CHAPTER

05

Permutations and Combinations

In everyday life, we need to know about the number of ways of doing certain work from given number of available options. *For example,* Three persons A, B and C are applying for a job in which only one post is vacant. Clearly, vacant post can be filled either by A or B or C i.e., total number of ways doing this work is three.

Again, let two persons A and B are to be seated in a row, then only two possible ways of arrangement is AB or BA. In two arrangements, persons are same but their order is different. Thus, in arranging things, order of things is important.

Session 1

Fundamental Principle of Counting, Factorial Notation

Fundamental Principle of Counting

(i) Multiplication Principle

If an operation can be performed in 'm' different ways, following which a second operation can be performed in 'n' different ways, then the two operations in succession can be performed in $m \times n$ ways. This can be extended to any finite number of operations.

Note For AND $\rightarrow$ '×' (multiply)

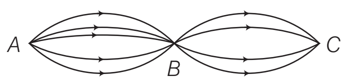
Example 1. A hall has 12 gates. In how many ways can a man enter the hall through one gate and come out through a different gate?

Sol. Since, there are 12 ways of entering into the hall. After entering into the hall, the man come out through a different gate in 11 ways.

Hence, by the fundamental principle of multiplication, total number of ways is $12 \times 11 = 132$ ways.

Example 2. There are three stations A, B and C, five routes for going from station A to station B and four routes for going from station B to station C. Find the number of different ways through which a person can go from A to C via B.

Sol. Since, there are five routes for going from A to B. So, there are four routes for going from B to C.



Hence, by the fundamental principle of multiplication, total number of different ways

$= 5 \times 4$ [i.e., A to B and then B to C]

$= 20$ ways

(ii) Addition Principle

If an operation can be performed in 'm' different ways and another operation, which is independent of the first operation, can be performed in 'n' different ways. Then, either of the two operations can be performed in $(m + n)$ ways. This can be extended to any finite number of mutually exclusive operation.

Note For OR $\rightarrow$ '+' (Addition)

Example 3. There are 25 students in a class in which 15 boys and 10 girls. The class teacher select either a boy or a girl for monitor of the class. In how many ways the class teacher can make this selection?

Sol. Since, there are 15 ways to select a boy, so there are 10 ways to select a girl.

Hence, by the fundamental principle of addition, either a boy or a girl can be performed in $15 + 10 = 25$ ways.

Example 4. There are 4 students for Physics, 6 students for Chemistry and 7 students for Mathematics gold medal. In how many ways one of these gold medals be awarded?

Sol. The Physics, Chemistry and Mathematics student's gold medal can be awarded in 4, 6 and 7 ways, respectively.

Hence, by the fundamental principle of addition, number ways of awarding one of the three gold medals.

$= 4 + 6 + 7 = 17$ ways.

Factorial Notation

Let n be a positive integer. Then, the continued product of first 'n' natural numbers is called factorial n, to be denoted by $n!$ or n i.e., $n! = n(n-1)(n-2)\ldots 3 \cdot 2 \cdot 1$

Note When n is negative or a fraction, $n!$ is not defined.

Some Important Properties

(i) $n! = n(n-1)! = n(n-1)(n-2)!$

(ii) $0! = 1! = 1$

(iii) $(2n)! = 2^n n![1\cdot 3\cdot 5\ldots(2n-1)]$

(iv) $\dfrac{n!}{r!} = n(n-1)(n-2)\ldots(r+1)$ $[r<n]$

(v) $\dfrac{n!}{(n-r)!} = n(n-1)(n-2)\ldots(n-r+1)$ $[r<n]$

(vi) $\dfrac{1}{n!} + \dfrac{1}{(n+1)!} = \dfrac{\lambda}{(n+2)!}$, then $\lambda = (n+2)^2$

(vii) If $x! = y! \Rightarrow x = y$ or $x = 0, y = 1$ or $x = 1, y = 0$

Example 5. Find n, if $(n+2)! = 60\times(n-1)!$.

Sol. $\because (n+2)! = (n+2)(n+1)n(n-1)!$

$$\Rightarrow \frac{(n+2)!}{(n-1)!} = (n+2)(n+1)n$$

$$\Rightarrow 60 = (n+2)(n+1)n \quad \text{[given]}$$

$$\Rightarrow 5\times 4\times 3 = (n+2)\times(n+1)\times n$$

$$\therefore n = 3$$

Example 6. Evaluate $\sum_{r=1}^{n} r\times r!$.

Sol. We have, $\sum_{r=1}^{n} r\times r! = \sum_{r=1}^{n}\{(r+1)-1\}r! = \sum_{r=1}^{n}(r+1)! - r!$

$$= (n+1)! - 1!$$

[put $r = n$ in $(r+1)!$ and $r = 1$ is $r!$]

$$= (n+1)! - 1$$

Example 7. Find the remainder when $\sum_{r=1}^{n} r!$ is divided by 15, if $n \geq 5$.

Sol. Let

$$N = \sum_{r=1}^{n} r! = 1! + 2! + 3! + 4! + 5! + 6! + 7! + \ldots + n!$$

$$= (1! + 2! + 3! + 4!) + (5! + 6! + 7! + \ldots + n!)$$

$$= 33 + (5! + 6! + 7! + \ldots + n!)$$

$$\Rightarrow \frac{N}{15} = \frac{33}{15} + \frac{(5! + 6! + 7! + \ldots + n!)}{15}$$

$$= 2 + \frac{3}{15} + \text{Integer} \quad \text{[as 5!, 6!, … are divisible by 15]}$$

$$= \frac{3}{15} + \text{Integer}$$

Hence, remainder is 3.

Exponent of prime p in $n!$

Exponent of prime p in $n!$ is denoted by $E_p(n!)$, where p is prime number and n is a natural number. The last integer amongst 1, 2, 3, …, $(n-1)$, n which is divisible by p is $\left[\dfrac{n}{p}\right]p$, where $[\cdot]$ denotes the greatest integer function.

$$\therefore \quad E_p(n!) = E_p(1\cdot 2\cdot 3\ldots(n-1)\cdot n)$$

$$= E_p\left(p\cdot 2p\cdot 3p\ldots(n-1)p\cdot\left[\frac{n}{p}\right]\ p\right)$$

[because the remaining natural numbers from 1 to n are not divisible by p]

$$= \left[\frac{n}{p}\right] + E_p\left(1\cdot 2\cdot 3\ldots\left[\frac{n}{p}\right]\right) \quad \ldots\text{(i)}$$

Now, the last integer amongs 1, 2, 3, …, $\left[\dfrac{n}{p}\right]$ which is divisible by p is $\left[\dfrac{\left[\frac{n}{p}\right]}{p}\right] = \left[\dfrac{n}{p^2}\right]$. Now, from Eq. (i), we get

$$E_p(n!) = \left[\frac{n}{p}\right] + E_p\left(p, 2p, 3p, \ldots, \left[\frac{n}{p^2}\right]p\right)$$

[because the remaining natural numbers from 1 to $\left[\dfrac{n}{p}\right]$ are not divisible by p]

$$\therefore \quad E_p(n!) = \left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] + E_p\left(1\cdot 2\cdot 3\ldots\left[\frac{n}{p^2}\right]\right)$$

Similarly, we get

$$E_p(n!) = \left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] + \left[\frac{n}{p^3}\right] + \ldots + \left[\frac{n}{p^s}\right]$$

where, s is the largest natural number such that

$$p^s \leq n < p^{s+1}$$

Note Number of zeroes at the end of $n! = E_5(n!)$.

Example 8. ***Find the exponent of*** 3 ***in*** 100! .

Sol. In terms of prime factors 100! can be written as $2^a\cdot 3^b\cdot 5^c\cdot 7^d\ldots$

Now, $b = E_3(100!)$

$$= \left[\frac{100}{3}\right] + \left[\frac{100}{3^4}\right] + \left[\frac{100}{3^3}\right] + \left[\frac{100}{3^4}\right] + \ldots$$

$$= 33 + 11 + 3 + 1 + 0 + \ldots = 48$$

Hence, exponent of 3 is 48.

Aliter

$\because \quad 100! = 1 \times 2 \times 3 \times 4 \times 5 \times \ldots \times 98 \times 99 \times 100$

$$= (1 \times 2 \times 4 \times 5 \times 7 \times \ldots \times 98 \times 100)(3 \times 6 \times 9 \times \ldots \times 96 \times 99)$$

$$= k \times 3^{33}(1 \times 2 \times 3 \times \ldots \times 32 \times 33)$$

$$= [k(1 \times 2 \times 4 \times 5 \times \ldots \times 31 \times 32)](3 \times 6 \times 9 \times \ldots \times 30 \times 33)$$

$$= 3^{33} k_1 \times 3^{11}(1 \times 2 \times 3 \times \ldots \times 10 \times 11)$$

$$= 3^{44}\ [k_1\,(1 \times 2 \times 4 \times 5 \times \ldots \times 10 \times 11)](3 \times 6 \times 9)$$

$$= k_2 \times 3^{44} \times 3^4 \times 2 = k_3 \times 3^{48}$$

Hence, exponent of 3 in 100! is 48.

Example 9. ***Prove that 33! is divisible by 2^{19} and what is the largest integer n such that 33! is divisible by 2^n?***

Sol. In terms of prime factors, 33! can be written as $2^a \cdot 3^b \cdot 5^c \cdot 7^d \cdot \ldots$

Now, $$E_2(33!) = \left[\frac{33}{2}\right] + \left[\frac{33}{2^2}\right] + \left[\frac{33}{2^3}\right] + \left[\frac{33}{2^4}\right] + \left[\frac{33}{2^5}\right] + \ldots$$

$$= 16 + 8 + 4 + 2 + 1 + 0 + \ldots$$

$$= 31$$

Hence, the exponent of 2 in 33! is 31. Now, 33! is divisible by 2^{31} which is also divisible by 2^{19}.

$\therefore$ Largest value of n is 31.

Example 10. ***Find the number of zeroes at the end of 100!.***

Sol. In terms of prime factors, 100! can be written as $2^a \cdot 3^b \cdot 5^c \cdot 7^d \ \ldots$

Now, $$E_2(100!) = \left[\frac{100}{2}\right] + \left[\frac{100}{2^2}\right] + \left[\frac{100}{2^3}\right] + \left[\frac{100}{2^4}\right] + \left[\frac{100}{2^5}\right] + \left[\frac{100}{2^6}\right]$$

$$= 50 + 25 + 12 + 6 + 3 + 1 = 97$$

and $$E_5(100!) = \left[\frac{100}{5}\right] + \left[\frac{100}{5^2}\right]$$

$$= 20 + 4 = 24$$

$$\therefore \quad 100! = 2^{97} \cdot 3^b \cdot 5^{24} \cdot 7^d \ \ldots = 2^{73} \cdot 3^b \cdot (2 \times 5)^{24} \cdot 7^d \ \ldots$$

$$= 2^{73} \cdot 3^b \cdot (10)^{24} \cdot 7^d \ \ldots$$

Hence, number of zeroes at the end of 100! is 24.

or Exponent of 10 in 100! = min (97, 24) = 24.

Aliter

Number of zeroes at the end of 100!

$$= E_5(100!) = \left[\frac{100}{5}\right] + \left[\frac{100}{5^2}\right] + \ldots$$

$$= 20 + 4 + 0 + \ldots = 24$$

Example 11. For how many positive integral values of n does $n!$ end with precisely 25 zeroes?

Sol. $\because$ Number of zeroes at the end of $n! = 25$ [given]

$$\Rightarrow \quad E_5(x!) = 25$$

$$\Rightarrow \quad \left[\frac{n}{5}\right] + \left[\frac{n}{25}\right] + \left[\frac{n}{125}\right] + \ldots = 25$$

It's easy to see that $n = 105$ is the smallest satisfactory value of n. The next four values of n will also work (i.e., $n = 106$, 107, 108, 109). Hence, the answer is 5.

Example 12. Find the exponent of 80 in 180!.

Sol. $\because \ 80 = 2^4 \times 5$

$$\therefore \quad E_2(180!) = \left[\frac{180}{2}\right] + \left[\frac{180}{2^2}\right] + \left[\frac{180}{2^3}\right] + \left[\frac{180}{2^4}\right] + \left[\frac{180}{2^5}\right] + \left[\frac{180}{2^6}\right] + \left[\frac{180}{2^7}\right] + \ldots$$

$$= 90 + 45 + 22 + 11 + 5 + 2 + 1 = 176$$

and $$E_5(180!) = \left[\frac{180}{5}\right] + \left[\frac{180}{5^2}\right] + \left[\frac{180}{5^3}\right] + \ldots$$

$$= 36 + 7 + 1 + 0 + \ldots$$

$$= 44$$

Now, exponent of 16 in 180! is $\left[\frac{176}{4}\right] = 44$, where $[\cdot]$ denotes the greatest integer function. Hence, the exponent of 80 in 180! is 44.

Exercise for Session 1

1. There are three routes: air, rail and road for going from Chennai to Hyderabad. But from Hyderabad to Vikarabad, there are two routes, rail and road. The number of routes from Chennai to Vikarabad via Hyderabad is

(a) 4 (b) 5 (c) 6 (d) 7

2. There are 6 books on Mathematics, 4 books on Physics and 5 books on Chemistry in a book shop. The number of ways can a student purchase either a book on Mathematics or a book on Chemistry, is

(a) 10 (b) 11 (c) 9 (d) 15

3. If a, b and c are three consecutive positive integers such that $a < b < c$ and $\frac{1}{a!} + \frac{1}{b!} = \frac{\lambda}{c!}$, the value of $\sqrt{\lambda}$ is

(a) a (b) b (c) c (d) $a + b + c$

4. If $n!$, $3 \times n!$ and $(n + 1)!$ are in GP, then $n!$, $5 \times n!$ and $(n + 1)!$ are in

(a) AP (b) GP (c) HP (d) AGP

5. Sum of the series $\sum_{r=1}^{n} (r^2 + 1)\, r!$ is

(a) $(n + 1)!$ (b) $(n + 2)! - 1$ (c) $n \cdot (n + 1)!$ (d) $n \cdot (n + 2)!$

6. If $15! = 2^{\alpha} \cdot 3^{\beta} \cdot 5^{\gamma} \cdot 7^{\delta} \cdot 11^{\theta} \cdot 13^{\phi}$, the value of $\alpha - \beta + \gamma - \delta + \theta - \phi$ is

(a) 4 (b) 6 (c) 8 (d) 10

7. The number of naughts standing at the end of 125! is

(a) 29 (b) 30 (c) 31 (d) 32

8. The exponent of 12 in 100! is

(a) 24 (b) 25 (c) 47 (d) 48

9. The number 24! is divisible by

(a) 6^{24} (b) 24^{6} (c) 12^{12} (d) 48^{5}

10. The last non-zero digit in 20! is

(a) 2 (b) 4 (c) 6 (d) 8

11. The number of prime numbers among the numbers $105! + 2, 105! + 3, 105! + 4, \ldots, 105! + 104$ and $105! + 105$ is

(a) 31 (b) 32 (c) 33 (d) None of these

Answers

Exercise for Session 1

1. (c)	**2.** (b)	**3.** (c)	**4.** (a)	**5.** (c)	**6.** (b)
7. (c)	**8.** (c)	**9.** (b)	**10.** (b)	**11.** (d)	