infty} \frac{(4r+5)5^{-r}}{r(5r+5)}$ *is*

(a) $\frac{1}{5}$ (b) $\frac{2}{5}$ (c) $\frac{1}{25}$ (d) $\frac{2}{125}$

Sol. (a) $\sum_{r=1}^{\infty} \frac{(4r+5)5^{-r}}{r(5r+5)} = \lim_{n\to\infty} \sum_{r=1}^{n} \left(\frac{(5r+5)-r}{r(5r+5)}\right)\cdot\frac{1}{5^r}$

$$= \lim_{n\to\infty} \sum_{r=1}^{n} \left(\frac{1}{r} - \frac{1}{5r+5}\right)\frac{1}{5^r}$$

$$= \lim_{n\to\infty} \sum_{r=1}^{n} \left(\frac{1}{r\cdot 5^{-r}} - \frac{1}{(r+1)5^{r+1}}\right)$$

$$= \lim_{n\to\infty} \sum_{r=1}^{n} \left(\frac{1}{5} - \frac{1}{(n+1)5^{n+1}}\right) = \frac{1}{5} - 0 = \frac{1}{5}$$

● **Ex. 9** *Let* λ *be the greatest integer for which* $5p^2 - 16, 2p\lambda, \lambda^2$ *are distinct consecutive terms of an AP, where* $p \in R$. *If the common difference of the AP is* $\left(\frac{m}{n}\right), m, n \in N$ *and m, n are relative prime, the value of* $m + n$ *is*

(a) 133 (b) 138 (c) 143 (d) 148

Sol. (c) $\because 5p^2 - 16, 2p\lambda, \lambda^2$ are in AP, then

$$4p\lambda = 5p^2 - 16 + \lambda^2$$

$\Rightarrow \quad 5p^2 - 4p\lambda + \lambda^2 - 16 = 0 \quad$ ...(i)

$\therefore \quad B - 4AC \geq 0 \quad [\because p \in R]$

$\Rightarrow \quad 16\lambda^2 - 4\cdot 5\cdot(\lambda^2 - 16) \geq 0$

$\Rightarrow \quad -\lambda^2 + 80 \geq 0$ or $\lambda^2 \geq 80$

$\Rightarrow \quad -\sqrt{80} \leq \lambda \leq \sqrt{80}$

$\therefore \quad \lambda = 8 \quad$ [greatest integer]

From Eq. (i), $\quad 5p^2 - 32p + 48 = 0$

$\Rightarrow \quad (p-4)(5p-12) = 0$

$\therefore \quad p = 4, p = \frac{12}{5}$

$\Rightarrow \quad p = \frac{12}{5}, p \neq 4$

[for $p = 4$ all terms are equal]

Now, common difference $= \lambda^2 - 2p\lambda$

$$= 64 - 16 \times \frac{12}{5} = 64\left(1 - \frac{3}{5}\right) = \frac{128}{5} = \frac{m}{n} \quad \text{[given]}$$

$\therefore \quad m = 128$ and $n = 5$

Hence, $\quad m + n = 143$

● **Ex. 10** *If* $2\lambda, \lambda$ *and* $[\lambda^2 - 14], \lambda \in R - \{0\}$ *and [.] denotes the greatest integer function are the first three terms of a GP in order, then the 51th term of the sequence,* $1, 3\lambda, 6\lambda, 10\lambda, \ldots$ *is*

(a) 5104 (b) 5304
(c) 5504 (d) 5704

Sol. (b) $\because 2\lambda, \lambda, [\lambda^2 - 14]$ are in GP, then

$$\lambda^2 = 2\lambda[\lambda^2 - 14]$$

$\Rightarrow \quad \frac{\lambda}{2} = [\lambda^2 - 14]$

$\therefore$ λ must be an even integer

Hence, $\quad \lambda = 4$

Now, required sequence 1,12, 24, 40, ...

or $1, 4(1+2), 4(1+2+3), 4(1+2+3+4), \ldots$

$\therefore \quad$ 51th term $= 4(1+2+3+\ldots+51)$

$$= 4\cdot\frac{51}{2}(1+51) = 4\cdot 51\cdot 26 = 5304$$

JEE Type Solved Examples : More than One Correct Option Type Questions

- This section contains **5 multiple choice examples.** Each example has four choices (a), (b), (c) and (d) out of **which more than one** may be correct.

Ex. 11 *The first three terms of a sequence are* 3, −1, −1. *The next terms are*

(a) 2 (b) 3 (c) $-\frac{5}{27}$ (d) $-\frac{5}{9}$

Sol. (b, d) The given sequence is not an AP or GP or HP. It is an AGP, $3, (3+d)r, (3+2d)r^2, \ldots$

$$\Rightarrow \quad (3+d)r = -1, (3+2d)r^2 = -1$$

Eliminating r, we get $(3+d)^2 = -(3+2d)$

$$\Rightarrow \quad d^2 + 8d + 12 = 0 \Rightarrow d = -2, -6,$$

then $r = -1, \frac{1}{3}$

$\therefore$ Next term is $(3+3d)r^3 = 3, -\frac{5}{9}$

Ex. 12 *There are two numbers a and b whose product is 192 and the quotient of AM by HM of their greatest common divisor and least common multiple is* $\frac{169}{48}$. *The smaller of a and b is*

(a) 2 (b) 4 (c) 6 (d) 12

Sol. (b, d) If $G =$ GED of a and b, $L =$ LCM of a and b, we have $GL = ab = 192$ ...(i)

$$\frac{\text{AM}}{\text{HM}} \text{ of } G \text{ and } L \text{ is } \left(\frac{G+L}{2}\right)\left(\frac{G+L}{2GL}\right) = \frac{169}{48}$$

$$\Rightarrow \quad (G+L)^2 = \frac{169}{12}GL = \frac{169}{12} \times 192 = 13^2 \cdot 4^2$$

$$\Rightarrow \quad G + L = 52 \text{ but } GL = 192$$

$$\Rightarrow \quad G = 4, L = 48 \Rightarrow a = 4, b = 48 \text{ or } a = 12, b = 16$$

Ex. 13 *Consider a series* $\frac{1}{2} + \frac{1}{2^2} + \frac{2}{2^3} + \frac{3}{2^4} + \frac{5}{2^5} + \ldots + \frac{\lambda n}{2^n}$. *If* S_n *denotes its sum to n terms, then* S_n *cannot be*

(a) 2 (b) 3 (c) 4 (d) 5

Sol. (a, b, c, d)

$$\because \quad Sn = \frac{1}{2} + \frac{1}{2^2} + \frac{2}{2^3} + \frac{3}{2^4} + \frac{5}{2^5} + \ldots + \frac{\lambda_n}{2^n}$$

$$= \frac{3}{4} + \frac{1}{4}\left(\frac{1}{2} + \frac{1}{2^2} + \frac{2}{2^3} + \frac{3}{2^4} + \frac{5}{2^5} + \ldots + \frac{\lambda_n}{2^n}\right)$$

$$+ \frac{1}{2}\left(\frac{1}{2} + \frac{1}{2^2} + \frac{2}{2^3} + \ldots + \frac{\lambda_n}{2^n}\right) - \frac{1}{4} - \frac{\lambda_n}{2^{n+2}} - \frac{\lambda_n}{2^{n+1}}$$

$$\Rightarrow \quad S_n = \frac{3}{4} + \frac{1}{4}S_n + \frac{1}{2}S_n - \frac{1}{4} - \frac{\lambda_n}{2^{n+2}} - \frac{\lambda_n}{2^{n+1}}$$

$$\Rightarrow \frac{1}{4}S_n = \frac{1}{2} - \frac{\lambda_n}{2^{n+2}} - \frac{\lambda_n}{2^{n+1}} \Rightarrow S_n = 2 - \frac{\lambda_n}{2^{n+1}} - \frac{\lambda_n}{2^{n-1}} < 2$$

Ex. 14 *If* $S_r = \sqrt{r + \sqrt{r + \sqrt{r + \sqrt{\ldots\infty}}}}$, $r > 0$ *then which the following is/are correct.*

(a) S_r, S_6, S_{12}, S_{20} are in AP

(b) S_4, S_9, S_{16} are irrational

(c) $(2S_{4-1})^2, (2S_{5-1})^2 (2S_{6-1})^2$ are in AP

(d) S_2, S_{12}, S_{56} are in GP

Sol. (a, b, c, d)

$$\because S_r = \sqrt{r + \sqrt{r + \sqrt{+\sqrt{\ldots\infty}}}} = \sqrt{r + S_r}$$

$$\Rightarrow \quad S_r^2 - S_r - r = 0$$

$$\therefore \quad S_r = \frac{1 + \sqrt{(1+4r)}}{2} \qquad [\because r > 0]$$

Alternate (a) S_2, S_6, S_{12}, S_{20} i.e., 2, 3, 4, 5 are in AP.

Alternate (b) S_4, S_9, S_{16} i.e., $\frac{1+\sqrt{17}}{2}, \frac{1+\sqrt{37}}{2}, \frac{1+\sqrt{65}}{2}$ are irrationals.

Alternate (c) $(2S_{4-1})^2, (2S_{5-1})^2, (2S_{6-1})^2$ i.e., 17, 21, 25 are in AP

Alternate (d) S_2, S_{12}, S_{56} i.e., 2, 4, 8 are in GP.

Ex. 15 *If* $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ *are in AP and a, b, −2c are in GP, where a, b, c are non-zero, then*

(a) $a^3 + b^3 + c^3 = 3abc$ (b) $-2a, b, -2c$ are in AP

(c) $-2a, b, -2c$ are in GP (d) $a^2, b^2, 4c^2$ are in GP

Sol. (a, b, d)

$\because \quad \frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in AP $\Rightarrow a, b, c$ are in HP

$$\therefore \quad b = \frac{2ab}{a+c} \qquad \text{...(i)}$$

and $a, b, -2c$ are in GP, then $b^2 = -2ac$ ...(ii)

From Eqs. (i) and (ii), we get

$$b = \frac{-b^2}{a+c} \Rightarrow a + b + c = 0 \qquad [\because b \neq 0]$$

$\therefore \quad a^3 + b^3 + c^3 = 3abc$ and $a, b, -2c$ are in GP

$\Rightarrow \quad a^2, b^2, 4c^2$ are also in GP and $a + b + c = 0$

$$\Rightarrow \quad 2b = -2a - 2c$$

$\therefore \quad -2a, b, -2c$ are in AP.

JEE Type Solved Examples :
Passage Based Questions

- This section contains **3 solved passages** based upon each of the passage **3 multiple choice** examples have to be answered. Each of these examples has four choices (a), (b), (c) and (d) out of which **ONLY ONE** is correct.

Passage I
(Ex. Nos. 16 to 18)

Consider a sequence whose sum to n terms is given by the quadratic function $S_n = 3n^2 + 5n$.

16. *The nature of the given series is*

(a) AP (b) GP (c) HP (d) AGP

Sol. (*a*) $\because \quad S_n = 3n^2 + 5n$

$\therefore \quad T_n = S_n - S_{n-1}$

$= (3n^2 + 5n) - [3(n-1)^2 + 5(n-1)]$

$= 3(2n - 1) + 5 = 6n + 2$

The nth term is a linear function in n. Hence, sequence must be an AP.

17. *For the given sequence, the number 5456 is the*

(a) 153 th term (b) 932 th term
(c) 707 th term (d) 909 th term

Sol. (*d*) Given, $T_n = 5456$

$\Rightarrow \quad 6n + 2 = 5456 \Rightarrow 6n = 5454$

$\therefore \quad n = 909$

$\therefore$ The number 5456 is the 909 th term.

18. *Sum of the squares of the first 3 terms of the given series is*

(a) 1100 (b) 660 (c) 799 (d) 1000

Sol. (*b*) $T_1^2 + T_2^2 + T_3^2 = 8^2 + 14^2 + 20^2 = 64 + 196 + 400 = 660$

Passage II
(Ex. Nos. 19 to 21)

Let r be the number of identical terms in the two AP's. Form the sequence of identical terms, it will be an AP, then the rth term of this AP make $t_r \le$ *the smaller of the last term of the two AP's.*

19. *The number of terms common to two AP's* 3, 7, 11, …, 407 *and* 2, 9, 16, …, 709 *is*

(a) 14 (b) 21 (c) 28 (d) 35

Sol. (*a*) Sequence 3, 7, 11, … , 407 has common difference = 4 and sequence 2, 9, 16, … , 709 has common difference = 7.

Hence, the sequence with common terms has common difference LCM of 4 and 7 which is 28.

The first common term is 23.

Hence, the sequence is 23, 51, 79, … , 387 which has 14 terms.

Aliter By inspection, first common term to both the series is 23, second common term is 51, third common term is 79 and so on. These numbers form an AP 23, 51, 79, …

Since, $T_{14} = 23 + 13(28) = 387 < 407$

and $T_{15} = 23 + 14(28) = 415 > 407$

Hence, number of common terms = 14

20. *The 10th common term between the series* $3 + 7 + 11 + \ldots$ *and* $1 + 6 + 11 + \ldots$ *is*

(a) 189 (b) 191 (c) 211 (d) 213

Sol. (b) Series 3 + 7 + 11 + … has common difference = 4 and series 1 + 6 + 11 + … has common difference = 5

Hence, the series with common terms has common difference LCM of 4 and 5 which is 20.

The first common terms is 11.

Hence, the series is 11 + 31 + 51 + 71 + …

$\therefore \quad t_{10} = 11 + (10 - 1)(20) = 191$

Aliter t_n for $3 + 7 + 11 + \ldots = 3 + (n-1)(4) = 4n - 1$

and t_m for $1 + 6 + 11 + \ldots = 1 + (m-1)(5) = 5m - 4$

For a common term, $4n - 1 = 5m - 4$ i.e., $4n = 5m - 3$

For $m = 3$, $n = 3$ gives the first common term i.e., 11.

For $m = 7$, $n = 8$ gives the second common term i.e., 31.

For $m = 11$, $n = 13$ gives the third common term i.e., 51.

Hence, the common term series is 11 + 31 + 51 + …

$\therefore \quad t_{10} = 11 + (10 - 1)20 = 191$

21. *The value of largest term common to the sequences* 1, 11, 21, 31, … *upto* 100 *terms and* 31, 36, 41, 46, … *upto* 100 *terms, is*

(a) 281 (b) 381 (c) 471 (d) 521

Sol. (*d*) Sequence 1, 11, 21, 31, … has common difference = 10 and sequence 31, 36, 41, 46, … has common difference = 5. Hence, the sequence with common terms has common difference LCM of 10 and 5 which is 10.

The first common term is 31.

Hence, the sequence is 31, 41, 51, 61, 71, … …(i)

Now, t_{100} of first sequence $= 1 + (100 - 1)10 = 991$

and t_{100} of second sequence $= 31 + (100 - 1)5 = 526$

Value of largest common term < 526

$\therefore t_n$ of Eq. (i) is $31 + (n - 1)10 = 10n + 21$

$t_{50} = 10 \times 50 + 21 = 521$

is the value of largest common term.

Aliter Let mth term of the first sequence be equal to the nth term of the second sequence, then

$$1 + (m - 1)10 = 31 + (n - 1)5$$

$\Rightarrow \quad 10m - 9 = 5n + 26 \Rightarrow 10m - 35 = 5n$

$\Rightarrow \quad 2m - 7 = n \le 100 \Rightarrow 2m \le 107$

$\Rightarrow \quad m \le 53\frac{1}{2}$

$\therefore$ Largest value of $m = 53$

$\therefore$ Value of largest term $= 1 + (53 - 1)\,10 = 521$

Passage III

(Ex. Nos. 22 to 24)

We are giving the concept of arithmetic mean of mth power. Let $a_1, a_2, a_3, \ldots, a_n$ be n positive real numbers (not all equal) and m be a real number. Then,

$$\frac{a_1^m + a_2^m + a_3^m + \ldots + a_n^m}{n} > \left(\frac{a_1 + a_2 + a_3 + \ldots + a_n}{n}\right)^m,$$

if $m \in R \sim [0, 1]$

However, if $m \in (0, 1)$, then

$$\frac{a_1^m + a_2^m + a_3^m + \ldots + a_n^m}{n} < \left(\frac{a_1 + a_2 + a_3 + \ldots + a_n}{n}\right)^m$$

Obviously, if $m = \{0, 1\}$, then

$$\frac{a_1^m + a_2^m + a_3^m + \ldots + a_n^m}{n} = \left(\frac{a_1 + a_2 + a_3 + \ldots + a_n}{n}\right)^m.$$

22. *If $x > 0, y > 0, z > 0$ and $x + y + z = 1$, the minimum value of $\frac{x}{2-x} + \frac{y}{2-y} + \frac{z}{2-z}$, is*

(a) 0.2 (b) 0.4

(c) 0.6 (d) 0.8

Sol. (*c*) Since, AM of (-1)th powers $\ge (-1)$th powers of AM

$$\therefore \frac{(2-x)^{-1} + (2-y)^{-1} + (2-z)^{-1}}{3} \ge \left(\frac{2 - x + 2 - y + 2 - z}{3}\right)^{-1}$$

$$= \left[\frac{6 - (x + y + z)}{3}\right]^{-1} = \left(\frac{6-1}{3}\right)^{-1} = \frac{3}{5} \qquad [\because x + y + z = 1]$$

$$\Rightarrow \frac{(2-x)^{-1} + (2-y)^{-1} + (2-z)^{-1}}{3} \ge \frac{3}{5}$$

or $$\frac{1}{3}\left[\frac{1}{2-x} + \frac{1}{2-y} + \frac{1}{2-z}\right] \ge \frac{3}{5}$$

$$\Rightarrow \frac{1}{2-x} + \frac{1}{2-y} + \frac{1}{2-z} \ge \frac{9}{5}$$

or $$\frac{2}{2-x} + \frac{2}{2-y} + \frac{2}{2-z} \ge \frac{18}{5}$$

or $$1 + \frac{x}{2-x} + 1 + \frac{y}{2-y} + 1 + \frac{z}{2-z} \ge \frac{18}{5}$$

or $$\frac{x}{2-x} + \frac{y}{2-y} + \frac{z}{2-z} \ge \frac{18}{5} - 3$$

Hence, $$\frac{x}{2-x} + \frac{y}{2-y} + \frac{z}{2-z} \ge \frac{3}{5} = 0.6$$

$$\Rightarrow \frac{x}{2-x} + \frac{y}{2-y} + \frac{z}{2-z} \ge 0.6$$

Thus, minimum value of $\frac{x}{2-x} + \frac{y}{2-y} + \frac{z}{2-z}$ is 0.6.

23. *If $\sum_{i=1}^{n} a_i^2 = \lambda, \forall\, a_i \ge 0$ and if greatest and least values of $\left(\sum_{i=1}^{n} a_i\right)^2$ are λ_1 and λ_2 respectively, then $(\lambda_1 - \lambda_2)$ is*

(a) $n\lambda$ (b) $(n-1)\lambda$

(c) $(n+2)\lambda$ (d) $(n+1)\lambda$

Sol. (*b*) $\because$ AM of 2nd powers $\ge$ 2nd power of AM

$$\therefore \frac{a_1^2 + a_2^2 + a_3^2 + \ldots + a_n^2}{n} \ge \left(\frac{a_1 + a_2 + a_3 + \ldots + a_n}{n}\right)^2$$

$$\Rightarrow \frac{\lambda}{n} \ge \left(\frac{\sum_{i=1}^{n} a_i}{n}\right)^2 \qquad \therefore \left(\sum_{i=1}^{n} a_i\right)^2 \le n\lambda \qquad \ldots\text{(i)}$$

Also, $(a_1 + a_2 + a_3 + \ldots + a_n)^2 = a_1^2 + a_2^2 + a_3^2 + \ldots + a_n^2 + 2\sum a_1 a_2$

$$= \lambda + 2\sum a_1 a_2 \ge \lambda$$

$$\therefore \left(\sum_{i=1}^{n} a_i\right)^2 \ge \lambda \qquad \ldots\text{(ii)}$$

From Eqs. (i) and (ii), we get

$$\lambda \le \left(\sum_{i=1}^{n} a_i\right)^2 \le n\lambda$$

$\therefore \quad \lambda_1 = n\lambda$ and $\lambda_2 = \lambda$

Then, $\quad \lambda_1 - \lambda_2 = (n-1)\lambda$

24. *If sum of the mth powers of first n odd numbers is λ, $\forall\, m > 1$, then*

(a) $\lambda < n^m$ (b) $\lambda > n^m$ (c) $\lambda < n^{m+1}$ (d) $\lambda > n^{m+1}$

Sol. (*d*) $\because m > 1$

$$\therefore \frac{1^m + 3^m + 5^m + \ldots + (2n-1)^m}{n} > \left(\frac{1 + 3 + 5 + \ldots + (2n-1)}{n}\right)^m$$

$$= \left(\frac{\frac{n}{2}(1 + 2n - 1)}{n}\right)^m = n^m$$

$\therefore 1^m + 3^m + 5^m + \ldots + (2n-1)^m > n^{m+1}$

Hence, $\quad \lambda > n^{m+1}$

JEE Type Solved Examples :
Single Integer Answer Type Questions

- This section contains **2 examples.** The answer to each example is a **single digit integer** ranging from **0** to **9** (both inclusive).

● ***Ex. 25*** *A sequence of positive terms $A_1, A_2, A_3, \ldots, A_n$ satisfies the relation $A_{n+1} = \frac{3(1+A_n)}{(3+A_n)}$. Least integral value of A_1 for which the sequence is decreasing can be*

Sol. (2) $\because \quad A_{n+1} = \frac{3(1+A_n)}{(3+A_n)}$. For $n = 1$, $A_2 = \frac{3(1+A_1)}{(3+A_1)}$

For $n = 2$, $A_3 = \frac{3(1+A_2)}{(3+A_2)}$

$$= \frac{3\left(1+\frac{3(1+A_1)}{(3+A_1)}\right)}{3+\frac{3(1+A_1)}{(3+A_1)}} = \frac{6+4A_1}{4+2A_1} = \frac{3+2A_1}{2+A_1}$$

$\because$ Given, sequence can be written as

$$A_1, \frac{3(1+A_1)}{(3+A_1)}, \frac{(3+2A_1)}{(2+A_1)}, \ldots$$

Given, $A_1 > 0$ and sequence is decreasing, then

$$A_1 > \frac{3(1+A_1)}{(3+A_1)}, \frac{3(1+A_1)}{(3+A_1)} > \frac{(3+2A_1)}{(2+A_1)}$$

$\Rightarrow \quad A_1^2 > 3 \quad \text{or} \quad A_1 > \sqrt{3}$

$\therefore \quad A_1 = 2$ [least integral value of A_1]

● ***Ex. 26*** *When the ninth term of an AP is divided by its second term we get 5 as the quotient, when the thirteenth term is divided by sixth term the quotient is 2 and the remainder is 5, then the second term is*

Sol. (7) Let a be the first term and d be the common difference, then $T_9 = 5\,T_2$

$\Rightarrow \quad (a+8d) = 5(a+d)$

$\therefore \quad 4a = 3d$ ...(i)

and $\quad T_{13} = T_6 \times 2 + 5$

$\Rightarrow \quad a + 12d = 2(a+5d) + 5$

$\Rightarrow \quad 2d = a + 5$ ...(ii)

From Eqs. (i) and (ii), we get

$a = 3$ and $d = 4$

$\therefore \quad T_2 = a + d = 7$

JEE Type Solved Examples :
Matching Type Questions

- This section contains **2 examples.** Examples 27 has three statements (A, B and C) given in **Column I** and four statements (p, q, r and s) in **Column II** example 28 has four statements (A, B, C and D) given in **Column I** and five statements (p, q, r, s and t) in **Column II**. Any given statement in **Column I** can have correct matching with one or more statement(s) given in **Column II.**

● ***Ex. 27***

	Column I		Column II
(A)	If $a_1, a_2, a_3, \ldots$ are in AP and $a_1 + a_6 + a_{10} + a_{21} + a_{25} + a_{30} = 120$, then $\sum_{i=1}^{30} a_i$ is	(p)	400
(B)	If $a_1, a_2, a_3, \ldots$ are in AP and $a_1 + a_5 + a_9 + a_{13} + a_{17} + a_{21} + a_{25} = 112$, then $\sum_{i=1}^{25} a_i$ is	(q)	600
(C)	If $a_1, a_2, a_3, \ldots$ are in AP and $a_1 + a_4 + a_7 + a_{10} + a_{13} + a_{16} = 375$, then $\sum_{i=1}^{16} a_i$ is	(r)	800
		(s)	1000

Sol. (A) → (q); (B) → (p); (C) → (s)

(A) $\because a_1, a_2, a_3, \ldots$ are in AP.

$\therefore \quad a_1 + a_{30} = a_6 + a_{25} = a_{10} + a_{21} = \lambda$ [say]

$\because \quad a_1 + a_6 + a_{10} + a_{21} + a_{25} + a_{30} = 120$

$\therefore \quad 3\lambda = 120$

$\Rightarrow \quad \lambda = 40$

Then, $\quad \sum_{i=1}^{30} a_i = \frac{30}{2}(a_1 + a_{30}) = 15 \times \lambda = 15 \times 40 = 600$

(B) $\because a_1, a_2, a_3, \ldots$ are in AP.

$\therefore \quad a_1 + a_{25} = a_5 + a_{21}$

$= a_9 + a_{17} = a_{13} + a_{13} = \lambda$ [say]

$\because \quad a_1 + a_5 + a_9 + a_{13} + a_{17} + a_{21} + a_{25} = 112$

$\therefore \quad 3\lambda + \frac{\lambda}{2} = 112$

$\Rightarrow \quad \frac{7\lambda}{2} = 112$

$\Rightarrow \quad \lambda = 32$

Then, $\quad \sum_{i=1}^{25} a_i = \frac{25}{2}(a_1 + a_{25}) = \frac{25}{2} \times 32 = 400$

(C) $\because a_1, a_2, a_3, \ldots$ are in AP.

$\therefore \quad a_1 + a_{16} = a_4 + a_{13} = a_7 + a_{10} = \lambda$ [say]

$\because \quad a_1 + a_4 + a_7 + a_{10} + a_{13} + a_{16} = 375$

$\therefore \quad 3\lambda = 375 \quad \therefore \quad \lambda = 125$

Then, $$\sum_{i=1}^{16} a_i = \frac{16}{2}(a_1 + a_{16})$$
$$= 8 \times \lambda = 8 \times 125 = 1000$$

Ex. 28

	Column I		Column II
(A)	If $a > 0, b > 0, c > 0$ and the minimum value of $a(b^2 + c^2) + b(c^2 + a^2) + c(a^2 + b^2)$ is $\lambda\, abc$, then λ is	(p)	2
(B)	If a, b, c are positive, $a + b + c = 1$ and the minimum value of $\left(\frac{1}{a} - 1\right)\left(\frac{1}{b} - 1\right)\left(\frac{1}{c} - 1\right)$ is λ, then λ is	(q)	4
(C)	If $a > 0, b > 0, c > 0, s = a + b + c$ and the minimum value of $\frac{2s}{s-a} + \frac{2s}{s-b} + \frac{2s}{s-c}$ is $(\lambda - 1)$, then λ is	(r)	6
(D)	If $a > 0, b > 0, c > 0, a, b, c$ are in GP and the the minimum value of $\left(\frac{a}{b}\right)^\lambda + \left(\frac{c}{b}\right)^\lambda$ is 2, then λ is	(s)	8
		(t)	10

Sol. (A) $\rightarrow$ (r); (B) $\rightarrow$ (s); (C) $\rightarrow$ (t); (D) $\rightarrow$ (p, q, r, s, t)

(A) $\because$ AM $\geq$ GM

$$\therefore \frac{ab^2 + ac^2 + bc^2 + ba^2 + ca^2 + cb^2}{6}$$
$$\geq (ab^2 \cdot ac^2 \cdot bc^2 \cdot ba^2 \cdot ca^2 \cdot cb^2)^{1/6} = abc$$

$\therefore \quad a(b^2 + c^2) + b(c^2 + a^2) + c(a^2 + b^2) \geq 6\, abc$

$\Rightarrow \quad \lambda = 6$

(B) $\because$ AM $\geq$ GM

For b, c, we get $\dfrac{(b + c)}{2} \geq \sqrt{bc}$

$\Rightarrow \quad (b + c) \geq 2\sqrt{bc}$ ...(i)

For c, a, we get
$$\frac{(c + a)}{2} \geq \sqrt{ca}$$

$\Rightarrow \quad (c + a) \geq 2\sqrt{ca}$ ...(ii)

and for a, b, we get
$$\frac{(a + b)}{2} \geq \sqrt{ab}$$

$\Rightarrow \quad (a + b) \geq 2\sqrt{ab}$ ...(iii)

On multiplying Eqs. (i), (ii) and (iii), we get
$$(b + c)(c + a)(a + b) \geq 8abc$$

$\Rightarrow \quad (1 - a)(1 - b)(1 - c) \geq 8abc \quad [\because a + b + c = 1]$

$\Rightarrow \quad \left(\frac{1}{a} - 1\right)\left(\frac{1}{b} - 1\right)\left(\frac{1}{c} - 1\right) \geq 8$

$\therefore \quad \lambda = 8$

(C) $\because$ AM $\geq$ HM

$$\frac{(s - a) + (s - b) + (s - c)}{3} \geq \frac{3}{\frac{1}{s-a} + \frac{1}{s-b} + \frac{1}{s-c}}$$

$$\Rightarrow \frac{3s - (a + b + c)}{3} \geq \frac{3}{\frac{1}{s-a} + \frac{1}{s-b} + \frac{1}{s-c}}$$

$$\Rightarrow \frac{3s - s}{3} \geq \frac{3}{\left(\frac{1}{s-a} + \frac{1}{s-b} + \frac{1}{s-c}\right)}$$

$$\Rightarrow \frac{2s}{s-a} + \frac{2s}{s-b} + \frac{2s}{s-c} \geq 9$$

Here, $\lambda - 1 = 9$

$\therefore \quad \lambda = 10$

(D) If a, b, c are in GP.

Then, $a^\lambda, b^\lambda, c^\lambda$ are also in GP.

Then, AM $\geq$ GM
$$\frac{a^\lambda + c^\lambda}{2} \geq b^\lambda$$

$\Rightarrow \quad a^\lambda + c^\lambda \geq 2b^\lambda$

$\Rightarrow \quad \left(\frac{a}{b}\right)^\lambda + \left(\frac{c}{b}\right)^\lambda \geq 2$

$\therefore \quad \lambda \in R$

Hence, $\lambda = 2, 4, 6, 8, 10$

JEE Type Solved Examples : Statement I and II Type Questions

- **Directions** Example numbers 29 to 32 are Assertion-Reason type examples. Each of these examples contains two statements:
Statement-1 (Assertion) and **Statement-2** (Reason)
Each of these examples also has four alternative choices, only one of which is the correct answer. You have to select the correct choice as given below.
(a) Statement-1 is true, Statement-2 is true; Statement-2 is a correct explanation for Statement-1
(b) Statement-1 is true, Statement-2 is true; Statement-2 is not a correct explanation for Statement-1
(c) Statement-1 is true, Statement-2 is false
(d) Statement-1 is false, Statement-2 is true

Ex. 29. Statement 1 *The sum of first n terms of the series* $1^2 - 2^2 + 3^2 - 4^2 + 5^2 - \ldots$ can be $= \pm \frac{n(n+1)}{2}$.

Statement 2 *Sum of first n natural numbers is* $\frac{n(n+1)}{2}$.

Sol. (a) Clearly, nth term of the given series is negative or positive according as n is even or odd, respectively.

***Case* I** When n is even, in this case the given series is

$$1^2 - 2^2 + 3^2 - 4^2 + 5^2 - 6^2 + \ldots + (n-1)^2 - n^2$$
$$= (1^2 - 2^2) + (3^2 - 4^2) + (5^2 - 6^2) + \ldots + [(n-1)^2 - n^2]$$
$$= (1-2)(1+2) + (3-4)(3+4) + (5-6)(5+6) + \ldots + (n-1-n)(n-1+n)$$
$$= -(1+2+3+4+5+6+\ldots+(n-1)+n) = \frac{n(n+1)}{2}$$

***Case* II** When n is odd, in this case the given series is

$$1^2 - 2^2 + 3^2 - 4^2 + 5^2 - 6^2 + \ldots + (n-2)^2 - (n-1)^2 + n^2$$
$$= (1^2 - 2^2) + (3^2 - 4^2) + (5^2 - 6^2) + \ldots + [(n-2)^2 - (n-1)^2] + n^2$$
$$= (1-2)(1+2) + (3-4)(3+4) + (5-6)(5+6) + \ldots + [(n-2)-(n-1)][(n-2)+(n-1)] + n^2$$
$$= -[1+2+3+4+5+6+\ldots+(n-2)+(n-1)] + n^2$$
$$= -\frac{(n-1)(n-1+1)}{2} + n^2 = \frac{n(n+1)}{2}$$

It is clear that Statement-1 is true, Statement-2 is true and Statement-2 is correct explanation for Statement-1.

Ex. 30 Statement 1 *If a, b, c are three positive numbers in GP, then* $\left(\frac{a+b+c}{3}\right)\left(\frac{3abc}{ab+bc+ca}\right) = (abc)^{2/3}$.

Statement-2 *(AM) (HM)=(GM)^2 is true for positive numbers.*

Sol. (c) If a, b be two real, positive and unequal numbers, then

$$\text{AM} = \frac{a+b}{2}, \text{GM} = \sqrt{ab} \text{ and HM} = \frac{2ab}{a+b}$$

$\therefore$ AM) (HM) = (GM)^2

This result will be true for n numbers, if they are in GP.
Hence, Statement-1 is true and Statement-2 is false.

Ex. 31 *Consider an AP with a as the first term and d is the common difference such that S_n denotes the sum to n terms and a_n denotes the nth term of the AP. Given that for some $m, n \in N$, $\frac{S_m}{S_n} = \frac{m^2}{n^2}$ $(m \neq n)$.*

Statement 1 *$d = 2a$ because*

Statement 2 $\frac{a_m}{a_n} = \frac{2m+1}{2n+1}$

Sol. (c) $\because$ $\frac{S_m}{S_n} = \frac{m^2}{n^2}$

Let $S_m = m^2 k$, $S_n = n^2 k$

$\therefore$ $a_m = S_m - S_{m-1} = m^2 k - (m-1)^2 k$

$\Rightarrow$ $a_m = (2m-1)k$

Similarly, $a_n = (2n-1)k$ $\therefore$ $\frac{a_m}{a_n} = \frac{2m-1}{2n-1}$

Statement-2 is false.

Also, $\because$ $a_1 = k, a_2 = 3k, a_3 = 5k, \ldots$

Given, $a_1 = a = k$

$\therefore$ $a_1 = a, a_2 = 3a, a_3 = 5a, \ldots$

$\therefore$ Common difference $d = a_2 - a_1 = a_3 - a_2 = \ldots$

$\Rightarrow$ $d = 2a$

$\therefore$ Statement-1 is true.

Ex. 32 Statement-1 *1, 2, 4, 8, ... is a GP, 4, 8, 16, 32, ... is a GP and $1+4, 2+8, 4+16, 8+32, \ldots$ is also a GP.*

Statement-2 *Let general term of a GP with common ratio r be T_{k+1} and general term of another GP with common ratio r be T'_{k+1}, then the series whose general term $T''_{k+1} = T_{k+1} + T'_{k+1}$ is also a GP with common ratio r.*

Sol. (a) 1, 2, 4, 8, ...

Common ratio $r = 2$

$\therefore$ $T_{k+1} = 1 \cdot (2)^{k+1-1} = 2^k$

and 4, 8, 16, 32, ...

Common ratio, $r = 2$

$\therefore$ $T'_{k+1} = 4 \cdot (2)^{k+1-1} = 4 \cdot 2^k$

Then, $T_{k+1} + T'_{k+1} = 5 \cdot 2^k = T''_{k+1}$

Common ratio of $T''_{k+1} = \frac{5 \cdot 2^k}{5 \cdot 2^{k-1}} = 2$, which is true.

Hence, Statement-1 and Statement-2 both are true and Statement-2 is the correct explanation of Statement-1.

Subjective Type Examples

- In this section, there are **24 subjective** solved examples.

Ex. 33 *In a set of four numbers, the first three are in GP and the last three are in AP with a common difference of 6. If the first number is same as the fourth, then find the four numbers.*

Sol. Let the last three numbers in AP, be $a, a+6, a+12$.

[$\because$ 6 is the common difference]

If first number is b, then four numbers are

$$b, a, a+6, a+12$$

But given, $b = a + 12$

$\therefore$ Four numbers are $a + 12, a, a+6, a+12$ ...(i)

Since, first three numbers are in GP.

Then, $a^2 = (a+12)(a+6)$

$\Rightarrow \quad a^2 = a^2 + 18a + 72$

$\Rightarrow \quad 18a + 72 = 0$

$\therefore \quad a = -4$ [from Eq. (i)]

Hence, four numbers are 8, − 4, 2, 8.

Ex. 34 *Find the natural number a for which* $\sum_{k=1}^{n} f(a+k) = 16(2^n - 1)$, *where the function f satisfies* $f(x+y) = f(x)\,f(y)$ *for all natural numbers x, y and further* $f(1) = 2$.

Sol. Given, $f(x+y) = f(x)\,f(y)$ ...(i)

and $f(1) = 2$ ...(ii)

On putting $x = y = 1$ in Eq. (i), we get

$$f(1+1) = f(1)\,f(1) = 2\cdot 2$$

$\therefore \quad f(2) = 2^2$ ...(iii)

Now, on putting $x = 1, y = 2$ in Eq. (i), we get

$f(1+2) = f(1)\,f(2) = 2\cdot 2^2$ [from Eqs. (ii) and (iii)]

$\therefore \quad f(3) = 2^3$

On putting $x = y = 2$ in Eq. (i), we get

$f(2+2) = f(2)\,f(2) = 2^2 \cdot 2^2$ [from Eq. (iii)]

$\therefore \quad f(4) = 2^4$

$\vdots \quad \vdots \quad \vdots$

Similarly, $f(\lambda) = 2^{\lambda}, \lambda \in N$

$\therefore \quad f(a+k) = 2^{a+k}, a+k \in N$

$\because \quad \sum_{k=1}^{n} f(a+k) = 16(2^n - 1) \Rightarrow \sum_{k=1}^{n} 2^{a+k} = 16(2^n - 1)$

$\Rightarrow \quad 2^a \sum_{k=1}^{n} 2^k = 16(2^n - 1)$

$\Rightarrow \quad 2^a(2^1 + 2^2 + 2^3 + \ldots + 2^n) = 16(2^n - 1)$

$\Rightarrow \quad 2^a \cdot \dfrac{2(2^n - 1)}{(2-1)} = 16(2^n - 1)$

$\Rightarrow \quad 2^{a+1} = 16 = 2^4$

$\Rightarrow \quad a + 1 = 4$

$\therefore \quad a = 3$

Ex. 35 *If n is a root of* $x^2(1-ac) - x(a^2 + c^2) - (1+ac) = 0$ *and if n harmonic means are inserted between a and c, find the difference between the first and the last means.*

Sol. Let $H_1, H_2, H_3, \ldots, H_n$, are n harmonic means, then $a, H_1, H_2, H_3, \ldots, H_n, b$ are in HP.

$\therefore \dfrac{1}{a}, \dfrac{1}{H_1}, \dfrac{1}{H_2}, \dfrac{1}{H_3}, \ldots, \dfrac{1}{H_n}, \dfrac{1}{b}$ are in AP.

If d be the common difference, then $\dfrac{1}{c} = \dfrac{1}{a} + (n + 2 - 1)\,d$

$\therefore \quad d = \dfrac{(a-c)}{ac\,(n+1)}$ ...(i)

$\Rightarrow \quad \dfrac{1}{h_1} = \dfrac{1}{a} + d$ and $\dfrac{1}{h_n} = \dfrac{1}{c} - d$

$\therefore \quad h_1 - h_n = \dfrac{a}{1+ad} - \dfrac{c}{1-cd} = \dfrac{a}{1 + \dfrac{(a-c)}{c(n+1)}} - \dfrac{c}{1 - \dfrac{(a-c)}{a(n+1)}}$

$= \dfrac{ac\,(n+1)}{cn+a} - \dfrac{ac\,(n+1)}{an+c} = ac\,(n+1)\left(\dfrac{1}{cn+a} - \dfrac{1}{an+c}\right)$

$= ac\,(n+1)\left(\dfrac{an + c - cn - a}{acn^2 + (a^2+c^2)n + ac}\right)$

$= \dfrac{ac\,(a-c)(n^2-1)}{acn^2 + (a^2+c^2)\,n + ac}$ ...(ii)

But given n is a root of

$$x^2(1-ac) - x(a^2+c^2) - (1+ac) = 0.$$

Then, $n^2(1-ac) - n(a^2+c^2) - (1+ac) = 0$

or $\quad acn^2 + (a^2+c^2)n + ac = n^2 - 1,$

then from Eq. (ii), $h_1 - h_n = \dfrac{ac\,(a-c)(n^2-1)}{(n^2-1)} = ac\,(a-c)$

Ex. 36 *A number consists of three-digits which are in GP the sum of the right hand and left hand digits exceeds twice the middle digit by 1 and the sum of the left hand and middle digits is two third of the sum of the middle and right hand digits. Find the number.*

Sol. Let the three digits be a, ar and ar^2, then number is

$$100a + 10ar + ar^2 \quad \text{...(i)}$$

Given, $a + ar^2 = 2ar + 1$

or $a(r^2 - 2r + 1) = 1$

or $a(r-1)^2 = 1$ ...(ii)

Also, given $a + ar = \frac{2}{3}(ar + ar^2) \Rightarrow 3 + 3r = 2r + 2r^2$

or $2r^2 - r - 3 = 0$ or $(r+1)(2r-3) = 0$

$\therefore$ $r = -1, \frac{3}{2}$

For $r = -1$, $a = \frac{1}{(r-1)^2} = \frac{1}{4} \notin I$

$\therefore$ $r \neq -1$

For $r = \frac{3}{2}, a = \frac{1}{\left(\frac{3}{2} - 1\right)^2} = 4$ [from Eq. (ii)]

From Eq. (i), number is $400 + 10 \cdot 4 \cdot \frac{3}{2} + 4 \cdot \frac{9}{4} = 469$

● **Ex. 37** *Find the value of the expression* $\sum_{i=1}^{n} \sum_{j=1}^{i} \sum_{k=1}^{j} 1$.

Sol. $\sum_{i=1}^{n} \sum_{j=1}^{i} \sum_{k=1}^{j} 1 = \sum_{i=1}^{n} \sum_{j=1}^{i} j = \sum_{i=1}^{n} \frac{i(i+1)}{2}$

$$= \frac{1}{2}\left[\sum_{i=1}^{n} i^2 + \sum_{i=1}^{n} i\right] = \frac{1}{2}\left[\sum n^2 + \sum n\right]$$

$$= \frac{1}{2}\left[\frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}\right]$$

$$= \frac{n(n+1)}{12}[2n+1+3] = \frac{n(n+1)(n+2)}{6}$$

● **Ex. 38** *Three numbers are in GP whose sum is 70. If the extremes be each multiplied by 4 and the mean by 5, then they will be in AP. Find the numbers.*

Sol. Let the three numbers in GP be $\frac{a}{r}, a, ar$.

Given, $\frac{a}{r} + a + ar = 70$ ...(i)

and $\frac{4a}{r}, 5a, 4ar$ are in AP.

$\therefore$ $10a = \frac{4a}{r} + 4ar$ or $\frac{10a}{4} = \frac{a}{r} + ar$

or $\frac{5a}{2} = 70 - a$ [from Eq. (i)]

or $5a = 140 - 2a$ or $7a = 140$

$\therefore$ $a = 20$

From Eq. (i), we get

$$\frac{20}{r} + 20 + 20r = 70$$

or $\frac{20}{r} + 20r = 50$

or $2 + 2r^2 = 5r$ or $2r^2 - 5r + 2 = 0$

or $(r-2)(2r-1) = 0$ $\therefore$ $r = 2$ or $\frac{1}{2}$

Hence, the three numbers are 10, 20, 40 or 40, 20, 10.

● **Ex. 39** *If the sum of m terms of an AP is equal to the sum of either the next n terms or the next p terms, then prove that* $(m+n)\left(\frac{1}{m} - \frac{1}{p}\right) = (m+p)\left(\frac{1}{m} - \frac{1}{n}\right)$.

Sol. Let the AP be $a, a+d, a+2d, \ldots$

Given, $T_1 + T_2 + \ldots + T_m = T_{m+1} + T_{m+2} + \ldots + T_{m+n}$ ...(i)

On adding $T_1 + T_2 + \ldots + T_m$ both sides in Eq. (i), we get

$2(T_1 + T_2 + \ldots + T_m) = T_1 + T_2 + \ldots + T_m + T_{m+1} + \ldots + T_{m+n}$

$\Rightarrow$ $2S_m = S_{m+n}$

$\therefore$ $2 \cdot \frac{m}{2}[2a + (m-1)d] = \frac{m+n}{2}[2a + (m+n-1)d]$

Let $2a + (m-1)d = x$

$\Rightarrow$ $mx = \frac{m+n}{2}\{x + nd\}$

$\Rightarrow$ $(m-n)x = (m+n)nd$ ...(ii)

Again, $T_1 + T_2 + \ldots + T_m = T_{m+1} + T_{m+2} + \ldots + T_{m+p}$

Similarly, $(m-p)x = (m+p)pd$ ...(iii)

On dividing Eq. (ii) by Eq. (iii), we get

$$\frac{m-n}{m-p} = \frac{(m+n)n}{(m+p)p}$$

$\Rightarrow$ $(m-n)(m+p)p = (m-p)(m+n)n$

On dividing both sides by mnp, we get

$$(m+p)\left(\frac{1}{n} - \frac{1}{m}\right) = (m+n)\left(\frac{1}{p} - \frac{1}{m}\right)$$

Hence, $(m+n)\left(\frac{1}{m} - \frac{1}{p}\right) = (m+p)\left(\frac{1}{m} - \frac{1}{n}\right)$

● **Ex. 40** *Find the sum of the products of every pair of the first n natural numbers.*

Sol. We find that

$S = 1\cdot2 + 1\cdot3 + 1\cdot4 + \ldots + 2\cdot3 + 2\cdot4 + \ldots + 3\cdot4 + 3\cdot5 + \ldots + (n-1)\cdot n$ ...(i)

$\because$ $[1 + 2 + 3 + \ldots + (n-1) + n]^2 = 1^2 + 2^2 + 3^2 + \ldots + (n-1)^2 + n^2 + 2[1\cdot2 + 1\cdot3 + 1\cdot4 + \ldots + 2\cdot3 + 2\cdot4 + \ldots + 3\cdot4 + 3\cdot5 + \ldots + (n-1)\cdot n]$

$(\sum n)^2 = \sum n^2 + 2S$ [from Eq. (i)]

$$\Rightarrow \quad S = \frac{(\sum n)^2 - \sum n^2}{2}$$

$$= \frac{\left\{\frac{n(n+1)}{2}\right\}^2 - \frac{n(n+1)(2n+1)}{6}}{2}$$

$$= \frac{\frac{n^2(n+1)^2}{4} - \frac{n(n+1)(2n+1)}{6}}{2}$$

$$= \frac{n(n+1)}{24}[3n(n+1) - 2(2n+1)]$$

$$= \frac{n(n+1)(3n^2 - n - 2)}{24}$$

Hence, $S = \frac{(n-1)n(n+1)(3n+2)}{24}$

● **Ex. 41** *If* $I_n = \int_0^{\pi/4} \tan^n x\, dx$, *show that*

$\frac{1}{I_2 + I_4}, \frac{1}{I_3 + I_5}, \frac{1}{I_4 + I_6}, \frac{1}{I_5 + I_7}, \ldots$ *form an AP. Find its common difference.*

Sol. We have,

$$I_n + I_{n+2} = \int_0^{\pi/4} (\tan^n x + \tan^{n+2} x)\, dx$$

$$= \int_0^{\pi/4} \tan^n x (1 + \tan^2 x)\, dx$$

$$= \int_0^{\pi/4} \tan^n x \cdot \sec^2 x\, dx = \left[\frac{\tan^{n+1} x}{n+1}\right]_0^{\pi/4} = \frac{1}{n+1}$$

Hence, $\frac{1}{I_n + I_{n+2}} = n + 1$

On putting $n = 2, 3, 4, 5, \ldots$

$\therefore \quad \frac{1}{I_2 + I_4} = 3, \frac{1}{I_3 + I_5} = 4, \frac{1}{I_4 + I_6} = 5, \frac{1}{I_5 + I_7} = 6, \ldots$

Hence, $\frac{1}{I_2 + I_4}, \frac{1}{I_3 + I_5}, \frac{1}{I_4 + I_6}, \frac{1}{I_5 + I_7}, \ldots$ are in AP with common difference 1.

● **Ex. 42** *If the sum of the terms of an infinitely decreasing GP is equal to the greatest value of the function* $f(x) = x^3 + 3x - 9$ *on the interval* $[-5, 3]$ *and the difference between the first and second terms is* $f'(0)$, *then show that the common ratio of the progression is* $\frac{2}{3}$.

Sol. Given, $f(x) = x^3 + 3x - 9$

$\therefore \quad f'(x) = 3x^2 + 3$

Hence, $f'(x) > 0$ in $[-5, 3]$, then $f(x)$ is an increasing function on $[-5, 3]$ and therefore, $f(x)$ will have greatest value at $x = 3$.

Thus, greatest value of $f(x)$ is

$$f(x) = 3^3 + 3 \cdot 3 - 9 = 27$$

Let $a, ar, ar^2, \ldots$ be a GP with common ratio $|r| < 1$ [$\because$ given infinitely GP]

and also given $S_\infty = 27$

so, $\frac{a}{1-r} = 27$ ...(i)

and $a - ar = f'(0)$

$\Rightarrow \quad a(1-r) = f'(0) = 3$ [$\because f'(0) = 3$]

$\therefore \quad a(1-r) = 3$ ...(ii)

From Eqs. (i) and (ii), we get

$$(1-r)^2 = \frac{1}{9} \Rightarrow 1 - r = \pm\frac{1}{3}$$

$\therefore \quad r = 1 \pm \frac{1}{3}$

So, $r = \frac{4}{3}, \frac{2}{3} \Rightarrow r \neq \frac{4}{3}$ [$\because |r| < 1$]

Hence, $r = \frac{2}{3}$

● **Ex. 43** *Solve the following equations for x and y*

$$\log_{10} x + \frac{1}{2}\log_{10} x + \frac{1}{4}\log_{10} x + \ldots = y$$

and $\frac{1 + 3 + 5 + \ldots + (2y - 1)}{4 + 7 + 10 + \ldots + (3y + 1)} = \frac{20}{7\log_{10} x}$.

Sol. From the first equation

$$\log_{10} x \left\{1 + \frac{1}{2} + \frac{1}{4} + \ldots + \infty\right\} = y$$

$$\Rightarrow \quad \log_{10} x \left\{\frac{1}{1 - \frac{1}{2}}\right\} = y$$

$$\Rightarrow \quad 2\log_{10} x = y \quad \text{...(i)}$$

From the second equation

$$\frac{1 + 3 + 5 + \ldots + (2y - 1)}{4 + 7 + 10 + \ldots + (3y + 1)} = \frac{20}{7\log_{10} x}$$

$$\Rightarrow \quad \frac{\frac{y}{2}(1 + 2y - 1)}{\frac{y}{2}(4 + 3y + 1)} = \frac{20}{7\log_{10} x}$$

$$\Rightarrow \quad \frac{2y}{3y + 5} = \frac{20}{7\log_{10} x}$$

$\Rightarrow \quad 7y(2\log_{10} x) = 60y + 100$

$\Rightarrow \quad 7y(y) = 60y + 100$ [from Eq. (i)]

$\Rightarrow \quad 7y^2 - 60y - 100 = 0$

$\therefore \quad (y - 10)(7y + 10) = 0$

$\therefore \quad y = 10, y \neq \frac{-10}{7}$

[because y being the number of terms in series $\Rightarrow y \in N$]

From Eq. (i), we have

$$2\log_{10} x = 10 \Rightarrow \log_{10} x = 5$$

$\therefore \quad x = 10^5$

Hence, required solution is $x = 10^5, y = 10$

Ex. 44 *If* $0 < x < \dfrac{\pi}{2}$,

$\exp[(\sin^2 x + \sin^4 x + \sin^6 x + \ldots + \infty) \log_e 2]$ *satisfies the quadratic equation* $x^2 - 9x + 8 = 0$, *find the value of* $\dfrac{\sin x - \cos x}{\sin x + \cos x}$.

Sol. $0 < x < \dfrac{\pi}{2}$

$\therefore \quad 0 < \sin^2 x < 1$

Then, $\sin^2 x + \sin^4 x + \sin^6 x + \ldots + \infty$

$$= \frac{\sin^2 x}{1 - \sin^2 x} = \tan^2 x$$

$\therefore \exp[(\sin^2 x + \sin^4 x + \sin^6 x + \ldots + \infty) \log_e 2]$

$$= \exp(\tan^2 x \cdot \log_e 2) = \exp(\log_e 2^{\tan^2 x})$$

$$= e^{\log_e 2^{\tan^2 x}} = 2^{\tan^2 x}$$

Let $y = 2^{\tan^2 x}$

Because y satisfies the quadratic equation.

Then, $y^2 - 9y + 8 = 0$

So, $y = 1, 8$

if $y = 1 = 2^{\tan^2 x}$

$\Rightarrow 2^{\tan^2 x} = 2^0$

$\Rightarrow \tan^2 x = 0$

$\therefore x = 0$ [impossible] [$\because x > 0$]

Now, if $y = 8 = 2^{\tan^2 x}$

$\Rightarrow 2^{\tan^2 x} = 2^3$

$\Rightarrow \tan^2 x = 3$

$\therefore \tan x = \sqrt{3}$

$$\therefore \frac{\sin x - \cos x}{\sin x + \cos x} = \frac{\tan x - 1}{\tan x + 1} = \frac{\sqrt{3} - 1}{\sqrt{3} + 1} \times \frac{\sqrt{3} - 1}{\sqrt{3} - 1}$$

$$= \frac{(\sqrt{3} - 1)^2}{3 - 1} = \frac{3 + 1 - 2\sqrt{3}}{2}$$

Hence, $\dfrac{\sin x - \cos x}{\sin x + \cos x} = 2 - \sqrt{3}$

Ex. 45 *The natural numbers are arranged in the form given below*

			1				
		2		3			
	4	5		6	7		
8	9	10	11	12	13	14	15

....

....

The rth group containing 2^{r-1} numbers. Prove that sum of the numbers in the nth group is $2^{n-2}[2^n + 2^{n-1} - 1]$.

Sol. Let 1st term of the r th group be T_r and the 1st terms of successive rows are 1, 2, 4, 8, ..., respectively.

$$T_r = 1 \cdot 2^{r-1} = 2^{r-1}$$

Hence, the sum of the numbers in the r th group is

$$= \frac{2^{r-1}}{2}\{2 \cdot 2^{r-1} + (2^{r-1} - 1) \cdot 1\}$$

[$\because$ number of terms in rth group is 2^{r-1}]

$$= 2^{r-2}\{2^r + 2^{r-1} - 1\}$$

Hence, sum of the numbers in the nth group is $2^{n-2}[2^n + 2^{n-1} - 1]$.

Ex. 46 *If* a, b, c *are in HP, then prove that* $\dfrac{a+b}{2a-b} + \dfrac{c+b}{2c-b} > 4$.

Sol. Since, a, b, c are in HP.

$$\therefore \quad \frac{2}{b} = \frac{1}{a} + \frac{1}{c} \quad \text{...(i)}$$

and let $P = \dfrac{a+b}{2a-b} + \dfrac{c+b}{2c-b}$

$$= \frac{a + \frac{2ac}{a+c}}{2a - \frac{2ac}{a+c}} + \frac{c + \frac{2ac}{a+c}}{2c - \frac{2ac}{a+c}} \quad \text{[from Eq. (i)]}$$

$$= \frac{a + 3c}{2a} + \frac{3a + c}{2c} = 1 + \frac{3}{2}\left(\frac{c}{a} + \frac{a}{c}\right) \quad \text{...(ii)}$$

$\because$ AM > GM [$\because a \neq c$]

$\therefore \left(\dfrac{c}{a} + \dfrac{a}{c}\right) > 2$

$\Rightarrow \dfrac{3}{2}\left(\dfrac{c}{a} + \dfrac{a}{c}\right) > 3$

or $1 + \dfrac{3}{2}\left(\dfrac{c}{a} + \dfrac{a}{c}\right) > 1 + 3$ or $P > 4$

Hence, $\dfrac{a+b}{2a-b} + \dfrac{c+b}{2c-b} > 4$

Ex. 47 *Find the sum of n terms of the series* $\dfrac{1}{1+1^2+1^4} + \dfrac{2}{1+2^2+2^4} + \dfrac{3}{1+3^2+3^4} + \ldots$.

Sol. The n th term of the given series is $T_n = \dfrac{n}{(1 + n^2 + n^4)}$

$$\therefore \text{Sum of } n \text{ terms} = S_n = \sum T_n = \sum \frac{n}{(1 + n^2 + n^4)}$$

$$= \sum \frac{n}{(1 + n + n^2)(1 - n + n^2)}$$

$$= \frac{1}{2} \sum \left(\frac{1}{1 - n + n^2} - \frac{1}{1 + n + n^2} \right)$$

$$= \frac{1}{2} \left(\frac{1}{1 - 1 + 1} - \frac{1}{1 + n + n^2} \right) \quad \text{[by property]}$$

$$= \frac{(n + n^2)}{2(1 + n + n^2)} = \frac{n(n+1)}{2(n^2 + n + 1)}$$

Ex. 48 *The value of xyz is 55 or $\frac{343}{55}$ according as the series a, x, y, z, b is an AP or HP. Find the values of a and b given that they are positive integers.*

Sol. If a, x, y, z, b are in AP.

Then, b = Fifth term $= a + (5-1)d$

where, d is common difference]

$$\therefore \quad d = \frac{b - a}{4}$$

$$\therefore \quad x \cdot y \cdot z = (a + d)(a + 2d)(a + 3d) = 55 \quad \text{[given]}$$

$$\Rightarrow \quad \left(\frac{b + 3a}{4} \right) \left(\frac{2a + 2b}{4} \right) \left(\frac{a + 3b}{4} \right) = 55$$

$$\Rightarrow \quad (a + 3b)(a + b)(3a + b) = 55 \times 32 \quad \text{...(i)}$$

If they are in HP.

The common difference of the associated AP is $\frac{1}{4}\left(\frac{1}{b} - \frac{1}{a}\right)$.

i.e. $\quad \frac{(a - b)}{4ab}$

$$\therefore \quad \frac{1}{x} = \frac{1}{a} + \frac{(a - b)}{4ab}$$

$$\Rightarrow \quad x = \frac{4ab}{a + 3b}$$

$$\therefore \quad \frac{1}{y} = \frac{1}{a} + \frac{2(a - b)}{4ab}$$

$$\Rightarrow \quad y = \frac{4ab}{2a + 2b} = \frac{2ab}{a + b}$$

and $\quad \frac{1}{z} = \frac{1}{a} + \frac{3(a - b)}{4ab}$

$$\Rightarrow \quad z = \frac{4ab}{3a + b}$$

$$\therefore \quad xyz = \frac{4ab}{(a + 3b)} \cdot \frac{2ab}{(a + b)} \cdot \frac{4ab}{(3a + b)} = 343 \quad \text{[given]}$$

$$\Rightarrow \quad \frac{32 a^3 b^3}{55 \times 32} = \frac{343}{55} \quad \text{[from Eq. (i)]}$$

or $\quad a^3 b^3 = 343$

$\Rightarrow \quad ab = 7$

Hence, $\quad a = 7, b = 1$

or $\quad a = 1, b = 7$

Ex. 49 *Find the sum of the first n terms of the series* $1^3 + 3 \cdot 2^2 + 3^3 + 3 \cdot 4^2 + 5^3 + 3 \cdot 6^2 + \dots$ *If (i) n is even, (ii) n is odd.*

Sol. ***Case* I** If n is even.

Let $\quad n = 2m$

$$\therefore \quad S = 1^3 + 3 \cdot 2^2 + 3^3 + 3 \cdot 4^2 + 5^3 + 3 \cdot 6^2 + \dots + (2m - 1)^3 + 3(2m)^2$$

$$= \{1^3 + 3^3 + 5^3 + \dots + (2m - 1)^3\} + 3\{2^2 + 4^2 + 6^2 + \dots + (2m)^2\}$$

$$= \sum_{r=1}^{m} (2r - 1)^3 + 3 \cdot 4 \sum_{r=1}^{m} r^2$$

$$= \sum_{r=1}^{m} \{8r^3 - 12r^2 + 6r - 1\} + 12 \sum_{r=1}^{m} r^2$$

$$= 8 \sum_{r=1}^{m} r^3 - 12 \sum_{r=1}^{m} r^2 + 6 \sum_{r=1}^{m} r - \sum_{r=1}^{m} 1 + 12 \sum_{r=1}^{m} r^2$$

$$= 8 \sum_{r=1}^{m} r^3 + 6 \sum_{r=1}^{m} r - \sum_{r=1}^{m} 1$$

$$= 8 \cdot \frac{m^2 (m + 1)^2}{4} + 6 \frac{m(m + 1)}{2} - m$$

$$= 2m^2 (m + 1)^2 + 3m(m + 1) - m$$

$$= m[2m^3 + 4m^2 + 5m + 2]$$

$$= \frac{n}{2} \left[2\left(\frac{n}{2}\right)^3 + 4\left(\frac{n}{2}\right)^2 + 5\left(\frac{n}{2}\right) + 2 \right] \quad \left[\because m = \frac{n}{2} \right]$$

Hence, $S = \frac{n}{8}(n^3 + 4n^2 + 10n + 8) \quad$ ...(i)

***Case* II** If n is odd.

Then, $(n + 1)$ is even in the case

Sum of first n terms = Sum of first $(n + 1)$ terms $- (n + 1)$ th term

$$= \frac{(n + 1)}{8} [(n + 1)^3 + 4(n + 1)^2 + 10(n + 1) + 8] - 3(n + 1)^2$$

$$= \frac{1}{8}(n + 1)[n^3 + 3n^2 + 3n + 1 + 4n^2 + 8n + 4 + 10n + 10 + 8 - 24n - 24]$$

Hence, $S = \frac{1}{8}(n + 1)[n^3 + 7n^2 - 3n - 1]$

Ex. 50 *Find out the largest term of the sequence* $\frac{1}{503}, \frac{4}{524}, \frac{9}{581}, \frac{16}{692}, \dots$.

Sol. General term can be written as $T_n = \frac{n^2}{500 + 3n^3}$

Let $\quad U_n = \frac{1}{T_n} = \frac{500}{n^2} + 3n$

Then, $\quad \frac{dU_n}{dn} = -\frac{1000}{n} + 3$

and $\quad \frac{d^2U_n}{dn^2} = \frac{3000}{n^4}$

For maxima or minima of U_n, we have

$$\frac{dU_n}{dn} = 0 \Rightarrow n^3 = \frac{1000}{3}$$

$$\Rightarrow \quad n = \left(\frac{1000}{3}\right)^{1/3} \text{ (not an integer) and } 6 < \left(\frac{1000}{3}\right)^{1/3} < 7$$

But n is an integer, therefore for the maxima or minima of U_n we will take n as the nearest integer to $\left(\frac{1000}{3}\right)^{1/3}$.

Since, $\left(\frac{1000}{3}\right)^{1/3}$ is more close to 7 than to 6. Thus, we take $n = 7$.

Further $\frac{d^2U^n}{dn^2} = +ve$, then U_n will be minimum and therefore, T_n will be maximum for $n = 7$.

Hence, T_7 is largest term. So, largest term in the given sequence is $\frac{49}{1529}$.

● ***Ex. 51*** *If $f(r) = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{r}$ and $f(0) = 0$, find $\sum_{r=1}^{n}(2r+1)\, f(r)$.*

Sol. Since, $\sum_{r=1}^{n}(2r+1)\, f(r)$

$$= \sum_{r=1}^{n}(r^2 + 2r + 1 - r^2)\, f(r) = \sum_{r=1}^{n}\{(r+1)^2 - r^2\}\, f(r)$$

$$= \sum_{r=1}^{n}\{(r+1)^2 f(r) - (r+1)^2 f(r+1) + (r+1)^2 f(r+1) - r^2 f(r)\}$$

$$= \sum_{r=1}^{n}(r+1)^2\{f(r) - f(r+1)\} + \sum_{r=1}^{n}\{(r+1)^2 f(r+1) - r^2 f(r)\}$$

$$= -\sum_{r=1}^{n}\frac{(r+1)^2}{(r+1)} + \sum_{r=1}^{n-1}(r+1)^2 f(r+1) + (n+1)^2 f(n+1) - \sum_{r=1}^{n} r^2 f(r) \quad \left[\because f(r+1) - f(r) = \frac{1}{r+1}\right]$$

$$= -\sum_{r=1}^{n}(r+1) + \{2^2 f(2) + 3^2 f(3) + \ldots + n^2 f(n)\} + (n+1)^2 f(n+1) - \{1^2 f(1) + 2^2 f(2) + 3^2 f(3) + \ldots + n^2 f(n)\}$$

$$= -\sum_{r=1}^{n} r - \sum_{r=1}^{n} 1 + (n+1)^2 f(n+1) - 1^2 f(1)$$

$$= -\frac{n(n+1)}{2} - n + (n+1)^2 f(n+1) - f(1)$$

$$= (n+1)^2 f(n+1) - \frac{n(n+3)}{2} - 1 \quad [\because f(1) = 1]$$

$$= (n+1)^2 f(n+1) - \frac{(n^2+3n+2)}{2}$$

Hence, this is the required result.

● ***Ex. 52*** *If the equation $x^4 - 4x^3 + ax^2 + bx + 1 = 0$ has four positive roots, find the values of a and b.*

Sol. Let x_1, x_2, x_3, x_4 are the roots of the equation

$$x^4 - 4x^3 + ax^2 + bx + 1 = 0 \quad \ldots\text{(i)}$$

$\therefore\ x_1 + x_2 + x_3 + x_4 = 4$ and $x_1 x_2 x_3 x_4 = 1$

$\because \quad \text{AM} = \frac{x_1 + x_2 + x_3 + x_4}{4} = \frac{4}{4} = 1$

and $\text{GM} = (x_1 x_2 x_3 x_4)^{1/4} = (1)^{1/4} = 1$

i.e., AM = GM

which is true only when $x_1 = x_2 = x_3 = x_4 = 1$

Hence, given equation has all roots identical, equal to 1 i.e., equation have form

$$(x-1)^4 = 0$$

$$\Rightarrow \quad x^4 - 4x^3 + 6x^2 - 4x + 1 = 0 \quad \ldots\text{(ii)}$$

On comparing Eqs. (i) and (ii), we get

$$a = 6,\ b = -4$$

● ***Ex. 53*** *Evaluate* $\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{m^2 n}{3^m(n\cdot 3^m + m\cdot 3^n)}$.

Sol. Let $S = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{m^2 n}{3^m(n\cdot 3^m + m\cdot 3^n)}$

$$= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{\left(\frac{3^m}{m}\right)\left(\frac{3^m}{m} + \frac{3^n}{n}\right)}$$

Now, let $\quad a_m = \frac{3^m}{m}$ and $a_n = \frac{3^n}{n}$

Then, $\quad S = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{a_m(a_m + a_n)} \quad \ldots\text{(i)}$

By interchanging m and n, then

$$S = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{a_n(a_n + a_m)} \quad \ldots\text{(ii)}$$

On adding Eqs. (i) and (ii), we get

$$2S = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{a_m a_n} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{mn}{3^m 3^n}$$

$$=\left(\sum_{n=1}^{\infty}\frac{n}{3^n}\right)^2=\left[1\left(\frac{1}{3}\right)+2\left(\frac{1}{3}\right)^2+3\left(\frac{1}{3}\right)^3+\dots\right]^2$$

$$=(S')^2 \qquad \dots\text{(iii)}$$

where, $S'=1\left(\frac{1}{3}\right)+2\left(\frac{1}{3}\right)^2+3\left(\frac{1}{3}\right)^3+\dots+\infty$

$$\frac{1}{3}S'=\quad 1\left(\frac{1}{3}\right)^2+2\left(\frac{1}{3}\right)^3+\dots+\infty$$

$$\frac{2}{3}S'=\frac{1}{3}+\left(\frac{1}{3}\right)^2+\left(\frac{1}{3}\right)^3+\dots+\infty$$

$$=\frac{\frac{1}{3}}{1-\frac{1}{3}}=\frac{1}{2}$$

$$\therefore \quad S'=\frac{3}{4}$$

From Eq. (iii), we get $2S=\left(\frac{3}{4}\right)^2$

$$\therefore \quad S=\frac{9}{32}$$

● **Ex. 54** *Find the value of* $\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{1}{3^i\,3^j\,3^k}$ $(i\neq j\neq k)$

Sol. Let $S=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{1}{3^i\,3^j\,3^k}$ $\qquad [i\neq j\neq k]$

We will first of all find the sum without any restriction on i, j, k.

Let $\quad S_1=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{1}{3^i\,3^j\,3^k}=\left(\sum_{i=0}^{\infty}\frac{1}{3^i}\right)^3$

$$=\left(\frac{3}{2}\right)^3=\frac{27}{8}$$

***Case* I** If $i=j=k$

Let $\quad S_2=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{1}{3^i\,3^j\,3^k}$

$$=\sum_{i=0}^{\infty}\frac{1}{3^{3i}}=1+\frac{1}{3^3}+\frac{1}{3^6}+\dots=\frac{1}{1-\frac{1}{3^3}}=\frac{27}{26}$$

***Case* II** If $i=j\neq k$

Let $\quad S_3=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{1}{3^i\,3^j\,3^k}=\left(\sum_{i=0}^{\infty}\frac{1}{3^{2i}}\right)\left(\sum_{k=0}^{\infty}\frac{1}{3^k}\right)$

$$[\because k\neq i]$$

$$=\sum_{i=0}^{\infty}\frac{1}{3^{2i}}\left(\frac{3}{2}-\frac{1}{3^i}\right)=\sum_{i=0}^{\infty}\frac{3}{2}\cdot\frac{1}{3^{2i}}-\sum_{i=0}^{\infty}\frac{1}{3^{3i}}$$

$$=\frac{3}{2}\cdot\frac{9}{8}-\frac{27}{26}=\frac{135}{208}$$

Hence required sum, $S=S_1-S_2-3S_3$

$$=\frac{27}{8}-\frac{27}{26}-3\left(\frac{135}{208}\right)=\frac{27\times26-27\times8-3\times135}{208}=\frac{81}{208}$$

● **Ex. 55** *Let S_n, $n=1,2,3,\dots$ be the sum of infinite geometric series, whose first term is n and the common ratio is* $\frac{1}{n+1}$. *Evaluate*

$$\lim_{n\to\infty}\frac{S_1S_n+S_2\,S_{n-1}+S_3S_{n-2}+\dots+S_nS_1}{S_1^2+S_2^2+\dots+S_n^2}.$$

Sol. $\because \quad S_n=\frac{n}{1-\frac{1}{n+1}} \Rightarrow S_n=n+1$

$$\therefore S_1S_n+S_2S_{n-1}+S_3S_{n-2}+\dots+S_nS_1$$

$$=\sum_{r=1}^{n}S_r\,S_{n-r+1}=\sum_{r=1}^{n}(r+1)(n-r+2)$$

$$=\sum_{r=1}^{n}[(n+1)r-r^2+(n+2)]$$

$$=(n+1)\sum_{r=1}^{n}r-\sum_{r=1}^{n}r^2+(n+2)\sum_{r=1}^{n}1$$

$$=(n+1)\sum n-\sum n^2+(n+2)\cdot n$$

$$=\frac{(n+1)\,n\,(n+1)}{2}-\frac{n(n+1)(2n+1)}{6}+(n+2)\,n$$

$$=\frac{n}{6}(n^2+9n+14) \qquad \dots\text{(i)}$$

and $S_1^2+S_2^2+\dots+S_n^2=\sum_{r=1}^{n}S_r^2=\sum_{r=1}^{n}(r+1)^2=\sum_{r=0}^{n}(r+1)^2-1^2$

$$=\frac{(n+1)(n+2)(2n+3)}{6}-1$$

$$=\frac{n}{6}(2n^2+9n+13) \qquad \dots\text{(ii)}$$

From Eqs. (i) and (ii), we get

$$\lim_{n\to\infty}\frac{S_1S_n+S_2\,S_{n-1}+S_3S_{n-2}+\dots+S_nS_1}{S_1^2+S_2^2+\dots+S_n^2}$$

$$=\lim_{n\to\infty}\frac{\frac{n}{6}(n^2+9n+14)}{\frac{n}{6}(2n^2+9n+13)}=\lim_{n\to\infty}\frac{\left(1+\frac{9}{n}+\frac{14}{n^2}\right)}{\left(2+\frac{9}{n}+\frac{13}{n^2}\right)}$$

$$=\frac{1+0+0}{2+0+0}=\frac{1}{2}$$

Ex. 56 *The nth term of a series is given by* $t_n = \dfrac{n^5 + n^3}{n^4 + n^2 + 1}$ *and if sum of its n terms can be expressed as* $S_n = a_n^2 + a + \dfrac{1}{b_n^2 + b}$*, where* a_n *and* b_n *are the nth terms of some arithmetic progressions and a, b are some constants, prove that* $\dfrac{b_n}{a_n}$ *is a constant.*

Sol. Since, $t_n = \dfrac{n^5 + n^3}{n^4 + n^2 + 1}$

$$= n - \frac{n}{n^4 + n^2 + 1}$$

$$= n + \frac{1}{2(n^2 + n + 1)} - \frac{1}{2(n^2 - n + 1)}$$

Sum of n terms $S_n = \sum t_n$

$$= \sum n + \frac{1}{2}\left\{\sum\left(\frac{1}{n^2+n+1} - \frac{1}{n^2-n+1}\right)\right\}$$

$$= \frac{n(n+1)}{2} + \frac{1}{2}\left(\frac{1}{n^2 + n + 1} - 1\right) \quad \text{[by property]}$$

$$= \frac{n^2}{2} + \frac{n}{2} - \frac{1}{2} + \frac{1}{2n^2 + 2n + 2}$$

$$= \left(\frac{n}{\sqrt{2}} + \frac{1}{2\sqrt{2}}\right)^2 - \frac{1}{8} - \frac{1}{2} + \frac{1}{\left(n\sqrt{2} + \frac{1}{\sqrt{2}}\right)^2 + \frac{3}{2}}$$

$$= \left(\frac{n}{\sqrt{2}} + \frac{1}{2\sqrt{2}}\right)^2 - \frac{5}{8} + \frac{1}{\left(n\sqrt{2} + \frac{1}{\sqrt{2}}\right)^2 + \frac{3}{2}}$$

but given, $S_n = a_n^2 + a + \dfrac{1}{b_n^2 + b}$

On comparing, we get

$$a_n = \frac{n}{\sqrt{2}} + \frac{1}{2\sqrt{2}},\ a = -\frac{5}{8},\ b_n = \left(n\sqrt{2} + \frac{1}{\sqrt{2}}\right),\ b = \frac{3}{2}$$

$\therefore \quad \dfrac{b_n}{a_n} = 2$, which is constant.