Mathematical Induction Exercise 1 :
Single Option Correct Type Questions

- This section contains 3 **multiple choice questions**. Each question has four choices (a), (b), (c) and (d), out of which **ONLY ONE** is correct.

1. If $a_n = \sqrt{7 + \sqrt{7 + \sqrt{7 + \dots}}}$ having n radical signs. Then, by mathematical induction which one is true?
(a) $a_n > 7, \forall\ n \ge 1$
(b) $a_n > 3, \forall\ n \ge 1$
(c) $a_n < 4, \forall\ n \ge 1$
(d) $a_n < 3, \forall\ n \ge 1$

2. If $P(n) = 2 + 4 + 6 + \dots + 2n, n \in N$, then
$P(k) = k(k+1) + 2$
$\Rightarrow P(k+1) = (k+1)(k+2) + 2, \forall\ k \in N$. So, we can conclude that $P(n) = n(n+1) + 2$ for
(a) all $n \in N$ (b) $n > 1$
(c) $n > 2$ (d) Nothing can be said

3. The value of the natural number n such that the inequality $2^n > 2n + 1$ is valid, is
(a) for $n \ge 3$ (b) for $n < 3$ (c) for all n (d) for mn

Mathematical Induction Exercise 2 :
Statement I and II Type Questions

- **Directions** Question Number 4 to 6 Assertion-Reason type questions. Each of these questions contains two statements.
Statement-1 (Assertion) and
Statement-2 (Reason)
Each of these questions also four alternative choices, only one of which is the correct answer. You have to select the correct choice as given below:
(a) Statement-1 is true, Statement-2 is true; Statement-2 is correct explanation for Statement-1
(b) Statement-1 is true, Statement-2 is true; Statement-2 is not correct explanation for Statement-1
(c) Statement-1 is true, Statement-2 is false
(d) Statement-1 is false, Statement-2 is true

4. **Statement-1** If $a_1 = 1, a_2 = 5$, then $a_n = 3^n - 2^n, \forall\ n \in N$ and $n \ge 1$.
Statement-2 $a_{n+2} = 5a_{n+1} - 6a_n, n \ge 1$.

5. **Statement-1** For all natural numbers n, $2 \cdot 7^n + 3 \cdot 5^n - 5$ is divisible by 24.
Statement-2 If $f(x)$ is divisible by x, then $f(x+1) - f(x)$ is divisible by $x + 1, \forall\ x \in N$.

6. **Statement-1** For all natural numbers n,
$0 \cdot 5 + 0 \cdot 55 + 0 \cdot 555 + \dots$ upto n terms$= \frac{5}{9}\left\{n - \frac{1}{9}\left(1 - \frac{1}{10^n}\right)\right\}$
Statement-2 $a + ar + ar^2 + \dots + ar^{n-1} = \frac{a(1 - r^n)}{(1 - r)}$, for $0 < r < 1$.

Mathematical Induction Exercise 3 :
Subjective Type Questions

- In this section, there are **10 subjective** questions.

7. Prove the following by using induction for all $n \in N$.
(i) $11^{n+2} + 12^{2n+1}$ is divisible by 133.
(ii) $n^7 - n$ is divisible by 42.
(iii) $3^{2n} + 24n - 1$ is divisible by 32.
(iv) $n(n+1)(n+5)$ is divisible by 6.
(v) $(25)^{n+1} - 24n + 5735$ is divisible by $(24)^2$.
(vi) $x^{2n} - y^{2n}$ is divisible by $x + y$.

8. Prove by induction that if n is a positive integer not divisible by 3, then $3^{2n} + 3^n + 1$ is divisible by 13.

9. Prove by induction that the product of three consecutive positive integers is divisible by 6.

10. Prove by induction that the sum of three successive natural numbers is divisible by 9.

11. Prove by induction that the even power of every odd integer when divided by 8 leaves the remainder 1.

12. Prove the following by using induction for all $n \in N$:

(i) $1 + 2 + 3 + \ldots + n = \dfrac{n(n+1)}{2}$

(ii) $1^2 + 2^2 + 3^2 + \ldots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$

(iii) $1 \cdot 3 + 3 \cdot 5 + 5 \cdot 7 + \ldots + (2n-1)(2n+1) = \dfrac{n(4n^2 + 6n - 1)}{3}$

(iv) $\dfrac{1}{2 \cdot 5} + \dfrac{1}{5 \cdot 8} + \dfrac{1}{8 \cdot 11} + \ldots + \dfrac{1}{(3n-1)(3n+2)} = \dfrac{n}{6n+4}$

(v) $1 \cdot 4 \cdot 7 + 2 \cdot 5 \cdot 8 + 3 \cdot 6 \cdot 9 + \ldots$ upto n terms $= \dfrac{n}{4}(n+1)(n+6)(n+7)$

(vi) $\dfrac{1^2}{1 \cdot 3} + \dfrac{2^2}{3 \cdot 5} + \ldots + \dfrac{n^2}{(2n-1)(2n+1)} = \dfrac{n(n+1)}{2(2n+1)}$

13. Let $a_0 = 2, a_1 = 5$ and for $n \geq 2, a_n = 5a_{n-1} - 6a_{n-2}$, then prove by induction that $a_n = 2^n + 3^n, \forall\, n \geq 0, n \in N$.

14. If $a_1 = 1, a_{n+1} = \dfrac{1}{n+1} a_n, n \geq 1$, then prove by induction that $a_{n+1} = \dfrac{1}{(n+1)!}, n \in N$.

15. If a, b, c, d, e and f are six real numbers such that

$$a + b + c = d + e + f$$

$$a^2 + b^2 + c^2 = d^2 + e^2 + f^2$$

and $a^3 + b^3 + c^3 = d^3 + e^3 + f^3$, prove by mathematical induction that

$a^n + b^n + c^n = d^n + e^n + f^n, \forall\, n \in N$.

16. Using mathematical induction, prove that

$$\tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{7}\right) + \ldots + \tan^{-1}\left(\frac{1}{n^2+n+1}\right) = \tan^{-1}\left(\frac{n}{n+2}\right).$$

Mathematical Induction Exercise 4 :

Questions Asked in Previous 13 Year's Exam

- This section contains questions asked in **IIT-JEE, AIEEE, JEE Main** & **JEE Advanced** from year **2005** to year **2017**.

17. Statement-1 For every natural number $n \geq 2 \dfrac{1}{\sqrt{1}} + \dfrac{1}{\sqrt{2}} + \ldots + \dfrac{1}{n} > \sqrt{n}$,

Statement-2 For every natural number $n \geq 2 \sqrt{n(n+1)} < n+1$

(a) Statement-1 is true, Statement-2 is true; Statement-2 is correct explanation for Statement-1

(b) Statement-1 is true, Statement-2 is true; Statement-2 is not a correct explanation for Statement-1

(c) Statement-1 is true, Statement-2 is false

(d) Statement-1 is false, Statement-2 is true **[AIEEE 2008, 3M]**

18. Statement-1 For each natural number $n, (n+1)^7 - n^7 - 1$ is divisible by 7.

Statement-2 For each natural number $n, n^7 - n$ is divisible by 7.

(a) Statement-1 is false, Statement-2 is true

(b) Statement-1 is true, Statement-2 is true; Statement-2 is correct explanation for Statement-1

(c) Statement-1 is true, Statement-2 is true; Statement-2 is not a correct explanation for Statement-1

(d) Statement-1 is true, Statement-2 is false **[AIEEE 2011, 4M]**

Answers

Chapter Exercise

1. (c) **2.** (d) **3.** (a) **4.** (a) **5.** (c) **6.** (b) **17.** (b) **18.** (b)

Solutions

1. Let $P(n): a_n = \sqrt{7+\sqrt{7+\sqrt{7+\dots}}}$ (n radical sign)

***Step* I** For $n = 1$,

$$P(1): a_1 = \sqrt{7} < 4$$

***Step* II** Assume that $a_k < 4$ for all natural number, $n = k$

***Step* III** For $n = k+1$,

$$P(k+1): a_{k+1} = \sqrt{7+\sqrt{7+\sqrt{7+\dots}}}$$

[($k+1$) radical sign]

$$= \sqrt{7+a_k} < \sqrt{7+4} \quad [\because a_k < 4]$$

$$< 4 \quad \text{[by assumption]}$$

This shows that, $a_{k+1} < 4$, i.e. the result is true for $n = k+1$.
Hence, by the principle of mathematical induction

$$a_n < 4, \forall n \geq 1$$

2. It is obvious.

3. Check through options, the condition $2^n > 2n+1$ is valid for $n \geq 3$.

4. Let $P(n): a_n = 3^n - 2^n$

***Step* I** For $n = 1$,

LHS $= a_1 = 1$ [given]

and RHS $= 3^1 - 2^1 = 1$

$\therefore$ LHS = RHS

Hence, $P(1)$ is true.

For $n = 2$,

LHS $= a_2 = 5$ [given]

and RHS $= 3^2 - 2^2 = 5$

$\therefore$ LHS = RHS

Hence, $P(2)$ is also true.

Thus, $P(1)$ and $P(2)$ are true.

***Step* II** Let $P(k)$ and $P(k-1)$ are true

$\therefore$ $a_k = 3^k - 2^k$ and $a_{k-1} = 3^{k-1} - 2^{k-1}$

***Step* III** For $n = k+1$,

$$a_{k+1} = 5a_k - 6a_{k-1} \quad \text{[from Statement-2]}$$
$$= 5(3^k - 2^k) - 6(3^{k-1} - 2^{k-1})$$
$$= 5\cdot 3^k - 5\cdot 2^k - 2\cdot 3^k + 3\cdot 2^k$$
$$= 3\cdot 3^k - 2\cdot 2^k = 3^{k+1} - 2^{k+1}$$

which is true for $n = k+1$.

Hence, both statements are true and Statement-2 is a correct explanation of Statement-1.

5. Let $P(n): 2\cdot 7^n + 3\cdot 5^n - 5$

***Step* I** For $n = 1$,

$$P(1): 2\cdot 7^1 + 3\cdot 5^1 - 5$$

: 24 is divisible by 24.

***Step* II** Assume $P(k)$ is divisible by 24, then

$P(k): 2\cdot 7^k + 3\cdot 5^k - 5 = 24\lambda$, λ is positive integer.

***Step* III** For $n = k+1$,

$$P(k+1) - P(k) = (2\cdot 7^{k+1} + 3\cdot 5^{k+1} - 5) - (2\cdot 7^k + 3\cdot 5^k - 5)$$
$$= 2\cdot 7^k(7-1) + 3\cdot 5^k(5-1)$$
$$= 12(7^k + 5^k)$$
$$= \text{divisible by } 24$$
$$= 24\mu, \forall \mu \in I$$

[$\because 7^k + 5^k$ is always divisible by 24]

$$\therefore \quad P(k+1) = P(k) + 24\mu = 24\lambda + 24\mu$$
$$= 24(\lambda + \mu)$$

Hence, $P(k+1)$ is divisible by 24.

Hence, Statement-1 is true and Statement-2 is false.

6. ***Step* I** For $n = 1$,

LHS = 0.5 and RHS $= \frac{5}{9}\left\{1 - \frac{1}{9}\left(1 - \frac{1}{10}\right)\right\} = \frac{5}{9}\left(1 - \frac{1}{10}\right) = \frac{5}{10} = 0.5$

$\therefore$ LHS = RHS

which is true for $n = 1$.

***Step* II** Assume it is true for $n = k$, then $0.5 + 0.55 + 0.555 + \dots$ + upto k terms

$$= \frac{5}{9}\left\{k - \frac{1}{9}\left(1 - \frac{1}{10^k}\right)\right\}$$

***Step* III** For $n = k+1$,

LHS $= 0.5 + 0.55 + 0.555 + \dots$ + upto $(k+1)$ terms

$$= \frac{5}{9}\left\{k - \frac{1}{9}\left(1 - \frac{1}{10^k}\right)\right\} + (k+1) \text{ th terms}$$

$$= \frac{5}{9}\left\{k - \frac{1}{9}\left(1 - \frac{1}{10^k}\right)\right\} + \underbrace{0.555\dots5}_{(k+1)\text{ times}}$$

$$= \frac{5}{9}\left\{k - \frac{1}{9}\left(1 - \frac{1}{10^k}\right)\right\} + \frac{1}{10^{k+1}}\left(\underbrace{555\dots5}_{(k+1)\text{ times}}\right)$$

$$= \frac{5}{9}\left\{k - \frac{1}{9}\left(1 - \frac{1}{10^k}\right)\right\} + \frac{5}{10^{k+1}}(1 + 10 + 10^2 + \dots + 10^k)$$

$$= \frac{5}{9}\left\{k - \frac{1}{9}\left(1 - \frac{1}{10^k}\right)\right\} + \frac{5\cdot(10^{k+1}-1)}{10^{k+1}\cdot(10-1)}$$

$$= \frac{5}{9}\left\{k - \frac{1}{9}\left(1 - \frac{1}{10^k}\right) + \frac{10^{k+1}-1}{10^{k+1}}\right\}$$

$$= \frac{5}{9}\left\{(k+1) - \frac{1}{9} + \frac{1}{9\cdot 10^k} - \frac{1}{10^{k+1}}\right\}$$

$$= \frac{5}{9}\left\{(k+1) - \frac{1}{9} + \frac{(10-9)}{9\cdot 10^{k+1}}\right\}$$

$$= \frac{5}{9}\left\{(k+1) - \frac{1}{9}\left(1 - \frac{1}{10^{k+1}}\right)\right\} = \text{RHS}$$

which is true for $n = k+1$.

Hence, both statements are true but Statement-2 is not a correct explanation for Statement-1.

7. (i) Let $P(n) = 11^{n+2} + 12^{2n+1}$

***Step* I** For $n = 1$,

$$P(1) = 11^{1+2} + 12^{2\times1+1} = 11^3 + 12^3$$
$$= (11 + 12)(11^2 - 11 \times 12 + 12^2)$$
$$= 23 \times 133 \text{, which is divisible by 133.}$$

Therefore, the result is true for $n = 1$.

***Step* II** Assume that the result is true for $n = k$, then

$$P(k) = 11^{k+2} + 12^{2k+1} \text{ is divisible by 133.}$$
$$\Rightarrow \quad P(k) = 133r, \text{ where } r \text{ is an integer.}$$

***Step* III** For $n = k + 1$,

$$\therefore\ P(k+1) = 11^{(k+1)+2} + 12^{2(k+1)+1} = 11^{k+3} + 12^{2k+3}$$
$$= 11^{(k+1)+1} \cdot 11 + 12^{2k+1} \cdot 12^2$$
$$= 11 \cdot 11^{k+2} + 144 \cdot 12^{2k+1}$$

Now, $11^{k+2} + 12^{2k+1}\,\big)\,11 \cdot 11^{k+2} + 144 \cdot 12^{2k+1}\,\big(\,11$

$$11 \cdot 11^{k+2} + 11 \cdot 12^{2k+1}$$
$$- \qquad -$$
$$133 \cdot 12^{2k+1}$$

$$\therefore\ 11 \cdot 11^{(k+2)} + 144 \cdot 12^{2k+1}$$
$$= 11(11^{k+2} + 12^{2k+1}) + 133 \cdot 12^{2k+1}$$

i.e. $\quad P(k+1) = 11\,P(k) + 133 \cdot 12^{2k+1}$

But we know that, $P(k)$ is divisible by 133. Also, $133 \cdot 12^{2k+1}$ is divisible by 133.

Hence, $P(k + 1)$ is divisible by 133. This shows that, the result is true for $n = k + 1$.

Hence, by the principle of mathematical induction, the result is true for all $n \in N$.

(ii) Let $P(n) = n^7 - n$

***Step* I** For $n = 1$,

$$P(1) = 1^7 - 1 = 0, \text{ which is divisible by 42.}$$

Therefore, the result is true for $n = 1$.

***Step* II** Assume that the result is true for $n = k$. Then,

$$P(k) = k^7 - k \text{ is divisible by 42.}$$
$$\Rightarrow \quad P(k) = 42r, \text{ where } r \text{ is an integer.}$$

***Step* III** For $n = k + 1$,

$$P(k+1) = (k+1)^7 - (k+1)$$
$$= (1+k)^7 - (k+1)$$
$$= 1 + {}^7C_1 k + {}^7C_2 k^2 + {}^7C_3 k^3 + {}^7C_4 k^4 + {}^7C_5 k^5 + {}^7C_6 k^6 + {}^7C_7 k^7 - (k+1)$$
$$= (k^7 - k) + ({}^7C_1 k + {}^7C_2 k^2 + {}^7C_3 k^3 + {}^7C_4 k^4 + {}^7C_5 k^5 + {}^7C_6 k^6)$$

But by assumption $k^7 - k$ is divisible by 42.

Also, ${}^7C_1 k + {}^7C_2 k^2 + {}^7C_3 k^3 + {}^7C_4 k^4 + {}^7C_5 k^5 + {}^7C_6 k^6$ is divisible by 42. $\quad [\because\ {}^7C_r, 1 \le r \le 6 \text{ is divisible by } 7]$

Hence, $P(k + 1)$ is divisible by 42. This shows that, the result is true for $n = k + 1$.

$\therefore$ By the principle of mathematical induction, the result is true for all $n \in N$.

(iii) Let $P(n) = 3^{2n} + 24n - 1$

***Step* I** For $n = 1$,

$$P(1) = 3^{2\times1} + 24 \times 1 - 1 = 3^2 + 24 - 1 = 9 + 24 - 1$$
$$= 32, \text{ which is divisible by 32.}$$

Therefore, the result is true for $n = 1$.

***Step* II** Assume that the result is true for $n = k$. Then,

$$P(k) = 3^{2k} + 24k - 1 \text{ is divisible by 32.}$$
$$\Rightarrow \quad P(k) = 32r, \text{ where } r \text{ is an integer.}$$

***Step* III** For $n = k + 1$,

$$P(k+1) = 3^{2(k+1)} + 24(k+1) - 1$$
$$= 3^{2k+2} + 24k + 24 - 1$$
$$= 3^2 \cdot 3^{2k} + 24k + 23$$
$$= 9 \cdot 3^{2k} + 24k + 23$$

Now, $3^{2k} + 24k - 1\,\big)\,9 \times 3^{2k} + 24k + 23\,\big(\,9$

$$9 \cdot 3^{2k} + 216k - 9$$
$$- \quad - \quad +$$
$$-192k + 32$$

$$\therefore \quad P(k+1) = 9(3^{2k} + 24k - 1) - 32(6k - 1)$$
$$= 9\,P(k) - 32(6k - 1)$$
$$\therefore \quad P(k+1) = 9(32r) - 32(6k - 1) \quad \text{[by assumption step]}$$
$$= 32(9r - 6k + 1),$$

which is divisible by 32, as $9r - 6k + 1$ is an integer.

Therefore, $P(k + 1)$ is divisible by 32. Hence, by the principle of mathematical induction $P(n)$ is divisible by 32, $\forall\ n \in N$.

(iv) Let $\quad P(n) = n(n+1)(n+5)$

***Step* I** For $n = 1$,

$$P(1) = 1 \cdot (1+1)(1+5) = 1 \cdot 2 \cdot 6$$
$$= 12, \text{ which is divisible by 6.}$$

Therefore, the result is true for $n = 1$.

Step II Assume that the result is true for $n = k$. Then,

$$\Rightarrow \left.\begin{array}{l} P(k) = k(k+1)(k+5) \text{ is divisible by 6.} \\ P(k) = 6r, r \text{ is an integer.} \end{array}\right\}$$

Step III For $n = k + 1$,

$$P(k+1) = (k+1)(k+1+1)(k+1+5)$$
$$= (k+1)(k+2)(k+6)$$

Now, $P(k+1) - P(k) = (k+1)(k+2)(k+6) - k(k+1)(k+5)$

$$= (k+1)\{k^2 + 8k + 12 - k^2 - 5k\}$$
$$= (k+1)(3k+12)$$
$$= 3(k+1)(k+4)$$
$$\Rightarrow \quad P(k+1) = P(k) + 3(k+1)(k+4)$$

which is divisible by 6 as $P(k)$ is divisible by 6 [by assumption step]

and clearly $3(k + 1)(k + 4)$ is divisible by 6, $\forall\ k \in N$.

Hence, the result is true for $n = k + 1$.

Therefore, by the principle of mathematical induction, the result is true for all $n \in N$.

(v) Let $P(n) = (25)^{n+1} - 24n + 5735$

***Step* I** For $n = 1$,

$$P(1) = (25)^2 - 24 + 5735 = 625 - 24 + 5735 = 6336$$
$$= 11 \times (24)^2, \text{ which is divisible by } (24)^2.$$

Therefore, the result is true for $n = 1$.

***Step* II** Assume that the result is true for $n = k$. Then,

$P(k) = (25)^{k+1} - 24k + 5735$ is divisible by $(24)^2$.

$\Rightarrow \quad P(k) = (24)^2 r$, where r is an integer.

***Step* III** For $n = k + 1$,

$$P(k+1) = (25)^{(k+1)+1} - 24(k+1) + 5735$$
$$= (25)^{k+2} - 24k + 5711$$
$$= (25)(25)^{k+1} - 24k + 5711$$

Now, $P(k+1) - P(k)$

$$= \{(25)(25)^{k+1} - 24k + 5711\} - \{(25)^{k+1} - 24k + 5735\}$$
$$= (24)(25)^{k+1} - 24$$
$$= 24\{(25)^{k+1} - 1\}$$

$$\Rightarrow \quad P(k+1) = P(k) + 24\{(25)^{k+1} - 1\}$$

But by assumption $P(k)$ is divisible by $(24)^2$. Also, $24\{(25)^{k+1} - 1\}$ is clearly divisible by $(24)^2$, $\forall\, k \in N$. This shows that, the result is true for $n = k + 1$.

Hence, by the principle of mathematical induction, result is true for all $n \in N$.

(vi) Let $P(n) = x^{2n} - y^{2n}$

***Step* I** For $n = 1$,

$$P(1) = x^2 - y^2$$
$$= (x - y)(x + y) \text{ which is divisible by } (x + y).$$

Therefore, the result is true for $n = 1$.

***Step* II** Assume that the result is true for $n = k$. Then,

$P(k) = x^{2k} - y^{2k}$ is divisible by $x + y$.

$\Rightarrow \quad P(k) = (x + y)r$, where r is an integer.

***Step* III** For $n = k + 1$,

$$= x^2 \cdot x^{2k} - y^2 \cdot y^{2k}$$
$$= x^2x^{2k} - x^2y^{2k} + x^2y^{2k} - y^2y^{2k}$$
$$= x^2(x^{2k} - y^{2k}) + y^{2k}(x^2 - y^2)$$
$$= x^2(x + y)r + y^{2k}(x - y)(x + y)$$

[by assumption step]

$$= (x + y)\{x^2r + y^{2k}(x - y)\}$$

which is divisible by $(x + y)$ as $x^2r + y^{2k}(x - y)$ is an integer.

This shows that the result is true for $n = k + 1$. Hence, by the principle of mathematical induction, the result is true for all $n \in N$.

8. Let $P(n) = 3^{2n} + 3^n + 1$, $\forall\, n$ is a positive integer not divisible by 3.

***Step* I** For $n = 1$,

$$P(1) = 3^2 + 3 + 1 = 9 + 3 + 1$$
$$= 13, \text{ which is divisible by } 13.$$

Therefore, $P(1)$ is true.

***Step* II** Assume $P(n)$ is true for $n = k$, k is a positive integer not divisible by 3, then

$P(k) = 3^{2k} + 3^k + 1$, is divisible by 13.

$\Rightarrow \quad P(k) = 13r$, where r is an integer.

***Step* III** For $n = k + 1$,

$$P(k+1) = 3^{2(k+1)} + 3^{k+1} + 1$$
$$= 3^2 \cdot 3^{2k} + 3 \cdot 3^k + 1$$

Now, $\quad 3^{2k} + 3^k + 1 \;\big)\; 3^2 \cdot 3^{2k} + 3 \cdot 3^k + 1 \;\big(\; 3^2$

$\qquad 3^2 \cdot 3^{2k} + 3^2 \cdot 3^k + 3^2$

$\qquad -\qquad -\qquad -$

$\qquad -6 \cdot 3^k - 8$

$$\Rightarrow \quad P(k+1) = 3^2(3^{2k} + 3^k + 1) - 6 \cdot 3^k - 8$$
$$= 9\,P(k) - 2(3^{k+1} + 4)$$
$$= 9(13r) - 2(3^{k+1} + 4) \qquad \text{[by assumption step]}$$

which is divisible by 13 as $3^{k+1} + 4$ is also divisible by 13, $\forall\, k \in N$ and not divisible by 3. This shows that the result is true for $n = k + 1$. Hence, by the principle of mathematical induction, the result is true for all natural numbers not divisible by 3.

9. Let $P(n) = n(n+1)(n+2)$, where n is a positive integer.

***Step* I** For $n = 1$,

$$P(1) = 1(1 + 1)(1 + 2) = 1 \cdot 2 \cdot 3$$
$$= 6, \text{ which is divisible by } 6.$$

Therefore, the result is true for $n = 1$.

***Step* II** Let us assume that the result is true for $n = k$, where k is a positive integer.

Then, $\quad P(k) = k(k+1)(k+2)$ is divisible by 6.

$\Rightarrow \quad P(k) = 6r$, where r is an integer.

$\big) a^2\; a^k + b^2\, b^k \;\big(a^2$ [infact positive integer]

***Step* III** For $n = k + 1$, where k is a positive integer.

$$P(k+1) = (k+1)(k+1+1)(k+2+1)$$
$$= (k+1)(k+2)(k+3)$$

Now, $P(k+1) - P(k) = (k+1)(k+2)(k+3) - k(k+1)(k+2)$

$$= (k+1)(k+2)(k+3-k)$$
$$= 3(k+1)(k+2)$$

$$\Rightarrow \quad P(k+1) = P(k) + 3(k+1)(k+2)$$

But we know that, $P(k)$ is divisible by 6. Also, $3(k+1)(k+2)$ is divisible by 6 for all positive integer. This shows that the result is true for $n = k + 1$. Hence, by the principle of mathematical induction, the result is true for all positive integer.

10. Let $P(n) = n^3 + (n+1)^3 + (n+2)^3$, where $n \in N$.

***Step* I** For $n = 1$,

$$P(1) = 1^3 + 2^3 + 3^3 = 1 + 8 + 27$$
$$= 36, \text{ which is divisible by } 9.$$

***Step* II** Assume that $P(n)$ is true for $n = k$, then

$P(k) = k^3 + (k+1)^3 + (k+2)^3$, where $k \in N$.

$\Rightarrow \quad P(k) = 9r$, where r is a positive integer.

***Step* III** For $n = k + 1$,

$$P(k+1) = (k+1)^3 + (k+2)^3 + (k+3)^3$$

Now, $P(k+1) - P(k) = (k+1)^3 + (k+2)^3 + (k+3)^3 - \{k^3 + (k+1)^3 + (k+2)^3\}$

$$= (k+3)^3 - k^3$$
$$= k^3 + 9k^2 + 27k + 27 - k^3$$
$$= 9(k^2 + 3k + 3)$$

$\Rightarrow \quad P(k+1) = P(k) + 9(k^2 + 3k + 3)$

$$= 9r + 9(k^2 + 3k + 3)$$
$$= 9(r + k^2 + 3k + 3)$$

which is divisible by 9 as $(r + k^2 + 3k + 3)$ is a positive integer. Hence, by the principle mathematical induction, $P(n)$ is divisible by 9 for all $n \in N$.

11. Let $P(n) : (2r+1)^{2n}$, $\forall\ n \in N$ and $r \in I$.

***Step* I** For $n = 1$,

$$P(1) : (2r+1)^2 = 4r^2 + 4r + 1$$
$$= 4r(r+1) + 1 = 8p + 1,\ p \in I$$

[$\because r(r+1)$ is an even integer]

Therefore, $P(1)$ is true.

***Step* II** Assume $P(n)$ is true for $n = k$, then

$P(k) : (2r+1)^{2k}$ is divisible by 8 leaves remainder 1.

$\Rightarrow \quad P(k) = 8m + 1, n \in I$, where m is a positive integer.

***Step* III** For $n = k + 1$,

$\therefore \quad P(k+1) = (2r+1)\,2(k+1)$

$$= (2r+1)^{2k}(2r+1)^2$$
$$= (8m+1)(8p+1) \qquad \text{[from Steps I and II]}$$
$$= 64\ mp + 8\ (m + p) + 1$$
$$= 8\ (8mp + m + p) + 1$$

which is true for $n = k + 1$ as $8mp + m + p$ is an integer. Hence, by the principle of mathematical induction, when $P(n)$ is divided by 8 leaves the remainder 1 for all $n \in N$.

12. (i) Let $P(n) : 1 + 2 + 3 + \ldots + n = \dfrac{n(n+1)}{2}$ ...(i)

***Step* I** For $n = 1$,

LHS of Eq. (i) $= 1$

RHS of Eq. (i) $= \dfrac{1(1+1)}{2} = 1$

LHS = RHS

Therefore, $P(1)$ is true.

***Step* II** Let us assume that the result is true for $n = k$. Then,

$$P(k) : 1 + 2 + 3 + \ldots + k = \frac{k(k+1)}{2}$$

***Step* III** For $n = k + 1$, we have to prove that

$$P(k+1) = 1 + 2 + 3 + \ldots + k + (k+1) = \frac{(k+1)(k+2)}{2}$$

$$\text{LHS} = 1 + 2 + 3 + \ldots + k + (k+1)$$
$$= \frac{k(k+1)}{2} + k + 1 \qquad \text{[by assumption step]}$$
$$= (k+1)\left(\frac{k}{2} + 1\right) = (k+1)\left(\frac{k+2}{2}\right)$$
$$= \frac{(k+1)(k+2)}{2}$$
$$= \text{RHS}$$

This shows that the result is true for $n = k + 1$. Therefore, by the principle of mathematical induction, the result is true for all $n \in N$.

(ii) Let $P(n) : 1^2 + 2^2 + 3^2 + \ldots + n^2 = \dfrac{(n+1)(2n+1)}{6}$ ...(i)

***Step* I** For $n = 1$,

LHS of Eq. (i) $= 1^2 = 1$

RHS of Eq. (i) $= \dfrac{1(1+1)(2 \times 1 + 1)}{6}$

$$= \frac{1 \cdot 2 \cdot 3}{6} = 1$$

LHS = RHS

Therefore, $P(1)$ is true.

***Step* II** Let us assume that the result is true for $n = k$. Then,

$$P(k) : 1^2 + 2^2 + 3^2 + \ldots + k^2 = \frac{k(k+1)(2k+1)}{6}$$

***Step* III** For $n = k + 1$, we have to prove that

$$P(k+1) : 1^2 + 2^2 + 3^2 + \ldots + k^2 + (k+1)^2 = \frac{(k+1)(k+2)(2k+3)}{6}$$

$$\text{LHS} = 1^2 + 2^2 + 3^2 + \ldots + k^2 + (k+1)^2$$
$$= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \qquad \text{[by assumption step]}$$
$$= (k+1)\left\{\frac{k(2k+1)}{6} + (k+1)\right\}$$
$$= (k+1)\left\{\frac{2k^2 + 7k + 6}{6}\right\}$$
$$= (k+1)\left\{\frac{(k+2)(2k+3)}{6}\right\} = \frac{(k+1)(k+2)(2k+3)}{6}$$
$$= \text{RHS}$$

This shows that the result is true for $n = k + 1$. Therefore, by the principle of mathematical induction, the result is true for all $n \in N$.

(iii) Let $P(n) : 1 \cdot 3 + 3 \cdot 5 + 5 \cdot 7 + \ldots + (2n-1)(2n+1)$

$$= \frac{n(4n^2 + 6n - 1)}{3} \qquad \text{...(i)}$$

***Step* I** For $n = 1$,

LHS of Eq. (i) $= 1 \cdot 3 = 3$

RHS of Eq. (i) $= \dfrac{1(4 \times 1^2 + 6 \times 1 - 1)}{3} = \dfrac{4 + 6 - 1}{3} = 3$

$\therefore \quad$ LHS = RHS

Therefore, $P(1)$ is true.

***Step* II** Assume that the result is true for $n = k$. Then,

$$P(k) : 1 \cdot 3 + 3 \cdot 5 + 5 \cdot 7 + \ldots + (2k-1)(2k+1) = \frac{k(4k^2 + 6k - 1)}{3}$$

***Step* III** For $n = k + 1$, we have to prove that

$$P(k+1): 1\cdot 3 + 3\cdot 5 + 5\cdot 7 + \ldots + (2k-1)(2k+1) + (2k+1)(2k+3)$$

$$= \frac{(k+1)[4(k+1)^2 + 6(k+1) - 1]}{3}$$

$$= \frac{(k+1)(4k^2 + 14k + 9)}{3}$$

$$\text{LHS} = 1\cdot 3 + 3\cdot 5 + 5\cdot 7 + \ldots + (2k-1)(2k+1) + (2k+1)(2k+3)$$

$$= \frac{k(4k^2 + 6k - 1)}{3} + (2k+1)(2k+3)$$ [by assumption step]

$$= \frac{4k^3 + 6k^2 - k}{3} + (4k^2 + 8k + 3)$$

$$= \frac{4k^3 + 18k^2 + 23k + 9}{3}$$

$$= \frac{(k+1)(4k^2 + 14k + 9)}{3} = \text{RHS}$$

This shows that the result is true for $n = k + 1$. Therefore, by the principle of mathematical induction, the result is true for all $n \in N$.

(iv) Let $P(n): \frac{1}{2\cdot 5} + \frac{1}{5\cdot 8} + \frac{1}{8\cdot 11} + \ldots + \frac{1}{(3n-1)(3n+2)}$

$$= \frac{n}{6n+4} \quad \ldots\text{(i)}$$

***Step* I** For $n = 1$,

LHS of Eq. (i) $= \frac{1}{2\cdot 5} = \frac{1}{10}$

RHS of Eq. (i) $= \frac{1}{6 \times 1 + 4} = \frac{1}{10}$

LHS = RHS

Therefore, $P(1)$ is true.

***Step* II** Let us assume that the result is true for $n = k$. Then,

$$P(k): \frac{1}{2\cdot 5} + \frac{1}{5\cdot 8} + \frac{1}{8\cdot 11} + \ldots + \frac{1}{(3k-1)(3k+2)} = \frac{k}{6k+4}$$

***Step* III** For $n = k + 1$, we have to prove that

$$P(k+1): \frac{1}{2\cdot 5} + \frac{1}{5\cdot 8} + \frac{1}{8\cdot 11} + \ldots + \frac{1}{(3k-1)(3k+2)} + \frac{1}{(3k+2)(3k+5)}$$

$$= \frac{(k+1)}{6(k+1)+4} = \frac{(k+1)}{6k+10}$$

$$\text{LHS} = \frac{1}{2\cdot 5} + \frac{1}{5\cdot 8} + \frac{1}{8\cdot 11} + \ldots + \frac{1}{(3k-1)(3k+2)} + \frac{1}{(3k+2)(3k+5)}$$

$$= \frac{k}{6k+4} + \frac{1}{(3k+2)(3k+5)}$$ [by assumption step]

$$= \frac{k(3k+5) + 2}{2(3k+2)(3k+5)} = \frac{3k^2 + 5k + 2}{2(3k+2)(3k+5)}$$

$$= \frac{(k+1)(3k+2)}{2(3k+2)(3k+5)} = \frac{k+1}{6k+10}$$

$= \text{RHS}$

This shows that the result is true for $n = k + 1$. Therefore, by the principle of mathematical induction, the result is true for all $n \in N$.

(v) Let $P(n): 1\cdot 4\cdot 7 + 2\cdot 5\cdot 8 + 3\cdot 6\cdot 9 + \ldots +$ upto n terms

$$= \frac{n}{4}(n+1)(n+6)(n+7)$$

i.e., $P(n): 1\cdot 4\cdot 7 + 2\cdot 5\cdot 8 + 3\cdot 6\cdot 9 + \ldots + n(n+3)(n+6)$

$$= \frac{n}{4}(n+1)(n+6)(n+7) \quad \ldots\text{(i)}$$

***Step* I** For $n = 1$,

LHS of Eq. (i) $= 1\cdot 4\cdot 7 = 28$

RHS of Eq. (i) $= \frac{1}{4}(1+1)(1+6)(1+7) = \frac{2\cdot 7\cdot 8}{4} = 28$

LHS = RHS

Therefore, $P(1)$ is true.

***Step* II** Let us assume that the result is true for $n = k$. Then,

$$P(k): 1\cdot 4\cdot 7 + 2\cdot 5\cdot 8 + 3\cdot 6\cdot 9 + \ldots + k(k+3)(k+6) = \frac{k}{4}(k+1)(k+6)(k+7)$$

***Step* III** For $n = k + 1$, we have to prove that

$$P(k+1): 1\cdot 4\cdot 7 + 2\cdot 5\cdot 8 + 3\cdot 6\cdot 9 + \ldots + k(k+3)(k+6) + (k+1)(k+4)(k+7)$$

$$= \frac{(k+1)}{4}(k+2)(k+7)(k+8)$$

$$\text{LHS} = 1\cdot 4\cdot 7 + 2\cdot 5\cdot 8 + 3\cdot 6\cdot 9 + \ldots + k(k+3)(k+6) + (k+1)(k+4)(k+7)$$

$$= \frac{k}{4}(k+1)(k+6)(k+7) + (k+1)(k+4)(k+7)$$

[by assumption step]

$$= (k+1)(k+7)\left\{\frac{k}{4}(k+6) + (k+4)\right\}$$

$$= (k+1)(k+7)\left\{\frac{k^2 + 6k + 4k + 16}{4}\right\}$$

$$= (k+1)(k+7)\left\{\frac{k^2 + 10k + 16}{4}\right\}$$

$$= (k+1)(k+7)\left\{\frac{(k+2)(k+8)}{4}\right\}$$

$$= \frac{(k+1)}{4}(k+2)(k+7)(k+8) = \text{RHS}$$

This shows that the result is true for $n = k + 1$. Hence, by the principle of mathematical induction, the result is true for all $n \in N$.

(vi) Let $P(n): \frac{1^2}{1\cdot 3} + \frac{2^2}{3\cdot 5} + \ldots + \frac{n^2}{(2n-1)(2n+1)}$

$$= \frac{n(n+1)}{2(2n+1)} \quad \ldots\text{(i)}$$

Step I For $n = 1$,

LHS of Eq. (i) $= \dfrac{1^2}{1\cdot 3} = \dfrac{1}{3}$

RHS of Eq. (i) $= \dfrac{1(1+1)}{2(2\times 1+1)} = \dfrac{2}{2(3)} = \dfrac{1}{3}$

$$\text{LHS} = \text{RHS}$$

Therefore, $P(1)$ is true.

Step II Let us assume that the result is true for $n = k$, then

$$P(k) = \frac{1^2}{1\cdot 3} + \frac{2^2}{3\cdot 5} + \ldots + \frac{k^2}{(2k-1)(2k+1)} = \frac{k(k+1)}{2(2k+1)}$$

Step III For $n = k + 1$, we have to prove that

$$P(k+1): \frac{1^2}{1\cdot 3} + \frac{2^2}{3\cdot 5} + \ldots + \frac{k^2}{(2k-1)(2k+1)} + \frac{(k+1)^2}{(2k+1)(2k+3)} = \frac{(k+1)(k+2)}{2(2k+3)}$$

$$\text{LHS} = \frac{1^2}{1\cdot 3} + \frac{2^2}{3\cdot 5} + \ldots + \frac{k^2}{(2k-1)(2k+1)} + \frac{(k+1)^2}{(2k+1)(2k+3)}$$

$$= \frac{k(k+1)}{2(2k+1)} + \frac{(k+1)^2}{(2k+1)(2k+3)} \quad \text{[by assumption step]}$$

$$= \frac{(k+1)}{(2k+1)}\left\{\frac{k}{2} + \frac{k+1}{(2k+3)}\right\} = \frac{(k+1)}{(2k+1)}\left\{\frac{2k^2+5k+2}{2(2k+3)}\right\}$$

$$= \frac{(k+1)}{(2k+1)}\cdot\frac{(k+2)(2k+1)}{2(2k+3)} = \frac{(k+1)(k+2)}{2(2k+3)}$$

$$= \text{RHS}$$

This shows that, the result is true for $n = k + 1$. Therefore, by the principle of mathematical induction the result is true for all $n \in N$.

13. Let $P(n): a_n = 2^n + 3^n,\ \forall\ n \geq 0, n \in N$

and $a_0 = 2, a_1 = 5$ and for $n \geq 2$; $a_n = 5a_{n-1} - 6a_n - 2$

Step I For $n = 0$,

$$a_0 = 2^0 + 3^0 = 1 + 10 = 2$$

which is true as $a_0 = 2$. [given]

Also, for $n = 1$, $\quad a_1 = 2^1 + 3^1 = 2 + 3 = 5$

which is also true as $a_1 = 5$. [given]

Hence, $P(0)$ and $P(1)$ are true.

Step II Assume that $P(k-1)$ and $P(k)$ are true. Then,

$$a_{k-1} = 2^{k-1} + 3^{k-1} \quad \text{...(i)}$$

where $\quad a_{k-1} = 5a_{k-2} - 6a_{k-3}$ and $a_k = 2^k + 3^k$ ...(ii)

where $\quad a_k = 5a_{k-1} - 6a_{k-2}$

Step III For $n = k + 1$,

$$P(k+1): a_{k+1} = 2^{k+1} + 3^{k+1},\ \forall\ k \geq 0, k \in N.$$

where $\quad a_{k+1} = 5a_k - 6a_{k-1}$

Now, $\quad a_{k+1} = 5a_k - 6\,a_{k-1}$

$$= 5(2^k + 3^k) - 6\,(2^{k-1} + 3^{k-1})$$

[by using Eqs. (i) and (ii)]

$$= 5\cdot 2^k + 5\cdot 3^k - 6\cdot 2^{k-1} - 6\cdot 3^{k-1}$$

$$= 2^{k-1}(5\cdot 2 - 6) + 3^{k-1}(5\cdot 3 - 6)$$

$$= 2^{k-1}\cdot 4 + 3^{k-1}\cdot 9 = 2^{k+1} + 3^{k+1}$$

$\Rightarrow \quad a_{k+1} = 2^{k+1} + 3^{k+1}$

where $\quad a_{k+1} = 5a_k - 6a_{k-1}$

This shows that the result is true for $n = k + 1$. Hence, by the second principle of mathematical induction, the result is true for $n \in N, n \geq 0$.

14. Let $P(n): a_{n+1} = \dfrac{1}{(n+1)!}, n \in N$ ...(i)

where $a_1 = 1$ and $a_{n+1} = \dfrac{1}{(n+1)} a_n, n \geq 1$ ...(ii)

Step I For $n = 1$, from Eq. (i), we get

$$a_2 = \frac{1}{(1+1)!} = \frac{1}{2\,!}$$

But from Eq. (ii), we get $a_2 = \dfrac{1}{(1+1)}, a_1 = \dfrac{1}{2}(1) = \dfrac{1}{2}$

which is true.

Also, for $n = 2$ from Eq. (i), we get

$$a_3 = \frac{1}{3!} = \frac{1}{6}$$

But from Eq. (ii), we get $\quad a_3 = \dfrac{1}{3}, a_2 = \dfrac{1}{3}\cdot\dfrac{1}{2} = \dfrac{1}{6}$

which is also true.

Hence, $P(1)$ and $P(2)$ are true.

Step II Assume that $P(k-1)$ and $P(k)$ are true. Then,

$$P(k-1): a_k = \frac{1}{k!} \quad \text{...(ii)}$$

where, $a_k = \dfrac{1}{k} a_{k-1}, \quad k \geq 1$ ...(iv)

and $\quad P(k): a_{k+1} = \dfrac{1}{(k+1)!}$ ...(v)

where $\quad a_{k+1} = \dfrac{1}{k+1} a_k, k \geq 1$ ...(vi)

Step III For $n = k + 1$,

$$P(k+1): a_{k+2} = \frac{1}{(k+2)!} \quad \text{...(vii)}$$

where $\quad a_{k+2} = \dfrac{1}{(k+2)} a_{k+1}$ ...(viii)

Now, LHS of Eq. (vii) $= a_{k+2}$

$$= \frac{1}{(k+2)} a_{k+1} \quad \text{[using Eq. (viii)]}$$

$$= \frac{1}{(k+2)}\cdot\frac{1}{(k+1)} a_k \quad \text{[using Eq. (vi)]}$$

$$= \frac{1}{k+2}\cdot\frac{1}{k+1}\cdot\frac{1}{k} a_{k-1} \quad \text{[using Eq. (iv)]}$$

$$= \frac{1}{k+2}\cdot\frac{1}{k+1}\cdot\frac{1}{k}\cdot\frac{1}{k!} \quad \text{[using Eq. (iii)]}$$

$$= \frac{1}{(k+2)!} = \text{ RHS of Eq. (vii)}$$

This shows that the result is true for $n = k + 1$. Hence, by the second principle of mathematical induction, the result is true for all $n \geq 1, n \in N$.

15. Let $\quad P(n): a^n + b^n + c^n = d^n + e^n + f^n, \forall n \in N \qquad \text{...(i)}$

where $\quad a + b + c = d + e + f \qquad \text{...(ii)}$

$a^2 + b^2 + c^2 = d^2 + e^2 + f^2 \qquad \text{...(iii)}$

and $\quad a^3 + b^3 + c^3 = d^3 + e^3 + f^3 \qquad \text{...(iv)}$

***Step* I** For $n = 1$ from Eq. (i), we get

$P(1): a + b + c = d + e + f \qquad$ [given]

Hence, the result is true for $n = 1$.

Also, for $n = 2$ from Eq. (i), we get

$P(2): a^2 + b^2 + c^2 = d^2 + e^2 + f^2 \qquad$ [given]

Hence, the result is true for $n = 2$.

Also, for $x = 3$, from Eq. (i), we get

$P(3): a^3 + b^3 + c^3 = d^3 + e^3 + f^3 \qquad$ [given]

Hence, the result is true for $n = 3$.

Therefore, $P(1)$, $P(2)$ and $P(3)$ are true.

***Step* II** Assume that $P(k-2)$, $P(k-1)$ and $P(k)$ are true, then

$P(k-2): a^{k-2} + b^{k-2} + c^{k-2} = d^{k-2} + e^{k-2} + f^{k-2} \qquad \text{...(v)}$

$P(k-1): a^{k-1} + b^{k-1} + c^{k-1} = d^{k-1} + e^{k-1} + f^{k-1} \qquad \text{...(vi)}$

and $\quad P(k): a^k + b^k + c^k = d^k + e^k + f^k \qquad \text{...(vii)}$

***Step* III** For $xn = k + 1$, we shall to prove that

$P(k+1): a^{k+1} + b^{k+1} + c^{k+1} = d^{k+1} + e^{k+1} + f^{k+1}$

LHS $= a^{k+1} + b^{k+1} + c^{k+1}$

$= (a^k + b^k + c^k)(a + b + c) - (a^{k-1} + b^{k-1} + c^{k-1})$

$(ab + bc + ca) + abc\,(a^{k-2} + b^{k-2} + c^{k-2})$

$= (d^k + e^k + f^k)(d + e + f) - (d^{k-1} + e^{k-1} + f^{k-1})$

$(de + ef + fd) + def(d^{k-2} + e^{k-2} + f^{k-2})$

[using Eqs. (ii), (iii), (iv), (v), (vi), (vii)]

$\therefore \quad (a + b + c)^2 = (d + e + f)^2$

$\Rightarrow \quad a^2 + b^2 + c^2 + 2(ab + bc + ca)$

$= d^2 + e^2 + f^2 + 2(de + ef + fd)$

$\Rightarrow \quad ab + bc + ca = de + ef + fd$

$[\because a^2 + b^2 + c^2 = d^2 + e^2 + f^2]$

and $\quad a^3 + b^3 + c^3 - 3abc$

$= (d + e + f)(d^2 + e^2 + f^2 - de - ef - fd)$

$= d^3 + e^3 + f^3 - 3def$

$\Rightarrow \quad abc = def \qquad [\because a^3 + b^3 + c^3 = d^3 + e^3 + f^3]$

$= d^{k+1} + e^{k+1} + f^{k+1} = \text{RHS}$

This shows that the result is true for $n = k + 1$. Hence, by second principle of mathematical induction, the result is true for all $n \in N$.

16. Let $P(n): \tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{7}\right) + \ldots + \tan^{-1}\left(\frac{1}{n^2 + n + 1}\right)$

$$= \tan^{-1}\left(\frac{n}{n+2}\right) \qquad \text{...(i)}$$

***Step* I** For $n = 1$,

$$\text{LHS of Eq. (i)} = \tan^{-1}\left(\frac{1}{3}\right) = \tan^{-1}\left(\frac{1}{1+2}\right)$$

$= \text{RHS of Eq. (i)}$

Therefore, $P(1)$ is true.

***Step* II** Assume that $P(k)$ is true. Then,

$$P(k): \tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{7}\right) + \ldots + \tan^{-1}\left(\frac{1}{k^2 + k + 1}\right)$$

$$= \tan^{-1}\left(\frac{k}{k+2}\right)$$

***Step* III** For $n = k + 1$,

$$P(k+1): \tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{7}\right) + \ldots + \tan^{-1}\left(\frac{1}{k^2 + k + 1}\right)$$

$$+ \tan^{-1}\left(\frac{1}{(k+1)^2 + (k+1) + 1}\right)$$

$$= \tan^{-1}\left(\frac{k+1}{k+3}\right) \qquad \text{...(ii)}$$

LHS of Eq. (ii)

$$= \tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{7}\right) + \ldots + \tan^{-1}\left(\frac{1}{k^2 + k + 1}\right)$$

$$+ \tan^{-1}\left(\frac{1}{(k+1)^2 + (k+1) + 1}\right)$$

$$= \tan^{-1}\left(\frac{k}{k+2}\right) + \tan^{-1}\left(\frac{1}{(k+1)^2 + (k+1) + 1}\right)$$

[by assumption step]

$$= \tan^{-1}\left(\frac{k}{1 + (k+1)}\right) + \tan^{-1}\left(\frac{1}{k^2 + 3k + 3}\right)$$

$$= \tan^{-1}\left(\frac{k}{1 + (k+1)}\right) + \tan^{-1}\left(\frac{1}{1 + (k+1)(k+2)}\right)$$

$$= \tan^{-1}\left(\frac{(k+1) - 1}{1 + (k+1) \cdot 1}\right) + \tan^{-1}\left(\frac{(k+2) - (k+1)}{1 + (k+2)(k+1)}\right)$$

$$= \tan^{-1}(k+1) - \tan^{-1} 1 + \tan^{-1}(k+2) - \tan^{-1}(k+1)$$

$$= \tan^{-1}(k+2) - \tan^{-1} 1$$

$$= \tan^{-1}\left(\frac{k + 2 - 1}{1 + (k+2) \cdot 1}\right) = \tan^{-1}\left(\frac{k+1}{k+3}\right) = \text{RHS of Eq. (ii)}$$

This shows that the result is true for $n = k + 1$. Hence, by the principle of mathematical induction, the result is true for all $n \in N$.

17. Let $P(n) = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \ldots + \frac{1}{\sqrt{n}}$

$$\therefore \; P(2) = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} = 1.707 > \sqrt{2}$$

Let us assume that

$$P(k) = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \dots + \frac{1}{\sqrt{k}} > \sqrt{k} \text{ is true for } n = k + 1.$$

$$\text{LHS} = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \dots + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}}$$

$$> \sqrt{k} + \frac{1}{\sqrt{k+1}} = \frac{\sqrt{k(k+1)+1}}{\sqrt{(k+1)}} > \frac{k+1}{\sqrt{(k+1)}}$$

$$[\because \sqrt{k(k+1)+1} > k, \forall\, k \geq 0]$$

$\therefore \quad P(k+1) > \sqrt{(k+1)}$

By mathematical induction Statement-1 is true, $\forall\, n \geq 2$.

Now, let $\alpha(n) = \sqrt{n(n+1)}$

$\therefore \quad \alpha(2) = \sqrt{2(2+1)} = \sqrt{6} < 3$

Let us assume that

$$\alpha(k) = \sqrt{k(k+1)} < (k+1) \text{ is true}$$

for $\quad n = k + 1$

$$\text{LHS} = \sqrt{(k+1)(k+2)} < (k+2)$$

$$[\because (k+1) < (k+2)]$$

By mathematical induction Statement-2 is true but Statement-2 is not a correct explanation for Statement-1.

18. Let $\quad P(n) = n^7 - n$

By mathematical induction for $n = 1$, $P(1) = 0$, which is divisible by 7

for $\quad n = k$, $P(k) = k^7 - k$

Assume $P(k)$ is divisible by 7

$$\therefore \quad k^7 - k = 7\lambda, \lambda \in I \quad \text{...(i)}$$

For $n = k + 1$,

$$P(k+1) = (k+1)^7 - (k+1)$$

$$= ({}^7C_0 k^7 + {}^7C_1 k^6 + {}^7C_2 k^5 + {}^7C_3 k^4 + \dots + {}^7C_6 k + {}^7C_7) - (k+1)$$

$$= (k^7 - k) + 7(k^6 + 3k^5 + \dots + k)$$

$$= 7\lambda + 7(k^6 + 3k^5 + \dots + k) = \text{Divisible by 7}$$

$\therefore$ Statement-2 is true.

Also, let $F(n) = (n+1)^7 - n^7 - 1$

$$= \{(n+1)^7 - (n+1)\} - (n^7 - n)$$

= Divisible by 7 from Statement-2

Hence, both statements are true and Statement-2 is correct explanation of Statement-1.