JEE Type Solved Examples :
Single Option Correct Type Questions

- This section contains **10 multiple choice examples.** Each example has four choices (a), (b), (c) and (d) out of which **ONLY ONE** is correct.

Ex. 1 *If α and $\beta (\alpha < \beta)$, are the roots of the equation $x^2 + bx + c = 0$, where $c < 0 < b$, then*

(a) $0 < \alpha < \beta$ (b) $\alpha < 0 < \beta < |\alpha|$
(c) $\alpha < \beta < 0$ (d) $\alpha < 0 < |\alpha| < \beta$

Sol. (*b*) $\because \quad \alpha + \beta = -b, \ \alpha\beta = c \quad \ldots(i)$

$\because \quad c < 0 \Rightarrow \alpha\beta < 0$

Let $\quad \alpha < 0, \beta > 0$

$\therefore \quad |\alpha| = -\alpha$ and $\alpha < 0 < \beta \quad [\because \alpha < \beta] \ldots(ii)$

From Eq. (i), we get $-|\alpha| + \beta < 0$

$\Rightarrow \quad \beta < |\alpha| \quad \ldots(iii)$

From Eqs. (ii) and (iii), we get

$$\alpha < 0 < \beta < |\alpha|$$

Ex. 2 *Let α, β be the roots of the equation $x^2 - x + p = 0$ and γ, δ be the roots of the equation $x^2 - 4x + q = 0$. If α, β, γ and δ are in GP, the integral values of p and q respectively, are*

(a) $-2, -32$ (b) $-2, 3$
(c) $-6, 3$ (d) $-6, -32$

Sol. (*a*) Let r be the common ratio of the GP, then

$$\beta = \alpha r, \gamma = \alpha r^2 \text{ and } \delta = \alpha r^3$$

$\therefore \quad \alpha + \beta = 1 \Rightarrow \alpha + \alpha r = 1$

or $\quad \alpha(1 + r) = 1 \quad \ldots(i)$

and $\quad \alpha\beta = p \Rightarrow \alpha(\alpha r) = p$

or $\quad \alpha^2 r = p \quad \ldots(ii)$

and $\quad \gamma + \delta = 4 \Rightarrow \alpha r^2 + \alpha r^3 = 4$

or $\quad \alpha r^2(1 + r) = 4 \quad \ldots(iii)$

and $\quad \gamma\delta = q$

$\Rightarrow \quad (\alpha r^2)(\alpha r^3) = q$

or $\quad \alpha^2 r^5 = q \quad \ldots(iv)$

On dividing Eq. (iii) by Eq. (i), we get

$$r^2 = 4 \Rightarrow r = -2, 2$$

If we take $r = 2$, then α is not integer, so we take $r = -2$.
On substituting $r = -2$ in Eq. (i), we get $\alpha = -1$

Now, from Eqs. (ii) and (iv), we get

$$p = \alpha^2 r = (-1)^2(-2) = -2$$

and $\quad q = \alpha^2 r^5 = (-1)^2(-2)^5 = -32$

Hence, $\quad (p, q) = (-2, -32)$

Ex. 3 *Let $f(x) = \int_1^x \sqrt{(2 - t^2)}\, dt$, the real roots of the equation $x^2 - f'(x) = 0$ are*

(a) ± 1 (b) $\pm \frac{1}{\sqrt{2}}$
(c) $\pm \frac{1}{2}$ (d) 0 and 1

Sol. (*a*) We have, $f(x) = \int_1^x \sqrt{(2 - t^2)}\, dt$

$\Rightarrow \quad f'(x) = \sqrt{(2 - x^2)}$

$\therefore \quad x^2 - f'(x) = 0$

$\Rightarrow \quad x^2 - \sqrt{(2 - x^2)} = 0 \Rightarrow x^4 + x^2 - 2 = 0$

$\Rightarrow \quad x^2 = 1, -2$

$\Rightarrow \quad x = \pm 1 \quad$ [only for real value of x]

Ex. 4 *If $x^2 + 3x + 5 = 0$ and $ax^2 + bx + c = 0$ have a common root and $a, b, c \in N$, the minimum value of $a + b + c$ is*

(a) 3 (b) 9
(c) 6 (d) 12

Sol. (*b*) $\because$ Roots of the equation $x^2 + 3x + 5 = 0$ are non-real. Thus, given equations will have two common roots.

$\Rightarrow \quad \frac{a}{1} = \frac{b}{3} = \frac{c}{5} = \lambda \quad$ [say]

$\therefore \quad a + b + c = 9\lambda$

Thus, minimum value of $a + b + c = 9 \quad [\because a, b, c \in N]$

Ex. 5 *If $x_1, x_2, x_3, \ldots, x_n$ are the roots of the equation $x^n + ax + b = 0$, the value of $(x_1 - x_2)(x_1 - x_3)(x_1 - x_4)\ldots(x_1 - x_n)$ is*

(a) $nx_1 + b$
(b) $n(x_1)^{n-1}$
(c) $n(x_1)^{n-1} + a$
(d) $n(x_1)^{n-1} + b$

Sol. (*c*) $\because x^n + ax + b = (x - x_1)(x - x_2)(x - x_3)\ldots(x - x_n)$

$\Rightarrow \quad (x - x_2)(x - x_3)\ldots(x - x_n) = \frac{x^n + ax + b}{x - x_1}$

On taking $\lim_{x \to x_1}$ both sides, we get

$$(x_1 - x_2)(x_1 - x_3)\ldots(x_1 - x_n) = \lim_{x \to x_1} \frac{x^n + ax + b}{x - x_1} \left[\frac{0}{0} \text{form}\right]$$

$$= \lim_{x \to x_1} \frac{nx^{n-1} + a}{1} = n(x_1)^{n-1} + a$$

Ex. 6 *If α, β are the roots of the equation $ax^2 + bx + c = 0$ and $A_n = \alpha^n + \beta^n$, then $aA_{n+2} + bA_{n+1} + cA_n$ is equal to*

(a) 0 (b) 1 (c) $a + b + c$ (d) abc

Sol. (*a*) $\because \alpha + \beta = -\frac{b}{a}$ and $\alpha\beta = \frac{c}{a}$

$$\therefore \quad A_{n+2} = \alpha^{n+2} + \beta^{n+2}$$
$$= (\alpha + \beta)(\alpha^{n+1} + \beta^{n+1}) - \alpha\beta^{n+1} - \beta\alpha^{n+1}$$
$$= (\alpha + \beta)(\alpha^{n+1} + \beta^{n+1}) - \alpha\beta(\alpha^n + \beta^n)$$
$$= -\frac{b}{a} A_{n+1} - \frac{c}{a} A_n$$
$$\Rightarrow \quad aA_{n+2} + bA_{n+1} + cA_n = 0$$

Ex. 7 *If x and y are positive integers such that $xy + x + y = 71$, $x^2y + xy^2 = 880$, then $x^2 + y^2$ is equal to*

(a) 125 (b) 137 (c) 146 (d) 152

Sol. (*c*) $\because$ $xy + x + y = 71 \Rightarrow xy + (x + y) = 71$

and $x^2y + xy^2 = 880 \Rightarrow xy(x + y) = 880$

$\Rightarrow$ xy and $(x + y)$ are the roots of the quadratic equation.

$$t^2 - 71t + 880 = 0$$
$$\Rightarrow \quad (t - 55)(t - 16) = 0$$
$$\Rightarrow \quad t = 55, 16$$
$$\therefore \quad x + y = 16 \text{ and } xy = 55$$

So, $x^2 + y^2 = (x + y)^2 - 2xy = (16)^2 - 110 = 146$

Ex. 8 *If α, β are the roots of the equation $x^2 - 3x + 5 = 0$ and γ, δ are the roots of the equation $x^2 + 5x - 3 = 0$, then the equation whose roots are $\alpha\gamma + \beta\delta$ and $\alpha\delta + \beta\gamma$, is*

(a) $x^2 - 15x - 158 = 0$ (b) $x^2 + 15x - 158 = 0$

(c) $x^2 - 15x + 158 = 0$ (d) $x^2 + 15x + 158 = 0$

Sol. (*d*) $\because$ $\alpha + \beta = 3, \alpha\beta = 5, \gamma + \delta = (-5), \gamma\delta = (-3)$

Sum of roots $= (\alpha\gamma + \beta\delta) + (\alpha\delta + \beta\gamma)$
$$= (\alpha + \beta)(\gamma + \delta) = 3 \times (-5) = (-15)$$

Product of roots $= (\alpha\gamma + \beta\delta)(\alpha\delta + \beta\gamma)$
$$= \alpha^2\gamma\delta + \alpha\beta\gamma^2 + \beta\alpha\delta^2 + \beta^2\gamma\delta$$
$$= \gamma\delta(\alpha^2 + \beta^2) + \alpha\beta(\gamma^2 + \delta^2)$$
$$= -3(\alpha^2 + \beta^2) + 5(\gamma^2 + \delta^2)$$
$$= -3[(\alpha + \beta)^2 - 2\alpha\beta] + 5[(\gamma + \delta)^2 - 2\gamma\delta]$$
$$= -3[9 - 10] + 5[25 + 6] = 158$$

$\therefore$ Required equation is $x^2 + 15x + 158 = 0$.

Ex. 9 *The number of roots of the equation $\frac{1}{x} + \frac{1}{\sqrt{(1 - x^2)}} = \frac{35}{12}$ is*

(a) 0 (b) 1

(c) 2 (d) 3

Sol. (*d*) Let $\frac{1}{x} = u$ and $\frac{1}{\sqrt{(1 - x^2)}} = v$, then

$$u + v = \frac{35}{12} \text{ and } u^2 + v^2 = u^2v^2$$
$$\Rightarrow \quad (u + v)^2 = \left(\frac{35}{12}\right)^2$$
$$\Rightarrow \quad u^2 + v^2 + 2uv = \left(\frac{35}{12}\right)^2$$
$$\Rightarrow \quad u^2v^2 + 2uv = \left(\frac{35}{12}\right)^2 \quad [\because u^2 + v^2 = u^2v^2]$$
$$\Rightarrow \quad u^2v^2 + 2uv - \left(\frac{35}{12}\right)^2 = 0$$
$$\Rightarrow \quad \left(uv + \frac{49}{12}\right)\left(uv - \frac{25}{12}\right) = 0$$
$$\Rightarrow \quad uv = -\frac{49}{12}, uv = \frac{25}{12}$$

***Case* I** If $uv = -\frac{49}{12}$, then

$$\frac{1}{x} \cdot \frac{1}{\sqrt{(1 - x^2)}} = -\frac{49}{12} \quad [\text{here } x < 0]$$
$$\Rightarrow \quad x^4 - x^2 + \frac{(12)^2}{(49)^2} = 0$$
$$\Rightarrow \quad x = -\frac{(5 + \sqrt{73})}{14}$$

***Case* II** If $uv = \frac{25}{12}$, then

$$\frac{1}{x} \cdot \frac{1}{\sqrt{(1 - x^2)}} = \frac{25}{12} \quad [\text{here } x > 0]$$
$$\Rightarrow \quad x^4 - x^2 + \frac{(12)^2}{(25)^2} = 0$$
$$\Rightarrow \quad \left(x^2 - \frac{9}{25}\right)\left(x^2 - \frac{16}{25}\right) = 0 \Rightarrow x = \frac{3}{5}, \frac{4}{5}$$

On combining both cases,

$$x = -\frac{(5 + \sqrt{73})}{14}, \frac{3}{5}, \frac{4}{5}$$

Hence, number of roots = 3

Ex. 10 *The sum of the roots of the equation $2^{33x - 2} + 2^{11x + 2} = 2^{22x + 1} + 1$ is*

(a) $\frac{1}{11}$ (b) $\frac{2}{11}$ (c) $\frac{3}{11}$ (d) $\frac{4}{11}$

Sol. (*b*) Let $2^{11x} = t$, given equation reduces to

$$\frac{t^3}{4} + 4t = 2t^2 + 1$$
$$\Rightarrow \quad t^3 - 8t^2 + 16t - 4 = 0 \Rightarrow t_1 \cdot t_2 \cdot t_3 = 4$$
$$\Rightarrow \quad 2^{11x_1} \cdot 2^{11x_2} \cdot 2^{11x_3} = 4 \Rightarrow 2^{11(x_1 + x_2 + x_3)} = 2^2$$
$$\Rightarrow \quad 11(x_1 + x_2 + x_3) = 2$$
$$\therefore \quad x_1 + x_2 + x_3 = \frac{2}{11}$$

JEE Type Solved Examples :
More than One Correct Option Type Questions

- This section contains **5 multiple choice examples.** Each example has four choices (a), (b), (c) and (d) out of **which more than one** may be correct.

Ex. 11 *For the equation* $2x^2 - 6\sqrt{2}x - 1 = 0$

(a) roots are rational

(b) roots are irrational

(c) if one root is $(p + \sqrt{q})$, the other is $(-p + \sqrt{q})$

(d) if one root is $(p + \sqrt{q})$, the other is $(p - \sqrt{q})$

Sol. (b,c) As the coefficients are not rational, irrational roots need not appear in conjugate pair.

Here, $\alpha + \beta = 3\sqrt{2}$ and $\alpha\beta = -\frac{1}{2}$

Let $\alpha = p + \sqrt{q}$, then prove that other root $\beta = -p + \sqrt{q}$.

Ex. 12 *Given that* α, γ *are roots of the equation* $Ax^2 - 4x + 1 = 0$ *and* β, δ *the roots of the equation* $Bx^2 - 6x + 1 = 0$*, such that* α, β, γ *and* δ *are in HP then*

(a) $A = 3$ (b) $A = 4$ (c) $B = 2$ (d) $B = 8$

Sol. (a,d) Since, α, β, γ and δ are in HP, hence $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$ and $\frac{1}{\delta}$ are in AP and they may be taken as $a - 3d$, $a - d$, $a + d$ and $a + 3d$. Replacing x by $\frac{1}{x}$, we get the equation whose roots are $a - 3d$, $a + d$ is $x^2 - 4x + A = 0$ and equation whose roots are $a - d$, $a + 3d$ is $x^2 - 6x + B = 0$, then

$$(a - 3d) + (a + d) = 4 \Rightarrow 2(a - d) = 4$$

and $$(a - d) + (a + 3d) = 6 \Rightarrow 2(a + d) = 6$$

$$\therefore \quad a = \frac{5}{2} \text{ and } d = \frac{1}{2}$$

Now, $A = (a - 3d)(a + d) = \left(\frac{5}{2} - \frac{3}{2}\right)\left(\frac{5}{2} + \frac{1}{2}\right) = 3$

and $B = (a - d)(a + 3d) = \left(\frac{5}{2} - \frac{1}{2}\right)\left(\frac{5}{2} + \frac{3}{2}\right) = 8$

Ex. 13 *If* $|ax^2 + bx + c| \le 1$ *for all x in [0, 1], then*

(a) $|a| \le 8$ (b) $|b| > 8$

(c) $|c| \le 1$ (d) $|a| + |b| + |c| \le 17$

Sol. (a,c,d) On putting $x = 0, 1$ and $\frac{1}{2}$, we get

$$|c| \le 1 \quad \text{...(i)}$$

$$|a + b + c| \le 1 \quad \text{...(ii)}$$

and $$|a + 2b + 4c| \le 4 \quad \text{...(iii)}$$

From Eqs. (i), (ii) and (iii), we get

$$|b| \le 8 \text{ and } |a| \le 8$$

$$\Rightarrow \quad |a| + |b| + |c| \le 17$$

Ex. 14 *If* $\cos^4\theta + p$, $\sin^4\theta + p$ *are the roots of the equation* $x^2 + a(2x + 1) = 0$ *and* $\cos^2\theta + q$, $\sin^2\theta + q$ *are the roots of the equation* $x^2 + 4x + 2 = 0$ *then a is equal to*

(a) -2 (b) -1 (c) 1 (d) 2

Sol. (b,d)

$\because \quad \cos^4\theta - \sin^4\theta = \cos 2\theta$

$\Rightarrow \quad \cos^4\theta - \sin^4\theta = \cos^2\theta - \sin^2\theta$

$\Rightarrow \quad (\cos^4\theta + p) - (\sin^4\theta + p) = (\cos^2\theta + q) - (\sin^2\theta + q)$

$\Rightarrow \quad \frac{\sqrt{4a^2 - 4a}}{1} = \frac{\sqrt{16 - 8}}{1} \quad \left[\because \alpha - \beta = \frac{\sqrt{D}}{a}\right]$

$\Rightarrow \quad 4a^2 - 4a = 8$ or $a^2 - a - 2 = 0$

or $\quad (a - 2)(a + 1) = 0$ or $a = 2, -1$

Ex. 15 *If* α, β, γ *are the roots of* $x^3 - x^2 + ax + b = 0$ *and* β, γ, δ *are the roots of* $x^3 - 4x^2 + mx + n = 0$*. If* α, β, γ *and* δ *are in AP with common difference d then*

(a) $a = m$ (b) $a = m - 5$

(c) $n = b - a - 2$ (d) $b = m + n - 3$

Sol. (b,c,d)

$\because a, \beta, \gamma, \delta$ are in AP with common difference d, then

$$\beta = \alpha + d, \ \gamma = \alpha + 2d \text{ and } \delta = \alpha + 3d \quad \text{...(i)}$$

Given, a, β, γ are the roots of $x^3 - x^2 + ax + b = 0$, then

$$\alpha + \beta + \gamma = 1 \quad \text{...(ii)}$$

$$\alpha\beta + \beta\gamma + \gamma\alpha = a \quad \text{...(iii)}$$

$$\alpha\beta\gamma = -b \quad \text{...(iv)}$$

Also, β, γ, δ are the roots of $x^3 - 4x^2 + mx + n = 0$, then

$$\beta + \gamma + \delta = 4 \quad \text{...(v)}$$

$$\beta\gamma + \gamma\delta + \delta\beta = m \quad \text{...(vi)}$$

$$\beta\gamma\delta = -n \quad \text{...(vii)}$$

From Eqs. (i) and (ii), we get

$$3\alpha + 3d = 1 \quad \text{...(viii)}$$

and from Eqs. (i) and (v), we get

$$3\alpha + 6d = 4 \quad \text{...(ix)}$$

From Eqs. (viii) and (ix), we get

$$d = 1, \alpha = -\frac{2}{3}$$

Now, from Eq. (i), we get

$$\beta = \frac{1}{3}, \gamma = \frac{4}{3} \text{ and } \delta = \frac{7}{3}$$

From Eqs. (iii), (iv), (vi) and (vii), we get

$$a = -\frac{2}{3}, b = \frac{8}{27}, m = \frac{13}{3}, n = -\frac{28}{27}$$

$$\therefore \quad a = m - 5, n = b - a - 2 \text{ and } b = m + n - 3$$

JEE Type Solved Examples :
Passage Based Questions

- This section contains **2 solved passages** based upon each of the passage **3 multiple choice** examples have to be answered. Each of these examples has four choices (a), (b), (c) and (d) out of which **ONLY ONE** is correct.

Passage I
(Ex. Nos. 16 to 18)

If G and L are the greatest and least values of the expression $\dfrac{x^2 - x + 1}{x^2 + x + 1}, x \in R$ *respectively, then*

16. The least value of $G^5 + L^5$ is

(a) 0 (b) 2 (c) 16 (d) 32

Sol. (*b*) Let $y = \dfrac{x^2 - x + 1}{x^2 + x + 1}$

$\Rightarrow \quad x^2y + xy + y = x^2 - x + 1$

$\Rightarrow \quad (y - 1)x^2 + (y + 1)x + y - 1 = 0 \quad [\because x \in R]$

$\therefore \quad (y + 1)^2 - 4\cdot(y - 1)(y - 1) \geq 0 \quad [\because b^2 - 4ac \geq 0]$

$\Rightarrow \quad (y + 1)^2 - (2y - 2)^2 \geq 0$

$\Rightarrow \quad (3y - 1)(y - 3) \leq 0$

$\therefore \quad \dfrac{1}{3} \leq y \leq 3 \Rightarrow G = 3$ and $L = \dfrac{1}{3} \quad \therefore GL = 1$

$$\frac{G^5 + L^5}{2} \geq (GL)^{1/5} = (1)^{1/5} = 1$$

$\Rightarrow \quad \dfrac{G^5 + L^5}{2} \geq 1$ or $G^5 + L^5 \geq 2$

$\therefore$ Minimum value of $G^5 + L^5$ is 2.

17. G and L are the roots of the equation

(a) $3x^2 - 10x + 3 = 0$ (b) $4x^2 - 17x + 4 = 0$
(c) $x^2 - 7x + 10 = 0$ (d) $x^2 - 5x + 6 = 0$

Sol. (*a*) Equation whose roots are G and L, is

$$x^2 - (G + L)x + GL = 0$$

$\Rightarrow \quad x^2 - \dfrac{10}{3}x + 1 = 0$ or $3x^2 - 10x + 3 = 0$

18. If $L < \lambda < G$ and $\lambda \in N$, the sum of all values of λ is

(a) 2 (b) 3 (c) 4 (d) 5

Sol. (*b*) $\because L < \lambda < G \Rightarrow \dfrac{1}{3} < \lambda < 3 \quad \therefore \lambda = 1, 2$

Sum of values of $\lambda = 1 + 2 = 3$

Passage II
(Ex. Nos. 19 to 21)

Let a, b, c and d *are real numbers in GP. Suppose* u, v, w *satisfy the system of equations* $u + 2v + 3w = 6, 4u + 5v + 6w = 12$ *and* $6u + 9v = 4$. *Further, consider the expressions*

$$f(x) = \left(\frac{1}{u} + \frac{1}{v} + \frac{1}{w}\right)x^2 + [(b - c)^2 + (c - a)^2 + (d - b)^2]$$
$$x + u + v + w = 0 \text{ and } g(x) = 20x^2 + 10(a - d)^2x - 9 = 0.$$

19. $(b - c)^2 + (c - a)^2 + (d - b)^2$ is equal to

(a) $a - d$ (b) $(a - d)^2$ (c) $a^2 - d^2$ (d) $(a + d)^2$

Sol. (*b*) Let $b = ar, c = ar^2$ and $d = ar^3$

Now, $(b - c)^2 + (c - a)^2 + (d - b)^2$

$= (ar - ar^2)^2 + (ar^2 - a)^2 + (ar^3 - ar)^2$

$= a^2r^2(1 - r)^2 + a^2(r^2 - 1)^2 + a^2r^2(r^2 - 1)^2$

$= a^2(1 - r)^2\{r^2 + (r + 1)^2 + r^2(r + 1)^2\}$

$= a^2(1 - r)^2(r^4 + 2r^3 + 3r^2 + 2r + 1)$

$= a^2(1 - r)^2(1 + r + r^2)^2 = a^2(1 - r^3)^2$

$= (a - ar^3)^2 = (a - d)^2$

20. $(u + v + w)$ is equal to

(a) 2 (b) $\dfrac{1}{2}$ (c) 20 (d) $\dfrac{1}{20}$

Sol. (*a*) Now, $u + 2v + 3w = 6 \quad$ ...(i)

$4u + 5v + 6w = 12 \quad$ ...(ii)

and $6u + 9v = 4 \quad$ ...(iii)

From Eqs. (i) and (ii), we get

$2u + v = 0 \quad$ ...(iv)

Solving Eqs. (iii) and (iv), we get

$$u = -\frac{1}{3}, v = \frac{2}{3}$$

Now, from Eq. (i), we get $w = \dfrac{5}{3}$

$\therefore \quad v + u + w = -\dfrac{1}{3} + \dfrac{2}{3} + \dfrac{5}{3} = 2$

21. If roots of $f(x) = 0$ be α, β, the roots of $g(x) = 0$ will be

(a) α, β (b) $-\alpha, -\beta$ (c) $\dfrac{1}{\alpha}, \dfrac{1}{\beta}$ (d) $-\dfrac{1}{\alpha}, -\dfrac{1}{\beta}$

Sol. (*c*) Now, $f(x) = \left(\dfrac{1}{u} + \dfrac{1}{v} + \dfrac{1}{w}\right)x^2 + [(b - c)^2 + (c - a)^2 + (d - b)^2]x + u + v + w = 0$

$\Rightarrow \quad f(x) = -\dfrac{9}{10}x^2 + (a - d)^2x + 2 = 0$

$\Rightarrow \quad f(x) = -9x^2 + 10(a - d)^2x + 20 = 0 \quad$ ...(v)

Given, roots of $f(x) = 0$ are α and β.

Now, replace x by $\dfrac{1}{x}$ in Eq. (v), then

$$\frac{-9}{x^2} + \frac{10(a - d)^2}{x} + 20 = 0$$

$\Rightarrow \quad 20x^2 + 10(a - d)^2x - 9 = 0$

$g(x) = 0$

$\therefore \quad$ Roots of $g(x) = 0$ are $\dfrac{1}{\alpha}, \dfrac{1}{\beta}$.

JEE Type Solved Examples :
Single Integer Answer Type Questions

- This section contains **2 examples.** The answer to each example is a **single digit integer** ranging from **0** to **9** (both inclusive).

Ex. 22 *If the roots of the equation* $10x^3 - cx^2 - 54x - 27 = 0$ *are in harmonic progression, the value of c is*

Sol. (9) Given, roots of the equation

$$10x^3 - cx^2 - 54x - 27 = 0 \text{ are in HP.} \quad \text{...(i)}$$

Now, replacing x by $\frac{1}{x}$ in Eq. (i), we get

$$27x^3 + 54x^2 + cx - 10 = 0 \quad \text{...(ii)}$$

Hence, the roots of Eq. (ii) are in AP.

Let $a - d, a$ and $a + d$ are the roots of Eq. (ii).

Then, $a - d + a + a + d = -\frac{54}{27}$

$$\Rightarrow \quad a = -\frac{2}{3} \quad \text{...(iii)}$$

Since, a is a root of Eq. (ii), then

$$27a^3 + 54a^2 + ca - 10 = 0$$

$$\Rightarrow \quad 27\left(-\frac{8}{27}\right) + 54\left(\frac{4}{9}\right) + c\left(-\frac{2}{3}\right) - 10 = 0 \quad \text{[from Eq. (iii)]}$$

$$\Rightarrow \quad 6 - \frac{2c}{3} = 0 \text{ or } c = 9$$

Ex. 23 *If a root of the equation* $n^2 \sin^2 x - 2\sin x - (2n+1) = 0$ *lies in* $[0, \pi/2]$, *the minimum positive integer value of n is*

Sol. (3) $\because \; n^2 \sin^2 x - 2\sin x - (2n+1) = 0$

$$\Rightarrow \quad \sin x = \frac{2 \pm \sqrt{4 + 4n^2(2n+1)}}{2n^2}$$

[by Shridharacharya method]

$$= \frac{1 \pm \sqrt{(2n^3 + n^2 + 1)}}{n^2}$$

$$\because \quad 0 \le \sin x \le 1 \quad [\because x \in [0, \pi/2]]$$

$$\Rightarrow \quad 0 \le \frac{1 + \sqrt{(2n^3 + n^2 + 1)}}{n^2} \le 1$$

$$\Rightarrow \quad 0 \le 1 + \sqrt{(2n^3 + n^2 + 1)} \le n^2$$

$$\Rightarrow \sqrt{(2n^3 + n^2 + 1)} \le (n^2 - 1) \quad [\because n > 1]$$

On squaring both sides, we get

$$2n^3 + n^2 + 1 \le n^4 - 2n^2 + 1$$

$$\Rightarrow \quad n^4 - 2n^3 - 3n^2 \ge 0$$

$$\Rightarrow \quad n^2 - 2n - 3 \ge 0 \quad \Rightarrow \quad (n-3)(n+1) \ge 0$$

$$\Rightarrow \quad n \ge 3$$

$$\therefore \quad n = 3, 4, 5, \ldots$$

Hence, the minimum positive integer value of n is 3.

JEE Type Solved Examples :
Matching Type Questions

- This section contains **2 examples.** Examples 24 and 25 have three statements (A, B and C) given in Column I and four statements (p, q, r and s) in **Column II.** Any given statement in **Column I** can have correct matching with one or more statement(s) given in **Column II.**

Ex. 24 *Column* I *contains rational algebraic expressions and Column* II *contains possible integers which lie in their range. Match the entries of Column* I *with one or more entries of the elements of Column* II.

	Column I		Column II
(A)	$y = \frac{x^2 - 2x + 9}{x^2 + 2x + 9}, x \in R$	(p)	1
(B)	$y = \frac{x^2 - 3x - 2}{2x - 3}, x \in R$	(q)	3
(C)	$y = \frac{2x^2 - 2x + 4}{x^2 - 4x + 3}, x \in R$	(r)	−4
		(s)	−9

Sol. (A) → (p); (B) → (p, q, r, s); (C) → (p, q, s)

(A) $y = \frac{x^2 - 2x + 9}{x^2 + 2x + 9} \Rightarrow x^2 y + 2xy + 9y = x^2 - 2x + 9$

$$\Rightarrow \quad (y-1)x^2 + 2x(y+1) + 9(y-1) = 0$$

$$\because \quad x \in R$$

$$\therefore \quad 4(y+1)^2 - 4 \cdot 9 \cdot (y-1)^2 \ge 0$$

$$\Rightarrow \quad (y+1)^2 - (3y-3)^2 \ge 0$$

$$\Rightarrow \quad (4y-2)(-2y+4) \ge 0$$

$$\Rightarrow \quad (2y-1)(y-2) \le 0$$

$$\therefore \quad \frac{1}{2} \le y \le 2 \quad \Rightarrow \quad y = 1, 2 \text{ (p)}$$

(B) $\because \quad y = \frac{x^2 - 3x - 2}{2x - 3} \Rightarrow 2xy - 3y = x^2 - 3x - 2$

$$\Rightarrow \quad x^2 - x(3 + 2y) + 3y - 2 = 0 \quad \because \; x \in R$$

$$\therefore \quad (3 + 2y)^2 - 4 \cdot 1 \cdot (3y - 2) \ge 0$$

$$\Rightarrow \quad 4y^2 + 17 \ge 0$$

$$\therefore \quad y \in R \text{ (p, q, r, s)}$$

(C) $\because \quad y = \dfrac{2x^2 - 2x + 4}{x^2 - 4x + 3}$

$$\Rightarrow \quad x^2y - 4xy + 3y = 2x^2 - 2x + 4$$
$$\Rightarrow \quad x^2(y-2) + 2x(1-2y) + 3y - 4 = 0$$
$$\because \quad x \in R$$
$$\therefore \quad 4(1-2y)^2 - 4(y-2)(3y-4) \geq 0$$
$$\Rightarrow \quad (4y^2 - 4y + 1) - (3y^2 - 10y + 8) \geq 0$$
$$\Rightarrow \quad y^2 + 6y - 7 \geq 0$$
$$\Rightarrow \quad (y+7)(y-1) \geq 0$$
$$\therefore \quad y \leq -7 \text{ or } y \geq 1 \text{(p,q,s)}$$

● **Ex. 25** *Entries of Column I are to be matched with one or more entries of Column II.*

	Column I		Column II
(A)	If $a + b + 2c = 0$ but $c \neq 0$, then $ax^2 + bx + c = 0$ has	(p)	atleast one root in $(-2, 0)$
(B)	If $a, b, c \in R$ such that $2a - 3b + 6c = 0$, then equation has	(q)	atleast one root in $(-1, 0)$
(C)	Let a, b, c be non-zero real numbers such that	(r)	atleast one root in $(-1, 1)$
	$\int_0^1 (1 + \cos^8 x)(ax^2 + bx + c)dx = \int_0^2 (1 + \cos^8 x)(ax^2 + bx + c)dx$, the equation $ax^2 + bx + c = 0$ has	(s)	atleast one root in $(0, 1)$
		(t)	atleast one root in $(0, 2)$

Sol. (A) → (r, s, t); (B) → (p, q, r); (C) → (r, s, t)

(A) Let $f(x) = ax^2 + bx + c$

Then, $f(1) = a + b + c = -c$ $\quad [\because a + b + 2c = 0]$

and $f(0) = c$

$\therefore \quad f(0)\,f(1) = -c^2 < 0$ $\quad [\because c \neq 0]$

$\therefore$ Equation $f(x) = 0$ has a root in $(0, 1)$.

$\therefore \quad f(x)$ has a root in $(0, 2)$ as well as in $(-1, 1)$ (r)

(B) Let $f'(x) = ax^2 + bx + c$

$$\therefore \quad f(x) = \frac{ax^3}{3} + \frac{bx^2}{2} + cx + d$$

$\therefore \quad f(0) = d$

and $\quad f(-1) = -\dfrac{a}{3} + \dfrac{b}{2} + c + d = -\left(\dfrac{2a - 3b + 6c}{6}\right) + d$

$= 0 + d = d$ $\quad [\because 2a - 3b + 6c = 0]$

Hence, $f(0) = f(-1)$

Hence, $f'(x) = 0$ has atleast one root in $(-1, 0)$ (q)

$\therefore \quad f(x) = 0$ has a root in $(-2, 0)$ (p) as well as $(-1, 1)$ (r)

(C) Let $f(x) = \int (1 + \cos^8 x)(ax^2 + bx + c)dx$

Given, $f(1) - f(0) = f(2) - f(0)$

$\Rightarrow \quad f(1) = f(2)$

$\Rightarrow \quad f'(x) = 0$ has atleast one root in $(0, 1)$.

$\Rightarrow (1 + \cos^8 x)(ax^2 + bx + c) = 0$ has atleast one root in $(0, 1)$.

$\Rightarrow \quad ax^2 + bx + c = 0$ has atleast one root in $(0, 1)$ (s)

$\therefore \; ax^2 + bx + c = 0$ has a root in $(0, 2)$ (t) as well as in $(-1, 1)$(r)

JEE Type Solved Examples :
Statement I and II Type Questions

■ **Directions** Example numbers 26 and 27 are Assertion-Reason type examples. Each of these examples contains two statements:
Statement-1 (Assertion) and **Statement-2** (Reason)
Each of these examples also has four alternative choices, only one of which is the correct answer. You have to select the correct choice as given below.

(a) Statement-1 is true, Statement-2 is true; Statement-2 is a correct explanation for Statement-1
(b) Statement-1 is true, Statement-2 is true; Statement-2 is not a correct explanation for Statement-1
(c) Statement-1 is true, Statement-2 is false
(d) Statement-1 is false, Statement-2 is true

● **Ex. 26** **Statement 1** *Roots of* $x^2 - 2\sqrt{3}x - 46 = 0$ *are rational.*

Statement 2 *Discriminant of* $x^2 - 2\sqrt{3}x - 46 = 0$ *is a perfect square.*

Sol. (*d*) In $ax^2 + bx + c = 0$, $a, b, c \in Q$

[here Q is the set of rational number]

If $D > 0$ and is a perfect square, then roots are real, distinct and rational.

But, here $\quad 2\sqrt{3} \notin Q$

$\therefore$ Roots are not rational.

Here, roots are $\dfrac{2\sqrt{3} \pm \sqrt{(12 + 184)}}{2}$

i.e. $\sqrt{3} \pm 7$. [irrational]

But $\quad D = 12 + 184 = 196 = (14)^2$

$\therefore$ Statement-1 is false and Statement-2 is true.

● **Ex. 27** **Statement 1** *The equation* $a^x + b^x + c^x - d^x = 0$ *has only one real root, if* $a > b > c > d$.

Statement 2 *If* $f(x)$ *is either strictly increasing or decreasing function, then* $f(x) = 0$ *has only one real root.*

Sol. (*c*) $\because \; a^x + b^x + c^x - d^x = 0$

$$\Rightarrow \quad a^x + b^x + c^x = d^x$$

Let $f(x)=\left(\frac{a}{d}\right)^x+\left(\frac{b}{d}\right)^x+\left(\frac{c}{d}\right)^x-1$

$\therefore \quad f'(x)=\left(\frac{a}{d}\right)^x\ln\left(\frac{a}{d}\right)+\left(\frac{b}{d}\right)^x\ln\left(\frac{b}{d}\right)+\left(\frac{c}{d}\right)^x\ln\left(\frac{c}{d}\right)>0$

and $f(0)=2$

$\therefore f(x)$ is increasing function and $\lim_{x\to-\infty} f(x)=-1$

$\Rightarrow f(x)$ has only one real root.

But Statement-2 is false.

For example, $f(x)=e^x$ is increasing but $f(x)=0$ has no solution.

Subjective Type Examples

- In this section, there are **24 subjective** solved examples.

Ex. 28 *If* α,β *are roots of the equation* $x^2-p(x+1)-c=0$*, show that* $(\alpha+1)(\beta+1)=1-c$*. Hence, prove that* $\frac{\alpha^2+2\alpha+1}{\alpha^2+2\alpha+c}+\frac{\beta^2+2\beta+1}{\beta^2+2\beta+c}=1.$

Sol. Since, α and β are the roots of the equation,

$$x^2-px-p-c=0$$

$\therefore \quad \alpha+\beta=p$

and $\quad \alpha\beta=-p-c$

Now, $(\alpha+1)(\beta+1)=\alpha\beta+\alpha+\beta+1$

$$=-p-c+p+1=1-c$$

Hence, $\quad (\alpha+1)(\beta+1)=1-c \quad$ ...(i)

Second Part LHS $=\frac{\alpha^2+2\alpha+1}{\alpha^2+2\alpha+c}+\frac{\beta^2+2\beta+1}{\beta^2+2\beta+c}$

$$=\frac{(\alpha+1)^2}{(\alpha+1)^2-(1-c)}+\frac{(\beta+1)^2}{(\beta+1)^2-(1-c)}$$

$$=\frac{(\alpha+1)^2}{(\alpha+1)^2-(\alpha+1)(\beta+1)}+\frac{(\beta+1)^2}{(\beta+1)^2-(\alpha+1)(\beta+1)} \quad \text{[from Eq. (i)]}$$

$$=\frac{\alpha+1}{\alpha-\beta}+\frac{\beta+1}{\beta-\alpha}=\frac{\alpha-\beta}{\alpha-\beta}=1=\text{RHS}$$

Hence, RHS = LHS

Ex. 29 *Solve the equation* $x^2+px+45=0$*. It is given that the squared difference of its roots is equal to 144.*

Sol. Let α,β be the roots of the equation $x^2+px+45=0$ and given that

$$(\alpha-\beta)^2=144$$

$\Rightarrow \quad p^2-4\cdot1\cdot45=144 \quad \left[\because \alpha-\beta=\frac{\sqrt{D}}{a}\right]$

$\Rightarrow \quad p^2=324$

$\therefore \quad p=(\pm18)$

On substituting $p=18$ in the given equation, we obtain

$$x^2+18x+45=0$$

$\Rightarrow \quad (x+3)(x+15)=0$

$\Rightarrow \quad x=-3,5$

and substituting $p=-18$ in the given equation, we obtain

$$x^2-18x+45=0$$

$$(x-3)(x-15)=0$$

$\Rightarrow \quad x=3,15$

Hence, the roots of the given equation are $(-3),(-15)$, 3 and 15.

Ex. 30 *If the roots of the equation* $ax^2+bx+c=0\,(a\neq0)$ *be* α *and* β *and those of the equation* $Ax^2+Bx+C=0$ $(A\neq0)$ *be* $\alpha+k$ *and* $\beta+k$*. Prove that*

$$\frac{b^2-4ac}{B^2-4AC}=\left(\frac{a}{A}\right)^2.$$

Sol. $\because \alpha-\beta=(\alpha+k)-(\beta+k)$

$\Rightarrow \quad \frac{\sqrt{b^2-4ac}}{a}=\frac{\sqrt{(B^2-4AC)}}{A} \quad \left[\because \alpha-\beta=\frac{\sqrt{D}}{a}\right]$

$\Rightarrow \quad \sqrt{\left(\frac{b^2-4ac}{B^2-4AC}\right)}=\left(\frac{a}{A}\right)$

On squaring both sides, then we get

$$\frac{b^2-4ac}{B^2-4AC}=\left(\frac{a}{A}\right)^2$$

Ex. 31 *Let* a,b *and* c *be real numbers such that* $a+2b+c=4$*. Find the maximum value of* $(ab+bc+ca)$*.*

Sol. Given, $\quad a+2b+c=4$

$\Rightarrow \quad a=4-2b-c$

Let $\quad y=ab+bc+ca=a(b+c)+bc$

$$=(4-2b-c)(b+c)+bc$$

$$=-2b^2+4b-2bc+4c-c^2$$

$\Rightarrow \quad 2b^2+2(c-2)b-4c+c^2+y=0$

Since, $b\in R$, so

$$4(c-2)^2-4\times2\times(-4c+c^2+y)\geq0$$

$$\Rightarrow \quad (c-2)^2 + 8c - 2c^2 - 2y \geq 0$$
$$\Rightarrow \quad c^2 - 4c + 2y - 4 \leq 0$$
Since, $c \in R$, so $16 - 4(2y-4) \geq 0 \Rightarrow y \leq 4$

Hence, maximum value of $ab + bc + ca$ is 4.

Aliter

$$\because \quad \text{AM} \geq \text{GM}$$
$$\Rightarrow \quad \frac{(a+b)+(b+c)}{2} \geq \sqrt{(a+b)(b+c)}$$
$$\Rightarrow \quad 2 \geq \sqrt{(ab+bc+ca+b^2)} \quad [\because a + 2b + c = 4]$$
$$\Rightarrow \quad ab + bc + ca \leq 4 - b^2$$
$\therefore$ Maximum value of $(ab + bc + ca)$ is 4.

● **Ex. 32** *Find a quadratic equation whose roots x_1 and x_2 satisfy the condition $x_1^2 + x_2^2 = 5, 3(x_1^5 + x_2^5) = 11(x_1^3 + x_2^3)$ (assume that x_1, x_2 are real).*

Sol. We have, $3(x_1^5 + x_2^5) = 11(x_1^3 + x_2^3)$
$$\Rightarrow \quad \frac{x_1^5 + x_2^5}{x_1^3 + x_2^3} = \frac{11}{3}$$
$$\Rightarrow \quad \frac{(x_1^2 + x_2^2)(x_1^3 + x_2^3) - x_1^2 x_2^2 (x_1 + x_2)}{(x_1^3 + x_2^3)} = \frac{11}{3}$$
$$\Rightarrow \quad (x_1^2 + x_2^2) - \frac{x_1^2 x_2^2 (x_1 + x_2)}{(x_1 + x_2)(x_1^2 + x_2^2 - x_1 x_2)} = \frac{11}{3}$$
$$[\because x_1^2 + x_2^2 = 5]$$
$$\Rightarrow \quad 5 - \frac{x_1^2 x_2^2}{5 - x_1 x_2} = \frac{11}{3}$$
$$\Rightarrow \quad \frac{4}{3} = \frac{x_1^2 x_2^2}{5 - x_1 x_2}$$
$$\Rightarrow \quad 3x_1^2 x_2^2 + 4x_1 x_2 - 20 = 0$$
$$\Rightarrow \quad 3x_1^2 x_2^2 + 10x_1 x_2 - 6x_1 x_2 - 20 = 0$$
$$\Rightarrow \quad (x_1 x_2 - 2)(3x_1 x_2 + 10) = 0$$
$$\therefore \quad x_1 x_2 = 2, \left(-\frac{10}{3}\right)$$
We have, $(x_1 + x_2)^2 = x_1^2 + x_2^2 + 2x_1 x_2 = 5 + 2x_1 x_2$
$$\therefore \quad (x_1 + x_2)^2 = 5 + 4 = 9 \quad [\text{if } x_1 x_2 = 2]$$
$$\therefore \quad x_1 + x_2 = \pm 3$$
and $(x_1 + x_2)^2 = 5 + 2\left(-\frac{10}{3}\right) = -\frac{5}{3}\left[\text{if } x_1 x_2 = -\frac{10}{3}\right]$

which is not possible, since x_1, x_2 are real.

Thus, required quadratic equations are $x^2 \pm 3x + 2 = 0$.

● **Ex. 33** *If each pair of the three equations $x^2 + ax + b = 0, x^2 + cx + d = 0$ and $x^2 + ex + f = 0$ has exactly one root in common, then show that $(a + c + e)^2 = 4(ac + ce + ea - b - d - f)$.*

Sol. Given equations are
$$x^2 + ax + b = 0 \quad \text{...(i)}$$
$$x^2 + cx + d = 0 \quad \text{...(ii)}$$
$$x^2 + ex + f = 0 \quad \text{...(iii)}$$
Let α, β be the roots of Eq. (i), β, γ be the roots of Eq. (ii) and γ, δ be the roots of Eq. (iii), then
$$\alpha + \beta = -a, \alpha\beta = b \quad \text{...(iv)}$$
$$\beta + \gamma = -c, \beta\gamma = d \quad \text{...(v)}$$
$$\gamma + \alpha = -e, \gamma\alpha = f \quad \text{...(vi)}$$
$\therefore$ LHS $= (a + c + e)^2 = (-\alpha - \beta - \beta - \gamma - \gamma - \alpha)^2$

[from Eqs. (iv), (v) and (vi)]
$$= 4(\alpha + \beta + \gamma)^2 \quad \text{...(vii)}$$
RHS $= 4(ac + ce + ea - b - d - f)$
$$= 4\{(\alpha + \beta)(\beta + \gamma) + (\beta + \gamma)(\gamma + \alpha) + (\gamma + \alpha)(\alpha + \beta) - \alpha\beta - \beta\gamma - \gamma\alpha\}$$
[from Eqs. (iv), (v) and (vi)]
$$= 4(\alpha^2 + \beta^2 + \gamma^2 + 2\alpha\beta + 2\beta\gamma + 2\gamma\alpha)$$
$$= 4(\alpha + \beta + \gamma)^2 \quad \text{...(viii)}$$
From Eqs. (vii) and (viii), then we get
$$(a + c + e)^2 = 4(ac + ce + ea - b - d - f)$$

● **Ex. 34** *If α, β are the roots of the equation $x^2 + px + q = 0$ and γ, δ are the roots of the equation $x^2 + rx + s = 0$, evaluate $(\alpha - \gamma)(\alpha - \delta)(\beta - \gamma)(\beta - \delta)$ in terms of p, q, r and s. Deduce the condition that the equations have a common root.*

Sol. $\because \alpha, \beta$ are the roots of the equation
$$x^2 + px + q = 0$$
$$\therefore \quad \alpha + \beta = -p, \alpha\beta = q \quad \text{...(i)}$$
and γ, δ are the roots of the equation $x^2 + rx + s = 0$
$$\therefore \quad \gamma + \delta = -r, \gamma\delta = s \quad \text{...(ii)}$$
Now, $(\alpha - \gamma)(\alpha - \delta)(\beta - \gamma)(\beta - \delta)$
$$= [\alpha^2 - \alpha(\gamma + \delta) + \gamma\delta][\beta^2 - \beta(\gamma + \delta) + \gamma\delta]$$
$$= (\alpha^2 + r\alpha + s)(\beta^2 + r\beta + s) \quad \text{[from Eq. (ii)]}$$
$$= \alpha^2\beta^2 + r\alpha\beta(\alpha + \beta) + r^2\alpha\beta + s(\alpha^2 + \beta^2) + sr(\alpha + \beta) + s^2$$
$$= \alpha^2\beta^2 + r\alpha\beta(\alpha + \beta) + r^2\alpha\beta + s[(\alpha + \beta)^2 - 2\alpha\beta] + sr(\alpha + \beta) + s^2$$
$$= q^2 - pqr + r^2 q + s(p^2 - 2q) + sr(-p) + s^2$$
$$= (q - s)^2 - rpq + r^2 q + sp^2 - prs$$
$$= (q - s)^2 - rq(p - r) + sp(p - r)$$
$$= (q - s)^2 + (p - r)(sp - rq) \quad \text{...(iii)}$$
For a common root (let $\alpha = \gamma$ or $\beta = \delta$),

then $(\alpha - \gamma)(\alpha - \delta)(\beta - \gamma)(\beta - \delta) = 0 \quad \text{...(iv)}$

From Eqs. (iii) and (iv), we get

$$(q-s)^2+(p-r)(sp-rq)=0$$

$\Rightarrow (q-s)^2=(p-r)(rq-sp)$, which is the required condition .

Ex. 35 *Find all integral values of a for which the quadratic Expresion $(x-a)(x-10)+1$ can be factored as a product $(x+\alpha)(x+\beta)$ of two factors and $\alpha, \beta \in I$.*

Sol. We have, $(x-a)(x-10)+1=(x+\alpha)(x+\beta)$

On putting $x=-\alpha$ in both sides, we get

$$(-\alpha-a)(-\alpha-10)+1=0$$

$\therefore \quad (\alpha+a)(\alpha+10)=-1$

$\alpha+a$ and $\alpha+10$ are integers. $[\because a, \alpha \in I]$

$\therefore \quad \alpha+a=-1$ and $\alpha+10=1$

or $\quad \alpha+a=1$ and $\alpha+10=-1$

(i) If $\alpha+10=1$

$\therefore \quad \alpha=-9$, then $a=8$

Similarly, $\beta=-9$

Here, $(x-8)(x-10)+1=(x-9)^2$

(ii) If $\alpha+10=-1$

$\therefore \quad \alpha=-11$, then $a=12$

Similarly, $\beta=12$

Here, $(x-12)(x-10)+1=(x-11)^2$

Hence, $a=8, 12$

Ex. 36 *Solve the equation*

$$\sqrt{x+3-4\sqrt{(x-1)}}+\sqrt{x+8-6\sqrt{(x-1)}}=1.$$

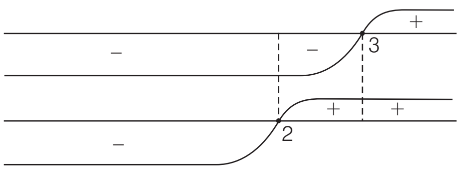
Sol. Let $\sqrt{(x-1)}=t$

We have, $x=t^2+1, t \geq 0$

The given equation reduce in the form

$$\sqrt{(t^2+4-4t)}+\sqrt{(t^2+9-6t)}=1$$

$\Rightarrow \quad |t-2|+|t-3|=1$



$\therefore \quad 2 \leq t \leq 3$

$\Rightarrow \quad 4 \leq t^2 \leq 9$

$\Rightarrow \quad 4 \leq x-1 \leq 9$

$\Rightarrow \quad 5 \leq x \leq 10$

$\therefore$ Solution of the original equation is $x \in [5, 10]$.

Ex. 37 *Solve for 'x'*

$1!+2!+3!+\ldots+(x-1)!+x!=k^2$ and $k \in I$.

Sol. For $x<4$, the given equation has the only solutions $x=1, k=\pm 1$ and $x=3, k=\pm 3$. Now, let us prove that there are no solutions for $x \geq 4$. The expressions

$$\left.\begin{array}{lr} 1!+2!+3!+4! & =33 \\ 1!+2!+3!+4!+5! & =153 \\ 1!+2!+3!+4!+5!+6! & =873 \\ 1!+2!+3!+4!+5!+6!+7! & =5913 \end{array}\right\} \text{ends with the digit}$$

3.

Now, for $x \geq 4$ the last digit of the sum $1!+2!+\ldots+x!$ is equal to 3 and therefore, this sum cannot be equal to a square of a whole number k (because a square of a whole number cannot end with 3).

Ex. 38 *Find the real roots of the equation*

$$\underbrace{\sqrt{x+2\sqrt{x+2\sqrt{x+\ldots+2\sqrt{x+2\sqrt{3x}}}}}}_{n \text{ radical signs}}=x$$

Sol. Rewrite the given equation

$$\sqrt{x+2\sqrt{x+2\sqrt{x+\ldots+2\sqrt{x+2\sqrt{x+2x}}}}}=x \qquad \ldots(i)$$

On replacing the last letter x on the LHS of Eq. (i) by the value of x expressed by Eq. (i), we get

$$x=\underbrace{\sqrt{x+2\sqrt{x+2\sqrt{x+\ldots+\sqrt{x+2x}}}}}_{2n \text{ radical signs}}$$

Further, let us replace the last letter x by the same expression again and again yields.

$$\therefore \quad x=\underbrace{\sqrt{x+2\sqrt{x+2\sqrt{x+\ldots+2\sqrt{x+2x}}}}}_{3n \text{ radical signs}}$$

$$=\underbrace{\sqrt{x+2\sqrt{x+2\sqrt{x+\ldots+2\sqrt{x+2x}}}}}_{4n \text{ radical signs}}=\ldots$$

We can write,

$$x=\sqrt{x+2\sqrt{x+2\sqrt{x+\ldots}}}$$

$$=\underbrace{\lim_{N \to \infty}\sqrt{x+2\sqrt{x+2\sqrt{x+\ldots+2\sqrt{x+2x}}}}}_{N \text{ radical signs}}$$

If follows that

$$x=\sqrt{x+2\sqrt{x+2\sqrt{x+\ldots}}}$$

$$=\sqrt{x+2(\sqrt{x+2\sqrt{x+\ldots}})}=\sqrt{(x+2x)}$$

Hence, $\quad x^2=x+2x$

$\Rightarrow \quad x^2-3x=0$

$\therefore \quad x=0, 3$

Ex. 39 *Solve the inequation,* $(x^2+x+1)^x<1$.

Sol. Taking logarithm both sides on base 10,

then $\quad x\log(x^2+x+1)<0$

which is equivalent to the collection of systems

$$\left[\begin{array}{l}\begin{cases}x>0,\\ \log(x^2+x+1)<0,\end{cases}\\ \begin{cases}x<0,\\ \log(x^2+x+1)>0,\end{cases}\end{array}\right. \Rightarrow \left[\begin{array}{l}\begin{cases}x>0,\\ x^2+x+1<1,\end{cases}\\ \begin{cases}x<0,\\ x^2+x+1>1,\end{cases}\end{array}\right.$$

$$\Rightarrow \left[\begin{array}{l}\begin{cases}x>0,\\ x(x+1)<0,\end{cases}\\ \begin{cases}x<0,\\ x(x+1)>0\end{cases}\end{array}\right. \Rightarrow \left[\begin{array}{l}\begin{cases}x>0,\\ -1<x<0\end{cases}\\ \begin{cases}x<0,\\ x>0 \text{ and } x<(-1)\end{cases}\end{array}\right.$$

$$\Rightarrow \begin{cases}x\in\phi,\\ x<(-1)\end{cases}$$

Consequently, the interval $x\in(-\infty,-1)$ is the set of all solutions of the original inequation.

Remark

When the inequation is in power, then it is better to take log.

Ex. 40 *Solve the equation*

$$1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\ddots\,1+\cfrac{1}{x}}}}=x$$

When in expression on left hand side the sign of a fraction is repeated n times.

Sol. Given equation is

$$1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\ddots\,1+\cfrac{1}{x}}}}=x$$

Let us replace x on the LHS of the given equation by the expression of x. This result in an equation of the same form, which however involves $2n$ fraction lines. Continuing this process on the basis of this transformation, we can write

$$x=1+\lim_{n\to\infty}1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\ddots\,1+\cfrac{1}{x}}}} \qquad [n \text{ fractions}]$$

$$\Rightarrow \quad x=1+\frac{1}{x}\Rightarrow x^2-x-1=0$$

$$\therefore \quad x=\frac{1\pm\sqrt{5}}{2}$$

$$\therefore \quad x_1=\frac{1+\sqrt{5}}{2},\ x_2=\frac{1-\sqrt{5}}{2}$$

satisfy the given equation and this equation has no other roots.

Ex. 41 *Solve the system of equations*

$$\begin{cases}|x-1|+|y-2|=1\\ y=2-|x-1|\end{cases}.$$

Sol. On substituting $|x-1|=2-y$ from second equation in first equation of this system, we get

$$2-y+|y-2|=1$$

Now, consider the following cases

If $\quad y\geq 2,$

then $\quad 2-y+y-2=1 \Rightarrow 0=1$

No value of y for $y\geq 2$.

If $\quad y<2,$

then $\quad 2-y+2-y=1 \Leftrightarrow y=\frac{3}{2}$, which is true.

From the second equation of this system,

$$\frac{3}{2}=2-|x-1|$$

$$\Rightarrow \quad |x-1|=\frac{1}{2}\Rightarrow x-1=\pm\frac{1}{2}$$

$$\Rightarrow \quad x=1\pm\frac{1}{2}\Rightarrow x=\frac{1}{2},\frac{3}{2}$$

Consequently, the set of all solutions of the original system is the set of pairs (x,y), where $x=\frac{1}{2},\frac{3}{2}$ and $y=\frac{3}{2}$.

Ex. 42 *Let a,b,c be real and $ax^2+bx+c=0$ has two real roots α and β, where $\alpha<-1$ and $\beta>1$, then show that* $1+\frac{c}{a}+\left|\frac{b}{a}\right|<0$.

Sol. Since, $\alpha<-1$ and $\beta>1$

$$\alpha+\lambda=-1 \text{ and } \beta=1+\mu \qquad [\lambda,\mu>0]$$

Now, $\quad 1+\frac{c}{a}+\left|\frac{b}{a}\right|=1+\alpha\beta+|\alpha+\beta|$

$$=1+(-1-\lambda)(1+\mu)+|-1-\lambda+1+\mu|$$

$$=1-1-\mu-\lambda-\lambda\mu+|\mu-\lambda|$$

$$=-\mu-\lambda-\lambda\mu+\mu-\lambda \qquad [\text{if } \mu>\lambda]$$

and $\quad =-\mu-\lambda-\lambda\mu+\lambda-\mu \qquad [\text{if } \lambda>\mu]$

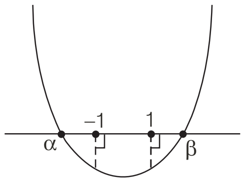
$$\therefore \quad 1+\frac{c}{a}+\left|\frac{b}{a}\right|=-2\lambda-\lambda\mu \text{ or } -2\mu-\lambda\mu$$

On both cases, $\quad 1+\frac{c}{a}+\left|\frac{b}{a}\right|<0 \qquad [\because \lambda,\mu>0]$

Aliter

$\because \quad ax^2 + bx + c = 0, a \neq 0$

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$



Let $\quad f(x) = x^2 + \frac{b}{a}x + \frac{c}{a}$

$f(-1) < 0$ and $f(1) < 0$

$\Rightarrow \quad 1 - \frac{b}{a} + \frac{c}{a} < 0$ and $1 + \frac{b}{a} + \frac{c}{a} < 0$

Then, $1 + \left|\frac{b}{a}\right| + \frac{c}{a} < 0$

● **Ex. 43** *Solve the equation* $x\left(\frac{3-x}{x+1}\right)\left(x + \frac{3-x}{x+1}\right) = 2.$

Sol. Hence, $x + 1 \neq 0$

and let $\quad x\left(\frac{3-x}{x+1}\right) = u$ and $x + \frac{3-x}{x+1} = v$

$\therefore \quad uv = 2 \quad$ ...(i)

and $\quad u + v = x\left(\frac{3-x}{x+1}\right) + x + \left(\frac{3-x}{x+1}\right)$

$$= (x+1)\left(\frac{3-x}{x+1}\right) + x = 3 - x + x = 3$$

$\therefore \quad u + v = 3$ and $uv = 2$

Then, $\quad u = 2, v = 1$ or $u = 1, v = 2$

Given equation is equivalent to the collection

$$\therefore \quad \left\{\begin{array}{l} x\left(\frac{3-x}{x+1}\right) = 2 \\ x + \frac{3-x}{x+1} = 1 \end{array}\right. \text{ or } \left\{\begin{array}{l} x\left(\frac{3-x}{x+1}\right) = 1 \\ x + \frac{3-x}{x+1} = 2 \end{array}\right.$$

$$\Rightarrow \quad \left\{\begin{array}{l} x^2 - x + 2 = 0 \\ x^2 - x + 2 = 0 \end{array}\right. \text{ or } \left\{\begin{array}{l} x^2 - 2x + 1 = 0 \\ x^2 - 2x + 1 = 0 \end{array}\right.$$

$$\Rightarrow \quad \left\{\begin{array}{l} x^2 - x + 2 = 0 \\ x^2 - 2x + 1 = 0 \end{array}\right. \Rightarrow \left\{\begin{array}{l} \left(x - \frac{1}{2}\right)^2 + \frac{7}{4} \neq 0 \\ (x-1)^2 = 0 \end{array}\right.$$

$\therefore \quad (x-1)^2 = 0$

$\Rightarrow x = 1$ is a unique solution of the original equation.

● **Ex. 44** *Show that for any real numbers* $a_3, a_4, a_5, \ldots, a_{85}$, *the roots of the equation*
$a_{85}x^{85} + a_{84}x^{84} + \ldots + a_3x^3 + 3x^2 + 2x + 1 = 0$ *are not real.*

Sol. Let $\quad P(x) = a_{85}x^{85} + a_{84}x^{84}$

$$+ \ldots + a_3x^3 + 3x^2 + 2x + 1 = 0 \quad \text{...(i)}$$

Since, $P(0) = 1$, then 0 is not a root of Eq. (i).
Let $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_{85}$ be the complex roots of Eq. (i).

Then, the $\beta_i \left(\text{let} \frac{1}{\alpha_i}\right)$ the complex roots of the polynomial

$$Q(y) = y^{85} + 2y^{84} + 3y^{83} + a_3y^{82} + \ldots + a_{85}$$

It follows that

$$\sum_{i=1}^{85}\beta_i = -2 \text{ and } \sum_{1 \leq i <} \sum_{j \leq 85} \beta_i\beta_j = 3$$

Then, $$\sum_{i=1}^{85}\beta_i^2 = \left(\sum_{i=1}^{85}\beta_i\right)^2 - 2\sum_{1 \leq i <} \sum_{j \leq 85} \beta_i\beta_j$$

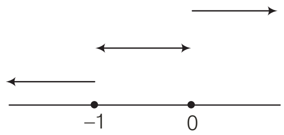
$$= 4 - 6 = -2 < 0$$

Thus, the β_i's is not all real and then α_i's are not all real.

● **Ex. 45** *Solve the equation*

$$2^{|x+1|} - 2^x = |2^x - 1| + 1.$$

Sol. Find the critical points :



$$x + 1 = 0, 2^x - 1 = 0$$

$\therefore \quad x = -1, x = 0$

Now, consider the following cases :

$\boldsymbol{x < -1}$

$$2^{-(x+1)} - 2^x = -(2^x - 1) + 1$$

$\Rightarrow \quad 2^{-(x+1)} = 2$

$\therefore \quad -(x+1) = 1$

$\therefore \quad x = -2 \quad$ ...(i)

$\boldsymbol{-1 \leq x < 0}$

$$2^{x+1} - 2^x = -(2^x - 1) + 1$$

$\Rightarrow \quad 2^{x+1} = 2$

$\therefore \quad x + 1 = 1$

$\therefore \quad x = 0$

$\quad x \neq 0 \quad [\because -1 \leq x < 0]$

$\boldsymbol{x \geq 0}$

$$2^{x+1} - 2^x = 2^x - 1 + 1$$

$\Rightarrow \quad 2^{x+1} = 2 \cdot 2^x$

$\Rightarrow \quad 2^{x+1} = 2^{x+1}$

which is true for $\quad x \geq 0. \quad$ ...(ii)

Now, combining all cases, we have the final solution as

$$x \in [0, \infty) \cup \{-2\}$$

● **Ex. 46** *Solve the inequation*

$$-|y| + x - \sqrt{(x^2 + y^2 - 1)} \geq 1.$$

Sol. We have, $-|y| + x - \sqrt{(x^2 + y^2 - 1)} \geq 1$

$\Rightarrow \quad x - |y| \geq 1 + \sqrt{(x^2 + y^2 - 1)}$

if $\quad x \geq |y|$,

then squaring both sides,

$$x^2 + y^2 - 2x|y| \geq 1 + x^2 + y^2 - 1 + 2\sqrt{(x^2 + y^2 - 1)}$$

$\Rightarrow \quad -x|y| \geq \sqrt{(x^2 + y^2 - 1)} \quad$ ...(i)

Since, $\quad x \geq |y| \geq 0 \quad$ ...(ii)

Then, LHS of Eq. (i) is non-positive and RHS of Eq. (ii) is non-negative. Therefore, the system is satisfied only, when both sides are zero.

$\therefore$ The inequality Eq. (i) is equivalent to the system.

$$\begin{cases} x|y| = 0 \\ x^2 + y^2 - 1 = 0 \end{cases}$$

The Eq.(i) gives $x = 0$ or $y = 0$. If $x = 0$, then we find $y = \pm 1$ from Eq. (ii) but $x \geq |y|$ which is impossible.

If $y = 0$, then from Eq. (ii), we find

$$x^2 = 1$$

$\therefore$ $$x = 1, -1$$

Taking $x = 1$ $\quad [\because x \geq |y|]$

$\therefore$ The pair (1, 0) satisfies the given inequation. Hence, (1, 0) is the solution of the original inequation.

Ex. 47 *If $a_1, a_2, a_3, \ldots, a_n (n \geq 2)$ are real and $(n-1)a_1^2 - 2na_2 < 0$, prove that atleast two roots of the equation $x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots + a_n = 0$ are imaginary.*

Sol. Let $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$ are the roots of the given equation.

Then, $\Sigma\alpha_1 = \alpha_1 + \alpha_2 + \alpha_3 + \ldots + \alpha_n = -a_1$

and $\Sigma\alpha_1\alpha_2 = \alpha_1\alpha_2 + \alpha_1\alpha_3 + \ldots + \alpha_{n-1}\alpha_n = a_2$

Now, $(n-1)a_1^2 - 2na_2 = (n-1)(\Sigma\alpha_1)^2 - 2n\Sigma\alpha_1\alpha_2$

$$= n\{(\Sigma\alpha_1)^2 - 2\Sigma\alpha_1\alpha_2\} - (\Sigma\alpha_1)^2$$

$$= n\Sigma\alpha_1^2 - (\Sigma a_1)^2$$

$$= \sum_{1 \leq i < j \leq n} \Sigma(\alpha_i - \alpha_j)^2$$

But given that $(n-1)a_1^2 - 2na_2 < 0$

$$\Rightarrow \quad \sum_{1 \leq i < j \leq n} \cdot \Sigma (\alpha_i - \alpha_j)^2 < 0$$

which is true only, when atleast two roots are imaginary.

Ex. 48 *Solve the inequation $|a^{2x} + a^{x+2} - 1| \geq 1$ for all values of $a\,(a > 0, a \neq 1)$.*

Sol. Using $a^x = t$,

the given inequation can be written in the form

$$|t^2 + a^2 t - 1| \geq 1 \quad \ldots\text{(i)}$$

$\because$ $a > 0$ and $a \neq 1$, then $a^x > 0$

$\therefore$ $$t > 0 \quad \ldots\text{(ii)}$$

Inequation (i) write in the forms,

$$t^2 + a^2 t - 1 \geq 1 \text{ and } t^2 + a^2 t - 1 \leq -1$$

$\therefore$ $$t \leq \frac{-a^2 - \sqrt{a^4 + 8}}{2}, t \geq \frac{-a^2 + \sqrt{(a^4 + 8)}}{2}$$

and $$-a^2 \leq t \leq 0$$

But $$t > 0 \quad \text{[from Eq. (ii)]}$$

$\therefore$ $$t \geq \frac{-a^2 + \sqrt{(a^4 + 8)}}{2}$$

$\therefore$ $$a^x \geq \frac{-a^2 + \sqrt{(a^4 + 8)}}{2}$$

For $0 < a < 1$,

$$x \leq \log_a\left(\frac{-a^2 + \sqrt{(a^4 + 8)}}{2}\right)$$

$\therefore$ $$x \in \left[-\infty, \log_a\left(\frac{-a^2 + \sqrt{(a^4 + 8)}}{2}\right)\right]$$

and for $a > 1$, $x \geq \log_a\left(\frac{-a^2 + \sqrt{(a^4 + 8)}}{2}\right)$

$\therefore$ $$x \in \left(\log_a\left(\frac{-a^2 + \sqrt{(a^4 + 8)}}{2}\right), \infty\right)$$

Ex. 49 *Solve the inequation*

$$\log_{|x|}(\sqrt{(9 - x^2)} - x - 1) \geq 1.$$

Sol. We rewrite the given inequation in the form,

$$\log_{|x|}(\sqrt{(9 - x^2)} - x - 1) \geq \log_{|x|}(|x|)$$

This inequation is equivalent to the collection of systems.

$$\begin{cases} \sqrt{(9 - x^2)} - x - 1 \geq |x|, \text{ if } |x| > 1 \\ \sqrt{(9 - x^2)} - x - 1 \leq |x|, \text{ if } \quad 0 < |x| < 1 \end{cases}$$

and $$\left[\begin{array}{l} \begin{cases} \text{For } x > 1 \\ \sqrt{(9 - x^2)} - x - 1 \geq x \end{cases} \\ \begin{cases} \text{For } x < -1 \\ \sqrt{(9 - x^2)} - x - 1 \geq -x \end{cases} \\ \begin{cases} \text{For } 0 < x < 1 \\ \sqrt{(9 - x^2)} - x - 1 \leq x \end{cases} \\ \begin{cases} \text{For } -1 < x < 0 \\ \sqrt{(9 - x^2)} - x - 1 \leq -x \end{cases} \end{array}\right. \Rightarrow \left[\begin{array}{l} \begin{cases} \text{For } x > 1 \\ \sqrt{(9 - x^2)} \geq 2x + 1 \end{cases} \\ \begin{cases} \text{For } x < -1 \\ \sqrt{(9 - x^2)} \geq 1 \end{cases} \\ \begin{cases} \text{For } 0 < x < 1 \\ \sqrt{(9 - x^2)} \leq 2x + 1 \end{cases} \\ \begin{cases} \text{For } -1 < x < 0 \\ \sqrt{(9 - x^2)} \leq 1 \end{cases} \end{array}\right.$$

$$\Rightarrow \left[\begin{array}{l} \begin{cases} \text{For } x > 1 \\ -\frac{2}{5}(\sqrt{11} + 1) \leq x \leq \frac{2}{5}(\sqrt{11} - 1) \\ \text{For } x < -1 \\ -2\sqrt{2} \leq x \leq 2\sqrt{2} \end{cases} \\ \begin{cases} \text{For } 0 < x < 1 \\ x \leq -\frac{2}{5}(\sqrt{11} + 1) \text{ and } x \geq \frac{2}{5}(\sqrt{11} - 1) \\ \text{For } -1 < x < 0 \\ x \leq -2\sqrt{2} \text{ and } x \geq 2\sqrt{2} \end{cases} \end{array}\right.$$

$$\Rightarrow \quad \left[\begin{array}{l} \begin{cases} x \in \phi \\ -2\sqrt{2} \le x < -1 \end{cases} \\ \begin{cases} \frac{2}{5}(\sqrt{11} - 1) \le x < 1 \\ x \in \phi \end{cases} \end{array}\right.$$

Hence, the original inequation consists of the intervals

$$-2\sqrt{2} \le x < -1 \text{ and } \frac{2}{5}(\sqrt{11} - 1) \le x < 1.$$

Hence, $x \in [-2\sqrt{2}, -1) \cup \left[\frac{2}{5}(\sqrt{11} - 1), 1\right)$

● **Ex. 50** *Find all values of 'a' for which the equation* $4^x - a2^x - a + 3 = 0$ has atleast one solution.

Sol. Putting $2^x = t > 0$, then the original equation reduced in the form

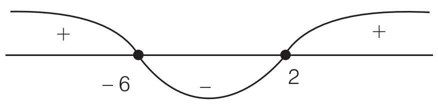
$$t^2 - at - a + 3 = 0 \quad \text{...(i)}$$

that the quadratic Eq. (i) should have atleast one positive root $(t > 0)$, then

Discriminant, $D = (-a)^2 - 4 \cdot 1 \cdot (-a + 3) \ge 0$

$$\Rightarrow \quad a^2 + 4a - 12 \ge 0$$

$$\Rightarrow \quad (a + 6)(a - 2) \ge 0$$



$$\therefore \quad a \in (-\infty, -6] \cup [2, \infty)$$

If roots of Eq. (i) are t_1 and t_2, then

$$\begin{cases} t_1 + t_2 = a \\ t_1 t_2 = 3 - a \end{cases}$$

For $\quad a \in (-\infty, -6]$

$t_1 + t_2 < 0$ and $t_1 t_2 > 0$. Therefore, both roots are negative and consequently, the original equation has no solutions.

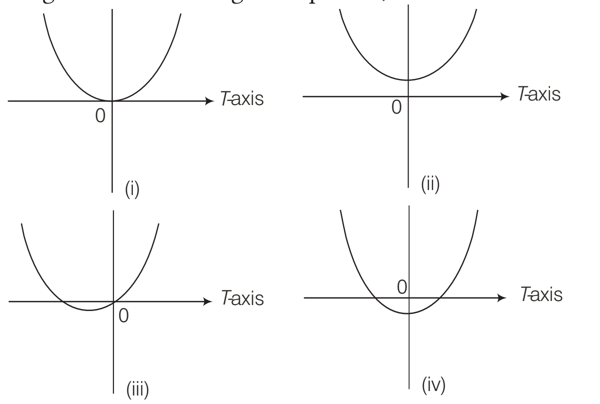
For $\quad a \in [2, \infty)$

$t_1 + t_2 > 0$ and $t_1 t_2 \gtreqless 0$, consequently, atleast one of the roots t_1 or t_2, is greater than zero.

Thus, for $a \in [2, \infty)$, the given equation has atleast one solution.

● **Ex. 51** *Find all the values of the parameter a for which the inequality* $a9^x + 4(a - 1)3^x + a > 1$, *is satisfied for all real values of* x.

Sol. Putting $t = 3^x$ in the original equation, then we obtain



$$at^2 + 4(a - 1)t + a > 1$$

$$\Rightarrow \quad at^2 + 4(a - 1)t + (a - 1) > 0 \quad [t > 0, \because 3^x > 0]$$

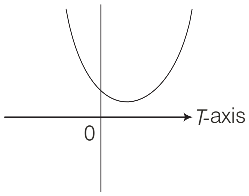
This is possible in two cases. First the parabola $f(t) = at^2 + 4(a - 1)t + (a - 1)$ opens upwards, with its vertex (turning point) lying in the non-positive part of the T-axis, as shown in the following four figures.

$\therefore \quad a > 0$ and sum of roots ≤ 0

$$\Rightarrow \quad -\frac{4(a-1)}{2a} \le 0 \text{ and } f(0) \ge 0$$

$$\therefore \quad a > 0, a - 1 \ge 0 \text{ and } a - 1 \ge 0$$

Hence, $\quad a \ge 1$



Second the parabola $f(t)$ opens upward, with its vertex lying in positive direction of t, then

$$a > 0, -\frac{4(a-1)}{2a} > 0 \text{ and } D \le 0$$

$$\Rightarrow \quad a > 0, (a - 1) < 0$$

and $\quad 16(a - 1)^2 - 4(a - 1)a \le 0$

$$\Rightarrow \quad a > 0, a < 1$$

and $\quad 4(a - 1)(3a - 4) \le 0$

$$\Rightarrow \quad a > 0, a < 1 \text{ and } 1 \le a \le \frac{4}{3}$$

These inequalities cannot have simultaneously.

Hence, $a \ge 1$ from Eq. (i).