Session 2

Arithmetic Progression (AP)

Types of Progression

Progressions are various types but in this chapter we will studying only three special types of progressions which are following :

1. Arithmetic Progression (AP)
2. Geometric Progression (GP)
3. Harmonic Progression (HP)

Arithmetic Progression (AP)

An arithmetic progression is a sequence in which the difference between any term and its just preceding term (i.e., term before it) is constant throughout. This constant is called the common difference (abbreviated as CD) and is generally denoted by 'd'.

Or

An arithmetic progression is a sequence whose terms increase or decrease by a fixed number. This fixed number is called the common difference of the AP.

A finite or infinite sequence $\{t_1, t_2, t_3, \dots, t_n\}$ or $\{t_1, t_2, t_3, \dots\}$ is said to be an arithmetic progression (AP), if $t_k - t_{k-1} = d$, a constant independent of k, for $k = 2, 3, 4, \dots, n$ or $k = 2, 3, 4, \dots$ as the case may be :
The constant d is called the common difference of the AP.

i.e. $\quad d = t_2 - t_1 = t_3 - t_2 = \dots = t_n - t_{n-1}$

Remarks

1. If a be the first term and d be the common difference, then AP can be written as
$$a, a + d, a + 2d, \dots, a + (n-1)d, \dots, \forall n \in N.$$
2. If we add the common difference to any term of AP, we get the next following term and if we subtract it from any term, we get the preceding term.
3. The common difference of an AP may be **positive**, **zero**, **negative** or **imaginary.**
4. **Constant AP** common difference of an AP is equal to zero.
5. **Increasing AP** common difference of an AP is greater than zero.
6. **Decreasing AP** common difference of an AP is less than zero.
7. **Imaginary AP** common difference of an AP is imaginary.

Algorithm to determine whether a sequence is an AP or not

***Step* I** Obtain t_n (the nth term of the sequence).

***Step* II** Replace n by $n-1$ in t_n to get t_{n-1}.

***Step* III** Calculate $t_n - t_{n-1}$.

If $t_n - t_{n-1}$ is independent of n, the given sequence is an AP otherwise it is not an AP.

Example 5.
(i) $1, 3, 5, 7, \dots$ (ii) $\pi, \pi + e^{\pi}, \pi + 2e^{\pi}, \dots$
(iii) $a, a-b, a-2b, a-3b, \dots$

Sol. (i) Here, 2nd term − 1st term = 3rd term − 2nd term = ...
$\Rightarrow$ $3 - 1 = 5 - 3 = \dots = 2$, which is a common difference.

(ii) Here, 2nd term − 1st term = 3rd term − 2nd term = ...
$$\Rightarrow (\pi + e^{\pi}) - \pi = (\pi + 2e^{\pi}) - (\pi + e^{\pi}) = \dots$$
$$= e^{\pi}, \text{ which is a common difference.}$$

(iii) Here, 2nd term − 1st term = 3rd term − 2nd term = ...
$$\Rightarrow (a-b) - a = (a-2b) - (a-b) = \dots$$
$$= -b, \text{ which is a common difference.}$$

Example 6. Show that the sequence $< t_n >$ defined by $t_n = 5n + 4$ is an AP, also find its common difference.

Sol. We have, $t_n = 5n + 4$

On replacing n by $(n-1)$, we get
$$t_{n-1} = 5(n-1) + 4$$
$$\Rightarrow \quad t_{n-1} = 5n - 1$$
$$\therefore \quad t_n - t_{n-1} = (5n+4) - (5n-1) = 5$$
Clearly, $t_n - t_{n-1}$ is independent of n and is equal to 5. So, the given sequence is an AP with common difference 5.

Example 7. Show that the sequence $< t_n >$ defined by $t_n = 3n^2 + 2$ is not an AP.

Sol. We have, $t_n = 3n^2 + 2$

On replacing n by $(n-1)$, we get
$$t_{n-1} = 3(n-1)^2 + 2$$
$$\Rightarrow \quad t_{n-1} = 3n^2 - 6n + 5$$
$$\therefore \quad t_n - t_{n-1} = (3n^2 + 2) - (3n^2 - 6n + 5)$$
$$= 6n - 3$$
Clearly, $t_n - t_{n-1}$ is not independent of n and therefore it is not constant. So, the given sequence is not an AP.

Remark

If the nth term of a sequence is an expression of first degree in n. For example, $t_n = An + B$, where A, B are constants, then that sequence will be in AP for $t_n - t_{n-1} = (An + B) - [A(n-1) + B]$ $= A[n-(n-1)] = A$ = constant = Common difference or coefficient of n in t_n Students are advised to consider the above point as a behaviour of standard result.

General Term of an AP

Let 'a' be the first term, 'd' be the common difference and 'l' be the last term of an AP having 'n' terms, where $n \in N$.

Then, AP can be written as $a, a+d, a+2d, \ldots, l-2d, l-d, l$

(i) nth Term of an AP from Beginning

1st term from beginning $= t_1 = a = a + (1-1)\, d$

2nd term from beginning $= t_2 = a + d = a + (2-1)\, d$

3rd term from beginning $= t_3 = a + 2d = a + (3-1)\, d$

$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$

n th term from beginning $= t_n = a + (n-1)\, d, \forall\, n \in N$

Hence, n th term of an AP from beginning

$= t_n = a + (n-1)\, d = l$ [last term]

(ii) nth Term of an AP from End

1st term from end $= t'_1 = l = l - (1-1)\, d$

2nd term from end $= t'_2 = l - d = l - (2-1)\, d$

3rd term from end $= t'_3 = l - 2d = l - (3-1)\, d$

$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$

nth term from end $= t'_n = l - (n-1)\, d, \forall\, n \in N$

Hence, n th term of an AP from end

$t'_n = l - (n-1)\, d = a$ [first term]

Now, it is clear that

$t_n + t'_n = a + (n-1)\, d + l - (n-1)\, d = a + l$

or $\quad t_n + t'_n = a + l$

i.e. In a finite AP, the sum of the terms equidistant from the beginning and end is always same and is equal to the sum of first and last term.

Remark

1. nth term is also called the general term.
2. If last term of AP is t_n and common difference be d, then terms of AP from end are $t_n, t_n - d, t_n - 2d, \ldots$
3. If in a sequence, the terms an alternatively positive and negative, then it cannot be an AP.
4. Common difference of AP $= \dfrac{l-a}{n+1}$, where, a = first term of AP, l = last term of AP and n = number of terms of AP.
5. If t_n, t_{n+1}, t_{n+2} are three consecutive terms of an AP, then $2t_{n+1} = t_n + t_{n+2}$. In particular, if a, b and c are in AP, then $2b = a + c$.

Example 8. Find first negative term of the sequence

$$20, 19\frac{1}{4}, 18\frac{1}{2}, 17\frac{3}{4}, \ldots$$

Sol. The given sequence is an AP in which first term, $a = 20$ and common difference, $d = -\frac{3}{4}$. Let the nth term of the given AP be the first negative term. Then,

$$t_n < 0 \Rightarrow a + (n-1)\, d < 0$$

$$\Rightarrow \quad 20 + (n-1)\left(-\frac{3}{4}\right) < 0$$

$$\Rightarrow \quad 80 - 3n + 3 < 0$$

$$\Rightarrow \quad n > \frac{83}{3} \text{ or } n > 27\frac{2}{3}$$

$$\Rightarrow \quad n = 28$$

Thus, 28th term of the given sequence is the first negative term.

Example 9. If the mth term of an AP is $\frac{1}{n}$ and the nth term is $\frac{1}{m}$, then find mnth term of an AP.

Sol. If A and B are constants, then rth term of AP is

$$t_r = Ar + B$$

Given, $\quad t_m = \dfrac{1}{n} \Rightarrow Am + B = \dfrac{1}{n}$ ...(i)

and $\quad t_n = \dfrac{1}{m} \Rightarrow An + B = \dfrac{1}{m}$ ...(ii)

From Eqs. (i) and (ii), we get $A = \dfrac{1}{mn}$ and $B = 0$

mn th term $= t_{mn} = Amn + B = \dfrac{1}{mn} \cdot mn + 0 = 1$

Hence, mn th term of the given AP is 1.

Example 10. If $|x-1|$, 3 and $|x-3|$ are first three terms of an increasing AP, then find the 6th term of on AP.

Sol. ***Case* I** For $x < 1$,

$$|x-1| = -(x-1)$$

and $\quad |x-3| = -(x-3)$

$\therefore 1 - x$, 3 and $3 - x$ are in AP.

$$\Rightarrow \quad 6 = 1 - x + 3 - x$$

$$\Rightarrow \quad x = -1$$

Then, first three terms are 2, 3, 4, which is an increasing AP.

$\therefore$ 6th term is 7. $\quad [\because d = 1]$

***Case* II** For $1 < x < 3$,

$$|x-1| = x - 1$$

and $\quad |x-3| = -(x-3) = 3 - x$

$\therefore x - 1$, 3 and $3 - x$ are in AP.

$$\Rightarrow \quad 6 = x - 1 + 3 - x$$

$$\Rightarrow \quad 6 = 2 \quad \text{[impossible]}$$

***Case* III** For $x > 3$, $|x-1| = x-1$ and $|x-3| = x-3$

$\therefore$ $x-1, 3$ and $x-3$ are in AP.

$\Rightarrow \quad 6 = x - 1 + x - 3 \Rightarrow x = 5$

Then, first three terms are 4, 3, 2, which is a decreasing AP.

Example 11. In the sequence 1, 2, 2, 3, 3, 3, 4, 4, 4, 4, ..., where n consecutive terms have the value n, find the 150th term of the sequence.

Sol. Let the 150th term $= n$

Then, $1 + 2 + 3 + \ldots + (n-1) < 150 < 1 + 2 + 3 + \ldots + n$

$$\Rightarrow \quad \frac{(n-1)\,n}{2} < 150 < \frac{n\,(n+1)}{2}$$

$$\Rightarrow \quad n\,(n-1) < 300 < n\,(n+1)$$

Taking first two members

$$n\,(n-1) < 300 \Rightarrow n^2 - n - 300 < 0$$

$$\Rightarrow \quad \left(n - \frac{1}{2}\right)^2 < 300 + \frac{1}{4}$$

$$\Rightarrow \quad 0 < n < \frac{1}{2} + \frac{\sqrt{1201}}{2}$$

$$\Rightarrow \quad 0 < n < 17.8 \quad \ldots(\text{i})$$

and taking last two members,

$$n\,(n+1) > 300$$

$$\Rightarrow \quad \left(n + \frac{1}{2}\right)^2 > 300 + \frac{1}{4}$$

$$\therefore \quad n > -\frac{1}{2} + \frac{\sqrt{1201}}{2}$$

$$\Rightarrow \quad n > 16.8 \quad \ldots(\text{ii})$$

From Eqs. (i) and (ii), we get

$$16.8 < n < 17.8$$

$$\Rightarrow \quad n = 17$$

Example 12. ***If a_1, a_2, a_3, a_4 and a_5 are in AP with common difference $\neq 0$, find the value of $\sum_{i=1}^{5} a_i$ when $a_3 = 2$.***

Sol. $\because$ a_1, a_2, a_3, a_4 and a_5 are in AP, we have

$$a_1 + a_5 = a_2 + a_4 = a_3 + a_3 \quad [\because t_n + t'_n = a + l]$$

$$a_1 + a_5 = a_2 + a_4 = 4 \quad [\because a_3 = 2]$$

$$a_1 + a_2 + a_3 + a_4 + a_5 = 4 + 2 + 4 = 10$$

$$\Rightarrow \quad \sum_{i=1}^{5} a_i = 10$$

Sum of a Stated Number of Terms of an Arithmetic Series

More than 200 yr ago, a class of German School Children was asked to find the sum of all integers from 1 to 100 inclusive. One boy in the class, an eight year old named **Carl Fredrick Gauss** (1777-1855) who later established his reputation as one of the greatest Mathematicians announced the answer almost at once. The teacher overawed at this asked Gauss to explain how he got this answer. Gauss explained that he had added these numbers in pairs as follows

$$(1+100), (2+99), (3+98), \ldots$$

There are $\frac{100}{2} = 50$ pairs. The answer can be obtained by multiplying 101 by 50 to get 5050.

Sum of n Terms of an AP

Let 'a' be the first term, 'd' be the common difference, 'l' be the last term of an AP having n terms and S_n be the sum of n terms, then

$$S_n = a + (a+d) + (a+2d) + \ldots + (l-2d) + (l-d) + l \quad \ldots(\text{i})$$

Reversing the right hand terms

$$S_n = l + (l-d) + (l-2d) + \ldots + (a+2d) + (a+d) + a \quad \ldots(\text{ii})$$

On adding Eqs. (i) and (ii), we get

$$2S_n = (a+l) + (a+l) + (a+l) + \ldots + (a+l) + (a+l) + (a+l)$$

$$= (a+l) + (a+l) + \ldots \text{ upto } n \text{ terms} = n(a+l)$$

$$\therefore \quad \boldsymbol{S_n = \frac{n}{2}(a+l)} \quad \ldots(\text{iii})$$

Now, if we substitute the value of l *viz.*, $l = a + (n-1)\,d$, in this formula, we get

$$S_n = \frac{n}{2}[a + a + (n-1)\,d] = \frac{n}{2}[2a + (n-1)d]$$

$$\therefore \quad S_n = \frac{n}{2}[2a + (n-1)\,d]$$

If we substitute the value of a *viz.*,

$$l = a + (n-1)\,d$$

or $\quad a = l - (n-1)\,d$ in Eq. (iii), then

$$S_n = \frac{n}{2}[2l - (n-1)\,d]$$

If we substitute the value of $a + l$ *viz.,*

$$t_n + t'_n = a + l \text{ in Eq. (iii), then}$$

$$\boldsymbol{S_n = \frac{n}{2}(t_n + t'_n)}$$

***Corollary* I** Sum of first n natural numbers

i.e. $1 + 2 + 3 + 4 + \ldots + n$

Here, $\quad a = 1$ and $d = 1$

$$\therefore \quad S = \frac{n}{2}[2 \cdot 1 + (n-1) \cdot 1]$$

$$= \frac{n\,(n+1)}{2}$$

***Corollary* II** Sum of first n odd natural numbers

i.e., $1+3+5+\ldots$

Here, $a=1$

and $d=2$

$\therefore \quad S=\frac{n}{2}[2\cdot 1+(n-1)\cdot 2] = n^2$

***Corollary* III** If sum of first n terms is S_n, then sum of next m terms is $S_{m+n}-S_n$.

Important Results with Proof

1. If S_n, t_n and d are sum of n terms, nth term and common difference of an AP respectively, then

$$\boldsymbol{d=t_n-t_{n-1}} \quad [n\ge 2]$$

$$\boldsymbol{t_n=S_n-S_{n-1}} \quad [n\ge 2]$$

$$\boldsymbol{d=S_n-2S_{n-1}+S_{n-2}} \quad [n\ge 3]$$

Proof

$\because \quad S_n=t_1+t_2+t_3+\ldots+t_{n-1}+t_n$

$\Rightarrow \quad S_n=S_{n-1}+t_n$

$\therefore \quad t_n=S_n-S_{n-1}$

but $\quad d=t_n-t_{n-1}$

$\quad =(S_n-S_{n-1})-(S_{n-1}-S_{n-2})$

$\therefore \quad d=S_n-2S_{n-1}+S_{n-2}$

2. A sequence is an AP if and only if the sum of its n terms is of the form An^2+Bn, where A and B are constants independent of n.

In this case, the nth term and common difference of the AP are $A(2n-1)+B$ and $2A$, respectively.

Proof As $S_n=An^2+Bn$

$\therefore \quad S_{n-1}=A(n-1)^2+B(n-1)$

$\therefore \quad t_n=S_n-S_{n-1}$

$\quad =(An^2+Bn)-[A(n-1)^2+B(n-1)]$

$\quad =A[n^2-(n-1)^2]+B$

$\quad t_n=A(2n-1)+B$

$\Rightarrow \quad t_{n-1}=A[2(n-1)-1]+B$

$\quad =A(2n-3)+B$

Now, $t_n-t_{n-1}=[A(2n-1)+B]-[A(2n-3)+B]$

$\quad =2A$ [a constant]

Hence, the sequence is an AP.

Conversely, consider an AP with first term a and common difference d.

Sum of first n terms $=\frac{n}{2}[2a+(n-1)d]$

$$=\frac{dn^2}{2}+\left(a-\frac{d}{2}\right)n=An^2+Bn,$$

where, $A=\frac{d}{2}$, $B=a-\frac{d}{2}$

Hence, $S_n=An^2+Bn$, where A and B are constants independent of n.

Hence, the converse is true.

Corollary $\because \quad S_n=An^2+Bn$

$\therefore \quad t_n=A(2n-1)+B$

$t_n=A$ (replacing n^2 by $2n-1$) + coefficient of n

and $\quad d=2A$

i.e. $\quad d=2$ [coefficient of n^2]

	S_n	t_n	d
1.	$5n^2+3n$	$5(2n-1)+3=10n-2$	10
2.	$-7n^2+2n$	$-7(2n-1)+2$ $=-14n+9$	-14
3.	$-9n^2-4n$	$-9(2n-1)-4$ $=-18n+5$	-18
4.	$4n^2-n$	$4(2n-1)-1=8n-5$	8

3. If $S_n=an^2+bn+c$, where S_n denotes the sum of n terms of a series, then whole series is not an AP. It is AP from the second term onwards.

Proof As $\quad S_n=an^2+bn+c$ for $n\ge 1$, we get

$$S_{n-1}=a(n-1)^2+b(n-1)+c \text{ for } n\ge 2$$

Now, $\quad t_n=S_n-S_{n-1}$

$\Rightarrow \quad t_n=a(2n-1)+b,\ n\ge 2$

$\therefore \quad t_{n-1}=a[2(n-1)-1]+b,\ n\ge 3$

$\Rightarrow \quad t_{n-1}=a(2n-3)+b,\ n\ge 3$

$\therefore \quad t_n-t_{n-1}=2a=\text{constant},\ n\ge 3$

$\therefore \quad t_3-t_2=t_4-t_3=t_5-t_4=\ldots$

But $\quad t_2-t_1=(S_2-S_1)-S_1=S_2-2S_1$

$\quad =(4a+2b+c)-2(a+b+c)$

$\quad =(2a-c)$ $[\because S_1=t_1]$

$\therefore \quad t_2-t_1\ne t_3-t_2$

Hence, the whole series is not an AP. It is AP from the second term onwards.

Ratio of Sums is Given

1. If ratio of the sums of m and n terms of an AP is given by

$$\frac{S_m}{S_n} = \frac{Am^2 + Bm}{An^2 + Bn}$$

where A, B are constants and $A \neq 0$.

$$\therefore \quad S_m = (Am^2 + Bm)\,k,$$
$$S_n = (An^2 + Bn)\,k$$
$$\Rightarrow \quad t_m = S_m - S_{m-1} = [A(2m-1) + B]k$$
$$t_n = S_n - S_{n-1} = [A(2n-1) + B]k$$
$$\therefore \quad \frac{t_m}{t_n} = \frac{A(2m-1)+B}{A(2n-1)+B}$$

Example 13. The ratio of sums of m and n terms of an AP is $m^2 : n^2$. The ratio of the mth and nth terms is

(a) $(2m+1):(2n-1)$ (b) $m:n$
(c) $(2m-1):(2n-1)$ (d) None of these

Sol. (c) Here, $\frac{S_m}{S_n} = \frac{m^2}{n^2}$ $[\because A = 1, B = 0]$

$$\therefore \quad \frac{t_m}{t_n} = \frac{(2m-1)}{(2n-1)}$$
$$\Rightarrow \quad t_m : t_n = (2m-1):(2n-1)$$

2. If ratio of the sums of n terms of two AP's is given by

$$\frac{S_n}{S'_n} = \frac{An + B}{Cn + D}$$

where, A, B, C, D are constants and $A, C \neq 0$

$$\therefore \quad S_n = n(An + B)\,k,\; S'_n = n(Cn + D)\,k$$
$$\Rightarrow \quad t_n = [A(2n-1) + B]k,\; t'_n = [C(2n-1) + D]k$$
$$\Rightarrow \quad d = t_n - t_{n-1} = 2A,\; d' = t'_n - t'_{n-1} = 2C$$
$$\therefore \quad \frac{t_n}{t'_n} = \frac{A(2n-1)+B}{C(2n-1)+D} \text{ and } \frac{d}{d'} = \frac{A}{C}$$

Note If $A = 0, C = 0$

Then, $\frac{S_n}{S'_n} = \frac{B}{D} \Rightarrow \frac{t_n}{t'_n} = \frac{B}{D}$ and $\frac{d}{d'} = \frac{0}{0}$ = not defined

Remark

If $\frac{t_n}{t'_n} = \frac{an + b}{cn + d}$

where, a, b, c, d are constants and $a, c \neq 0$, then

$$\frac{S_n}{S'_n} = \frac{a\left(\frac{n+1}{2}\right) + b}{c\left(\frac{n+1}{2}\right) + d}$$

Example 14. The sums of n terms of two arithmetic progressions are in the ratio $(7n+1):(4n+17)$. Find the ratio of their nth terms and also common differences.

Sol. Given, $S_n : S'_n = (7n + 1):(4n + 17)$

Here, $A = 7, B = 1, C = 4$ and $D = 17$

$$\therefore \quad \frac{t_n}{t'_n} = \frac{7(2n-1)+1}{4(2n-1)+17} = \frac{14n-6}{8n+13}$$

and $\frac{d}{d'} = \frac{A}{C} = \frac{7}{4}$

Hence, $t_n : t'_n = (14n - 6):(8n + 13)$ and $d : d' = 7 : 4$

Example 15. The sums of n terms of two AP's are in the ratio $(3n-13):(5n+21)$. Find the ratio of their 24th terms.

Sol. Given, $S_n : S'_n = (3n - 13):(5n + 21)$

Here, $A = 3, B = -13, C = 5$ and $D = 21$

$$\therefore \quad \frac{t_{24}}{t'_{24}} = \frac{3(2 \times 24 - 1) - 13}{5(2 \times 24 - 1) + 21} = \frac{128}{256} = \frac{1}{2}$$
$$\therefore \quad t_{24} : t'_{24} = 1:2$$

Example 16. How many terms of the series $20 + 19\frac{1}{3} + 18\frac{2}{3} + \dots$ must be taken to make 300? Explain the double answer.

Sol. Here, given series is an AP with first term $a = 20$ and the common difference, $d = -\frac{2}{3}$.

Let the sum of n terms of the series be 300.

Then, $S_n = \frac{n}{2}\{2a + (n-1)\,d\}$

$$\Rightarrow \quad 300 = \frac{n}{2}\left\{2 \times 20 + (n-1)\left(-\frac{2}{3}\right)\right\}$$
$$\Rightarrow \quad 300 = \frac{n}{3}\{60 - n + 1\}$$
$$\Rightarrow \quad n^2 - 61n + 900 = 0$$
$$\Rightarrow \quad (n - 25)(n - 36) = 0$$
$$\Rightarrow \quad n = 25 \text{ or } n = 36$$

$\therefore$ Sum of 25 terms = Sum of 36 terms = 300

Explanation of double answer

Here, the common difference is negative, therefore terms go on diminishing and $t_{31} = 20 + (31 - 1)\left(\frac{-2}{3}\right) = 0$ i.e., 31st term becomes zero. All terms after 31st term are negative. These negative terms $(t_{32}, t_{33}, t_{34}, t_{35}, t_{36})$ when added to positive terms $(t_{26}, t_{27}, t_{28}, t_{29}, t_{30})$, they cancel out each other i.e., sum of terms from 26th to 36th terms is zero. Hence, the sum of 25 terms as well as that of 36 terms is 300.

Example 17. Find the arithmetic progression consisting of 10 terms, if the sum of the terms occupying the even places is equal to 15 and the sum of those occupying the odd places is equal to $12\frac{1}{2}$.

Sol. Let the successive terms of an AP be $t_1, t_2, t_3, \ldots, t_9, t_{10}$.

By hypothesis,

$$t_2 + t_4 + t_6 + t_8 + t_{10} = 15$$

$$\Rightarrow \quad \frac{5}{2}(t_2 + t_{10}) = 15$$

$$\Rightarrow \quad t_2 + t_{10} = 6$$

$$\Rightarrow \quad (a + d) + (a + 9d) = 6$$

$$\Rightarrow \quad 2a + 10d = 6 \quad \ldots(\text{i})$$

and $\quad t_1 + t_3 + t_5 + t_7 + t_9 = 12\frac{1}{2}$

$$\Rightarrow \quad \frac{5}{2}(t_1 + t_9) = \frac{25}{2}$$

$$\Rightarrow \quad t_1 + t_9 = 5$$

$$\Rightarrow \quad a + a + 8d = 5$$

$$\Rightarrow \quad 2a + 8d = 5 \quad \ldots(\text{ii})$$

From Eqs. (i) and (ii), we get $d = \frac{1}{2}$ and $a = \frac{1}{2}$

Hence, the AP is $\frac{1}{2}, 1, 1\frac{1}{2}, 2, 2\frac{1}{2}, \ldots$

Example 18. ***If*** N, ***the set of natural numbers is partitioned into groups*** $S_1 = \{1\}, S_2 = \{2, 3\}$, $S_3 = \{4, 5, 6\}, \ldots$, ***find the sum of the numbers in*** S_{50}.

Sol. The number of terms in the groups are 1, 2, 3, ...

$\because$ The number of terms in the 50th group = 50

$\therefore$ The last term of 1st group = 1

The last term of 2nd group = 3 = 1 + 2

The last term of 3rd group = 6 = 1 + 2 + 3

$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$

The last term of 49th group = 1 + 2 + 3 + ... + 49

$\therefore$ First term of 50th group = 1 + (1 + 2 + 3 + ... + 49)

$$= 1 + \frac{49}{2}(1 + 49) = 1226$$

$$\therefore \quad S_{50} = \frac{50}{2}\{2 \times 1226 + (50 - 1) \times 1\}$$

$$= 25 \times 2501 = 62525$$

Example 19. Find the sum of first 24 terms of on AP $t_1, t_2, t_3, \ldots$, if it is known that

$$t_1 + t_5 + t_{10} + t_{15} + t_{20} + t_{24} = 225.$$

Sol. We know that, in an AP the sums of the terms equidistant from the beginning and end is always same and is equal to the sum of first and last term.

Then, $t_1 + t_{24} = t_5 + t_{20} = t_{10} + t_{15}$

but given

$$t_1 + t_5 + t_{10} + t_{15} + t_{20} + t_{24} = 225$$

$$\Rightarrow \quad (t_1 + t_{24}) + (t_5 + t_{20}) + (t_{10} + t_{15}) = 225$$

$$\Rightarrow \quad 3(t_1 + t_{24}) = 225$$

$$\Rightarrow \quad t_1 + t_{24} = 75$$

$$\therefore \quad S_{24} = \frac{24}{2}(t_1 + t_{24}) = 12 \times 75 = 900$$

Example 20. If $(1 + 3 + 5 + \ldots + p) + (1 + 3 + 5 + \ldots + q) = (1 + 3 + 5 + \ldots + r)$, where each set of parentheses contains the sum of consecutive odd integers as shown, then find the smallest possible value of $p + q + r$ (where, $p > 6$).

Sol. We know that, $1 + 3 + 5 + \ldots + (2n - 1) = n^2$

Thus, the given equation can be written as

$$\left(\frac{p+1}{2}\right)^2 + \left(\frac{q+1}{2}\right)^2 = \left(\frac{r+1}{2}\right)^2$$

$$\Rightarrow \quad (p + 1)^2 + (q + 1)^2 = (r + 1)^2$$

Therefore, $(p + 1, q + 1, r + 1)$ form a Pythagorean triplet as $p > 6 \Rightarrow p + 1 > 7$

The first Pythagorean triplet containing a number > 7 is (6, 8, 10).

$$\Rightarrow \quad p + 1 = 8, q + 1 = 6, r + 1 = 10$$

$$\Rightarrow \quad p + q + r = 21$$

Properties of Arithmetic Progression

1. If $a_1, a_2, a_3, \ldots$ are in AP with common difference d, then $a_1 \pm k, a_2 \pm k, a_3 \pm k, \ldots$ are also in AP with common difference d.

2. If $a_1, a_2, a_3, \ldots$ are in AP with common difference d, then $a_1k, a_2k, a_3k, \ldots$ and $\frac{a_1}{k}, \frac{a_2}{k}, \frac{a_3}{k}, \ldots$ are also in AP $(k \neq 0)$ with common differences kd and $\frac{d}{k}$, respectively.

3. If $a_1, a_2, a_3, \ldots$ and $b_1, b_2, b_3, \ldots$ are two AP's with common differences d_1 and d_2, respectively. Then, $a_1 \pm b_1, a_2 \pm b_2, a_3 \pm b_3, \ldots$ are also in AP with common difference $(d_1 \pm d_2)$.

4. If $a_1, a_2, a_3, \ldots$ and $b_1, b_2, b_3, \ldots$ are two AP's with common differences d_1 and d_2 respectively, then $a_1b_1, a_2b_2, a_3b_3, \ldots$ and $\frac{a_1}{b_1}, \frac{a_2}{b_2}, \frac{a_3}{b_3}, \ldots$ are not in AP.

5. If $a_1, a_2, a_3, \ldots, a_n$ are in AP, then

$$a_r = \frac{a_{r-k} + a_{r+k}}{2}, \forall\, k, 0 \le k \le n - r$$

6. If three numbers in AP whose sum is given are to be taken as $\alpha-\beta, \alpha, \alpha+\beta$ and if five numbers in AP whose sum is given, are to be taken as $\alpha-2\beta, \alpha-\beta, \alpha, \alpha+\beta, \alpha+2\beta$, etc.

In general, If $(2r+1)$ numbers in AP whose sum is given, are to be taken as $(r \in N)$.
$\alpha-r\beta, \alpha-(r-1)\beta, \ldots, \alpha-\beta, \alpha, \alpha+\beta, \ldots,$
$\alpha+(r-1)\beta, \alpha+r\beta$

Remark

1. Sum of three numbers $= 3\alpha$
   Sum of five numbers $= 5\alpha$
   ⋮ ⋮ ⋮ ⋮ ⋮
   Sum of $(2r+1)$ numbers $= (2r+1)\alpha$
2. From given conditions, find two equations in α and β and then solve them. Now, the numbers in AP can be obtained.

7. If four numbers in AP whose sum is given, are to be taken as
$\alpha-3\beta, \alpha-\beta, \alpha+\beta, \alpha+3\beta$ and if six numbers in AP, whose sum is given are to be taken as $\alpha-5\beta, \alpha-3\beta, \alpha-\beta, \alpha+\beta, \alpha+3\beta, \alpha+5\beta$, etc.

In general If $2r$ numbers in AP whose sum is given, are to be taken as $(r \in N)$.
$\alpha-(2r-1)\beta, \alpha-(2r-3)\beta, \ldots, \alpha-3\beta, \alpha-\beta,$
$\alpha+\beta, \alpha+3\beta, \ldots, \alpha+(2r-3)\beta, \alpha+(2r-1)\beta$

Remark

1. Sum of four numbers $= 4\alpha$
   Sum of six numbers $= 6\alpha$
   ⋮ ⋮ ⋮ ⋮ ⋮
   Sum of $2r$ numbers $= 2r\alpha$
2. From given conditions, find two conditions in α and β and then solve them. Now, the numbers in AP can be obtained.

Example 21. If $S_1, S_2, S_3, \ldots, S_p$ are the sums of n terms of p AP's whose first terms are $1, 2, 3, \ldots, p$ and common differences are $1, 2, 3, \ldots, (2p-1)$ respectively, show that $S_1+S_2+S_3+\ldots+S_p = \frac{1}{2}np(np+1)$.

Sol. $\because$ $1, 2, 3, \ldots, p$ are in AP.

Then, $2\cdot1, 2\cdot2, 2\cdot3, \ldots, 2p$ are also in AP. ...(i)
[multiplying 2 to each term]

and $1, 3, 5, \ldots, (2p-1)$ are in AP.

Then, $(n-1)\cdot1, (n-1)\cdot3, (n-1)\cdot5, \ldots, (n-1)(2p-1)$ are also in AP. ...(ii)
[multiplying $(n-1)$ to each term]

From Eqs. (i) and (ii), we get
$2\cdot1+(n-1)\cdot1, 2\cdot2+(n-1)\cdot3, 2\cdot3+(n-1)\cdot5, \ldots,$
$2p+(n-1)(2p-1)$ are also in AP. ...(iii)
[adding corresponding terms of Eqs. (i) and (ii)]

From Eq. (iii),
$$\frac{n}{2}\{2\cdot1+(n-1)\cdot1\}, \frac{n}{2}\{2\cdot2+(n-1)\cdot3\}, \frac{n}{2}\{2\cdot3+(n-1)\cdot5\}, \ldots,$$
$$\frac{n}{2}\{2p+(n-1)(2p-1)\}$$ are also in AP
[multiplying $\frac{n}{2}$ to each term]

i.e. $S_1, S_2, S_3, \ldots, S_p$ are in AP.
$$\therefore\ S_1+S_2+S_3+\ldots+S_p = \frac{p}{2}\{S_1+S_p\}$$
$$= \frac{p}{2}\left\{\frac{n}{2}[2\cdot1+(n-1)\cdot1]+\frac{n}{2}[2\cdot p+(n-1)(2p-1)]\right\}$$
$$= \frac{np}{4}\{2+(n-1)+2p+(n-1)(2p-1)\}$$
$$= \frac{np}{4}(2np+2) = \frac{1}{2}np(np+1)$$

Aliter

Here, $S_1 = 1+2+3+\ldots$ upto n terms $= \dfrac{n(n+1)}{2}$
$$S_2 = 2+5+8+\ldots \text{ upto } n \text{ terms} = \frac{n}{2}[2\cdot2+(n-1)3]$$
$$= \frac{n(3n+1)}{2}$$
Similarly, $S_3 = 3+8+13+\ldots$ upto n terms $= \dfrac{n(5n+1)}{2}$, etc.

Now, $S_1+S_2+S_3+\ldots+S_p$
$$= \frac{n(n+1)}{2}+\frac{n(3n+1)}{2}+\frac{n(5n+1)}{2}+\ldots \text{ upto } p \text{ terms}$$
$$= \frac{n}{2}[(n+3n+5n+\ldots \text{ upto } p \text{ terms}) + (1+1+1+\ldots \text{ upto } p \text{ terms})]$$
$$= \frac{n}{2}\left[\frac{p}{2}(2n+(p-1)2n)+p\right]$$
$$= \frac{np}{2}[n+n(p-1)+1] = \frac{1}{2}np(np+1)$$

Example 22. Let α and β be roots of the equation $x^2-2x+A=0$ and let γ and δ be the roots of the equation $x^2-18x+B=0$. If $\alpha<\beta<\gamma<\delta$ are in arithmetic progression, then find the values of A and B.

Sol. $\because$ $\alpha, \beta, \gamma, \delta$ are in AP.

Let $\beta = \alpha+d, \gamma = \alpha+2d, \delta = \alpha+3d, d>0$
[here, sum of $\alpha, \beta, \gamma, \delta$ is not given]

Given, $\alpha+\beta = 2, \alpha\beta = A$
$\Rightarrow$ $2\alpha+d = 2, \alpha\beta = A$ ...(i)
and $\gamma+\delta = 18, \gamma\delta = B$
$\Rightarrow$ $2\alpha+5d = 18, \gamma\delta = B$ ...(ii)

From Eqs. (i) and (ii), we get

$$d = 4, \alpha = -1$$

$\therefore \quad \beta = 3, \gamma = 7, \delta = 11$

$\Rightarrow \quad A = \alpha\beta = (-1)(3) = -3$

and $\quad B = \gamma\delta = (7)(11) = 77$

Example 23. The digits of a positive integer having three digits are in AP and their sum is 15. The number obtained by reversing the digits is 594 less than the original number. Find the number.

Sol. Let the digit in the unit's place be $a-d$, digit in the ten's place be a and the digit in the hundred's place be $a+d$.

Sum of digits $= a - d + a + a + d = 15$ [given]

$\Rightarrow \quad 3a = 15$

$\therefore \quad a = 5 \quad$ ...(i)

$\therefore$ Original number $= (a-d) + 10a + 100(a+d)$

$= 111a + 99d = 555 + 99d$

and number formed by reversing the digits

$= (a+d) + 10a + 100(a-d)$

$= 111a - 99d = 555 - 99d$

Given, $(555 + 99d) - (555 - 99d) = 594 \Rightarrow 198d = 594$

$\therefore \quad d = 3$

Hence, original number $= 555 + 99 \times 3 = 852$

Example 24. If three positive real numbers are in AP such that $abc = 4$, then find the minimum value of b.

Sol. $\because a, b, c$ are in AP.

Let $\quad a = A - D, b = A, c = A + D$

Then, $\quad a = b - D, c = b + D$

Now, $\quad abc = 4$

$(b - D)\, b\, (b + D) = 4$

$\Rightarrow \quad b(b^2 - D^2) = 4$

$\Rightarrow \quad b^2 - D^2 < b^2$

$\Rightarrow \quad b(b^2 - D^2) < b^3 \Rightarrow 4 < b^3$

$\therefore \quad b > (4)^{1/3}$ or $b > (2)^{2/3}$

Hence, the minimum value of b is $(2)^{2/3}$.

Example 25. If a, b, c, d are distinct integers form an increasing AP such that $d = a^2 + b^2 + c^2$, then find the value of $a + b + c + d$.

Sol. Here, sum of numbers i.e., $a + b + c + d$ is not given.

Let $\quad b = a + D, c = a + 2D, d = a + 3D, \forall\ D \in N$

According to hypothesis,

$$a + 3D = a^2 + (a+D)^2 + (a+2D)^2$$

$\Rightarrow \quad 5D^2 + 3(2a-1)D + 3a^2 - a = 0 \quad$ ...(i)

$$\therefore \quad D = \frac{-3(2a-1) \pm \sqrt{9(2a-1)^2 - 20(3a^2 - a)}}{10}$$

$$= \frac{-3(2a-1) \pm \sqrt{(-24a^2 - 16a + 9)}}{10}$$

Now, $\quad -24a^2 - 16a + 9 \geq 0$

$\Rightarrow \quad 24a^2 + 16a - 9 \leq 0$

$$\Rightarrow \quad -\frac{1}{3} - \frac{\sqrt{70}}{3} \leq a \leq -\frac{1}{3} + \frac{\sqrt{70}}{12}$$

$\Rightarrow \quad a = -1, 0 \quad [\because a \in I]$

When $a = 0$ from Eq. (i), $D = 0, \frac{3}{5}$ (not possible $\because D \in N$) and for $a = -1$

From Eq. (i), $\quad D = 1, \frac{4}{5}$

$\therefore \quad D = 1 \quad [\because D \in N]$

$\therefore \quad a = -1, b = 0, c = 1, d = 2$

Then, $\quad a + b + c + d = -1 + 0 + 1 + 2 = 2$

Exercise for Session 2

1. If nth term of the series $25 + 29 + 33 + 37 + \dots$ and $3 + 4 + 6 + 9 + 13 + \dots$ are equal, then n equals

(a) 11 (b) 12 (c) 13 (d) 14

2. The rth term of the series $2\frac{1}{2} + 1\frac{7}{13} + 1\frac{1}{9} + \frac{20}{23} + \dots$ is

(a) $\frac{20}{5r+3}$ (b) $\frac{20}{5r-3}$ (c) $20(5r+3)$ (d) $\frac{20}{5r^2+3}$

3. In a certain AP, 5 times the 5th term is equal to 8 times the 8th term, its 13th term is

(a) 0 (b) –1 (c) – 12 (d) –13

4. If the 9th term of an AP is zero, the ratio of its 29th and 19th terms is

(a) 1 : 2 (b) 2 : 1 (c) 1 : 3 (d) 3 : 1

5. If the pth, qth and rth terms of an AP are a, b and c respectively, the value of $a(q-r) + b(r-p) + c(p-q)$ is

(a) 1 (b) –1 (c) 0 (d) $\frac{1}{2}$

6. The 6th term of an AP is equal to 2, the value of the common difference of the AP which makes the product $a_1 a_4 a_5$ least is given by

(a) $\frac{8}{5}$ (b) $\frac{5}{4}$ (c) $\frac{2}{3}$ (d) $\frac{1}{3}$

7. The sum of first $2n$ terms of an AP is α and the sum of next n terms is β, its common difference is

(a) $\frac{\alpha - 2\beta}{3n^2}$ (b) $\frac{2\beta - \alpha}{3n^2}$ (c) $\frac{\alpha - 2\beta}{3n}$ (d) $\frac{2\beta - \alpha}{3n}$

8. The sum of three numbers in AP is – 3 and their product is 8, then sum of squares of the numbers is

(a) 9 (b) 10 (c) 12 (d) 21

9. Let S_n denote the sum of n terms of an AP, if $S_{2n} = 3S_n$, then the ratio $\frac{S_{3n}}{S_n}$ is equal to

(a) 9 (b) 6 (c) 16 (d) 12

10. The sum of the products of the ten numbers $\pm 1, \pm 2, \pm 3, \pm 4, \pm 5$ taking two at a time, is

(a) – 65 (b) 165 (c) – 55 (d) 95

11. If $a_1, a_2, a_3, \dots, a_n$ are in AP, where $a_i > 0$ for all i, the value of

$$\frac{1}{\sqrt{a_1} + \sqrt{a_2}} + \frac{1}{\sqrt{a_2} + \sqrt{a_3}} + \dots + \frac{1}{\sqrt{a_{n-1}} + \sqrt{a_n}} \text{ is}$$

(a) $\frac{1}{\sqrt{a_1} + \sqrt{a_n}}$ (b) $\frac{1}{\sqrt{a_1} - \sqrt{a_n}}$ (c) $\frac{n}{\sqrt{a_1} - \sqrt{a_n}}$ (d) $\frac{(n-1)}{\sqrt{a_1} + \sqrt{a_n}}$