Session 3

Examples on Largest Value of a Third Order Determinant, Multiplication of Two Determinants of the Same Order, System of Linear Equations, Cramer's Rule, Nature of Solutions of System of Linear Equations, System of Homogeneous Linear Equations

Examples on Largest Value of a Third Order Determinant

Example 17. ***Find the largest value of a third order determinant whose elements are 0 or 1.***

Sol. Let $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

$$\Delta = a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2)$$
$$= (a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2) - (b_1c_3a_2 + b_2c_1a_3 + b_3c_2a_1)$$

Since, each element of Δ is either 0 or 1, therefore the value of the Δ cannot exceed 3. But to attain this value, each expression with a positive sign must equal 1, while those with a negative sign must be 0. However, if $a_1b_2c_3 = a_2b_3c_1 = a_3b_1c_2 = 1$, every element of the determinant must be 1, making its value zero. Thus, noting that

$$\begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} = 2$$

The largest value of Δ is 2.

Example 18. Find the largest value of a third order determinant, whose elements are 1 or −1.

Sol. Let $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

$$\therefore \Delta = a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2)$$
$$= (a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2) - (b_1c_3a_2 + b_2c_1a_3 + b_3c_2a_1)$$

Since, each element of Δ is either 1 or −1, therefore the value of the Δ cannot exceed 6. But it can be 6 only if

$$a_1b_2c_3 = a_2b_3c_1 = a_3b_1c_2 = 1 \quad \text{...(i)}$$

and $$b_1c_3a_2 = b_2c_1a_3 = b_3c_2a_1 = -1 \quad \text{...(ii)}$$

In the first case, the product of the nine elements of the determinant equals 1, while it is −1 in the second case, so the two cannot occur simultaneously i.e., the determinant cannot equal 6. The following determinant satisfies the given conditions and equals the largest value

$$\begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix} = -1(1-1) - 1(-1-1) + 1(1+1) = 4$$

Example 19. Show that the value of a third order determinant whose all elements are 1 or −1 is an even number.

Sol. Let $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

Applying $R_2 \to R_2 - \dfrac{a_2}{a_1}R_1$ and $R_3 \to R_3 - \dfrac{a_3}{a_1}R_1$, then

$$\Delta = \begin{vmatrix} a_1 & \cdots & b_1 & \cdots & c_1 \\ \vdots & & & & \\ 0 & & b_2 - \dfrac{a_2}{a_1}b_1 & & c_2 - \dfrac{a_2}{a_1}c_1 \\ \vdots & & & & \\ 0 & & b_3 - \dfrac{a_3}{a_1}b_1 & & c_3 - \dfrac{a_3}{a_1}c_1 \end{vmatrix}$$

Expanding along C_1, we get

$$\Delta = a_1\left\{\left(b_2 - \frac{a_2}{a_1}b_1\right)\left(c_3 - \frac{a_3}{a_1}c_1\right) - \left(b_3 - \frac{a_3}{a_1}b_1\right)\left(c_2 - \frac{a_2}{a_1}c_1\right)\right\} \quad \text{...(i)}$$

Since, $a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3$ are 1 or −1.

$\therefore \quad b_2, \dfrac{a_2}{a_1}b_1, c_3, \dfrac{a_3}{a_1}c_1, b_3, \dfrac{a_3}{a_1}b_1, c_2, \dfrac{a_2}{a_1}c_1$ are 1 or −1

$\Rightarrow b_2 - \dfrac{a_2}{a_1}b_1, c_3 - \dfrac{a_3}{a_1}c_1, b_3 - \dfrac{a_3}{a_1}b_1, c_2 - \dfrac{a_2}{a_1}c_1$ are 2, −2 or 0.

$\therefore \quad \left(b_2 - \dfrac{a_2}{a_1}b_1\right)\left(c_3 - \dfrac{a_3}{a_1}c_1\right)$

and $\quad \left(b_3 - \dfrac{a_3}{a_1}b_1\right)\left(c_2 - \dfrac{a_2}{a_1}c_1\right)$ are 4, − 4

or $\quad$ 0 = an even number

From Eq. (i), Δ = an even number (a_1 = 1 or −1)

Multiplication of Two Determinants of the Same Order

Let the two determinants of third order be

$$\Delta_1 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \text{ and } \Delta_2 = \begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix}$$

Let Δ be their product.

Method of Multiplication (Row by Row)

Take the first row of Δ_1 and the first row of Δ_2 i.e., a_1, b_1, c_1 and $\alpha_1, \beta_1, \gamma_1$ multiplying the corresponding elements and add. The result is $a_1\alpha_1 + b_1\beta_1 + c_1\gamma_1$ is the first element of first row of Δ.

Now, similar product first row of Δ_1 and second row of Δ_2 gives $a_1\alpha_2 + b_1\beta_2 + c_1\gamma_2$ is the second element of first row of Δ and the product of first row of Δ_1 and third row of Δ_2 gives $a_1\alpha_3 + b_1\beta_3 + c_1\gamma_3$ is the third element of first row of Δ. The second row and third row of Δ is obtained by multiplying second row and third row of Δ_1 with 1st, 2nd, 3rd row of Δ_2 in the above manner.

$$\text{Hence, } \Delta = \Delta_1 \times \Delta_2 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \times \begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1\alpha_1 + b_1\beta_1 + c_1\gamma_1 & a_1\alpha_2 + b_1\beta_2 + c_1\gamma_2 & a_1\alpha_3 + b_1\beta_3 + c_1\gamma_3 \\ a_2\alpha_1 + b_2\beta_1 + c_2\gamma_1 & a_2\alpha_2 + b_2\beta_2 + c_2\gamma_2 & a_2\alpha_3 + b_2\beta_3 + c_2\gamma_3 \\ a_3\alpha_1 + b_3\beta_1 + c_3\gamma_1 & a_3\alpha_2 + b_3\beta_2 + c_3\gamma_2 & a_3\alpha_3 + b_3\beta_3 + c_3\gamma_3 \end{vmatrix}$$

Multiplication can also be performed row by column or column by row or column by column as required in the problem.

Example 20. Evaluate $\begin{vmatrix} 1 & 2 & 3 \\ -2 & 3 & 2 \\ 3 & 4 & -4 \end{vmatrix} \times \begin{vmatrix} -2 & 1 & 3 \\ 3 & -2 & 1 \\ 2 & 1 & -2 \end{vmatrix}$.

Using the concept of multiplication of determinants.

Sol. Let $\Delta = \begin{vmatrix} 1 & 2 & 3 \\ -2 & 3 & 2 \\ 3 & 4 & -4 \end{vmatrix} \times \begin{vmatrix} -2 & 1 & 3 \\ 3 & -2 & 1 \\ 2 & 1 & -2 \end{vmatrix}$

On multiplying row by row, we get

$$\Delta = \begin{vmatrix} -2+2+9 & 3-4+3 & 2+2-6 \\ 4+3+6 & -6-6+2 & -4+3-4 \\ -6+4-12 & 9-8-4 & 6+4+8 \end{vmatrix}$$

$$= \begin{vmatrix} 9 & 2 & -2 \\ 13 & -10 & -5 \\ -14 & -3 & 18 \end{vmatrix}$$

Applying $C_1 \to C_1 + C_3$ and $C_2 \to C_2 + C_3$, then

$$\Delta = \begin{vmatrix} 7 & 0 & -2 \\ 8 & -15 & -5 \\ 4 & 15 & 18 \end{vmatrix}$$

Applying $R_2 \to R_2 + R_3$, then

$$\Delta = \begin{vmatrix} 7 & 0 & -2 \\ & \vdots & \\ 12 & 0 & 13 \\ & \vdots & \\ 4 & \cdots 15 \cdots & 18 \end{vmatrix}$$

Expanding along C_2, we get

$$-15\begin{vmatrix} 7 & -2 \\ 12 & 13 \end{vmatrix} = -15(91+24) = -15 \times 115 = -1725$$

Example 21. If $ax_1^2 + by_1^2 + cz_1^2 = ax_2^2 + by_2^2 + cz_2^2 = ax_3^2 + by_3^2 + cz_3^2 = d$, $ax_2x_3 + by_2y_3 + cz_2z_3 = ax_3x_1 + by_3y_1 + cz_3z_1 = ax_1x_2 + by_1y_2 + cz_1z_2 = f$, then prove that

$$\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = (d-f)\left\{\frac{(d+2f)}{abc}\right\}^{1/2}$$

Sol. Let LHS $= \Delta = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$

$$\therefore \Delta^2 = \Delta \times \Delta = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} \times \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$$

$$= \frac{1}{abc}\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} \times \begin{vmatrix} ax_1 & by_1 & cz_1 \\ ax_2 & by_2 & cz_2 \\ ax_3 & by_3 & cz_3 \end{vmatrix}$$

$$= \frac{1}{abc}\begin{vmatrix} ax_1^2 + by_1^2 + cz_1^2 & ax_1x_2 + by_1y_2 + cz_1z_2 & ax_3x_1 + by_3y_1 + cz_3z_1 \\ ax_1x_2 + by_1y_2 + cz_1z_2 & ax_2^2 + by_2^2 + cz_2^2 & ax_2x_3 + by_2y_3 + cz_2z_3 \\ ax_3x_1 + by_3y_1 + cz_3z_1 & ax_2x_3 + by_2y_3 + cz_2z_3 & ax_3^2 + by_3^2 + cz_3^2 \end{vmatrix} \text{[multiplying row by row]}$$

$$= \frac{1}{abc}\begin{vmatrix} d & f & f \\ f & d & f \\ f & f & d \end{vmatrix} \quad \text{[given]}$$

Applying $C_1 \to C_1 + C_2 + C_3$, then

$$= \frac{1}{abc}\begin{vmatrix} d+2f & f & f \\ d+2f & d & f \\ d+2f & f & d \end{vmatrix} = \frac{(d+2f)}{abc}\begin{vmatrix} 1 & f & f \\ 1 & d & f \\ 1 & f & d \end{vmatrix}$$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, then

$$= \frac{(d+2f)}{abc}\begin{vmatrix} 1 & f & f \\ & \ddots & \\ 0 & d-f & 0 \\ & & \ddots \\ 0 & 0 & d-f \end{vmatrix} = \frac{(d+2f)}{abc}(d-f)^2$$

$$\therefore \Delta = (d-f)\left\{\frac{d+2f}{abc}\right\}^{1/2} = \text{RHS}$$

An Important Property

If A_1, B_1 and $C_1, \ldots$ are respectively the cofactors of the elements a_1, b_1 and $c_1, \ldots$ of the determinant.

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, \Delta \neq 0, \text{ then } \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} = \Delta^2$$

Proof Consider

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \times \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1A_1 + b_1B_1 + c_1C_1 & a_1A_2 + b_1B_2 + c_1C_2 & a_1A_3 + b_1B_3 + c_1C_3 \\ a_2A_1 + b_2B_1 + c_2C_1 & a_2A_2 + b_2B_2 + c_2C_2 & a_2A_3 + b_2B_3 + c_2C_3 \\ a_3A_1 + b_3B_1 + c_3C_1 & a_3A_2 + b_3B_2 + c_3C_2 & a_3A_3 + b_3B_3 + c_3C_3 \end{vmatrix}$$

[multiplying row by row]

$$= \begin{vmatrix} \Delta & 0 & 0 \\ 0 & \Delta & 0 \\ 0 & 0 & \Delta \end{vmatrix} = \Delta^3 \qquad \left(\text{as } a_iA_j + b_iB_j + c_iC_j = \begin{cases} \Delta, & i = j \\ 0, & i \neq j \end{cases}\right)$$

$$\Rightarrow \Delta\begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} = \Delta^3 \text{ or } \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} = \Delta^2$$

$[\because \Delta \neq 0]$

Note Let $\Delta \neq 0$ and Δ^c denotes the determinant formed by the cofactors of Δ and n is order of determinant, then

$$\Delta^c = \Delta^{n-1}$$

This is known as **power cofactor formula.**

Example 22. Show that

$$\begin{vmatrix} a^2+x^2 & ab-cx & ac+bx \\ ab+cx & b^2+x^2 & bc-ax \\ ac-bx & bc+ax & c^2+x^2 \end{vmatrix} = \begin{vmatrix} x & c & -b \\ -c & x & a \\ b & -a & x \end{vmatrix}^2.$$

Sol. Let $\Delta = \begin{vmatrix} x & c & -b \\ -c & x & a \\ b & -a & x \end{vmatrix}$

Cofactors of 1st row of Δ are $x^2 + a^2, cx + ab, ac - bx$, cofactors of 2nd row of Δ are $ab - cx, x^2 + b^2, ax + bc$ and cofactors of 3rd row of Δ are $ac + bx, bc - ax, x^2 + c^2$.

Hence, the determinant of the cofactors of Δ is

$$\Delta^c = \begin{vmatrix} a^2+x^2 & ab+cx & ac-bx \\ ab-cx & b^2+x^2 & bc+ax \\ ac+bx & bc-ax & c^2+x^2 \end{vmatrix}$$

Interchanging rows into columns, we get

$$\Delta^c = \begin{vmatrix} a^2+x^2 & ab-cx & ac+bx \\ ab+cx & b^2+x^2 & bc-ax \\ ac-bx & bc+ax & c^2+x^2 \end{vmatrix} = \begin{vmatrix} x & c & -b \\ -c & x & a \\ b & -a & x \end{vmatrix}^2 \quad [\because \Delta^c = \Delta^2]$$

Example 23. Prove the following by multiplication of determinants and power cofactor formula

$$\begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix}^2 = \begin{vmatrix} b^2+c^2 & ab & ac \\ ab & c^2+a^2 & bc \\ ac & bc & a^2+b^2 \end{vmatrix}$$

$$= \begin{vmatrix} -a^2 & ab & ac \\ ab & -b^2 & bc \\ ac & bc & -c^2 \end{vmatrix} = 4a^2b^2c^2$$

Sol. Let $\Delta = \begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix}$. Expanding along R_1, then

$$\Delta = 0 - c(0 - ab) + b(ac - 0) = 2abc$$

$$\therefore \begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix}^2 = \Delta^2 = (2abc)^2 = 4a^2b^2c^2 \quad \ldots\text{(i)}$$

Also, $$\begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix}^2 = \begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix} \times \begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix}$$

$$= \begin{vmatrix} b^2+c^2 & ab & ac \\ ab & c^2+a^2 & bc \\ ac & bc & a^2+b^2 \end{vmatrix} \quad \ldots\text{(ii)}$$

[multiplying row by row]

and $$\Delta^c = \begin{vmatrix} -a^2 & ab & ac \\ ab & -b^2 & bc \\ ac & bc & -c^2 \end{vmatrix} = \Delta^{3-1} = \Delta^2$$

$$= \begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix}^2 \qquad \text{...(iii)}$$

From Eqs. (i), (ii) and (iii), we get

$$\begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix}^2 = \begin{vmatrix} b^2+c^2 & ab & ac \\ ab & c^2+a^2 & bc \\ ac & bc & a^2+b^2 \end{vmatrix}$$

$$= \begin{vmatrix} -a^2 & ab & ac \\ ab & -b^2 & bc \\ ac & bc & -c^2 \end{vmatrix} = 4a^2b^2c^2$$

Express a Determinant Into Product of Two Determinants

Consider the determinant $\begin{vmatrix} a_1\alpha_1 + b_1\beta_1 & a_1\alpha_2 + b_1\beta_2 \\ a_2\alpha_1 + b_2\beta_1 & a_2\alpha_2 + b_2\beta_2 \end{vmatrix}$

Let $\Delta = \begin{vmatrix} a_1\alpha_1 + b_1\beta_1 & a_1\alpha_2 + b_1\beta_2 \\ a_2\alpha_1 + b_2\beta_1 & a_2\alpha_2 + b_2\beta_2 \end{vmatrix}$

By the property of determinant, Δ can be written as

$$\Delta = \begin{vmatrix} a_1\alpha_1 & a_1\alpha_2 + b_1\beta_2 \\ a_2\alpha_1 & a_2\alpha_2 + b_2\beta_2 \end{vmatrix} + \begin{vmatrix} b_1\beta_1 & a_1\alpha_2 + b_1\beta_2 \\ b_2\beta_1 & a_2\alpha_2 + b_2\beta_2 \end{vmatrix}$$

$$= \begin{vmatrix} a_1\alpha_1 & a_1\alpha_2 \\ a_2\alpha_1 & a_2\alpha_2 \end{vmatrix} + \begin{vmatrix} a_1\alpha_1 & b_1\beta_2 \\ a_2\alpha_1 & b_2\beta_2 \end{vmatrix} + \begin{vmatrix} b_1\beta_1 & a_1\alpha_2 \\ b_2\beta_1 & a_2\alpha_2 \end{vmatrix} + \begin{vmatrix} b_1\beta_1 & b_1\beta_2 \\ b_2\beta_1 & b_2\beta_2 \end{vmatrix}$$

$$= \alpha_1\alpha_2 \begin{vmatrix} a_1 & a_1 \\ a_2 & a_2 \end{vmatrix} + \alpha_1\beta_2 \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} + \beta_1\alpha_2 \begin{vmatrix} b_1 & a_1 \\ b_2 & a_2 \end{vmatrix} + \beta_1\beta_2 \begin{vmatrix} b_1 & b_1 \\ b_2 & b_2 \end{vmatrix}$$

$$= 0 + \alpha_1\beta_2 \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} - \beta_1\alpha_2 \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} + 0$$

$$= \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} (\alpha_1\beta_2 - \alpha_2\beta_1)$$

$$= \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \times \begin{vmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{vmatrix}$$

$$\therefore \begin{vmatrix} a_1\alpha_1 + b_1\beta_1 & a_1\alpha_2 + b_1\beta_2 \\ a_2\alpha_1 + b_2\beta_1 & a_2\alpha_2 + b_2\beta_2 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \times \begin{vmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{vmatrix}$$

Example 24. Prove that

$$\begin{vmatrix} a_1\alpha_1 + b_1\beta_1 & a_1\alpha_2 + b_1\beta_2 & a_1\alpha_3 + b_1\beta_3 \\ a_2\alpha_1 + b_2\beta_1 & a_2\alpha_2 + b_2\beta_2 & a_2\alpha_3 + b_2\beta_3 \\ a_3\alpha_1 + b_3\beta_1 & a_3\alpha_2 + b_3\beta_2 & a_3\alpha_3 + b_3\beta_3 \end{vmatrix} = 0.$$

Sol. LHS $= \begin{vmatrix} a_1\alpha_1 + b_1\beta_1 & a_1\alpha_2 + b_1\beta_2 & a_1\alpha_3 + b_1\beta_3 \\ a_2\alpha_1 + b_2\beta_1 & a_2\alpha_2 + b_2\beta_2 & a_2\alpha_3 + b_2\beta_3 \\ a_3\alpha_1 + b_3\beta_1 & a_3\alpha_2 + b_3\beta_2 & a_3\alpha_3 + b_3\beta_3 \end{vmatrix}$

$$= \begin{vmatrix} a_1 & b_1 & 0 \\ a_2 & b_2 & 0 \\ a_3 & b_3 & 0 \end{vmatrix} \times \begin{vmatrix} \alpha_1 & \beta_1 & 0 \\ \alpha_2 & \beta_2 & 0 \\ \alpha_3 & \beta_3 & 0 \end{vmatrix} \qquad \text{[row by row]}$$

$= 0 \times 0 = 0 =$ RHS

Example 25. Prove that

$$\begin{vmatrix} 2 & \alpha+\beta+\gamma+\delta & \alpha\beta+\gamma\delta \\ \alpha+\beta+\gamma+\delta & 2(\alpha+\beta)(\gamma+\delta) & \alpha\beta(\gamma+\delta)+\gamma\delta(\alpha+\beta) \\ \alpha\beta+\gamma\delta & \alpha\beta(\gamma+\delta)+\gamma\delta(\alpha+\beta) & 2\alpha\beta\gamma\delta \end{vmatrix} = 0.$$

Sol. LHS $= \begin{vmatrix} 2 & \alpha+\beta+\gamma+\delta & \alpha\beta+\gamma\delta \\ \alpha+\beta+\gamma+\delta & 2(\alpha+\beta)(\gamma+\delta) & \alpha\beta(\gamma+\delta)+\gamma\delta(\alpha+\beta) \\ \alpha\beta+\gamma\delta & \alpha\beta(\gamma+\delta)+\gamma\delta(\alpha+\beta) & 2\alpha\beta\gamma\delta \end{vmatrix}$

$$= \begin{vmatrix} 1 & 1 & 0 \\ \alpha+\beta & \gamma+\delta & 0 \\ \alpha\beta & \gamma\delta & 0 \end{vmatrix} \times \begin{vmatrix} 1 & 1 & 0 \\ \gamma+\delta & \alpha+\beta & 0 \\ \gamma\delta & \alpha\beta & 0 \end{vmatrix} \qquad \text{[row by row]}$$

$= 0 \times 0 = 0 =$ RHS

Example 26. ***Prove that***

$$\begin{vmatrix} \cos(A-P) & \cos(A-Q) & \cos(A-R) \\ \cos(B-P) & \cos(B-Q) & \cos(B-R) \\ \cos(C-P) & \cos(C-Q) & \cos(C-R) \end{vmatrix} = 0.$$

Sol. LHS $= \begin{vmatrix} \cos(A-P) & \cos(A-Q) & \cos(A-R) \\ \cos(B-P) & \cos(B-Q) & \cos(B-R) \\ \cos(C-P) & \cos(C-Q) & \cos(C-R) \end{vmatrix}$

$$= \begin{vmatrix} \cos A & \sin A & 0 \\ \cos B & \sin B & 0 \\ \cos C & \sin C & 0 \end{vmatrix} \times \begin{vmatrix} \cos P & \sin P & 0 \\ \cos Q & \sin Q & 0 \\ \cos R & \sin R & 0 \end{vmatrix} \qquad \text{[row by row]}$$

$= 0 \times 0 = 0 =$ RHS

Example 27. *If* α, β and γ *are real numbers, without expanding at any stage, prove that*

$$\begin{vmatrix} 1 & \cos(\beta-\alpha) & \cos(\gamma-\alpha) \\ \cos(\alpha-\beta) & 1 & \cos(\gamma-\beta) \\ \cos(\alpha-\gamma) & \cos(\beta-\gamma) & 1 \end{vmatrix} = 0.$$

Sol. LHS $= \begin{vmatrix} 1 & \cos(\beta-\alpha) & \cos(\gamma-\alpha) \\ \cos(\alpha-\beta) & 1 & \cos(\gamma-\beta) \\ \cos(\alpha-\gamma) & \cos(\beta-\gamma) & 1 \end{vmatrix}$

$$= \begin{vmatrix} \cos(\alpha-\alpha) & \cos(\beta-\alpha) & \cos(\gamma-\alpha) \\ \cos(\alpha-\beta) & \cos(\beta-\beta) & \cos(\gamma-\beta) \\ \cos(\alpha-\gamma) & \cos(\beta-\gamma) & \cos(\gamma-\gamma) \end{vmatrix}$$

$$= \begin{vmatrix} \cos\alpha & \sin\alpha & 0 \\ \cos\beta & \sin\beta & 0 \\ \cos\gamma & \sin\gamma & 0 \end{vmatrix} \times \begin{vmatrix} \cos\alpha & \sin\alpha & 0 \\ \cos\beta & \sin\beta & 0 \\ \cos\gamma & \sin\gamma & 0 \end{vmatrix}$$

$= 0 \times 0 = 0 =$ RHS

Example 28. If $a, b, c, x, y, z \in R$, prove that

$$\begin{vmatrix} (a-x)^2 & (b-x)^2 & (c-x)^2 \\ (a-y)^2 & (b-y)^2 & (c-y)^2 \\ (a-z)^2 & (b-z)^2 & (c-z)^2 \end{vmatrix} = \begin{vmatrix} (1+ax)^2 & (1+bx)^2 & (1+cx)^2 \\ (1+ay)^2 & (1+by)^2 & (1+cy)^2 \\ (1+az)^2 & (1+bz)^2 & (1+cz)^2 \end{vmatrix}.$$

Sol. LHS $= \begin{vmatrix} (a-x)^2 & (b-x)^2 & (c-x)^2 \\ (a-y)^2 & (b-y)^2 & (c-y)^2 \\ (a-z)^2 & (b-z)^2 & (c-z)^2 \end{vmatrix}$

$$= \begin{vmatrix} a^2-2ax+x^2 & b^2-2bx+x^2 & c^2-2cx+x^2 \\ a^2-2ay+y^2 & b^2-2by+y^2 & c^2-2cy+y^2 \\ a^2-2az+z^2 & b^2-2bz+z^2 & c^2-2cz+z^2 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 2x & x^2 \\ 1 & 2y & y^2 \\ 1 & 2z & z^2 \end{vmatrix} \times \begin{vmatrix} a^2 & -a & 1 \\ b^2 & -b & 1 \\ c^2 & -c & 1 \end{vmatrix}$$ [row by row]

$$= \begin{vmatrix} 1 & 2x & x^2 \\ 1 & 2y & y^2 \\ 1 & 2z & z^2 \end{vmatrix} \times (-1)(-1) \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

[$C_1 \leftrightarrow C_3$ and taking (-1) common from second determinant]

$$= \begin{vmatrix} 1 & 2x & x^2 \\ 1 & 2y & y^2 \\ 1 & 2z & z^2 \end{vmatrix} \times \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

$$= \begin{vmatrix} 1+2ax+a^2x^2 & 1+2bx+b^2x^2 & 1+2cx+c^2x^2 \\ 1+2ay+a^2y^2 & 1+2by+b^2y^2 & 1+2cy+c^2y^2 \\ 1+2az+a^2z^2 & 1+2bz+b^2z^2 & 1+2cz+c^2z^2 \end{vmatrix}$$

[multiplying row by row]

$$= \begin{vmatrix} (1+ax)^2 & (1+bx)^2 & (1+cx)^2 \\ (1+ay)^2 & (1+by)^2 & (1+cy)^2 \\ (1+az)^2 & (1+bz)^2 & (1+cz)^2 \end{vmatrix} = \text{RHS}$$

System of Linear Equations

(i) Consistent equations *Definite and unique solution* [Intersecting lines]

A system of (linear) equations is said to be consistent, if it has atleast one solution.

For example, System of equations $\left.\begin{matrix} x+y=2 \\ x-y=6 \end{matrix}\right\}$ is consistent because it has a solution $x = 4$, $y = -2$. Here, two lines intersect at one point.

i.e., intersecting lines.

(ii) Inconsistent equations *No solution* [Parallel lines]

A system of (linear) equations is said to be inconsistent, if it has no solution.

Let $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$, then $\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$

$\Rightarrow$ Given equations are inconsistent.

For example, System of equations $\left.\begin{matrix} x+y=2 \\ 2x+2y=5 \end{matrix}\right\}$ is inconsistent because it has no solution i.e., there is no value of x and y which satisfy both the equations. Here, the two lines are parallel.

(iii) Dependent equations *Infinite solutions* [Identical lines]

A system of (linear) equations is said to be dependent, if it has infinite solutions.

Let $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$, then $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2} \Rightarrow$ Given equations are dependent.

For example, System of equations $\left.\begin{matrix} x+2y=3 \\ 2x+4y=6 \end{matrix}\right\}$ is dependent because it has infinite solutions i.e., there are infinite values of x and y satisfy both the equations. Here, the two lines are identical.

Cramer's Rule

System of linear equations in two variables

Let us consider a system of equations be

$$\left.\begin{aligned} a_1x + b_1y + c_1 &= 0 \\ a_2x + b_2y + c_2 &= 0 \end{aligned}\right\} \text{ where } \frac{a_1}{a_2} \neq \frac{b_1}{b_2}$$

On solving by cross-multiplication, we get

$$\frac{x}{(b_1c_2 - b_2c_1)} = \frac{y}{(c_1a_2 - c_2a_1)} = \frac{1}{(a_1b_2 - a_2b_1)}$$

or $$\frac{x}{\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}} = \frac{y}{\begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}} = \frac{1}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

or $$x = \frac{\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}, y = \frac{\begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

System of Linear Equations in Three Variables

Let us consider a system of linear equations be

$$a_1x + b_1y + c_1z = d_1 \quad \text{...(i)}$$
$$a_2x + b_2y + c_2z = d_2 \quad \text{...(ii)}$$
$$a_3x + b_3y + c_3z = d_3 \quad \text{...(iii)}$$

Here, $$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, \Delta_1 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix},$$

$$\Delta_2 = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix} \text{ and } \Delta_3 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

If $\Delta \neq 0$, then

$$\Delta_1 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1x + b_1y + c_1z & b_1 & c_1 \\ a_2x + b_2y + c_2z & b_2 & c_2 \\ a_3x + b_3y + c_3z & b_3 & c_3 \end{vmatrix}$$

Applying $C_1 \to C_1 - yC_2 - zC_3$, then

$$\Delta_1 = \begin{vmatrix} a_1x & b_1 & c_1 \\ a_2x & b_2 & c_2 \\ a_3x & b_3 & c_3 \end{vmatrix} = x\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = x\Delta$$

$\therefore \quad x = \frac{\Delta_1}{\Delta}$, where $\Delta \neq 0$

Similarly, $\Delta_2 = y\Delta$ and $\Delta_3 = z\Delta$

$\therefore \quad y = \frac{\Delta_2}{\Delta}$ and $z = \frac{\Delta_3}{\Delta}$

Thus, $$x = \frac{\Delta_1}{\Delta}, y = \frac{\Delta_2}{\Delta}, z = \frac{\Delta_3}{\Delta}, \text{ where } \Delta \neq 0 \quad \text{...(iv)}$$

The rule given in Eq. (iv) to find the values of x, y and z is called the **CRAMER'S RULE**.

Remark

1. Δ_i is obtained by replacing elements of ith columns by d_1, d_2, d_3, where $i = 1, 2, 3$.
2. Cramer's rule can be used only when $\Delta \neq 0$.

Nature of Solution of System of Linear Equations

Let us consider a system of linear equations be

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

Now, there are two cases arise:

***Case* I** If $\Delta \neq 0$

In this case, $x = \frac{\Delta_1}{\Delta}, y = \frac{\Delta_2}{\Delta}, z = \frac{\Delta_3}{\Delta}$

Then, system will have unique finite solutions and so equations are **consistent**.

***Case* II** If $\Delta = 0$

(a) **When atleast one of $\Delta_1, \Delta_2, \Delta_3$ be non-zero**

(i) Let $\Delta_1 \neq 0$, then from $\Delta_1 = x\Delta$ will not be satisfied for any value of x because $\Delta = 0$ and $\Delta_1 \neq 0$ and hence no value of x is possible.

(ii) Let $\Delta_2 \neq 0$, then from $\Delta_2 = y\Delta$ will not be satisfied for any value of y because $\Delta = 0$ and $\Delta_2 \neq 0$ and hence no value of y is possible.

(iii) Let $\Delta_3 \neq 0$, then from $\Delta_3 = z\Delta$ will not be satisfied for any value of z because $\Delta = 0$ and $\Delta_3 \neq 0$ and hence no value of z is possible.

Thus, if $\Delta = 0$ and any of $\Delta_1, \Delta_2, \Delta_3$ is non-zero. Then, the system has no solution i.e., equations are **inconsistent.**

(b) **When $\Delta_1 = \Delta_2 = \Delta_3 = 0$**

In this case, $$\left.\begin{aligned} \Delta_1 &= x\Delta \\ \Delta_2 &= y\Delta \\ \Delta_3 &= z\Delta \end{aligned}\right\}$$ will be true for all values of x, y and z.

But, since $a_1x + b_1y + c_1z = d_1$, therefore only two of x, y and z will be independent and third will be dependent on the other two.

Thus, the system will have infinite number of solutions i.e., equations are **consistent**.

Remark

1. If $\Delta \neq 0$, the system will have unique finite solution and so equations are consistent.
2. If $\Delta = 0$ and atleast one of Δ_1, Δ_2, Δ_3 be non-zero, then the system has no solution i.e., equations are inconsistent.
3. If $\Delta = \Delta_1 = \Delta_2 = \Delta_3 = 0$, the equations will have infinite number of solutions i.e. equations are consistent.

Example 29. Solve the following system of equations by Cramer's rule.

$$x + y = 4 \text{ and } 3x - 2y = 9$$

Sol. Here, $\Delta = \begin{vmatrix} 1 & 1 \\ 3 & -2 \end{vmatrix} = -2 - 3 = -5 \neq 0$

$$\Delta_1 = \begin{vmatrix} 4 & 1 \\ 9 & -2 \end{vmatrix} = -8 - 9 = -17$$

and $\Delta_2 = \begin{vmatrix} 1 & 4 \\ 3 & 9 \end{vmatrix} = 9 - 12 = -3$

Then, by Cramer's rule

$$x = \frac{\Delta_1}{\Delta} = \frac{-17}{-5} = \frac{17}{5} \text{ and } y = \frac{\Delta_2}{\Delta} = \frac{-3}{-5} = \frac{3}{5}$$

$\therefore \quad x = \frac{17}{5}, y = \frac{3}{5}$

Example 30. Solve the following system of equations by Cramer's rule.

$$x + y + z = 9$$
$$2x + 5y + 7z = 52$$
$$2x + y - z = 0$$

Sol. Here, $\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 5 & 7 \\ 2 & 1 & -1 \end{vmatrix}$

Applying $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$, then

$$= \begin{vmatrix} 1 & 0 & 0 \\ 2 & 3 & 5 \\ 2 & -1 & -3 \end{vmatrix}$$

Expanding along R_1, then

$$\Delta = 1\begin{vmatrix} 3 & 5 \\ -1 & -3 \end{vmatrix} = -9 + 5 = -4 \neq 0, \quad \Delta_1 = \begin{vmatrix} 9 & 1 & 1 \\ 52 & 5 & 7 \\ 0 & 1 & -1 \end{vmatrix}$$

Applying $C_2 \to C_2 + C_3$, then

$$\Delta_1 = \begin{vmatrix} 9 & 2 & 1 \\ 52 & 12 & 7 \\ 0 & 0 & -1 \end{vmatrix}$$

Expanding along R_3, then

$$\Delta_1 = (-1)\begin{vmatrix} 9 & 2 \\ 52 & 12 \end{vmatrix} = -(108 - 104) = -4$$

$\Rightarrow \quad \Delta_2 = \begin{vmatrix} 1 & 9 & 1 \\ 2 & 52 & 7 \\ 2 & 0 & -1 \end{vmatrix}$

Applying $C_1 \to C_1 + 2C_3$, then

$$\Delta_2 = \begin{vmatrix} 3 & 9 & 1 \\ 16 & 52 & 7 \\ 0 & 0 & -1 \end{vmatrix}$$

Expanding along R_3, then

$$\Delta_2 = (-1)\begin{vmatrix} 3 & 9 \\ 16 & 52 \end{vmatrix} = -(156 - 144) = -12 \text{ and } \Delta_3 = \begin{vmatrix} 1 & 1 & 9 \\ 2 & 5 & 52 \\ 2 & 1 & 0 \end{vmatrix}$$

Applying $C_1 \to C_1 - 2C_2$, then

$$\Delta_3 = \begin{vmatrix} -1 & 1 & 9 \\ -8 & 5 & 52 \\ 0 & 1 & 0 \end{vmatrix}$$

Expanding along R_3, then

$$\Delta_3 = (-1)\begin{vmatrix} -1 & 9 \\ -8 & 52 \end{vmatrix}$$
$$= -(-52 + 72) = -20$$

Then, by Cramer's rule

$$x = \frac{\Delta_1}{\Delta} = \frac{-4}{-4} = 1, \; y = \frac{-12}{-4} = 3$$

and $\quad z = \frac{\Delta_3}{\Delta} = \frac{-20}{-4} = 5$

$\therefore \quad x = 1, y = 3, z = 5$

Example 31. ***For what values of p and q, the system of equations***

$$x + y + z = 6$$
$$x + 2y + 3z = 10$$
$$x + 2y + pz = q \text{ has}$$

(i) unique solution?
(ii) an infinitely many solutions?
(iii) no solution?

Sol. Given equations are

$$x + y + z = 6 \quad \Rightarrow \quad x + 2y + 3z = 10$$
$$x + 2y + pz = q$$

$$\therefore \quad \Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & p \end{vmatrix} = (p-3) \Rightarrow \Delta_1 = \begin{vmatrix} 6 & 1 & 1 \\ 10 & 2 & 3 \\ q & 2 & p \end{vmatrix}$$

$$= 6(2p-6) - 1(10p-3q) + (20-2q)$$

$$= 2p + q - 16$$

$$\Delta_2 = \begin{vmatrix} 1 & 6 & 1 \\ 1 & 10 & 3 \\ 1 & q & p \end{vmatrix}$$

$$= 1(10p-3q) - 6(p-3) + 1(q-10) = 4p - 2q + 8$$

$$\text{and } \Delta_3 = \begin{vmatrix} 1 & 1 & 6 \\ 1 & 2 & 10 \\ 1 & 2 & q \end{vmatrix}$$

$$= 1(2q-20) - 1(q-10) + 6(2-2) = q - 10$$

(i) For unique solution, $A \neq 0 \Rightarrow p \neq 3, q \in R$

(ii) For infinitely many solutions, $\Delta = \Delta_1 = \Delta_2 = \Delta_3 = 0$

$\therefore \quad p = 3, q = 10$

(iii) For no solution, $\Delta = 0$ and atleast one of $\Delta_1, \Delta_2, \Delta_3$ is non-zero is $p = 3$ and $q \neq 10$.

Condition for Consistency of Three Linear Equations in Two Unknowns

Let us consider a system of linear equations in x and y

$$a_1x + b_1y + c_1 = 0 \quad \text{...(i)}$$

$$a_2x + b_2y + c_2 = 0 \quad \text{...(ii)}$$

$$a_3x + b_3y + c_3 = 0 \quad \text{...(iii)}$$

will be consistent, the values of x and y obtained from any two equations satisfy the third equation.

On solving Eqs. (ii) and (iii) by Cramer's rule, we have

$$\frac{x}{\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}} = \frac{y}{\begin{vmatrix} c_2 & a_2 \\ c_3 & a_3 \end{vmatrix}} = \frac{1}{\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}}$$

These values of x and y will satisfy Eq. (i), then

$$a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} + b_1 \begin{vmatrix} c_2 & a_2 \\ c_3 & a_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} = 0$$

$$\Rightarrow \quad a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} = 0$$

$$\therefore \quad \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

which is the required condition.

Remark

For consistency of three linear equations in two knowns, the number of solution is one.

Example 32. Find the value of λ, if the following equations are consistent

$$x + y - 3 = 0$$

$$(1+\lambda)x + (2+\lambda)y - 8 = 0$$

$$x - (1+\lambda)y + (2+\lambda) = 0$$

Sol. The given equations in two unknowns are consistent, then

$$\begin{vmatrix} 1 & 1 & -3 \\ (1+\lambda) & (2+\lambda) & -8 \\ 1 & -(1+\lambda) & (2+\lambda) \end{vmatrix} = 0$$

Applying $C_2 \to C_2 - C_1$ and $C_3 \to C_3 + 3C_1$, then

$$\begin{vmatrix} 1 & \cdots & 0 & \cdots & 0 \\ \vdots & & & & \\ (1+\lambda) & & 1 & & (3\lambda - 5) \\ \vdots & & & & \\ 1 & & -(2+\lambda) & & (5+\lambda) \end{vmatrix} = 0$$

Expanding along R_1, then

$$1 \cdot \begin{vmatrix} 1 & 3\lambda - 5 \\ -(2+\lambda) & (5+\lambda) \end{vmatrix} = 0$$

$$\Rightarrow \quad (5+\lambda) + (2+\lambda)(3\lambda - 5) = 0$$

$$\Rightarrow \quad 3\lambda^2 + 2\lambda - 5 = 0 \text{ or } (3\lambda + 5)(\lambda - 1) = 0$$

$$\therefore \quad \lambda = 1, -5/3$$

System of Homogeneous Linear Equations

Let us consider a system of homogeneous linear equations in three unknown x, y and z be

$$a_1x + b_1y + c_1z = 0 \quad \text{...(i)}$$

$$a_2x + b_2y + c_2z = 0 \quad \text{...(ii)}$$

$$a_3x + b_3y + c_3z = 0 \quad \text{...(iii)}$$

Here, $$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

***Case* I** If $\Delta \neq 0$, then $x = 0, y = 0, z = 0$ is the only solution of above system. This solution is called a **Trivial solution**.

***Case* II** If $\Delta = 0$, atleast one of x, y and z is non-zero. This solution is called a **Non-trivial solution**.

Explanation From Eqs. (ii) and (iii), we get

$$\frac{x}{(b_2c_3 - b_3c_2)} = \frac{y}{(c_2a_3 - c_3a_2)} = \frac{z}{(a_2b_3 - a_3b_2)}$$

or $$\frac{x}{\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}} = \frac{y}{\begin{vmatrix} c_2 & a_2 \\ c_3 & a_3 \end{vmatrix}} = \frac{z}{\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}} = k\text{[say]} \ (\neq 0)$$

$$\therefore \quad x = k\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}, y = k\begin{vmatrix} c_2 & a_2 \\ c_3 & a_3 \end{vmatrix} \text{ and } z = k\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

On putting these values of x, y and z in Eq. (i), we get

$$a_1\left\{k\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}\right\} + b_1\left\{k\begin{vmatrix} c_2 & a_2 \\ c_3 & a_3 \end{vmatrix}\right\} + c_1\left\{k\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}\right\} = 0$$

$$\Rightarrow \quad a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} = 0 \quad [\because k \neq 0]$$

or
$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0 \text{ or } \Delta = 0$$

This is the condition for system have **Non-trivial solution.**

Remark

1. If $\Delta \neq 0$, the given system of equations has **only zero** solution for all its variables, then the given equations are said to have **Trivial solution**.
2. If $\Delta = 0$, the given system of equations has **no solution** or **infinite solutions** for all its variables, then the given equations are said to have **Non-trivial solution**.

Example 33. *Find all values of λ for which the equations*

$$(\lambda - 1)x + (3\lambda + 1)y + 2\lambda z = 0$$
$$(\lambda - 1)x + (4\lambda - 2)y + (\lambda + 3)z = 0$$
$$2x + (3\lambda + 1)y + 3(\lambda - 1)z = 0$$

possess non-trivial solution and find the ratios $x : y : z$, where λ has the smallest of these values.

Sol. The given system of linear equations has non-trivial solution, then we must have

$$\begin{vmatrix} \lambda - 1 & 3\lambda + 1 & 2\lambda \\ \lambda - 1 & 4\lambda - 2 & \lambda + 3 \\ 2 & 3\lambda + 1 & 3(\lambda - 1) \end{vmatrix} = 0$$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, then

$$\begin{vmatrix} \lambda - 1 & 3\lambda + 1 & 2\lambda \\ 0 & \lambda - 3 & -\lambda + 3 \\ 3 - \lambda & 0 & \lambda - 3 \end{vmatrix} = 0$$

Applying $C_3 \to C_3 + C_2$, then

$$\begin{vmatrix} \lambda - 1 & 3\lambda + 1 & 5\lambda + 1 \\ & \vdots & \\ 0 \cdots & \lambda - 3 & \cdots 0 \\ & \vdots & \\ 3 - \lambda & 0 & \lambda - 3 \end{vmatrix} = 0$$

Expanding along R_2, we get

$$(\lambda - 3)\begin{vmatrix} \lambda - 1 & 5\lambda + 1 \\ 3 - \lambda & \lambda - 3 \end{vmatrix} = 0$$

$$\Rightarrow \quad (\lambda - 3)[(\lambda - 1)(\lambda - 3) - (3 - \lambda)(5\lambda + 1)] = 0$$

$$\Rightarrow \quad (\lambda - 3)^2 \cdot 6\lambda = 0$$

$$\therefore \quad \lambda = 0, 3$$

Here, smallest value of λ is 0.

$\therefore$ The first two equations can be written as $x - y = 0$ and $x + 2y - 3z = 0$.

Using Cramer's rule, we get

$$\frac{x}{\begin{vmatrix} -1 & 0 \\ 2 & -3 \end{vmatrix}} = \frac{y}{\begin{vmatrix} 0 & 1 \\ -3 & 1 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 1 & -1 \\ 1 & 2 \end{vmatrix}}$$

$$\Rightarrow \quad \frac{x}{3} = \frac{y}{3} = \frac{z}{3} \Rightarrow \frac{x}{1} = \frac{y}{1} = \frac{z}{1}$$

$$\therefore \quad x : y : z = 1 : 1 : 1$$

Example 34. *Given, $x = cy + bz, y = az + cx$ and $z = bx + ay$, where x, y and z are not all zero, prove that $a^2 + b^2 + c^2 + 2abc = 1$.*

Sol. The given equation can be rewritten as

$$x - cy - bz = 0$$
$$-cx + y - az = 0$$
$$-bx - ay + z = 0$$

Since, x, y and z are not all zero, the system will have non-trivial solution, if

$$\begin{vmatrix} 1 & -c & -b \\ -c & 1 & -a \\ -b & -a & 1 \end{vmatrix} = 0$$

Applying $C_2 \to C_2 + cC_1$ and $C_3 \to C_3 + bC_1$, then

$$\begin{vmatrix} 1 \cdots & 0 \cdots & 0 \\ \vdots & & \\ -c & 1 - c^2 & -a - bc \\ \vdots & & \\ -b & -a - bc & 1 - b^2 \end{vmatrix} = 0$$

Expanding along R_1, we get

$$1\begin{vmatrix} 1 - c^2 & -a - bc \\ -a - bc & 1 - b^2 \end{vmatrix} = 0$$

$$\Rightarrow \quad (1 - c^2)(1 - b^2) - (a + bc)^2 = 0$$

$$\Rightarrow \quad 1 - b^2 - c^2 + b^2c^2 - a^2 - b^2c^2 - 2abc = 0$$

$$\Rightarrow \quad a^2 + b^2 + c^2 + 2abc = 1$$

Exercise for Session 3

1. Number of second order determinants which have maximum values whose each entry is either –1or 1 is equal to

(a) 2 (b) 4 (c) 6 (d) 8

2. Minimum value of a second order determinant whose each entry is either 1 or 2 is equal to

(a) 0 (b) –1 (c) –2 (d) –3

3. If $l_i^2 + m_i^2 + n_i^2 = 1, (i = 1, 2, 3)$ and $l_i l_j + m_i m_j + n_i n_j = 0, (i \neq j; i, j = 1, 2, 3)$ and $\Delta = \begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix}$, then

(a) $|\Delta| = 3$ (b) $|\Delta| = 2$ (c) $|\Delta| = 1$ (d) $|\Delta| = 0$

4. Let $\Delta_0 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ and Δ_1 denotes the determinant formed by the cofactors of elements of Δ_0 and Δ_2 denote the determinant formed by the cofactors of Δ_1 and so on. Δ_n denotes the determinant formed by the cofactors of Δ_{n-1}, the determinant value of Δ_n is

(a) Δ_0^{2n} (b) $\Delta_0^{2^n}$ (c) $\Delta_0^{n^2}$ (d) Δ_0^2

5. If $\begin{vmatrix} 1 & x & x^2 \\ x & x^2 & 1 \\ x^2 & 1 & x \end{vmatrix} = 3$, then the value of $\begin{vmatrix} x^3 - 1 & 0 & x - x^4 \\ 0 & x - x^4 & x^3 - 1 \\ x - x^4 & x^3 - 1 & 0 \end{vmatrix}$, is

(a) 6 (b) 9 (c) 18 (d) 27

6. The value of the determinant $\begin{vmatrix} (a_1 - b_1)^2 & (a_1 - b_2)^2 & (a_1 - b_3)^2 & (a_1 - b_4)^2 \\ (a_2 - b_1)^2 & (a_2 - b_2)^2 & (a_2 - b_3)^2 & (a_2 - b_4)^2 \\ (a_3 - b_1)^2 & (a_3 - b_2)^2 & (a_3 - b_3)^2 & (a_3 - b_4)^2 \\ (a_4 - b_1)^2 & (a_4 - b_2)^2 & (a_4 - b_3)^2 & (a_4 - b_4)^2 \end{vmatrix}$, is

(a) depends on $a_i, i = 1, 2, 3, 4$ (b) depends on $b_i, i = 1, 2, 3, 4$ (c) depends on $c_i, i = 1, 2, 3, 4$ (d) 0

7. Value of $\begin{vmatrix} 1 + x_1 & 1 + x_1 x & 1 + x_1 x^2 \\ 1 + x_2 & 1 + x_2 x & 1 + x_2 x^2 \\ 1 + x_3 & 1 + x_3 x & 1 + x_3 x^2 \end{vmatrix}$ depends upon

(a) only x (b) only x_1 (c) only x_2 (d) None of these

8. If the system of linear equations $x + y + z = 6$, $x + 2y + 3z = 14$ and $2x + 5y + \lambda z = \mu \ (\lambda, \mu \in R)$ has a unique solution, then

(a) $\lambda \neq 8$ (b) $\lambda = 8$ and $\mu \neq 36$ (c) $\lambda = 8$ and $\mu = 36$ (d) None of these

9. The system of equations $ax - y - z = a - 1,\ x - ay - z = a - 1,\ x - y - az = a - 1$ has no solution, if a is

(a) either –2 or 1 (b) –2 (c) 1 (d) not (–2)

10. The system of equations $x + 2y - 4z = 3$, $2x - 3y + 2z = 5$ and $x - 12y + 16z = 1$ has

(a) inconsistent solution (b) unique solution (c) infinitely many solutions (d) None of these

11. If $c < 1$ and the system of equations $x + y - 1 = 0$, $2x - y - c = 0$ and $-bx + 3by - c = 0$ is consistent, then the possible real values of b are

(a) $b \in \left(-3, \frac{3}{4}\right)$ (b) $b \in \left(-\frac{3}{2}, 4\right)$ (c) $b \in \left(-\frac{3}{4}, 3\right)$ (d) None of these

12. The equations $x + 2y = 3$, $y - 2x = 1$ and $7x - 6y + a = 0$ are consistent for

(a) $a = 7$ (b) $a = 1$ (c) $a = 11$ (d) None of these

13. Values of k for which the system of equations $x + ky + 3z = 0$, $kx + 2y + 2z = 0$ and $2x + 3y + 4z = 0$ possesses non-trivial solution

(a) $\left\{2, \frac{5}{4}\right\}$ (b) $\left\{-2, \frac{5}{4}\right\}$ (c) $\left\{2, -\frac{5}{9}\right\}$ (d) $\left\{-2, -\frac{5}{4}\right\}$

Answers

Exercise for Session 3

1. (b)	**2.** (c)	**3.** (c)	**4.** (b)	**5.** (b)	**6.** (d)
7. (d)	**8.** (a)	**9.** (b)	**10.** (c)	**11.** (c)	**12.** (a)
13. (a)					