# 5 Continuity and Differentiability

"Many real life events, such as the trajectory traced by a football, where you see Ronaldo hit the soccer ball, the angle and distance covered animation on the screen is shown to the viewers using technology can be described with the help of mathematical functions. The knowledge of Continuity and Differentiation is popularly used in finding speed, direction and other parameters from a given function.

## Topic Notes

- *Limits and Continuity of a Function*
- *Differentiability*
- *Derivatives*
- *Second Order Derivative*

# LIMITS AND CONTINUITY OF A FUNCTION

# 1

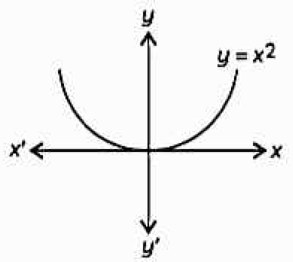
This Chapter is essentially a continuation of our study of limits and derivatives done in class XI. We have already learnt how to find out the limit of a function at a point and derivative of polynomial and all trigonometric functions. In this chapter, we will extend the concept of limits for finding the continuity of a function at a point and in an interval. The concept of limit is further used to check the differentiability of a function at a point. Also, we are going to learn various techniques which will help in finding the derivative of various exponential functions, logarithmic functions and many more types of functions.

For studying the concept of continuity and differentiability, we need to have mastery over the concept of limits that we have studied in class XI.

## TOPIC 1

## CONCEPT OF LIMITS

Consider the function $y = x^2$.




We observe that in the given figure, as $x$ takes values very much nearer to 0, the value of $f(x)$ also moves nearer to 0. We say:

$$\lim_{x \to 0} f(x) = 0$$

(to be read as "*limit of f(x) as x tends to 0, is equal to* 0").

The limit of $f(x)$ as $x$ tends to 0 is to be thought of as the value of $f(x)$ should assume at $x = 0$.

Let $a \in$ R and let $f$ be a real-valued function in real variable $x$ defined at the points in an open interval containing $a$ except possibly at $a$. Then, we say that the **limit** of the function $f(x)$ is a real number $l$, as $x$ tends to $a$, if the value of $f(x)$ approaches $l$ as $x$ approaches $a$. It is denoted by $\lim_{x \to a} f(x)$ and is read as "limit of $f(x)$ as $x$ tends to $a$".

Now we know that $x$ can approach to $a$ in two different ways: either from left or from right, *i.e.*, all values of $x$ nearer $a$ could be less than or greater than $a$, which leads us to two limits, the right hand limit and left hand limit. When $x$ takes values close to $a$ but less than $a$, if the corresponding values of $f$ move closer and closer to a number $l$, we say that $l$ is the left hand limit of $f$ at $x = a$ and write $\lim_{x \to a^-} f(x)$.

Similarly, when $x$ is very close to a number $a$ but remains greater than $a$, if the values of $f$ are closer to a number $l$, we say that $l$ is the right hand limit of $f$ at $x = a$ and write $\lim_{x \to a^+} f(x) = l$.

If the left and right hand limits coincide, we call the coinciding value as the limit of $f(x)$ at $x = a$ and denote it by $\lim_{x \to a} f(x)$.

**Important**

- *We say $\lim_{x \to a^-} f(x)$ is the expected value of f at $x = a$ given the values of near x to the left of a. This value is called the left hand limit of f at a.*
  *We say $\lim_{x \to a^+} f(x)$ is the expected value of f at $x = a$ given the values of near x to the right of a. This value is called the right hand limit of f at a. If the left and right hand limits coincide, we call the common value as the limit of f(x) at $x = a$ and denote it by $\lim_{x \to a} f(x)$.*

## TOPIC 2

## ALGEBRA OF LIMITS

Let $f$ and $g$ be two functions such that and $\lim_{x \to a} f(x) = l$ and $\lim_{x \to \pi^+} f(x) = m$. Then,

(1) $\lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x) = l + m$

(2) $\lim_{x \to a} [f(x) - g(x)] = \lim_{x \to a} f(x) - \lim_{x \to a} g(x) = l - m$

(3) $\lim_{x \to a} [f(x) \times g(x)] = \lim_{x \to a} f(x) \times \lim_{x \to a} g(x) = l \times m$

(4) $\lim_{x \to a} [kf(x)] = k \lim_{x \to a} f(x) = kl$

(5) $\lim_{x \to a}\left[\frac{1}{f(x)}\right] = \frac{1}{\lim_{x \to a} f(x)} = \frac{1}{l}$, provided $l \neq 0$

(6) $\lim_{x \to a}\left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} = \frac{l}{m}$, provided $m \neq 0$

(7) $\lim_{x \to a}[f(x)]^n = \left[\lim_{x \to a} f(x)\right]^n = l^n$, for all $n \in \mathrm{N}$

(8) $\lim_{x \to a} |f(x)| = \left|\lim_{x \to a} f(x)\right| = |l|$

## Some Standard Results on Limits

(1) $\lim_{x \to a}\left[\frac{x^n - a^n}{x - a}\right] = na^{n-1}$, for all $n \in \mathrm{N}$

(2) $\lim_{x \to 0}\left[\frac{\sin x}{x}\right] = 1$

(3) $\lim_{x \to 0}\left[\frac{1 - \cos x}{x}\right] = 0$

(4) $\lim_{x \to 0}\left[\frac{\tan x}{x}\right] = 1$

(5) $\lim_{x \to 0}\left[\frac{\log(1 + x)}{x}\right] = 1$

(6) $\lim_{x \to 0}\left[\frac{e^x - 1}{x}\right] = 1$

(7) $\lim_{x \to 0}\left[\frac{a^x - 1}{x}\right] = \log a$

## Methods of Evaluation of Limits

(1) **Direct substitution:** For finding $\lim_{x \to a} f(x)$, we put $x = a$ in the given function. If $f(a)$ is a real number, then $\lim_{x \to a} f(x) = f(a)$.

**Illustration:** $\lim_{x \to 0} \cos x = \cos 0 = 1$;

**Illustration:** $\lim_{x \to 1} (x^3 - 4x + 7) = (1)^3 - 4(1) + 7 = 4$

### Caution

➥ *But to find $\lim_{x \to 5} \frac{x^2 - 25}{x - 5}$, we cannot but $x = 5$ as in that case, $\frac{5^2 - 25}{5 - 5}$ will be $\frac{0}{0}$, which is an indeterminate form.*

*To evaluate such limits we use any of the below methods.*

(2) **Factorisation:** If we get an indeterminate value when we put $x = a$ in the given function and $f(x)$ is a rational function, then factorise the numerator and the denominator. Cancel out the common factors and then put $x = a$.

**Illustration:** $\lim_{x \to 5} \frac{x^2 - 25}{x - 5} = \lim_{x \to 5} \frac{(x + 5)(x - 5)}{x - 5}$

$= \lim_{x \to 5} (x + 5) = 5 + 5 = 10$

**Illustration:**

$\lim_{x \to 1} \frac{x^2 - 3x + 2}{x - 1} = \lim_{x \to 1} \frac{(x - 2)(x - 1)}{x - 1} = \lim_{x \to 1} (x - 2)$

$= 1 - 2 = -1$

(3) **Rationalisation:** If the given function has some expression within square root sign (or other surd) and on putting $x = a$, the value of the function becomes $\frac{0}{0}$ or non-real (*i.e.*, square root of a negative number), then follow the following procedure:

**Step 1:** Rationalise the given function, so that the denominator is made free of radicals. For example, in the expression $\frac{1}{5 - \sqrt{x}}$, the denominator can be rationalised by multiplying the numerator and the denominator by $5 + \sqrt{x}$ to give $\frac{5 + \sqrt{x}}{25 - x}$.

**Step 2:** Simplify and put $x = a$ and get the desired value of the limit.

**Illustration:** $\lim_{x \to 0} \frac{\sqrt{4 + x} - \sqrt{4 - x}}{x}$

$= \lim_{x \to 0} \frac{\sqrt{4 + x} - \sqrt{4 - x}}{x} \times \frac{\sqrt{4 + x} + \sqrt{4 - x}}{\sqrt{4 + x} + \sqrt{4 - x}}$

$= \lim_{x \to 0} \frac{(4 + x) - (4 - x)}{x[\sqrt{4 + x} + \sqrt{4 - x}]}$

$= \lim_{x \to 0} \frac{2x}{x[\sqrt{4 + x} + \sqrt{4 - x}]}$

$= \frac{2}{[\sqrt{4 + 0} + \sqrt{4 - 0}]}$

$= \frac{2}{4}$, or $\frac{1}{2}$

### Important

➥ *Sometimes, we need to make use of conjugate to simplify a function.*

# TOPIC 3

# INTUITIVE IDEA OF CONTINUITY OF A FUNCTION

A real function is continuous at a point iff the graph of the function has no break or jump at that point of consideration, *i.e.*, we can draw the graph of a function without lifting the pen. Continuity of a function is a local property of the function because it is defined at a point, *i.e.*, function may be continuous at one point, may not be continuous at some other point. Roughly speaking, a function is continuous if its graph is a single unbroken curve with no 'holes' or 'jumps'.

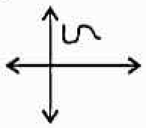


Fig. (i)

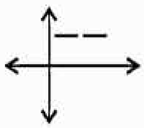


Fig. (ii)

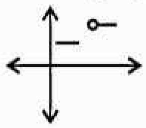


Fig. (iii)

From above intuitive idea, it is clear that the function shown in figure (i) is continuous. The function shown in figure (ii) has a hole/break at a point and hence is not continuous. The function shown in figure (iii) has a jump/break at a point and hence is not continuous.

## Continuity of a Function at a Point

Let D be an open interval in R and $a \in$ D.

A function $f$ : D $\to$ R is said to be right continuous (or continuous from right) at $x = a$, if $\lim\limits_{x \to a^+} f(x) = f(a)$.

A function $f$ : D $\to$ R is said to be left continuous (or continuous from left) at $x = a$, if $\lim\limits_{x \to a^-} f(x) = f(a)$.

A function $f$: D $\to$ R is said to be continuous at $x = a$ if it is both left continuous and right continuous at $x = a$ *i.e.*, $\lim\limits_{x \to a^-} f(x) = \lim\limits_{x \to a^+} f(x) = f(a)$.

A function $f$ : D $\to$ R is said to be discontinuous at $x = a$ if it is not continuous at $x = a$.

### Important

➥ *A function f may fail to be continuous at x = a for any of the following reasons:*

*(1) f is not defined at x = a, i.e., f(a) does not exist.*

*(2) Either* $\lim\limits_{x \to a^-} f(x)$ *does not exist or* $\lim\limits_{x \to a^+} f(x)$ *does not exist.*

*(3)* $\lim\limits_{x \to a^-} f(x) \neq \lim\limits_{x \to a^+} f(x)$

*(4)* $\lim\limits_{x \to a^-} f(x) = \lim\limits_{x \to a^+} f(x) \neq f(a)$

## Working Rule to Check Continuity of a Function at a Given Point

In order to check whether a given function $f$ : D $\to$ R is continuous at a point $x = a$, we follow the following steps:

**Step 1:** Write the value of $f(a)$, *i.e.*, $f(x)$ at $x = a$.

**Step 2:** Find $\lim\limits_{x \to a^-} f(x)$ by simplifying $\lim\limits_{h \to 0} f(a-h)$.

**Step 3:** Find $\lim\limits_{x \to a^+} f(x)$ by simplifying $\lim\limits_{h \to 0} f(a+h)$.

If $\lim\limits_{x \to a^-} f(x) = \lim\limits_{x \to a^+} f(x) = f(a)$, $f$ is continuous at $x = a$, otherwise not.

**Example 1.1:** Examine the continuity of the function $f(x) = 2x^2 - 1$ at $x = 3$. **[NCERT]**

**Ans.** Clearly, the given function $f(x) = 2x^2 - 1$ is defined at the given point $x = 3$ and its value is $2(3)^2 - 1$, *i.e.*, 17.

Also, $\text{LHL} = \lim\limits_{x \to 3^-} f(x) = \lim\limits_{h \to 0} f(3-h)$

$= \lim\limits_{h \to 0} [2(3-h)^2 - 1]$

$= 2(3)^2 - 1$

$= 17$

$\text{RHL} = \lim\limits_{x \to 3^+} f(x) = \lim\limits_{h \to 0} f(3+h)$

$= \lim\limits_{h \to 0} [2(3+h)^2 - 1]$

$= 2(3)^2 - 1$

$= 17$

Since $\lim\limits_{x \to 3^-} f(x) = \lim\limits_{x \to 3^+} f(x) = f(3)$,

$\therefore$ $f(x)$ is continuous at $x = 3$.

**Example 1.2:** Is the function $f(x) = \begin{cases} x, \text{if } x \le 1 \\ 5, \text{if } x > 1 \end{cases}$ continuous at $x = 0$? At $x = 1$? At $x = 2$? **[NCERT]**

**Ans. Continuity of $f(x)$ at $x = 0$:**

Clearly the given function $f(x)$ is defined at $x = 0$ and its value is 0, *i.e.*, $f(0) = 0$

$\text{LHL} = \lim\limits_{x \to 0^-} f(x)$

$= \lim\limits_{h \to 0} f(0-h)$

$= \lim\limits_{h \to 0} [0-h]$

$= 0$

$\text{RHL} = \lim\limits_{x \to 0^+} f(x) = \lim\limits_{h \to 0} f(0+h)$

$= \lim\limits_{h \to 0} [0+h]$

$= 0$

Since, $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = f(0)$

$\therefore$ $f(x)$ is continuous at $x = 0$.

**Continuity of $f(x)$ at $x = 1$:**

Clearly the given function $f(x)$ is defined at $x = 1$ and its value is 1, *i.e.*, $f(1) = 1$.

$$\text{LHL} = \lim_{x \to 1^-} f(x)$$
$$= \lim_{h \to 0} f(1-h)$$
$$= \lim_{h \to 0} [1-h]$$
$$= 1$$
$$\text{RHL} = \lim_{x \to 1^+} f(x)$$
$$= \lim_{h \to 0} f(1+h)$$
$$= \lim_{h \to 0} [5]$$
$$= 5$$

Since $\lim_{x \to 1^-} f(x) \neq \lim_{x \to 1^+} f(x)$,

$\therefore$ $f(x)$ is not continuous at $x = 1$.

**Continuity of $f(x)$ at $x = 2$:**

Clearly, the given function $f(x)$ is defined at $x = 2$ and its value is 5, *i.e.*, $f(2) = 2$.

$$\text{LHL} = \lim_{x \to 2^-} f(x)$$
$$= \lim_{h \to 0} f(2-h)$$
$$= \lim_{h \to 0} [5]$$
$$= 5$$
$$\text{RHL} = \lim_{x \to 2^+} f(x)$$
$$= \lim_{h \to 0} f(2+h)$$
$$= \lim_{h \to 0} [5]$$
$$= 5$$

Since $\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x) = f(2)$,

$\therefore$ $f(x)$ is continuous at $x = 2$.

## Continuity of a Function in its Domain

A function $f(x)$ is said to be continuous in its domain, if it is continuous at every point of the domain.

**Continuity of a Function in an Interval**

(1) A function $f(x)$ is said to be continuous in an open interval $(a, b)$, if it is continuous at every $x \in (a, b)$.

(2) A function $f(x)$ is said to be continuous in a closed interval $[a, b]$, if

- it is continuous at every $x \in (a, b)$
- it is continuous at $x = a$ from the right hand side, and
- it is continuous at $x = b$ from the left hand side.

## Noteworthy Results on Continuous Functions

(1) A constant function $f(x) = k$, is continuous everywhere.

(2) Identity function $f(x) = x$, is continuous everywhere.

(3) A polynomial function $f(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$, $n \in \mathrm{N}$, $x \in \mathrm{R}$, is continuous everywhere.

(4) The modulus function $f(x) = |x|$ is continuous everywhere.

(5) The logarithmic function is continuous in its domain.

(6) The exponential function $f(x) = a^x$, $a > 0$ is continuous everywhere.

(7) The sine function $f(x) = \sin x$ and cosine function $f(x) = \cos x$ are everywhere continuous.

(8) The tangent function, cotangent function, secant function and cosecant function are continuous in their respective domains.

(9) All the six inverse trigonometric functions are continuous in their respective domains.

(10) A rational function $f(x) = \dfrac{g(x)}{h(x)}, h(x) \neq 0$ is continuous at every point of its domain.

(11) The sum, difference, product and quotient of two continuous functions is a continuous function.

(12) The composition of continuous functions is a continuous function.

**Example 1.3: Find all points so discontinuity of $f$, where $f$ is defined by**

$$f(x) = \begin{cases} x^3 - 3, \text{ if } x \leq 2 \\ x^2 + 1, \text{ if } x > 2 \end{cases}$$ **[NCERT]**

**Ans.** Clearly, the given function is defined at every real number. The domain of the function is partitioned in three disjoint subsets of the real line.

Let $D_1 = \{x \in \mathrm{R} : x < 2\}$; $D_2 = \{2\}$; $D_3 = \{x \in \mathrm{R} : x > 2\}$

**Case 1:** At any point in $D_1$, we have $f(x) = x^3 - 3$ (a polynomial).

In view of Result 3 stated above (A polynomial function is continuous everywhere in its domain), $f(x)$ is continuous in $D_1$.

**Case 2:** At any point in $D_3$, we have $f(x) = x^2 + 1$ (a polynomial).

Again, in view of Result 3 stated above, $f(x)$ is continuous in $D_3$.

**Case 3:** We shall now check the continuity of $f(x)$ at $x = 2$

Clearly, the given function $f(x)$ is defined at $x = 2$ and its value is 5.

$$\text{LHL} = \lim_{x \to 2^-} f(x)$$
$$= \lim_{h \to 0} f(2-h)$$
$$= (2-h)^3 - 3 = 5$$
$$\text{RHL} = \lim_{x \to 2^+} f(x)$$
$$= \lim_{h \to 0} f(2+h)$$
$$= (2+h)^2 + 1 = 5$$

Since $\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x) = f(2)$,

$f(x)$ continuous at $x = 2$.

Thus, the function $f(x)$ is continuous everywhere, *i.e.*, there is no point of discontinuity.

**Example 1.4:** Determine if $f$ defined by

$$f(x) = \begin{cases} x^2 \sin\frac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

**is a continuous function?** **[NCERT]**

**Ans.** We note that domain of $f$ is R. Let $c$ be any real number . Then, two cases arise:

**Case 1:** If $c \neq 0$, then

$$\lim_{x \to c} f(x) = \lim_{x \to c} \left[x^2 \sin\frac{1}{x}\right] = c^2 \sin\frac{1}{c} = f(c)$$

$\Rightarrow f(x)$ is continuous for $c \in$ R, where $c \neq 0$.

**Case 2:** If $c = 0$, then $f(c) = f(0) = 0$

Here, $\lim_{x \to 0^+} f(x) = \lim_{h \to 0} f(0+h)$

$$= \lim_{h \to 0}\left[(0+h)^2 \sin\frac{1}{0+h}\right]$$
$$= \lim_{h \to 0}\left[(h^2)\sin\frac{1}{h}\right] = 0$$
$$\left\{\because \left|\sin\frac{1}{h}\right| \leq 1\right\}$$

and $\lim_{x \to 0^-} f(x) = \lim_{h \to 0} f(0-h)$

$$= \lim_{h \to 0}\left[(0-h)^2 \sin\frac{1}{0-h}\right]$$
$$= \lim_{h \to 0}\left[(h)^2 \sin\frac{1}{-h}\right] = 0$$
$$\left\{\because \left|\sin\frac{1}{h}\right| \leq 1\right\}$$

Hence, $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = f(0)$

So, $f(x)$ is continuous for all $x \in$ R.

**Example 1.5:** Find the value of $k$ so that the function $f$ defined by

$$f(x) = \begin{cases} kx + 1, & \text{if } x \leq \pi \\ \cos x, & \text{if } x > \pi \end{cases}$$

**is a continuous function at $x = \pi$.** **[NCERT]**

**Ans.** Since $f$ is continuous at $x = \pi$,

$$\therefore \quad \lim_{x \to \pi^-} f(x) = \lim_{x \to \pi^+} f(x) = f(\pi)$$

Now, $\lim_{x \to \pi^-} f(x) = \lim_{x \to \pi^+} f(x)$

$$\Rightarrow \lim_{h \to 0} [k(\pi - h) + 1] = \lim_{h \to 0} \cos(\pi + h)$$
$$\Rightarrow \quad k\pi + 1 = -1$$
$$\Rightarrow \quad k = -\frac{2}{\pi}$$

## OBJECTIVE Type Questions

[ **1** mark ]

## Multiple Choice Questions

1. The function $f(x) = [x]$, where $[x]$ denotes the greatest integer function, is continuous at:

   (a) 4 (b) 1.5
   (c) 1 (d) – 2

2. The value of $k$ $(k < 0)$ for which the function $f$ defined as:

$$f(x) = \begin{cases} \frac{1 - \cos kx}{x \sin x}, & x \neq 0 \\ \frac{1}{2}, & x = 0 \end{cases}$$

is continuous at $x = 0$, is:

(a) ±1 (b) − 1

(c) $\pm\frac{1}{2}$ (d) $\frac{1}{2}$

[CBSE Term-1 SQP 2021]

**Ans.** *(b)* −1

$$\lim_{x\to 0}\left(\frac{1-\cos kx}{x\sin x}\right) = \frac{1}{2}$$

$$\Rightarrow \lim_{x\to 0}\left(\frac{2\sin^2\frac{kx}{2}}{x\sin x}\right) = \frac{1}{2}$$

$$\Rightarrow \lim_{x\to 0} 2\left(\frac{k}{2}\right)^2\left(\frac{\sin\frac{kx}{2}}{\frac{kx}{2}}\right)^2\left(\frac{x}{\sin x}\right) = \frac{1}{2}$$

$\Rightarrow k^2 = 1$ and $k = \pm 1$ but $k > 0$ and $k = -1$

[CBSE Marking Scheme Term-1 SQP 2021]

**Explanation:** $\because$ $f(x)$ is continuous at $x = 0$,

$\therefore$ L.H.L at $x = 0 = f(0)$

$$\Rightarrow \lim_{h\to 0}\frac{1-\cos k(0-h)}{(0-h)\sin(0-h)} = \frac{1}{2}$$

$$\Rightarrow \lim_{h\to 0}\frac{1-\cos kh}{h\sin h} = \frac{1}{2}$$

$$\left[\because \begin{matrix}\sin(-\theta) = -\sin\theta \\ \cos(-\theta) = \cos\theta\end{matrix}\right]$$

$$\Rightarrow \lim_{h\to 0}\frac{2\sin^2\frac{kh}{2}}{h\sin h} = \frac{1}{2}$$

$$[\because 1-\cos 2\theta = 2\sin^2\theta]$$

$$\Rightarrow \lim_{h\to 0}\frac{2\left(\frac{\sin kh/2}{kh/2}\right)^2\times\left(\frac{kh}{2}\right)^2}{h\left(\frac{\sin h}{h}\right)\times h} = \frac{1}{2}$$

$$\Rightarrow \frac{2\times 1\times\frac{k^2h^2}{4}}{h^2} = \frac{1}{2}$$

$$\left[\because \lim_{h\to 0}\frac{\sin h}{h} = 1\right]$$

$$\Rightarrow \frac{k^2}{2} = \frac{1}{2}$$

$$\Rightarrow k^2 = 1$$

$$\Rightarrow k = \pm 1$$

But $k < 0$ [given]

$\therefore$ $k = -1$

**3.** The function $f(x) = \cot x$ is discontinuous on the set

(a) $\{x = n\pi : n \in Z\}$

(b) $\{x = 2n\pi : n \in Z\}$

(c) $\left\{x = (2n+1)\frac{\pi}{2} : n \in Z\right\}$

(d) $\left\{x = \frac{n\pi}{2}; n \in Z\right\}$ [NCERT Exemplar]

**4.** Let $f(x) = \begin{cases} -1+a, & x < 4 \\ a+b, & x = 4 \\ 1+b, & x > 4 \end{cases}$. Then, $f(x)$ is continuous at $x = 4$, when:

(a) $a = 0, b = 0$ (b) $a = 1, b = 1$

(c) $a = -1, b = 1$ (d) $a = 1, b = -1$

**Ans.** *(d)* $a = 1, b = -1$

**Explanation:** Here, $f(x)$ is given by

$$f(x) = \begin{cases} -1+a, & x < 4 \\ a+b, & x = 4 \\ 1+b, & x > 4 \end{cases}$$

Since $f(x)$ is continuous at $x = 4$,

$$\therefore \lim_{x\to 4^-} f(x) = \lim_{x\to 4^+} f(x) = f(4)$$

$$\Rightarrow \lim_{h\to 0}[-1+a] = \lim_{h\to 0}[1+b] = a+b$$

$$\Rightarrow -1+a = 1+b = a+b$$

$$\Rightarrow a = 1, b = -1$$

**5.** Let $f(x) = |x| + |x-1|$. Then, $f(x)$ is:

(a) continuous at $x = 0$, as well as $x = 1$

(b) continuous at $x = 0$, but not at $x = 1$

(c) continuous at $x = 1$, but not at $x = 0$

(d) none of these

**6.** If $f(x) = x^2 \sin\frac{1}{x}$, where $x \neq 0$, then the value of the function $f$ at $x = 0$, so that the function is continuous at $x = 0$, is:

(a) 0 (b) −1

(c) 1 (d) none of these

[NCERT Exemplar]

**7.** The value of $a$ for which the function $f(x) = \begin{cases} 5x-4, & \text{if } 0 < x \le 1 \\ 4x^2+3ax, & \text{if } 1 < x < 2 \end{cases}$ is continuous at every point of its domain, is:

(a) $\frac{13}{3}$ (b) 1

(c) −1 (d) 0

**Ans.** *(c)* −1

**Explanation:** Since the given function is continuous everywhere, so $f(x)$ is continuous at $x = 1$.

$\Rightarrow \quad \lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = f(1)$

$\Rightarrow \lim_{h \to 0} [5(1-h) - 4] = \lim_{h \to 0} [4(1+h)^2 + 3a(1+h)] = 1$ {Here, $f(1) = 1$}

$\Rightarrow \quad 1 = 4 + 3a = 1$

$\Rightarrow \quad a = -1$

**8.** If $f(x) = \frac{1}{1-x}$, then the point of discontinuity of the function $f(f(f(x)))$ is:

(a) {1} (b) {0, 1}

(c) {– 1, 1} (d) none of these

**Ans.** *(b) {0, 1}*

**Explanation:** Here, $f(x)$ is not defined at $x = 1$. So, $x = 1$ is a point of discontinuity of the function $f(f(f(x)))$.

Further, $f(f(x)) = \frac{1}{1-\frac{1}{1-x}} = \frac{1-x}{-x} = 1 - \frac{1}{x}$ is not defined at $x = 0$.

So, $x = 0$ is a point of discontinuity of the function $f(f(f(x)))$.

Thus, $x = 0$ and $x = 1$ are the two points of discontinuity of the function $f(f(f(x)))$.

**9.** The function $f(x) = |1 - x + |x||$ is continuous

(a) only at $x = 0$

(b) only at $x = 1$

(c) at $x = 0$ as well as at $x = 1$

(d) everywhere

**10.** If $f(x) = \begin{cases} mx+1, \text{if } x \le \frac{\pi}{2} \\ \sin x + n, \text{if } x > \frac{\pi}{2} \end{cases}$ is continuous at $x = \frac{\pi}{2}$, then:

(a) $m = 1, n = 0$ (b) $m = \frac{n\pi}{2} + 1$

(c) $n = \frac{m\pi}{2}$ (d) $m = n = \frac{\pi}{2}$

[NCERT Exemplar]

**Ans.** (c) $n = \frac{m\pi}{2}$

**Explanation:**

$$\text{LHL} = \lim_{x \to \frac{\pi^-}{2}} (mx + 1) = \lim_{h \to 0} \left[m\left(\frac{\pi}{2} - h\right) + 1\right]$$

$$= \frac{m\pi}{2} + 1$$

$$\text{RHL} = \lim_{x \to \frac{\pi^+}{2}} (\sin x + n) = \lim_{h \to 0} \left[\sin\left(\frac{\pi}{2} + h\right) + n\right]$$

$$= \lim_{h \to 0} \cos h + n = 1 + n$$

$\because f(x)$ is continuous at $x = \frac{\pi}{2}$,

$\therefore \quad$ LHL = RHL

$\Rightarrow \quad \frac{m\pi}{2} + 1 = n + 1$

$\Rightarrow \quad n = \frac{m\pi}{2}$

**11.** If $f(x) = \begin{cases} \frac{\sqrt{1+kx} - \sqrt{1-kx}}{x}, & -1 \le x < 0 \\ \frac{2x+1}{x-1}, & 0 \le x < 1 \end{cases}$

is continuous at $x = 0$, then the value of $k$ is

(a) 1 (b) 2

(c) –2 (d) –1

**12.** The point(s), at which the function $f$ given by $f(x) = \begin{cases} \frac{x}{|x|}, & x < 0 \\ -1, & x \ge 0 \end{cases}$ is continuous, is/are:

(a) $x \in R$ (b) $x = 0$

(c) $x \in R - \{0\}$ (d) $x = -1$ and 1

[CBSE Term-1 SQP 2021]

**Ans.** *(a) $x \in R$*

$$f(x) = \begin{cases} \frac{x}{-x} = -1 & x < 0 \\ -1 & x \ge 0 \end{cases}$$

$\Rightarrow \quad f(x) = -1 \ \forall \ x \in R$

$\Rightarrow f(x)$ is continuous $\forall \ x \in R$ as it is a constant function.

[CBSE Marking Scheme Term-1 SQP 2021]

**Explanation:** We have,

$$f(x) = \begin{cases} \frac{x}{|x|}, x < 0 \\ -1, x \ge 0 \end{cases}$$

$$= \begin{cases} \frac{x}{-x}, x < 0 \\ -1, x \ge 0 \end{cases}$$

$$= \begin{cases} -1, x < 0 \\ -1, x \ge 0 \end{cases}$$

$\Rightarrow \quad f(x) = -1, x \in R.$

$\Rightarrow f(x)$ is a constant function.

$\Rightarrow f(x)$ is a continuous function at $x \in R$.

13. If $g(x) = \begin{cases} k(x^2 + 3x), & \text{if } x \le 0 \\ 5x + 2, & \text{if } x > 0 \end{cases}$, then which one of the following is correct ?

(a) $g(x)$ is continuous at $x = 0$, for any value of $k$.

(b) $g(x)$ is discontinuous at $x = 0$, for any value of $k$.

(c) $g(x)$ is discontinuous at $x = 1$, for any value of $k$.

(d) None of these

14. Param observed that the shape of the mountain valley he had visited along with his friends while trekking was exactly like the English alphabet 'V'. He took a picture of the valley and on returning, suggested to his friends that the two lines a and b can be represented by functions as they are straight lines meeting at the Y-axis.

If $f(x) = \begin{cases} 1 + x, & \text{if } x \ge 0 \\ 1 - x, & \text{if } x < 0 \end{cases}$ is continuous at $x = 0$, then the value of $f(0)$ is

(a) 0 (b) 1

(c) 2 (d) -1

**Ans.** *(b) 1*

**Explanation:** Since $f(x)$ is continuous at $x = 0$,

$$\therefore \lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = f(0)$$

$$\Rightarrow \lim_{h \to 0} [1 - (0 - h)] = \lim_{h \to 0} [1 + (0 + h)] = f(0)$$

$$\Rightarrow 1 = 1 = f(0)$$

$$\Rightarrow f(0) = 1$$

## VERY SHORT ANSWER Type Questions (VSA)

[1 mark]

15. Examine the continuity of the function $f(x) = x^2 + 6$ at $x = -2$.

**Ans.** We have, $f(x) = x^2 + 6$

We have to check the continuity at $x = -2$.

Now, $f(-2) = (-2)^2 + 6$

$= 4 + 6 = 10$

$$\text{LHL} = \lim_{x \to -2^-} f(x) = \lim_{x \to -2^-} (x^2 + 6)$$

$$= \lim_{h \to 0} [(-2 - h)^2 + 6]$$

$$= (-2)^2 + 6$$

$$= 4 + 6 = 10$$

and $\text{RHL} = \lim_{x \to -2^+} f(x)$

$$= \lim_{x \to -2^+} (x^2 + 6)$$

$$= \lim_{h \to 0} [(-2 + h)^2 + 6]$$

$$= (-2)^2 + 6 = 10$$

Here, $f(-2) = \text{LHL} = \text{RHL}$, so $f(x)$ is continuous at $x = -2$.

16. Show that function $f(x) = [x]$, where $x$ is a greatest integer function, is continuous at $x = 2.7$.

17. If $f(x) = \begin{cases} 3x + 5, & \text{if } x \le 3 \\ 3x - 5, & \text{if } x > 3 \end{cases}$, then find point of discontinuity.

**Ans.** Given: $f(x) = \begin{cases} 3x + 5, & \text{if } x \le 3 \\ 3x - 5, & \text{if } x > 3 \end{cases}$

Given function is a polynomial function so it is continuous every where except at $x = 3$.

Now, we have to check the continuity only at $x = 3$.

$f(3) = 3(3) + 5$

$= 9 + 5 = 14$

$$\text{RHL} = \lim_{x \to 3^+} f(x)$$

$$= \lim_{x \to 3^+} (3x - 5)$$

$$= \lim_{h \to 0} [3(3-h) - 5]$$
$$= [3(3-0) - 5]$$
$$= 9 - 5 = 4$$

Here $f(3) \neq$ RHL

Hence, $f(x)$ is discontinuous at $x = 3$.

**18.** **Examine the continuity of the $f(x) = \dfrac{1}{x+4}, x \in R$.**

**19.** **Find the value of $\lambda$ so that the function $f$ defined by**

$$f(x) = \begin{cases} \lambda x, & \text{if } x \le \pi \\ \cos x, & \text{if } x > \pi \end{cases}$$

**is continuous at $x = \pi$.** **[CBSE 2020]**

**Ans.** We have, $f(x) = \begin{cases} \lambda x, & \text{if } x \le \pi \\ \cos x, & \text{if } x > \pi \end{cases}$

Since, $f(x)$ is continuous at $x = \pi$.

$\therefore \quad f(\pi) = \lim_{x \to \pi^-} f(x) = \lim_{x \to \pi^+} f(x)$ ...(i)

Now $f(\pi) = \lambda\pi$ ...(ii)

$$\text{RHL} = \lim_{x \to \pi^+} f(x)$$
$$= \lim_{h \to 0} f(\pi + h)$$
$$= \lim_{h \to 0} \cos(\pi + h)$$
$$= -\lim_{h \to 0} \cos h = -\cos 0$$
$$= -1 \quad \text{...(iii)}$$

Thus, from eqs. (i), (ii) and (iii), we get

$$\lambda\pi = -1$$
$$\Rightarrow \quad \lambda = -\frac{1}{\pi}$$

**20.** **Determine the value of $k$ for which the following function is continuous at $x = 4$.**

$$h(x) = \begin{cases} \dfrac{x^2-16}{x-4}, & x \neq 4 \\ k, & x = 4 \end{cases}$$

**21.** **Determine the value of the constant $k$ so that the function**

$$f(x) = \begin{cases} \dfrac{kx}{|x|}, & \text{if } x < 0 \\ 3, & \text{if } x \ge 0 \end{cases}$$

**is continuous at $x = 0$.** **[CBSE 2017]**

**Ans.** Given: $f(x) = \begin{cases} \dfrac{kx}{|x|}, & \text{if } x < 0 \\ 3, & \text{if } x \ge 0 \end{cases}$

Since $f(x)$ is continuous at $x = 0$

$\therefore \quad f(0) = \lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x)$ ...(i)

Now $f(0) = 3$ ...(ii)

$$\text{LHL} = \lim_{x \to 0^-} f(x)$$
$$= \lim_{x \to 0^-} \frac{kx}{|x|}$$
$$= \lim_{h \to 0^-} \frac{k(0-h)}{|0-h|}$$
$$= \lim_{h \to 0^-} \frac{-kh}{h} = \lim_{h \to 0} (-k)$$
$$= -k \quad \text{...(iii)}$$

Thus from eqs. (i), (ii) and (iii), we get

$$3 = -k \Rightarrow k = -3$$

**22.** **Find the value $k$, if**

$$f(x) = \begin{cases} |x-2|, & x \neq 3 \\ k, & x = 3 \end{cases}$$

**is continuous at $x = 3$.**

**23.** **Let R be the set of all real numbers. Find the value of $\lambda$ in which the function $f: R \to R$ defined by**

$$f(x) = \begin{cases} \dfrac{x^2-2x-3}{x+1}, & \text{where } x \neq -1 \\ \lambda, & \text{when } x = -1 \end{cases}$$

**is continuous at $x = -1$.**

**Ans.** We have, $f(x) = \begin{cases} \dfrac{x^2-2x-3}{x+1}, & \text{where } x \neq -1 \\ \lambda, & \text{when } x = -1 \end{cases}$

Since, $f(x)$ is continuous at $x = -1$.

$$\therefore \quad f(-1) = \lim_{x \to -1} f(x)$$
$$\Rightarrow \quad \lambda = \lim_{x \to -1} \frac{x^2-2x-3}{x+1}$$
$$= \lim_{x \to -1} \frac{(x-3)(x+1)}{(x+1)}$$
$$= \lim_{x \to -1} (x-3)$$
$$= -1 - 3$$

Hence, $\lambda = -4$

**24.** **Discuss the continuity of the function $f(x) = \sin|x|$.**

**25.** Determine the value of '$k$' for which the following function is continuous at $x = 3$.

$$f(x) = \begin{cases} \dfrac{(x+3)^2 - 36}{x-3} &, x \neq 3 \\ k &, x = 3 \end{cases}$$ [CBSE 2017]

**26.** If $g(x) = \begin{cases} \dfrac{\cos x}{x - \frac{\pi}{2}} &, x \neq \frac{\pi}{2} \\ \lambda &, x = \frac{\pi}{2} \end{cases}$ is continuous at $x = \frac{\pi}{2}$, then find the value of $\lambda$.

## SHORT ANSWER Type-I Questions (SA-I)

[ 2 marks ]

**27.** Discuss the continuity of the function $f(x)$ given by $f(x) = \begin{cases} 4 - x, & x < 4 \\ 4 + x, & x \geq 4 \end{cases}$ at $x = 4$.

**Ans.** We have $f(x) = \begin{cases} 4 - x, & x < 4 \\ 4 + x, & x \geq 4 \end{cases}$

$$\text{LHL} = \lim_{x \to 4^-} f(x)$$
$$= \lim_{x \to 4^-} (4 - x)$$
$$= \lim_{x \to 4^-} 4 - (4 - h)$$
$$= 0$$
$$\text{RHL} = \lim_{x \to 4^+} f(x)$$
$$= \lim_{x \to 4^+} (4 + x)$$
$$= \lim_{h \to 0} 4 + (h + 4)$$
$$= 8 + 0 = 8$$

Here LHL $\neq$ RHL. Hence, $f(x)$ is not continuous at $x = 4$.

**28.** Find whether the following function $f(x)$ is continuous or discontinuous at $x = 0$.

$$f(x) = \begin{cases} \dfrac{1 - \cos 2x}{x^2} &, \text{if } x \neq 0 \\ 5 &, \text{if } x = 0 \end{cases}$$

[NCERT Exemplar]

**Ans.** At $x = 0$,

$$\text{L.H.L} = \lim_{x \to 0^-} \frac{1 - \cos 2x}{x^2}$$
$$= \lim_{h \to 0} \frac{1 - \cos 2(0 - h)}{(0 - h)^2}$$
$$= \lim_{h \to 0} \frac{1 - \cos 2h}{h^2}$$
$$= \lim_{h \to 0} \frac{2\sin^2 h}{h^2}$$
$$= 2 \lim_{h \to 0} \left(\frac{\sin h}{h}\right)^2$$
$$= 2 \times 1 \qquad \left[\because \lim_{h \to 0} \frac{\sin h}{h} = 1\right]$$
$$= 2$$

But $f(0) = 5$

Since, L.H.L $\neq f(0)$

Hence, $f(x)$ is discontinuous at $x = 0$.

### Caution

➥ *A function is discontinuous if $\lim_{x \to a^-} f(x)$, $\lim_{x \to a^+} f(x)$ exist and are equal but not equal to $f(a)$.*

**29.** Text the continuity of the function at $x = 3$,

when $g(x) = \begin{cases} \dfrac{|x - 3|}{x - 3} &, x \neq 3 \\ 1 &, x = 3 \end{cases}$

**30.** Show that

$$g(x) = \begin{cases} 6x - 5 &, \text{when } 0 < x \leq 1 \\ 7x^3 - 6x &, \text{when } 1 < x < 2 \end{cases}$$

is continuous at $x = 1$.

**Ans.** We have, $g(x) = \begin{cases} 6x - 5 &, \text{when } 0 < x \leq 1 \\ 7x^3 - 6x &, \text{when } 1 < x < 2 \end{cases}$

Now $g(1) = 6(1) - 5$

$= 1$ ...(i)

$$\text{LHL} = \lim_{x \to 1^-} (6x - 5)$$
$$= \lim_{h \to 0} [6(1 - h) - 5]$$

$= 6(1 - 0) - 5 = 1$ ...(ii)

$$\text{RHL} = \lim_{x \to 1^+} (7x^3 - 6x)$$

$$= \lim_{h \to 0} [7(1+h)^3 - 6(1+h)]$$
$$= [7(1+0)^3 - 6(1+0)]$$
$$= 7 - 6 = 1 \quad ...(iii)$$

Thus, from eqs. (i), (ii) and (iii), we get

$g(1) = \text{LHL} = \text{RHL}$

Hence, $g(x)$ is continuous at $x = 1$.

**31. Show that the function $f(x)$ given by**

$$f(x) = \begin{cases} \dfrac{\tan x}{x} + \cos x & , x \neq 0 \\ 2 & , x = 0 \end{cases}$$

**is continuous at $x = 0$.**

**Ans.** We have, $f(x) = \begin{cases} \dfrac{\tan x}{x} + \cos x & , x \neq 0 \\ 2 & , x = 0 \end{cases}$

Now, $f(0) = 2 \quad ...(i)$

$$\text{LHL} = \lim_{x \to 0^-} f(x)$$
$$= \lim_{x \to 0^-} \left(\frac{\tan x}{x} + \cos x\right)$$
$$= \lim_{h \to 0} \left[\frac{\tan(0-h)}{(0-h)} + \cos(0-h)\right]$$
$$= \lim_{h \to 0} \left[-\frac{\tan h}{-h} + \cos h\right]$$
$$= \lim_{h \to 0} \frac{\tan h}{h} + \lim_{h \to 0} \cos h$$
$$= 1 + \cos(0) = 1 + 1$$
$$= 2 \quad ...(ii)$$
$$\text{RHL} = \lim_{x \to 0^+} f(x)$$
$$= \lim_{x \to 0^+} \left(\frac{\tan x}{x} + \cos x\right)$$
$$= \lim_{h \to 0} \left[\frac{\tan(0+h)}{0+h} + \cos(0+h)\right]$$
$$= \lim_{h \to 0} \frac{\tan h}{h} + \lim_{h \to 0} \cos h$$
$$= 1 + \cos(0) = 1 + 1$$
$$= 2 \quad ...(iii)$$

Thus, from eqs. (i), (ii) and (iii), we get

$f(0) = \text{LHL} = \text{RHL}$.

Hence proved.

**32. Find whether the given following function is continuous or discontinuous at $x = a$.**

$$f(x) = \begin{cases} |x-a| \sin \dfrac{1}{x-a} & , \text{if } x \neq a \\ 0 & , \text{if } x = 0 \end{cases}$$

**[NCERT Exemplar]**

**Ans.** $\text{L.H.L.} = \lim_{x \to a^-} |x-a| \sin \dfrac{1}{x-a}$

$$= \lim_{h \to 0} |a-h-a| \sin \frac{1}{a-h-a}$$
$$= \lim_{h \to 0} h . \sin \frac{1}{-h}$$
$$= \lim_{h \to 0} \left(-h \sin \frac{1}{h}\right)$$

$= 0 \times$ [a number oscillating between −1 and 1]

$= 0$

$$\text{R.H.L.} = \lim_{x \to a^+} |x-a| \sin \frac{1}{x-a}$$
$$= \lim_{h \to 0} |a+h-a| \sin \frac{1}{a+h-a}$$
$$= \lim_{h \to 0} h \sin \frac{1}{h} = 0$$

Also, $f(a) = \lim_{x \to a} f(x) = 0$

$\therefore \lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = \lim_{x \to a} f(x) = 0$

Hence, $f(x)$ is continuous at $x = a$.

**33. Show that the function defined by $f(x) = \sin x^2$ is a continuous function.**

## SHORT ANSWER Type-II Questions (SA-II)

[ 3 marks ]

**34. Find whether the following function $f(x)$ is continuous or discontinuous at $x = 2$.**

$$f(x) = \begin{cases} \dfrac{2x^2 - 3x - 2}{x-2}, & \text{if } x \neq 2 \\ 5, & \text{if } x = 2 \end{cases}$$

**[NCERT Exemplar]**

**Ans.** At $x = 2$,

$$\text{L.H.L.} = \lim_{x \to 2^-} \frac{2x^2 - 3x - 2}{x-2}$$
$$= \lim_{h \to 0} \frac{2(2-h)^2 - 3(2-h) - 2}{(2-h) - 2}$$
$$= \lim_{h \to 0} \frac{8 + 2h^2 - 8h - 6 + 3h - 2}{-h}$$

$$= \lim_{h\to 0} \frac{2h^2-5h}{-h}$$

$$= \lim_{h\to 0} \frac{h(2h-5)}{-h}$$

$$= \lim_{h\to 0} (-2h + 5)$$

$$= 5$$

$$\text{R.H.L.} = \lim_{x\to 2^+} \frac{2x^2-3x-2}{x-2}$$

$$= \lim_{h\to 0} \frac{2(2+h)^2-3(2+h)-2}{(2+h)-2}$$

$$= \lim_{h\to 0} \frac{8+2h^2+8h-6-3h-2}{h}$$

$$= \lim_{h\to 0} \frac{2h^2+5h}{h}$$

$$= \lim_{\to} (2h + 5) = 5$$

Also, $f(2) = 5$

$\therefore$ L.H.L = R.H.L. = $f(2)$

So, $f(x)$ is continuous at $x = 2$.

**35. Find the values of $p$ and $q$, for which**

$$f(x) = \begin{cases} \dfrac{1-\sin^3 x}{3\cos^2 x} & \text{, if } x < \dfrac{\pi}{2} \\ p & \text{, if } x = \dfrac{\pi}{2} \\ \dfrac{q(1-\sin x)}{(\pi-2x)^2} & \text{, if } x > \dfrac{\pi}{2} \end{cases}$$

**is continuous at $x = \dfrac{\pi}{2}$. [CBSE 2016]**

**Ans.** We have, $f(x) = \begin{cases} \dfrac{1-\sin^3 x}{3\cos^2 x} & \text{, if } x < \dfrac{\pi}{2} \\ p & \text{, if } x = \dfrac{\pi}{2} \\ \dfrac{q(1-\sin x)}{(\pi-2x)^2} & \text{, if } x > \dfrac{\pi}{2} \end{cases}$

Since, $f(x)$ is continuous at $\dfrac{\pi}{2}$,

$$\therefore \quad \lim_{x\to \frac{\pi^-}{2}} f(x) = \lim_{x\to \frac{\pi^+}{2}} f(x) = f\left(\frac{\pi}{2}\right) \quad \text{...(i)}$$

Now, $f = p$ ...(ii)

$$\text{LHL} = \lim_{x\to \frac{\pi^-}{2}} f(x)$$

$$= \lim_{h\to 0} f\left(\frac{\pi}{2}-h\right)$$

$$= \lim_{h\to 0} \frac{1-\sin^3\left(\frac{\pi}{2}-h\right)}{3\cos^2\left(\frac{\pi}{2}-h\right)}$$

$$= \lim_{h\to 0} \frac{1-\cos^3 h}{3\sin^2 h}$$

$$= \lim_{h\to 0} \frac{(1-\cos h)(1+\cos^2 h+\cos h)}{3(1-\cos h)(1+\cos h)}$$

$$= \frac{1+\cos^2 0+\cos 0}{3(1+\cos 0)} = \frac{1+1+1}{3(1+1)}$$

$$= \frac{1}{2} \quad \text{...(iii)}$$

and

$$\lim_{x\to \frac{\pi^+}{2}} f(x) = \lim_{h\to 0} f\left(\frac{\pi}{2}+h\right)$$

$$= \lim_{h\to 0} \frac{q\left[1-\sin\left(\frac{\pi}{2}+h\right)\right]}{\left[\pi-2\left(\frac{\pi}{2}+h\right)\right]^2}$$

$$= \lim_{h\to 0} \frac{q(1-\cos h)}{4h^2}$$

$$= \frac{q}{4}\times \lim_{h\to 0} \frac{2\sin^2\frac{h}{2}}{\frac{h^2}{4}\times 4}$$

$$= \frac{q}{4}\times\frac{2}{4} = \frac{q}{8} \quad \text{...(iv)}$$

Thus from eqs. (i), (ii), (iii) and (iv), we get

$$\frac{1}{2} = \frac{q}{8} = p$$

Consider $\dfrac{1}{2} = \dfrac{q}{8}, \dfrac{1}{2} = p$

$\Rightarrow \quad q = 4$ and $p = \dfrac{1}{2}$

**36. If the following function $f(x)$ is continuous at $x = 0$, then find the value of $k$. $f(x) =$**

$$\begin{cases} \dfrac{1-\cos kx}{x\sin x} & \text{, if } x \neq 0 \\ \dfrac{1}{2} & \text{, if } x = 0 \end{cases}$$

**[NCERT Exemplar]**

**Ans.** We have,

$$\lim_{x\to0^-} f(x) = \lim_{x\to0^-} \frac{1-\cos kx}{x\sin x}$$

$$= \lim_{h\to0} \frac{1-\cos k(0-h)}{(0-h)\sin(0-h)}$$

$$= \lim_{h\to0} \frac{1-\cos(-kh)}{-h\sin(-h)}$$

$$= \lim_{h\to0} \frac{1-\cos kh}{h\sin h}$$

$[\because \cos(-\theta) = \cos\theta, \sin(-\theta) = -\sin\theta]$

$$= \lim_{h\to0} \frac{2\sin^2\frac{kh}{2}}{h\sin h}$$

$$= 2\lim_{h\to0} \left(\frac{\sin\frac{kh}{2}}{\frac{kh}{2}}\right)^2 \times \left(\frac{kh}{2}\right)^2 \times \frac{1}{h} \times \frac{1}{\frac{\sin h}{h}\times h}$$

$$= 2\times \lim_{kh\to0} \left(\frac{\sin\frac{kh}{2}}{\frac{kh}{2}}\right)^2 \times \lim_{h\to0} \frac{1}{\frac{\sin h}{h}} \times \lim_{h\to0} \left(\frac{kh}{2}\right)^2 \times \frac{1}{h^2}$$

$$= 2\times1\times\frac{1}{1}\times\frac{k^2}{4}$$

$$= \frac{k^2}{2}$$

And, $\lim_{x\to0} f(x) = \frac{1}{2}$

Since the function is continuous at $x = 0$,

$\therefore \quad \lim_{x\to0^-} f(x) = \lim_{x\to0} f(x)$

$\Rightarrow \quad \frac{k^2}{2} = \frac{1}{2}$

$\Rightarrow \quad k = \pm1$

Hence, the value of $k$ is $\pm1$.

**37.** **Find the value of the constant $k$ so that the function $f$, defined below, is continuous at $x = 0$, where**

$$f(x) = \begin{cases} \dfrac{1-\cos 4x}{8x^2} & , \text{if } x \neq 0 \\ k & , \text{if } x = 0 \end{cases}$$

**[CBSE 2014]**

**38. Find whether the following function $f(x)$ is continuous or discontinuous at $x = 0$. $f(x) =$**

$$\begin{cases} \dfrac{e^{\frac{1}{x}}}{1+e^{\frac{1}{x}}} & ,\text{if } x \neq 0 \\ 0 & ,\text{if } x = 0 \end{cases}$$

**[NCERT Exemplar]**

**Ans.** $$\text{L.H.L.} = \lim_{x\to0^-} \frac{e^{\frac{1}{x}}}{1+e^{\frac{1}{x}}}$$

$$= \lim_{h\to0} \frac{e^{\frac{1}{0-h}}}{1+e^{\frac{1}{0-h}}}$$

$$= \lim_{h\to0} \frac{e^{-1/h}}{1+e^{-1/h}}$$

$$= \lim_{h\to0} \frac{1}{e^{1/h}(1-e^{-1/h})}$$

$$= \lim_{h\to0} \frac{1}{e^{1/h}-1} = \frac{1}{e^{1/0}-1}$$

$$= \frac{1}{e^{\infty}-1} = \frac{1}{0-1} = -1 \quad [\because e^{\infty} = 0]$$

$$\text{R.H.L.} = \lim_{x\to0^+} \frac{e^{\frac{1}{x}}}{1+e^{\frac{1}{x}}}$$

$$= \lim_{h\to0} \frac{e^{\frac{1}{0+h}}}{1+e^{\frac{1}{0+h}}} = \lim_{h\to0} \frac{e^{\frac{1}{h}}}{1+e^{\frac{1}{h}}}$$

$$= \lim_{h\to0} \frac{1}{e^{-1/h}(1+e^{1/h})}$$

$$= \lim_{h\to0} \frac{1}{e^{-1/h}+1}$$

$$= \frac{1}{e^{-\infty}+1} = \frac{1}{0+1} = 1 \quad [\because e^{-\infty} = 0]$$

Also, $\lim_{x\to0} f(x) = 0$

As $\lim_{x\to0^-} f(x) \neq \lim_{x\to0^+} f(x) \neq \lim_{x\to0} f(x)$

Hence, $f(x)$ is discontinuous at $x = 0$.

### ⚠ Caution

➥ *Calculate all the three limits $\lim_{x\to a^-} f(x)$, $\lim_{x\to a^+} f(x)$ and $\lim_{x\to a} f(x)$ for checking the discontinuity of the function.*

**39. Discuss the continuity of the function $f(x) = \cos x$.**

**40.** **For what value of $a$ is the function $f$ defined by** $f(x) = \begin{cases} a\sin\frac{\pi}{2}(x+1) & , x \le 0 \\ \frac{\tan x - \sin x}{x^3} & , x > 0 \end{cases}$

**continuous at $x = 0$?** **[CBSE 2011]**

**Ans.** We have, $f(x) = \begin{cases} a\sin\frac{\pi}{2}(x+1) & , x \le 0 \\ \frac{\tan x - \sin x}{x^3} & , x > 0 \end{cases}$

Since, $f(x)$ is continuous at $x = 0$,

$$\therefore \quad f(0) = \lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) \quad \text{...(i)}$$

Now,

$$f(0) = a\sin\frac{\pi}{2}(0+1)$$

$$= a\sin\frac{\pi}{2} = a \quad \text{...(ii)}$$

$$\text{RHL} = \lim_{x \to 0^+} f(x)$$

$$= \lim_{x \to 0^+} \frac{\tan x - \sin x}{x^3}$$

$$= \lim_{h \to 0} \frac{\tan(0+h) - \sin(0+h)}{(0+h)^3}$$

$$= \lim_{h \to 0} \frac{\tan h - \sin h}{h^3}$$

$$= \lim_{h \to 0} \frac{\frac{\sin h}{\cos h} - \sin h}{h^3}$$

$$= \lim_{h \to 0} \frac{\sin h\left(\frac{1}{\cos h} - 1\right)}{h^3}$$

$$= \lim_{h \to 0} \frac{\sin h}{h} \times \lim_{h \to 0} \frac{1 - \cos h}{(\cos h)h^2}$$

$$= \lim_{x \to 0} \frac{\sin h}{h} \times \lim_{h \to \frac{0}{2}} \frac{2\sin^2\frac{h}{2}}{h^2 \cos h}$$

$$= \frac{1}{2} \lim_{h \to 0} \frac{\sin h}{h} \times \lim_{x \to 0} \left(\frac{\sin\frac{h}{2}}{\frac{h}{2}}\right) \times \lim_{h \to 0} \frac{1}{\cos h}$$

$$= \frac{1}{2} \times 1 \times 1 \times 1$$

$$= \frac{1}{2} \quad \text{...(iii)}$$

Thus, from Eqs. (i), (ii) and (iii), we get

$$a = \frac{1}{2}$$

**41.** **If the function $f(x)$ given by**

$$f(x) = \begin{cases} 3ax + b & , \text{if } x > 1 \\ 11 & , \text{if } x = 1 \\ 5ax - 2b & , \text{if } x < 1 \end{cases}$$

**is continuous at $x = 1$, find the values of $a$ and $b$.** **[CBSE 2012]**

**42.** **Function** $f(x) = \dfrac{(5^x - 1)^3}{\sin\frac{x}{5}\log\left(1 + \frac{x^2}{3}\right)}$ **is continuous at $x = 0$, find the value of $f(0)$.**

**Ans.** We have, $f(x) = \dfrac{(5^x - 1)^3}{\sin\frac{x}{5}\log\left(1 + \frac{x^2}{3}\right)}$

Since, $f(x)$ is continuous at $x = 0$, therefore

$$f(0) = \lim_{x \to 0} f(x)$$

$$= \lim_{x \to 0} \frac{(5^x - 1)^3}{\sin\frac{x}{5}\log\left(1 + \frac{x^2}{3}\right)}$$

$$= \lim_{x \to 0} \frac{\left(\frac{5^x - 1}{x}\right)^3}{\left(\frac{\sin\frac{x}{5}}{5 \times \frac{x}{5}}\right)\left(\frac{\log\left(1 + \frac{x^2}{3}\right)}{\frac{x^2}{3} \times 3}\right)}$$

$$= \lim_{x \to 0} \frac{\left(\frac{5^x - 1}{x}\right)^3}{\frac{1}{5}\log_{x \to 0}\left(\frac{\sin\frac{x}{5}}{\sin\frac{x}{5}}\right) \times \frac{1}{3}\lim_{x \to 0} \frac{\log\left(1 + \frac{x^2}{3}\right)}{\frac{x^2}{3}}}$$

$$= \frac{(\log_e 5)^3}{\frac{1}{5} \times \frac{1}{3}} = 15(\log_e 5)^3$$

**43.** **If the function $g(x)$ defined by**

$$g(x) = \begin{cases} \frac{\log(1 + ax) - \log(1 - bx)}{x} & , \text{if } x \ne 0 \\ k & , \text{if } x = 0 \end{cases}$$

**is continuous at $x = 0$, then find the value of $k$.**

**Ans.** We have

$$f(x)=\begin{cases}\dfrac{\log(1+ax)-\log(1-bx)}{x} & ,\text{if } x\neq 0\\ k & ,\text{if } x=0\end{cases}$$

Since, $f(x)$ is continuous at $x = 0$

$\therefore \quad f(0) = \lim_{x\to 0} f(x) \quad$ ...(i)

Now $\quad f(0) = k \quad$ ...(ii)

and $\lim_{x\to 0} f(x)$

$$= \lim_{x\to 0}\frac{\log(1+ax)-\log(1-bx)}{x}$$

$$= \lim_{x\to 0}\frac{\log(1+ax)}{x} - \lim_{x\to 0}\frac{\log(1-bx)}{x}$$

$$= a\lim_{x\to 0}\frac{\log(1+ax)}{ax} - (-b)\lim_{x\to 0}\frac{(1-bx)}{(-bx)}$$

$$= a(1) + b(1)$$

$$= a + b \quad \text{...(iii)}$$

Thus from Eqs. (i), (ii) and (iii), we get

$$k = a + b$$

# LONG ANSWER Type Questions (LA)

[ 4 & 5 marks ]

**44. Find the relation between $a$ and $b$ in the following continuous functions**

**(A)** $g(x)=\begin{cases} ax+3b & ,3<x\le 8\\ 25x+2 & ,x>8\end{cases}$

**(B)** $f(x)=\begin{cases} ax+2 & ,x\le 4\\ bx+5 & ,x>4\end{cases}$

**45. Find the value of $k$ if the following functions are continuous at a given point**

**(A)** $g(x)=\begin{cases}\dfrac{x^3+x^2-16x+20}{(x-2)^2} & ,x\neq 2\\ k & ,x=2\end{cases}$

**at point $x = 2$.**

**(B)** $f(x)=\begin{cases}\dfrac{1-\cos 6x}{x^2} & ,x\neq 0\\ k & ,x=0\end{cases}$

**at $x = 0$**

**46. Prove that the function**

$$f(x)=\begin{cases}\dfrac{x}{|x|+2x^2} & ,\text{if } x\neq 0\\ k & ,\text{if } x=0\end{cases}$$

**is not continuous at $x = 0$ for any value of $k$.**

**Ans.** We have $f(x)=\begin{cases}\dfrac{x}{|x|+2x^2} & ,\text{if } x\neq 0\\ k & ,\text{if } x=0\end{cases}$

$$=\begin{cases}\dfrac{x}{-x+2x^2} & ,x<0\\ k & ,x=0\\ \dfrac{x}{x+2x^2} & ,x>0\end{cases}$$

$$=\begin{cases}\dfrac{1}{-1+2x} & ,x<0\\ k & ,x=0\\ \dfrac{1}{1+2x} & ,x>0\end{cases}$$

The condition for $f(x)$ to be continuous at $x = 0$ is

$$f(0) = \lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) \quad \text{...(i)}$$

Now $\quad f(0) = k$

$$\text{LHL} = \lim_{x\to 0^-} f(x)$$

$$= \lim_{x\to 0^-}\frac{1}{-1+2x}$$

$$= \lim_{h\to 0}\frac{1}{-1+2(0-h)}$$

$$= \frac{1}{-1+0} = -1 \quad \text{...(ii)}$$

$$\text{RHL} = \lim_{x\to 0^+} f(x)$$

$$= \lim_{x\to 0^+}\frac{1}{2x+1}$$

$$= \lim_{h\to 0}\frac{1}{2(0+h)+1}$$

$$= \frac{1}{2\times 0+1} = 1 \quad \text{...(iii)}$$

Thus, from Eqs. (i), (ii) and (iii), we get

$$\lim_{x\to 0^-} f(x) \neq \lim_{x\to 0^+} f(x)$$

Hence, $f(x)$ is not continuous at $x = 0$ for any value of $k$.

Hence proved.

# DIFFERENTIABILITY 2

## TOPIC 1

## DIFFERENTIABILITY OF A FUNCTION

Suppose $f$ is a real valued function and $a$ is a point in its domain. Then, the derivative of $f$ at $a$ defined by

$$\lim_{h \to 0} \frac{f(a+h)-f(a)}{h} \quad ...(i)$$

exists finitely.

As the existence of limit means the equality of R.H.L and L.H.L, $f$ is differentiable at a point $a$ in its domain if both $\lim_{h \to 0} \frac{f(a-h)-f(a)}{-h}$ and $\lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$ are finite and equal.

**⚠ Caution**

- *If the limit in (i) does not exist finitely, then we say $f$ is not differentiable at $a$.*
- $\lim_{h \to 0} \frac{f(a-h)-f(a)}{-h}$ *is called left hand derivative (LHD);*
- $\lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$ *is called right hand derivative (RHD).*
- *The derivative of $f(x)$ at $a$ is denoted by $f'(a)$ or $\frac{d}{dx}\{f(x)\}_a$*
- *If $y = f(x)$, then the derivative of $f(x)$ is denoted by $\frac{dy}{dx}$ or $y'$.*

**Illustration:** What is the derivative of

(1) $f(x) = x$ at $x = 1$?

$$\lim_{x \to 0} \frac{f(a+h)-f(a)}{h} = \lim_{x \to 0} \frac{(1+h)-(1)}{h}$$

$$= \lim_{x \to 0} \frac{h}{h} = 1$$

(2) $f(x) = x^2 - 2$ at $x = 10$?

$$\lim_{x \to 0} \frac{f(a+h)-f(a)}{h}$$

$$= \lim_{x \to 0} \frac{[(10+h)^2-2]-(10^2-2)}{h}$$

$$= \lim_{h \to 0} \frac{98+20h+h^2-98}{h}$$

$$= \lim_{h \to 0} (20+h) = 20$$

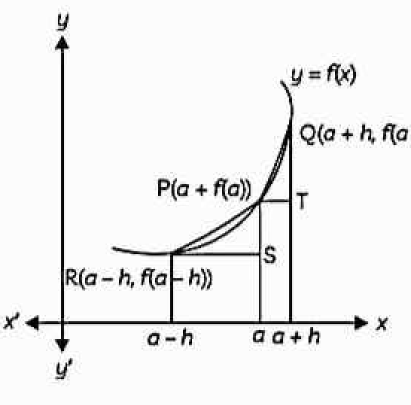
## TOPIC 2

## GEOMETRICAL INTERPRETATION OF DERIVATIVE

Let $y = f(x)$ be a function and $a$ be a point in its domain. Let P $(a, f(a))$ be a point on the curve and Q $(a+h, f(a+h))$, R $(a-h, f(a-h))$ be two points in right and left neighbourhood of P.

Slope of secant line PQ is $\frac{QT}{PT} = \frac{f(a+h)-f(a)}{a+h-a}$

$$= \frac{f(a+h)-f(a)}{h}$$




Now, if Q → P, then $h \to 0$. Consequently, chord QP becomes tangent at P.

In such limiting situation, the slope of chord QP will become the slope of the tangent at P to the curve $y = f(x)$.

⇒ Slope of the tangent at P = $\lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$

$= f'(a^+)$

Similarly,

Slope of secant line RP is $\frac{PS}{RS} = \frac{f(a)-f(a-h)}{a-a+h}$

$$= \frac{f(a)-f(a-h)}{h}$$

In limiting situation, when R moves closer and closer to P, *i.e.*, $h \to 0$, the secant line RP becomes tangent at P to the curve $y = f(x)$.

$\Rightarrow$ Slope of the tangent at P = $\lim_{h\to 0}\frac{f(a)-f(a-h)}{h}$

$= f'(a^-)$

If $f$ is differentiable at $x = a$, $f'(a^-)$ and $f'(a^+)$ must be equal.

$\Rightarrow$ the tangent lines obtained as limiting cases of secant lines RP and QP, will coincide.

$\Rightarrow$ If $f$ is differentiable at $x = a$, a unique tangent line can be drawn to the curve $y = f(x)$ at $x = a$ and the slope of tangent at P (for $x = a$) is given by $f'(a)$.

**Important**

*Any function f is differentiable at a point if and only if there exists a unique tangent at that point. This also means that this point must not be a corner point or at this point, curve must be smooth.*

**Example 2.1:** **Prove that the function $f$ given by**

$$f(x) = |x-1|, x \in R$$

**is not differentiable at $x = 1$.** **[NCERT]**

**Ans.** Here,

$$f(x) = |x-1| = \begin{cases} x-1, & \text{if } x \geq 1 \\ 1-x, & \text{if } x < 1 \end{cases}$$

LHD at $x = 1$,

$$\lim_{h\to 0}\frac{f(1-h)-f(1)}{-h} = \lim_{h\to 0}\frac{[1-(1-h)]-(1-1)}{-h}$$

$$= \lim_{h\to 0}\frac{h}{-h} = -1$$

RHD at $x = 1$,

$$\lim_{h\to 0}\frac{f(1+h)-f(1)}{h} = \lim_{h\to 0}\frac{[(1+h)-1]-[1-1]}{h}$$

$$= \lim_{h\to 0}\frac{h}{h} = 1$$

Since LHD $\neq$ RHD, $f$ is not differentiable at $x = 1$.

**Example 2.2:** **Prove that the function $f$ given by**

$$f(x) = [x], 0 < x < 3$$

**is not differentiable at $x = 1$ and $x = 2$.** **[NCERT]**

**Ans.** Here, $f(x) = [x]$

LHD at $x = 1$,

$$\lim_{h\to 0}\frac{f(1-h)-f(1)}{-h} = \lim_{h\to 0}\frac{0-1}{-h}$$

$$= \lim_{h\to 0}\frac{1}{h} = \infty \text{ (not defined)}$$

RHD at $x = 1$,

$$\lim_{h\to 0}\frac{f(1+h)-f(1)}{h} = \lim_{h\to 0}\frac{1-1}{h} = \lim_{h\to 0}\frac{0}{h} = 0$$

Since LHD $\neq$ RHD, $f$ is not differentiable at $x = 1$.

Further,

LHD at $x = 2$,

$$\lim_{h\to 0}\frac{f(2-h)-f(2)}{-h} = \lim_{h\to 0}\frac{1-2}{-h}$$

$$= \lim_{h\to 0}\frac{1}{h} = \infty \text{ (not defined)}$$

RHD at $x = 2$,

$$\lim_{h\to 0}\frac{f(2+h)-f(2)}{h} = \lim_{h\to 0}\frac{2-2}{h}$$

$$= \lim_{h\to 0}\frac{0}{h} = 0$$

Since LHD $\neq$ RHD, $f$ is not differentiable at $x = 2$.

**Example 2.3:** **Prove that the function $f$ given by**

$$f(x) = \sin x$$

**is differentiable at $x = \frac{\pi}{2}$.**

**Ans.** Here, $f(x) = \sin x$

LHD at $x = \frac{\pi}{2}$,

$$\lim_{h\to 0}\frac{f\left(\frac{\pi}{2}-h\right)-f\left(\frac{\pi}{2}\right)}{-h}$$

$$= \lim_{h\to 0}\frac{\sin\left(\frac{\pi}{2}-h\right)-\sin\frac{\pi}{2}}{-h}$$

$$= \lim_{h\to 0}\frac{2\cos\left(\frac{\pi-h}{2}\right)\sin\left(\frac{-h}{2}\right)}{-h}$$

$$= \lim_{h\to 0}2\cos\left(\frac{\pi-h}{2}\right)\lim_{h\to 0}\frac{\sin\left(\frac{-h}{2}\right)}{\left(\frac{-h}{2}\right)}\times\frac{1}{2}$$

$$= \lim_{h\to 0}2\cos\left(\frac{\pi-h}{2}\right)(1)\times\frac{1}{2}$$

$$= \cos\frac{\pi}{2}, i.e., 0$$

RHD at $x = \frac{\pi}{2}$,

$$\lim_{x\to 0}\frac{f\left(\frac{\pi}{2}+h\right)-f\left(\frac{\pi}{2}\right)}{h}$$

$$= \lim_{h\to 0}\frac{\sin\left(\frac{\pi}{2}+h\right)-\sin\frac{\pi}{2}}{h}$$

$$= \lim_{h\to 0}\frac{2\cos\left(\frac{\pi+h}{2}\right)\sin\left(\frac{h}{2}\right)}{h}$$

$$= \lim_{h\to 0} 2\cos\left(\frac{\pi+h}{2}\right)\lim_{h\to 0}\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}\times\frac{1}{2}$$

$$= \lim_{h\to 0} 2\cos\left(\frac{\pi+h}{2}\right)(1)\times\frac{1}{2}$$

$$= \cos\frac{\pi}{2},\ i.e.,\ 0$$

Since LHD = RHD, $f$ is differentiable at $x = \frac{\pi}{2}$.

**Noteworthy Results on Differentiable Functions**

(1) A constant function is differentiable at each real number.

(2) A polynomial function $f(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$, $n \in N$, $x \in R$ is differentiable at each real number.

(3) The logarithmic function is differentiable at each real number.

(4) The exponential function $f(x) = a^x$, $a > 0$ is differentiable at each real number.

(5) All trigonometric and inverse trigonometric functions are differentiable in their respective domains.

## TOPIC 3

## RELATION BETWEEN CONTINUITY AND DIFFERENTIABILITY

**Theorem:** If a function $f$ is differentiable at $x = a$, then it is continuous at $x = a$.

**Proof:** Let $f$ be differentiable at $x = a$. Then, $\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$ exists and finite.

Let $\lim_{x\to a}\frac{f(x)-f(a)}{x-a} = f'(a)$

In order to show that $f$ is continuous at $x = a$, we need to show that

$$\lim_{x\to a} f(x) = f(a)$$

Consider

$$\lim_{x\to a} f(x) = \lim_{x\to a}\frac{f(x)-f(a)}{x-a} \times \lim_{x\to a}(x-a) + \lim_{x\to a} f(a)$$

$$= f'(a)\times 0 + f(a)$$

$$= f(a)$$

Hence, $f$ is continuous at $x = a$.

**Important**

➥ *Converse of this theorem is not always true i.e. if a function is continuous at a point then it may not be differentiable at that point. For example, $f(x) = |x|$ is continuous at $x = 0$, but not differentiable at $x = 0$.*

# OBJECTIVE Type Questions

[ **1** mark ]

### Multiple Choice Questions

**1.** If $\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$ does not exist then

(a) $f$ is not differentiable at a point $a$

(b) $f$ is differentiable at a point $a$

(c) $f$ must be continuous at a point $a$

(d) None of these

**Ans.** *(a) f is not differentiable at point a*

**Explanation:** It is given that limit does not exist, then $f$ is not differentiable at the given point.

**2.** If a function $f$ is differentiable at C, then it is:

(a) continuous at C

(b) discontinuous at C

(c) not defined at C

(d) can not say anything

**3.** The set of points, in which $f(x) = \sec x$ is not differentiable, is:

(a) $\left\{x : x = \frac{(2n+1)\pi}{2}, n\in Z\right\}$

(b) $\left\{x : x = \frac{n\pi}{2}, n\in Z\right\}$

(c) $\{x : x = (2n+1)\pi, n\in Z\}$

(d) None of these

**Ans.** *(a)* $\left\{x : x = \frac{(2n+1)\pi}{2}, n \in Z\right\}$

**Explanation:** As we know that sec $x$ is not defined for $x = y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) - \{0\}$

Hence, $f(x)$ is not differentiable at $x = \frac{(2n+1)\pi}{2}, n \in Z$.

**4.** **Function $f(x) = \frac{1}{(x-1)^2}$ is not differentiable at point:**

(a) $x = 2$ (b) $x = 1$

(c) $x = -1$ (d) $x = 0$

**5. The derivative of $f(x) = |x|^3$ at $x = 0$ is:**

(a) $\frac{1}{2}$ (b) $-\frac{1}{2}$

(c) $-1$ (d) $0$

**Ans.** *(d)* 0

**Explanation:** $\lim_{x \to 0} \frac{f(a+h)-f(a)}{h}$

$$= \lim_{x \to 0} \frac{|0+h|^3 - |0|^3}{h}$$

$$= \lim_{x \to 0} h^2 = 0$$

**6. The function given below at $x = 4$ is:**

$$f(x) = \begin{cases} 2x + 3, x \le 4 \\ x^2 - 5, x > 4 \end{cases}$$

(a) Continuous but not differentiable

(b) Differentiable but not continuous

(c) Continuous as well as differentiable

(d) Neither continuous nor differentiable

**[Delhi Gov. 2022]**

**Ans.** *(a) Continuous but not differentiable*

**Explanation:** To check continuity at $x = 4$:

We have $f(4) = 2(4) + 3 = 11$

$$\text{LHL} = \lim_{x \to 4^-} (2x + 3)$$

$$= \lim_{h \to 0} [2(4 - h) + 3]$$

$$= 11 - 2(0) = 11$$

$$\text{RHL} = \lim_{x \to 4^+} (x^2 - 5)$$

$$= \lim_{h \to 0} [(4 + h)^2 - 5]$$

$$= \lim_{h \to 0} (11 + h + h^2)$$

$$= 11 + 8(0) + (0)^2$$

$$= 11$$

$\because$ LHL = RHL = $f(4)$

$\therefore$ $f(x)$ is continuas at $x = 4$.

To check differentiability at $x = 4$:

We have, $\text{LHD} = \lim_{h \to 0} \frac{f(4-h)-f(4)}{-h}$

$$= \lim_{h \to 0} \frac{[2(4-h)+3]-[2(4)+3]}{-h}$$

$$= \lim_{h \to 0} \frac{-2h}{-h} = 2$$

$$\text{RHD} = \lim_{h \to 0} \frac{f(4+h)-f(4)}{h}$$

$$= \lim_{h \to 0} \frac{[(4+h)^2-5]-[2(4)+3]}{h}$$

$$= \lim_{h \to 0} \frac{8h+h^2}{h}$$

$$= \lim_{h \to 0} (8 + h) = 8$$

$\because$ LHD $\ne$ RHD

$\therefore$ $f(x)$ is not differentiable at $x = 4$.

**7.** **The function $f(x) = |x|$ is:**

(a) continuous and differentiable everywhere

(b) continuous and differentiable no where

(c) continuous everywhere but differentiable everywhere except at $x = 0$

(d) continuous everywhere but differentiable no where

**8. The function $f(x) = |x| + |x + 5| + |x - 6|$ is not differentiable at:**

(a) $x= 0, 5, 6$ (b) $x = 0, -5, -6$

(c) $x = 0, -5, 6$ (d) $x = 0, 5, -6$

**Ans.** *(c)* $x = 0, -5, 6$

**Explanation:** We know that modulus function $f(x) = |x|$ is not differentiable at $x = 0$.

**9.** **The set points where the function $f$ given by $f(x) = |2x - 1| \sin x$ is differentiable, is:**

(a) R (b) $R - \left\{\frac{1}{2}\right\}$

(c) $(0, \infty)$ (d) None of these

**[NCERT Exemplar]**

**10. For what values $a$ and $b$, the function $f(x) = \begin{cases} x^2 & , \text{if } x \le 1 \\ ax + b & , \text{if } x > 1 \end{cases}$ is differentiable at all real points?**

(a) $a = 1, b = -1$ (b) $a = -1, b = 1$

(c) $a = 2, b = -1$ (d) $a = -2, b = 1$

**Ans.** *(c)* $a = 2, b = -1$

**Explanation:** Since differentiablity implies continuity, we have the function

$$f(x)=\begin{cases} x^2 & , \text{if } x \le 1 \\ ax+b & , \text{if } x > 1 \end{cases}$$ is continuous at all real points.

$\Rightarrow \quad \lim_{h\to 0} f(1-h) = \lim_{h\to 0} f(1+h) = f(1)$

$\Rightarrow \quad \lim_{h\to 0} (1-h)^2 = \lim_{h\to 0} [a(1+h)+b] = 1$

$\Rightarrow \quad 1 = a + b = 1$

$\Rightarrow \quad a + b = 1 \qquad ...(i)$

Further, $f(x)=\begin{cases} x^2 & , \text{if } x \le 1 \\ ax+b & , \text{if } x > 1 \end{cases}$ is differentiable at $x = 1$. So,

$$\text{LHD} = \lim_{h\to 0} \frac{f(1-h)-f(1)}{-h}$$

$$= \lim_{h\to 0} \frac{(1-h)^2-1}{-h}$$

$$= \lim_{h\to 0} (2-h) = 2$$

$$\text{RHD} = \lim_{h\to 0} \frac{f(1+h)-f(1)}{h}$$

$$= \lim_{h\to 0} \frac{\{a(1+h)+b\}-1}{h}$$

$$= \lim_{h\to 0} \frac{\{a+b\}+ah-1}{h}$$

$$= \lim_{h\to 0} \frac{ah}{h} \qquad \{\text{In view of (i) above}\}$$

$$= a$$

Thus, $a = 2, b = -1$.

**11.** **The function $f(x) = e^{|x|}$ is:**

**(a) continuous everywhere but not differentiable at $x = 0$.**

**(b) continuous and differentiable every where.**

**(c) not continuous at $x = 0$.**

**(d) none of these [NCERT Exemplar]**

**12.** **Let $f(x) = |\sin x|$. Then:**

**(a) $f$ is everywhere differentiable.**

**(b) $f$ is everywhere continuous but not differentiable at $x = n\pi, n \in Z$.**

**(c) $f$ is everywhere continuous but not differentiable at $x = (2n+1)\frac{\pi}{2}, n \in Z$.**

**(d) none of these**

**[NCERT Exemplar]**

**13.** **If $f(x)=\begin{cases} \cos x & , x \le \frac{\pi}{4} \\ k & , x > \frac{\pi}{4} \end{cases}$ is differentiable at $x = \frac{\pi}{4}$, then value of $k$ is:**

**(a) $-\frac{1}{\sqrt{2}}$ (b) 0**

**(c) $\frac{1}{\sqrt{2}}$ (d) 1**

**Ans.** *(c)* $\frac{1}{\sqrt{2}}$

**Explanation:** We have,

$$f(x)=\begin{cases} \cos x & , x \le \frac{\pi}{4} \\ k & , x > \frac{\pi}{4} \end{cases}$$

Now, $\text{LHD} = \lim_{x\to \frac{\pi}{4}^-} f(x)$

$$= \lim_{h\to 0} f\left(\frac{\pi}{4}-h\right)$$

$$= \lim_{h\to 0} \cos\left(\frac{\pi}{4}-h\right)$$

$$= \lim_{h\to 0} \left(\cos\frac{\pi}{4}\cos h + \sin\frac{\pi}{4}\sin h\right)$$

$$= \frac{1}{\sqrt{2}}\cos 0 + \frac{1}{\sqrt{2}}\sin 0$$

$$= \frac{1}{\sqrt{2}}\times 1 + \frac{1}{\sqrt{2}}\times 0 = \frac{1}{\sqrt{2}}$$

$$\text{RHD} = \lim_{h\to \frac{\pi}{4}^+} f(x)$$

$$= \lim_{h\to 0} f\left(\frac{\pi}{4}+h\right)$$

$$= \lim_{h\to 0} k = k$$

Since, $f(x)$ is differentiable, therefore

LHD = RHD

$\Rightarrow \quad \frac{1}{\sqrt{2}} = k$

**14.** **The set of points in which $f(x) = 3x^2 + 8$ is differentiable, is:**

**(a) $(-\infty, \infty)$ (b) $[0, \infty)$**

**(c) $(-\infty, 0]$ (d) None of these**

**15.** The exponential function $a^x$, $a > 0$ is differentiable at $x$ belongs to

(a) $(-\infty, 0)$ (b) $[0, \infty)$

(c) $(-\infty, \infty)$ (d) $(-\infty, \infty) - \{0\}$

**Ans.** (c) $(-\infty, \infty)$

**Explanation:** Let $f(x) = a^x$

As, we know that $f(x)$ is defined for every real values. So, $f(x)$ is differentiable $\forall\, x \in R$.

## VERY SHORT ANSWER Type Questions (VSA)

[ 1 mark ]

**16.** Find the non-differentiable point of $f(x) = |x - 2|$.

**17.** Find the differentiable value of a constant function.

**Ans.** Let $f(x) = c$

$$\text{LHD} = \lim_{h \to 0} \frac{f(0-h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{c - c}{h} = 0$$

and

$$\text{RHD} = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{c - c}{h} = 0$$

$\therefore$ LHD = RHD = 0, $\forall\, c \in R$

Hence, differentiable value of constant function is always zero.

**18.** Find the right hand derivative of $f(x) = |x|$ at $x = 3$.

**19.** The greatest integer function defined by $f(x) = [x]$, $0 < x < 2$ is not differentiable at which point? **[CBSE 2020]**

**Ans.** Since, the greatest integer function is not differentiable at integer value.

Hence, $f(x)$ is not differentiable at $x = 1$.

**20.** Find the right hand derivative of $f(x) = x^2 + 7$ at $x = 2$.

**21.** Find the non differentiable point of $f(x) = \dfrac{1}{(x+3)^2}$.

**22.** Prove that the greatest integer function defined by $f(x) = [x]$, $0 < x < 3$ is not differentiable at $x = 1$. **[CBSE 2020]**

**23.** Let $f(x) = x|x|$, for all $x \in R$. Check its differentiablity at $x = 0$. **[CBSE 2020]**

**Ans.** We have, $f(x) = x|x| = \begin{cases} x^2 & , x \geq 0 \\ -x^2 & , x < 0 \end{cases}$

To check the differentiability at $x = 0$

$$\text{LHD} = \lim_{h \to 0} \frac{f(0-h) - f(0)}{-h}$$

$$= \lim_{h \to 0} \frac{-(0-h)^2 - 0}{-h}$$

$$= \lim_{h \to 0} \frac{h^2}{h}$$

$$= \lim_{h \to 0} h = 0$$

$$\text{RHD} = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{(0+h)^2 - 0}{h}$$

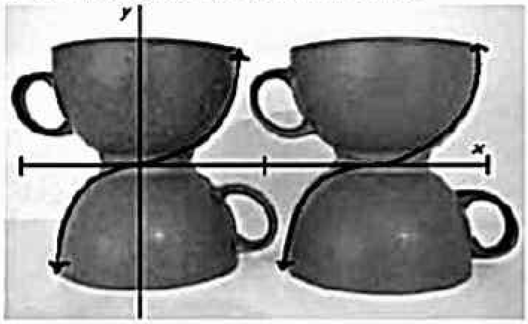
$$= \lim_{h \to 0} \frac{h^2}{h} = \lim_{h \to 0} h$$

$$= 0$$

Here LHD = RHD, hence $f(x)$ is differentiable at $x = 0$.

**24.** Find the left hand derivative of $f(x) = \tan x$ at $x = \dfrac{\pi}{4}$.

**25.** Sunaina took four cups and placed them as shown below. She then sketched the outlines of the cups and was quite amazed to note the striking resemblance it had to the graph of the trigonometric function $\tan x$.



Find the set of points in which $f(x) = \tan x$, $x \in \left[\frac{\pi}{3}, \frac{5\pi}{6}\right]$ is non-differentiable.

**Ans.** We have $f(x) = \tan x$.

We know that it is differentiable in its domain.

Since, $\tan x$ is not defined at $x = \frac{\pi}{2}$ in the interval $\left[\frac{\pi}{3}, \frac{5\pi}{6}\right]$

Hence, $f(x)$ is non-differentiable at point $x = \frac{\pi}{2}$.

## SHORT ANSWER Type-I Questions (SA-I)

[ 2 marks ]

**26. Find the differentiable value of $f(x) = x^2$ at $x = \frac{1}{2}$.**

**Ans.** Given: $f(x) = x^2$

Now, $\text{LHD} = Lf'\left(\frac{1}{2}\right)$

$$= \lim_{h \to 0} \frac{f\left(\frac{1}{2} - h\right) - f\left(\frac{1}{2}\right)}{-h}$$

$$= \lim_{h \to 0} \frac{\left(\frac{1}{2} - h\right)^2 - \left(\frac{1}{2}\right)^2}{-h}$$

$$= \lim_{h \to 0} \frac{\left(\frac{1}{2} - h - \frac{1}{2}\right)\left(\frac{1}{2} - h + \frac{1}{2}\right)}{-h}$$

$$= \lim_{h \to 0} \frac{-h(1-h)}{-h} = 1$$

$\text{RHD} = Rf'\left(\frac{1}{2}\right)$

$$= \lim_{h \to 0} \frac{f\left(\frac{1}{2} + h\right) - f\left(\frac{1}{2}\right)}{h}$$

$$= \lim_{h \to 0} \frac{\left(\frac{1}{2} + h\right)^2 - f\left(\frac{1}{2}\right)}{h}$$

$$= \lim_{h \to 0} \frac{\left(\frac{1}{2} + h - \frac{1}{2}\right)\left(\frac{1}{2} + h + \frac{1}{2}\right)}{h}$$

$$= \lim_{h \to 0} \frac{h(1+h)}{h}$$

$$= \lim_{h \to 0} (1+h)$$

$$= 1$$

Here, LHD = RHD = 1

Hence, differentiable value of $f(x)$ at $x = \frac{1}{2}$ is 1.

**27. Find the differentiable value of $f(x) = \log x$ at $x = 3$.**

**28. Show that function $f(x) = e^{|x|}$ is not differentiable at $x = 0$.**

**29. Discuss the differentiability of a function $f(x) = \log x$ at $x = e$.**

**Ans.** We have, $f(x) = \log x$

Now, differentiability at $x = e$,

$$\text{LHD} = \lim_{h \to 0} \frac{f(e-h) - f(e)}{-h}$$

$$= \lim_{h \to 0} \frac{\log(e-h) - \log e}{-h}$$

$$= \lim_{h \to 0} \frac{\log e\left(1 - \frac{h}{e}\right) - 1}{-h}$$

$$= \lim_{h \to 0} \frac{\log e + \log\left(1 - \frac{h}{e}\right) - 1}{-h}$$

$$= \lim_{h \to 0} \frac{1 + \log\left(1 - \frac{h}{e}\right) - 1}{-h}$$

$$= \lim_{h \to 0} \frac{\log\left(1 - \frac{h}{e}\right)}{e\left(-\frac{h}{e}\right)}$$

$$\left[\because \lim_{x \to 0} \frac{\log(1+x)}{x} = 1\right]$$

$$= \frac{1}{e}$$

and $\text{RHD} = \lim_{h \to 0} \frac{f(e+h) - f(e)}{h}$

$$= \lim_{h \to 0} \frac{\log(e+h) - \log e}{h}$$

$$= \lim_{h \to 0} \frac{\log e + \log\left(1 + \frac{h}{e}\right) - \log e}{h}$$

$$= \lim_{h \to 0} \frac{\log\left(1 + \frac{h}{e}\right)}{e\left(\frac{h}{e}\right)} = \frac{1}{e}$$

$\therefore$ LHD = RHD,

Hence, it is differentiable at $x = e$.

**30.** **Examine the differentiability of $f$, where $f$ is defined by $f(x) = \begin{cases} 1+x & , \text{if } x \le 2 \\ 5-x & , \text{if } x > 2 \end{cases}$ at $x = 2$.** **[NCERT Exemplar]**

**31.** **Discuss the differentiability of a function $g(x) = (x+2)^2$ at $x = 1$.**

**Ans.** Given, $g(x) = (x+2)^2$

Now, differentiability at $x = 1$,

$$\text{LHD} = \lim_{h \to 0} \frac{g(1-h) - g(1)}{-h}$$

$$= \lim_{h \to 0} \frac{(1-h+2)^2 - (1+2)^2}{-h}$$

$$= \lim_{h \to 0} \frac{(3-h)^2 - 3^2}{-h}$$

$$= \lim_{h \to 0} \frac{(3-h-3)(3-h+3)}{-h}$$

$$= \lim_{h \to 0} (6-h) = 6 - 0 = 6$$

$$\text{RHD} = \lim_{h \to 0} \frac{g(1+h) - g(1)}{h}$$

$$= \lim_{h \to 0} \frac{(1+h+2)^2 - (1+2)^2}{h}$$

$$= \lim_{h \to 0} \frac{(3+h)^2 - 3^2}{h}$$

$$= \lim_{h \to 0} \frac{(3+h-3)(3+h+3)}{h}$$

$$= \lim_{h \to 0} \frac{h(6+h)}{h}$$

$$= \lim_{h \to 0} (6+h)$$

$$= 6 + 0 = 6$$

Here LHD = RHD, hence $g(x)$ is differentiable at $x = 1$.

**32.** **Find the set of points in which $f(x) = |\sin x|$ is not differentiable.**

## SHORT ANSWER Type-II Questions (SA-II)

[ **3** marks ]

**33.** **Show that the function $g(x) = |x - 7|$, $x \in R$ is continuous but not differentiable at $x = 7$.**

**Ans.** We have, $g(x) = |x - 7|$

We know, modulus functions are continuous everywhere.

So, $g(x) = |x - 7|$, $x \in R$ is continuous.

At $x = 7$,

$$\text{LHD} = \lim_{h \to 0} \frac{g(7-h) - g(7)}{-h}$$

$$= \lim_{h \to 0} \frac{|7-h-7| - 0}{-h}$$

$$= \lim_{h \to 0} \frac{|-h|}{-h} = \lim_{h \to 0} \frac{h}{-h}$$

$$= -1$$

and

$$\text{RHD} = \lim_{h \to 0} \frac{g(7+h) - g(7)}{h}$$

$$= \lim_{h \to 0} \frac{|7+h-7| - 0}{h}$$

$$= \lim_{h \to 0} \frac{|h|}{h} = \lim_{h \to 0} \frac{h}{h}$$

$$= 1$$

Here, LHD $\neq$ RHD, hence, $g(x)$ is not differentiable at $x = 7$.

Hence, $g(x) = |x - 7|$ $x \in R$ is continuous but not differentiable at $x = 7$.

**34.** **Show that the function $f(x) = |x + 1|$, $+ |x - 1|$, for all $x \in R$, is not differentiable at the points $x = -1$ and $x = 1$.** **[CBSE 2015]**

**35.** **Discuss the differentiability of function $f(x) = |\sin x|$ at $x = 0$ in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.**

**Ans.** We have, $f(x) = |\sin x|$

$$= \begin{cases} \sin x & 0 \le x < \frac{\pi}{2} \\ -\sin x & -\frac{\pi}{2} < x < 0 \end{cases}$$

At point $x = 0$,

$$\text{LHD} = \lim_{h \to 0} \frac{f(0-h)-f(0)}{-h}$$

$$= \lim_{h \to 0} \frac{-\sin(0-h)-\sin 0}{-h}$$

$$= \lim_{h \to 0} \frac{\sin h - 0}{-h}$$

$$= \lim_{h \to 0} \frac{-\sin h}{h}$$

$$= -1$$

$$\text{RHD} = \lim_{h \to 0} \frac{f(0+h)-f(0)}{h}$$

$$= \lim_{h \to 0} \frac{\sin(0+h)-\sin 0}{h}$$

$$= \lim_{h \to 0} \frac{\sin h - 0}{h}$$

$$= \lim_{h \to 0} \frac{\sin h}{h} = 1$$

Here LHD ≠ RHD, hence, $f(x)$ is not differentiable at $x = 0$.

**36.** **Examine the differentiability of $f$, where $f$ is defined by $f(x) = \begin{cases} x[x], & \text{if } 0 \le x < 2 \\ (x-1)x, & \text{if } 2 \le x < 3 \end{cases}$ at $x = 2$.**

**37.** **For what value of $\lambda$, the function defined by $f(x) = \begin{cases} \lambda(x^2+2) & , \text{if } x \le 0 \\ 4x+6 & , x > 0 \end{cases}$ is continuous at $x = 0$? Hence, check the differentiability of $f(x)$ at $x = 0$.** **[CBSE 2015]**

**Ans.** We have, $f(x) = \begin{cases} \lambda(x^2+2) & , \text{if } x \le 0 \\ 4x+6 & , \text{if } x > 0 \end{cases}$

Since, $f(x)$ is continuous at $x = 0$. Therefore,

$$f(0) = \lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) \quad \text{...(i)}$$

Now $f(0) = \lambda(0^2 + 2)$

$$= 2\lambda \quad \text{...(ii)}$$

$$\text{RHL} = \lim_{x \to 0^+} f(x)$$

$$= \lim_{x \to 0^+} (4x+6)$$

$$= 4(0) + 6$$

$$= 6 \quad \text{...(iii)}$$

Thus from Eqs. (i), (ii) and (iii), we get

$$2\lambda = 6$$

$$\Rightarrow \quad \lambda = 3$$

Hence, given continuous function becomes

$$f(x) = \begin{cases} 3(x^2+2) & , \text{if } x \le 0 \\ 4x+6 & , \text{if } x > 0 \end{cases}$$

Now, we have to check the differentiability at $x = 0$.

$$\text{LHD} = \lim_{h \to 0} \frac{f(0-h)-f(0)}{-h}$$

$$= \lim_{h \to 0} \frac{3[(0-h)^2+2]-3(0+2)}{-h}$$

$$= \lim_{h \to 0} \frac{3[h^2+2]-6}{-h}$$

$$= \lim_{h \to 0} \frac{3h^2}{-h}$$

$$= \lim_{h \to 0} -3h = -3(0)$$

$$= 0$$

and $$\text{RHD} = \lim_{h \to 0} \frac{f(0+h)-f(0)}{h}$$

$$= \lim_{h \to 0} \frac{[4(0+h)+6]-3(0+2)}{h}$$

$$= \lim_{h \to 0} \frac{4h+6-6}{h} = \lim_{h \to 0} 4$$

$$= 4$$

Here, LHD ≠ RHD

Hence, $f(x)$ is not differentiable at $x = 0$.

**38.** **Examine the differentiability of $f$ where $f$ is defined by $f(x) = \begin{cases} x^2 \sin\frac{1}{x} & , \text{if } x \ne 0 \\ 0 & , \text{if } x = 0 \end{cases}$ at $x = 0$.** **[NCERT Exemplar]**

**Ans.** For continuity at $x = 0$,

$\lim_{x \to 0} x^2 \sin\frac{1}{x} = 0 \times$ [an oscillating number between −1 and 1]

$$= 0$$

Also, $f(0) = 0$

So, $f(x)$ is continuous at $x = 0$.

Now, for differentiability at $x = 0$,

$$Lf'(0) = \lim_{x \to 0^-} \frac{f(x)-f(0)}{x-0}$$

$$= \lim_{h \to 0} \frac{f(0-h)-0}{0-h}$$

$$= \lim_{h \to 0} \frac{(0-h)^2 \sin\frac{1}{0-h}}{0-h}$$

$$= \lim_{h \to 0} \frac{h^2 \sin\left(-\frac{1}{h}\right)}{-h}$$

$$= \lim_{h \to 0} h \sin\frac{1}{h}$$

$= 0 \times$ [an oscillating number between −1 and 1]

$= 0$

$$Rf'(0) = \lim_{x \to 0^+} \frac{f(x)-f(0)}{x-0}$$

$$= \lim_{x \to 0} \frac{f(0+h)-0}{0+h}$$

$$= \lim_{h \to 0} \frac{(0+h)^2 \sin\frac{1}{(0+h)}}{(0+h)}$$

$$= \lim_{h \to 0} h \sin\frac{1}{h}$$

$= 0 \times$ [an oscillating number between −1 and 1]

$= 0$

Since, $Lf'(0) = Rf'(0)$

So, $f(x)$ is differentiable at $x = 0$.

**39.** **Find the values of $a$ and $b$, if the function $f$ defined by**

$$f(x) = \begin{cases} x^2 + 3x + a & , x \le 1 \\ bx + 2 & , x > 1 \end{cases}$$

**is differentiable at $x = 1$. [CBSE 2016]**

**40.** **A function $f$: R → R satisfies the equation $f(x + y) = f(x) f(y)$ for all $x, y \in$ R, $f(x) \ne 0$. Suppose that the function is differentiable at $x = 0$ and $f'(0) = 2$. Prove that $f'(x) = 2f(x)$. [NCERT Exemplar]**

**Ans.** Let $f$: R → R satisfies the equation $f(x + y) = f(x) f(y)$ ∀ $x, y \in$ R, $f(x) \ne 0$.

Given that $f(x)$ is differentiable at $x = 0$ and $f'(0) = 2$.

Now, $$f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h \to 0} \frac{f(x)f(h)-h(x)}{h}$$

$[\because f(x + y) = f(x) f(y)]$

$$= f(x) \lim_{h \to 0} \frac{f(h)-1}{h}$$

Now, putting $x = y = 0$ in $f(x + y) = f(x) f(y)$, we get

$f(0 + 0) = f(0) f(0)$

$\therefore \quad f(0) = 1$ [As $f(x) \ne 0$]

$$\therefore \quad f'(x) = f(x) \lim_{h \to 0} \frac{f(h)-f(0)}{h}$$

$= f(x) f'(0)$

$\therefore \quad f'(x) = 2f(x)$ [$\because f'(0) = 2$(Given)]

Hence, proved.

## LONG ANSWER Type Questions (LA)

[ 4 & 5 marks ]

**41.** **Find the values of $p$ and $q$, so that**

$$f(x) = \begin{cases} x^2 + 3x + p, \text{if } x \le 1 \\ qx + 2, \text{if } x > 1 \end{cases}$$

**is differentiable at $x = 1$. [NCERT Exemplar]**

**Ans.** Given, $f(x) = \begin{cases} x^2+3x+p, \text{if } x \le 1 \\ qx+2, \text{if } x>1 \end{cases}$ is differentiable at $x = 1$

So, it must be continuous at $x = 1$.

$\therefore \quad f(1^-) = f(1^+) = f(1)$

$\Rightarrow 1 + 3 + p = q + 2 = 1 + 3 + p$

$\Rightarrow \quad p - q = -2$ ...(i)

$$\because \quad f'(x) = \begin{cases} 2x+3 & \text{if } x<1 \\ q, & \text{if } x>1 \end{cases}$$

We must have

$\therefore \quad Lf'(1) = Rf'(1)$

$\Rightarrow 2(1) + 3 = q$

$\Rightarrow \quad q = 5$

$\therefore \quad p = 3$ [from (i)]

Hence, the values of $p$ and $q$ are 3 and 5, respectively.

**42.** **Examine the continuity at $x = 1$ and differentiability at $x = 2$ of the function**

$$g(x) = \begin{cases} 5x - 4 & , 0 < x < 1 \\ 4x^2 - 3x & , 1 \le x < 2 \\ 3x + 4 & , \quad x \ge 2 \end{cases}$$

**Ans.** Given, $g(x) = \begin{cases} 5x - 4 & , 0 < x < 1 \\ 4x^2 - 3x & , 1 \le x < 2 \\ 3x + 4 & , \quad x \ge 2 \end{cases}$

Now, we examine the continuity at $x = 1$,

$$g(1) = 4(1)^2 - 3(1) = 1$$

$$\text{LHL} = \lim_{x \to 1^-} g(x)$$

$$= \lim_{x \to 1^-} (5x - 4)$$

$$= \lim_{h \to 0} [5(1 - h) - 4]$$

$$= 5(1 - 0) - 4 = 1$$

and $\text{RHL} = \lim_{x \to 1^+} g(x)$

$$= \lim_{x \to 1^+} (4x^2 - 3x)$$

$$= \lim_{h \to 0} [4(1 + h)^2 - 3(1 + h)]$$

$$= 4(1 + 0)^2 - 3(1 + 0)$$

$$= 4 - 3 = 1$$

Here, $g(1) = \text{LHL} = \text{RHL}$

Hence $g(x)$ is continuous at $x = 1$.

Now, we examine the differentiability at $x = 2$.

$$\text{LHD} = \lim_{h \to 0} \frac{f(2 - h) - f(2)}{-h}$$

$$= \lim_{h \to 0} \frac{[4(2 - h)^2 - 3(2 - h)] - [3(2) + 4]}{-h}$$

$$= \lim_{h \to 0} \frac{[4(4 + h^2 - 4h) - 6 + 3h] - 10}{-h}$$

$$= \lim_{h \to 0} \frac{16 - 16h + 4h^2 - 6 + 3h - 10}{-h}$$

$$= \lim_{h \to 0} \frac{4h^2 - 13h}{-h}$$

$$= \lim_{h \to 0} -(4h - 13)$$

$$= -(4 \times 0 - 13) = 13$$

and $\text{RHD} = \lim_{h \to 0} \frac{f(2 + h) - f(2)}{h}$

$$= \lim_{h \to 0} \frac{[3(2 + h) + 4] - [3 \times 2 + 4]}{h}$$

$$= \lim_{h \to 0} \frac{10 + 3h - 10}{h}$$

$$= \lim_{h \to 0} 3 = 3$$

Here, LHD $\ne$ RHD

Hence, $f$ is not differentiable at $x = 2$.

**43.** **Show that the function defined as follows is continuous at $x = 1, 2$ but not differentiable at $x = 2$.**

$$f(x) = \begin{cases} 3x - 2 & , 0 < x \le 1 \\ 2x^2 - x & , 1 < x \le 2 \\ 5x - 4 & , \quad x > 2 \end{cases}$$ **[CBSE 2010]**

**Ans.** We have, $f(x) = \begin{cases} 3x - 2 & , 0 < x \le 1 \\ 2x^2 - x & , 1 < x \le 2 \\ 5x - 4 & , \quad x > 2 \end{cases}$

First of all we will show that $f(x)$ is continuous at $x = 1, 2$.

**At point $x = 1$**

$$f(1) = 3(1) - 2$$

$$= 1 \quad \text{...(i)}$$

$$\text{LHL} = \lim_{h \to 0} f(1 - h)$$

$$= \lim_{h \to 0} [3(1 - h) - 2]$$

$$= \lim_{h \to 0} (3 - 3h - 2)$$

$$= \lim_{h \to 0} 1 - 3h$$

$$= 1 - 3(0) = 1 \quad \text{...(ii)}$$

$$\text{RHL} = \lim_{h \to 0} f(1 + h)$$

$$= \lim_{h \to 0} [2(1 + h)^2 - (1 + h)]$$

$$= \lim_{h \to 0} [2(1 + h^2 + 2h) - 1 - h]$$

$$= \lim_{h \to 0} (2h^2 + 3h + 1)$$

$$= 2(0)^2 + 3(0) + 1 = 1 \quad \text{...(iii)}$$

Thus from Eqs. (ii) and (iii), we get

$$f(1) = \text{LHL} = \text{RHL}$$

Hence, $f(x)$ is continuous at point $x = 1$.

**At point $x = 2$**

$$f(2) = 2(2)^2 - 2$$

$$= 8 - 2 = 6 \quad \text{...(iv)}$$

$$\text{LHL} = \lim_{h \to 0} f(2 - h)$$

$$= \lim_{h \to 0} [2(2 - h)^2 - (2 - h)]$$

$$= \lim_{h \to 0} 2[4 + h^2 - 4h] - 2 + h$$

$$= \lim_{h \to 0} 2h^2 - 7h + 6$$

$$= 2(0) - 7(0) + 6 = 6 \quad ...(v)$$

and $\text{RHL} = \lim_{h \to 0} f(2+h)$

$$= \lim_{h \to 0} [5(2+h) - 4]$$

$$= \lim_{h \to 0} (5h + 6)$$

$$= 5(0) + 6 = 6 \quad ...(vi)$$

Thus from Eqs. (iv), (v) and (vi), we get

$f(2) = \text{LHL} = \text{RHL}$

Hence, $f(x)$ is continuous at point $x = 2$.

Now, we will show that $f(x)$ is not differentiable at $x = 2$.

$$\text{LHD} = \lim_{h \to 0} \frac{f(2-h) - f(2)}{-h}$$

$$= \lim_{h \to 0} \frac{[2(2-h)^2 - (2-h)] - [8-2]}{-h}$$

$$= \lim_{h \to 0} \frac{[2(4 + h^2 - 4h) - 2 + h] - 6}{-h}$$

$$= \lim_{h \to 0} \frac{[2h^2 - 8h + 8 - 2 + h] - 6}{-h}$$

$$= \lim_{h \to 0} \frac{2h^2 - 7h}{-h}$$

$$= \lim_{h \to 0} -(2h - 7)$$

$$= -(2 \times 0 - 7) = 7$$

$$\text{RHD} = \lim_{h \to 0} \frac{f(2+h) - f(2)}{h}$$

$$= \lim_{h \to 0} \frac{[5(2+h) - 4] - [8-2]}{h}$$

$$= \lim_{h \to 0} \frac{6 + 5h - 6}{h}$$

$$= \lim_{h \to 0} 5 = 5$$

Here RHD $\neq$ LHD

Hence, $f(x)$ is not differentiable at $x = 2$.

**44.** Ⓐ**Discuss the continuity and differentiability of** $f(x) = \begin{cases} 1 - x & , x < 1 \\ (1-x)(2-x) & , 1 \le x \le 2 \\ 3 - x & , x > 2 \end{cases}$

# DERIVATIVES 3

## TOPIC 1

## DERIVATIVES OF FUNCTIONS

### Derivatives of Some Standard Functions

Here are derivatives of some standard functions studied in Class XI:

| | | |
|---|---|---|
| $\frac{d}{dx}(\text{constant}) = 0$ | $\frac{d}{dx}(x^n) = n\,x^{n-1}$ | $\frac{d}{dx}(kx^n) = k\,n\,x^{n-1}$ |
| $\frac{d}{dx}(\sin x) = \cos x$ | $\frac{d}{dx}(\cos x) = -\sin x$ | $\frac{d}{dx}(\tan x) = \sec^2 x$ |
| $\frac{d}{dx}(\text{cosec}\,x) = -\text{cosec}\,x\cot x$ | $\frac{d}{dx}(\sec x) = \sec x\tan x$ | $\frac{d}{dx}(\cot x) = -\text{cosec}^2 x$ |
| $\frac{d}{dx}(e^x) = e^x$ | $\frac{d}{dx}(a^x) = a^x . \log_e a, a > 0$ | $\frac{d}{dx}(\log_e x) = \frac{1}{x}, x > 0$ |

### Algebra of Derivatives

(1) Derivative of sum of two functions is equal to the sum of the derivatives of the two functions.

$$\frac{d}{dx}(f+g) = \frac{d}{dx}(f) + \frac{d}{dx}(g)$$

(2) Derivative of difference of two functions is equal to the difference of the derivatives of the two functions.

$$\frac{d}{dx}(f-g) = \frac{d}{dx}(f) - \frac{d}{dx}(g)$$

(3) Derivative of product of two functions is given by the following product rule:

$$\frac{d}{dx}(fg) = f\frac{d}{dx}(g) + g\frac{d}{dx}(f)$$

(4) Derivative of quotient of two functions is given by the following quotient rule:

$$\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{g\frac{d}{dx}(f) - f\frac{d}{dx}(g)}{(g)^2}$$

## TOPIC 2

## DERIVATIVES OF IMPLICIT FUNCTIONS

Until now we have been differentiating various functions given in the form $y = f(x)$. But it is not necessary that functions are expressed in this form. For example, consider the following relationships between $x$ and $y$:

(1) $x + 2y = 2$

(2) $xy + \sin y = 1$

In the first case, we can express $y$ in terms of $x$ as $y = \frac{2-x}{2}$. In the second case, it is not possible to express $y$ in terms of $x$.

When a relationship between $x$ and $y$ is such that it is easy to express $y$ in terms of $x$, we say that $y$ is an explicit function of $x$. In the latter case, $y$ is an implicit function of $x$. Now, we shall learn how to differentiate implicit functions.

Illustration: Let the implicit function be $2x + 3y = \sin x$

On differentiating both sides w.r.t. $x$, we get

$$2 + 3\frac{dy}{dx} = \cos x$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{\cos x - 2}{3}$$

**Example 3.1:** Find $\frac{dy}{dx}$ for the following implicit functions:

**(A)** $xy + y^2 = \tan x + y$

**(B)** $\sin^2 y + \cos(xy) = k$ **[NCERT]**

**Ans.** (A) Given function is $xy + y^2 = \tan x + y$

On differentiating both sides w.r.t. $x$, we get

$$\left(x\frac{dy}{dx} + y\right) + 2y\frac{dy}{dx} = \sec^2 x + \frac{dy}{dx}$$

$$\Rightarrow (x + 2y - 1)\frac{dy}{dx} = \sec^2 x - y$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{\sec^2 x - y}{x + 2y - 1}$$

(B) Given function is $\sin^2 y + \cos(xy) = k$

On differentiating both sides w.r.t. $x$, we get

$$\left(2\sin y \cos y\frac{dy}{dx}\right) - \sin(xy)\left\{x\frac{dy}{dx} + y\right\} = 0$$

$$\Rightarrow [2\sin y\cos y - x\sin(xy)]\frac{dy}{dx} = y\sin(xy)$$

$$\Rightarrow [\sin 2y - x\sin(xy)]\frac{dy}{dx} = y\sin(xy)$$

$$\Rightarrow \frac{dy}{dx} = \frac{y\sin(xy)}{\sin 2y - x\sin(xy)}$$

## TOPIC 3

## DERIVATIVES OF INVERSE TRIGONOMETRIC FUNCTIONS

In this section, we shall learn the method of determining the derivatives of inverse trigonometric functions.

**Results:**

(1) $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}, x \in (-1, 1)$

(2) $\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}, x \in (-1, 1)$

(3) $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{\sqrt{1+x^2}}, x \in R$

(4) $\frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1+x^2}, x \in R$

(5) $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}}, x \in R - [-1, 1]$

(6) $\frac{d}{dx}(\text{cosec}^{-1} x) = \frac{-1}{|x|\sqrt{x^2-1}}, x \in R - [-1, 1]$

**Proofs:**

(1) Let $y = \sin^{-1} x, x \in (-1, 1)$. Then, $y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ such that $\sin y = x$.

Differentiating both sides of $\sin y = x$ w.r.t. $x$, we have

$$\cos y \cdot \frac{dy}{dx} = 1$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{\cos y}$$

Since $y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, $\cos y > 0$

Therefore, $\frac{dy}{dx} = \frac{1}{\sqrt{1-\sin^2 y}}$

$$= \frac{1}{\sqrt{1-x^2}}$$

(2) $y = \cos^{-1} x, x \in (-1, 1)$.

Then, $y \in (0, \pi)$ such that $\cos y = x$.

Differentiating both sides w.r.t. $x$, we have

$$-\sin y \cdot \frac{dy}{dx} = 1$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{1}{\sin y}$$

Since, $y \in (0, \pi)$, $\sin y > 0$

Therefore, $\frac{dy}{dx} = -\frac{1}{\sqrt{1-\cos^2 y}}$

$$= -\frac{1}{\sqrt{1-x^2}}$$

| | |
|---|---|
| (3) Let $y = \tan^{-1}x, x \in R$. <br> Then, $y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ such that $\tan y = x$. <br> Differentiating both sides w.r.t. $x$, we have <br> $\sec^2 y \cdot \frac{dy}{dx} = 1$ <br> $\Rightarrow \quad \frac{dy}{dx} = \frac{1}{\sec^2 y}$ <br> $\Rightarrow \quad \frac{dy}{dx} = \frac{1}{1+\tan^2 y} = \frac{1}{1+x^2}$ | (4) Let $y = \cot^{-1}x, x \in R$. <br> Then, $y \in (0, \pi)$ such that $\cot y = x$. <br> Differentiating both sides w.r.t. $x$, we have <br> $-\text{cosec}^2 y \cdot \frac{dy}{dx} = 1$ <br> $\Rightarrow \quad \frac{dy}{dx} = -\frac{1}{\text{cosec}^2 y}$ <br> $\Rightarrow \quad \frac{dy}{dx} = -\frac{1}{1+\cot^2 y} = -\frac{1}{1+x^2}$ |
| (5) Let $y = \sec^{-1}x, x \in R - [-1, 1]$. <br> Then, $y \in (0, \pi) - \left\{\frac{\pi}{2}\right\}$ such that $\sec y = x$. <br> Differentiting both sides w.r.t. $x$, we have <br> $\sec y \tan y \cdot \frac{dy}{dx} = 1$ <br> $\Rightarrow \quad \frac{dy}{dx} = \frac{1}{\sec y \tan y}$ <br> Since $y \in (0, \pi) - \left\{\frac{\pi}{2}\right\}$, $\sec y \tan y > 0$ <br> Therefore, $\frac{dy}{dx} = \frac{1}{\|\sec y\|\|\tan y\|}$ <br> $= \frac{1}{\|\sec y\| \sqrt{\sec^2 y - 1}}$ <br> $= \frac{1}{\|x\| \sqrt{x^2 - 1}}$ | (6) Let $y = \text{cosec}^{-1}x, x \in R - [-1, 1]$. <br> Then, $y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) - \{0\}$ such that $\text{cosec}\, y = x$. <br> Differentiating both sides w.r.t. $x$, we have <br> $-\text{cosec}\, y \cot y \cdot \frac{dy}{dx} = 1$ <br> $\Rightarrow \quad \frac{dy}{dx} = -\frac{1}{\text{cosec}\, y \cot y}$ <br> Since $y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) - \{0\}$, $\text{cosec}\, y \cot y > 0$ <br> Therefore, $\frac{dy}{dx} = -\frac{1}{\|\text{cosec}\, y\|\|\cot y\|}$ <br> $= -\frac{1}{\|\text{cosec}\, y\| \sqrt{\text{cosec}^2 y - 1}}$ <br> $= -\frac{1}{\|x\| \sqrt{x^2 - 1}}$ |

## TOPIC 4

# DIFFERENTIATION USING TRIGONOMETRIC SUBSTITUTION

Consider a function $y = \tan^{-1}\left(\frac{3x - x^3}{1 - x^3}\right)$. In present form, it is difficult to differentiate $y$, but if we substitute $x = \tan\theta$ and simplify, then we get

$$y = \tan^{-1}\left(\frac{3x - x^3}{1 - 3x^2}\right)$$

$$= \tan^{-1}\left(\frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}\right)$$

$$= \tan^{-1}(\tan 3\theta) = 3\theta = 3\tan^{-1}x$$

Now it is quite easy to differentiate $y$ w.r.t. $x$.

$$\frac{dy}{dx} = 3\frac{d}{dx}(\tan^{-1}x) = \frac{3}{1+x^2}$$

Here, we are listing some important substitution and identities which are highly useful in the process of finding derivatives of inverse trignometric functions.

**Type-I**

| Form of exponent | Substitution | Form of exponent | Substitution |
|---|---|---|---|
| $a^2 - x^2$ | $x = a\sin\theta$ or $x = a\cos\theta$ | $1 - x^2$ | $x = \sin\theta$ or $x = \cos\theta$ |
| $a^2 + x^2$ | $x = a\tan\theta$ or $x = a\cot\theta$ | $1 + x^2$ | $x = \tan\theta$ or $x = \cot\theta$ |
| $x^2 - a^2$ | $x = a\sec\theta$ or $x = a\,\text{cosec}\,\theta$ | $x^2 - 1$ | $x = \sec\theta$ or $x = \text{cosec}\,\theta$ |
| $\sqrt{\frac{a+x}{a-x}}$ or $\sqrt{\frac{a-x}{a+x}}$ | $x = a\cos\theta$ | $\sqrt{\frac{1+x}{1-x}}$ or $\sqrt{\frac{1-x}{1+x}}$ | $x = \cos\theta$ |
| $\sqrt{\frac{a^2+x^2}{a^2-x^2}}$ or $\sqrt{\frac{a^2-x^2}{a^2+x^2}}$ | $x^2 = a^2\cos\theta$ | $\sqrt{\frac{1+x^2}{1-x^2}}$ or $\sqrt{\frac{1-x^2}{1+x^2}}$ | $x^2 = \cos\theta$ |

**Type-II**

| Form of $x$ in inverse exponent | Substitution | Form of $x$ in inverse exponent | Substitution |
|---|---|---|---|
| $\frac{2x}{1-x^2}$ | $x = \tan\theta$ | $\frac{2x}{1+x^2}$ | $x = \tan\theta$ |
| $2x^2 - 1$ | $x = \cos\theta$ | $1 - 2x^2$ | $x = \sin\theta$ |
| $\frac{x^2-1}{x^2+1}$ | $x = \cot\theta$ | $\frac{1-x^2}{1+x^2}$ | $x = \tan\theta$ |
| $3x - 4x^3$ | $x = \sin\theta$ | $4x^3 - 3x$ | $x = \cos\theta$ |
| $\frac{3x-x^3}{1-3x^2}$ | $x = \tan\theta$ | $2x\sqrt{1-x^2}$ | $x = \cos\theta$ or $\sin\theta$ |

**Example 3.2:** Find $\frac{dy}{dx}$ for each of the following:

(A) $y = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$

(B) $y = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right), 0 < x < 1$

(C) $y = \sin^{-1}(2x\sqrt{1-x^2}), -\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$

[NCERT]

**Ans. (A)** Let $y = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$ ...(i)

Putting $x = \tan\theta$ in (i), we get

$$y = \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)$$

$$= \sin^{-1}(\sin 2\theta) = 2\theta = 2\tan^{-1}x$$

Differentiating both sides w.r.t. $x$, we get

$$\frac{dy}{dx} = 2\frac{d}{dx}(\tan^{-1}x)$$

$$= 2\frac{1}{1+x^2}, \text{or } \frac{2}{1+x^2}$$

(B) Let $y = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$ ...(i)

Putting $x = \tan\theta$ in (i), we get

$$y = \cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)$$

$$= \cos^{-1}(\cos 2\theta) = 2\theta = 2\tan^{-1}x$$

Differentiating both sides w.r.t. $x$, we get

$$\frac{dy}{dx} = 2\frac{d}{dx}(\tan^{-1}x)$$

$$= 2\frac{1}{1+x^2}, \text{or } \frac{2}{1+x^2}$$

(C) Let $y = \sin^{-1}(2x\sqrt{1-x^2})$ ...(i)

Putting $x = \sin\theta$ in (i), we get

$$y = \sin^{-1}(2\sin\theta\sqrt{1-\sin^2\theta})$$
$$= \sin^{-1}(2\sin\theta\cos\theta)$$
$$= \sin^{-1}(\sin 2\theta) = 2\theta = 2\sin^{-1}x$$

Differentiating both sides w.r.t. $x$, we get

$$\frac{dy}{dx} = 2\frac{d}{dx}(\sin^{-1}x)$$
$$= 2\frac{1}{\sqrt{1-x^2}}, \text{ or } \frac{2}{\sqrt{1-x^2}}$$

# TOPIC 5

## LOGARITHMIC DIFFERENTIATION

If a function, whose derivative is to be determined, involves at least one of the following:

(1) Product of functions [e.g. $f(x)\,g(x)$];

(2) Quotient of functions $\left[\text{e.g. } \frac{f(x)}{g(x)}\right]$;

(3) Function raised to the power function [e.g. $\{f(x)\}^{g(x)}$].

Then, its derivative can be obtained easily by taking logarithm (with base $e$) on both sides and simplifying before differentiation. This procedure is called Logarithmic differentiation.

**Example 3.3: Differentiate w.r.t. $x$ each of the following:**

**(A)** $y = \cos x.\cos 2x.\cos 3x$

**(B)** $y = \sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}$

**(C)** $y = (\log x)^{\cos x}$ **[NCERT]**

**Ans. (A)** Let $y = \cos x.\cos 2x.\cos 3x$

Taking logarithm on both sides, we get

$$\log y = \log[\cos x.\cos 2x.\cos 3x]$$

or $\log y = \log[\cos x] + \log[\cos 2x] + \log[\cos 3x]$

$[\because \log ab = \log a + \log b]$

Differentiating both sides w.r.t. $x$, we get

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{\cos x}\frac{d}{dx}(\cos x) + \frac{1}{\cos 2x}\frac{d}{dx}(\cos 2x) + \frac{1}{\cos 3x}\frac{d}{dx}(\cos 3x)$$

$$= \frac{1}{\cos x}(-\sin x) + \frac{1}{\cos 2x}(-2\sin 2x) + \frac{1}{\cos 3x}(-3\sin 3x)$$

$$= -[\tan x + 2\tan 2x + 3\tan 3x]$$

$$\Rightarrow \frac{dy}{dx} = y\{-[\tan x + 2\tan 2x + 3\tan 3x]\}$$

$$= -\cos x.\cos 2x.\cos 3x(\tan x + 2\tan 2x + 3\tan 3x)$$

**(B)** Let $y = \sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}$, or $\left(\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}\right)^{1/2}$

Taking logarithm on both sides, we get

$$\log y = \frac{1}{2}\log\left[\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}\right]$$

$[\because \log a^b = b\log a]$

$$= \frac{1}{2}\{\log[(x-1)(x-2)] - \log[(x-3)(x-4)(x-5)]$$

$\left[\because \log\frac{a}{b} = \log a - \log b\right]$

$$= \frac{1}{2}\{\log(x-1) + \log(x-2) - \log(x-3) - \log(x-4) - \log(x-5)\}$$

Differentiating both sides w.r.t. $x$, we get

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{2}\left\{\frac{1}{x-1} + \frac{1}{x-2} - \frac{1}{x-3} - \frac{1}{x-4} - \frac{1}{x-5}\right\}$$

$$\Rightarrow \frac{dy}{dx}$$
$$= \frac{1}{2}y\left\{\frac{1}{x-1} + \frac{1}{x-2} - \frac{1}{x-3} - \frac{1}{x-4} - \frac{1}{x-5}\right\}$$
$$= \frac{1}{2}\sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}\left\{\frac{1}{x-1} + \frac{1}{x-2} - \frac{1}{x-3} - \frac{1}{x-4} - \frac{1}{x-5}\right\}$$

**(C)** $y = (\log x)^{\cos x}$

Taking logarithm on both sides, we get

$$\log y = \cos x.\log(\log x)$$

Differentiating both sides w.r.t. $x$, we get

$$\frac{1}{y}\frac{dy}{dx} = \cos x \cdot \frac{d}{dx}\{\log(\log x)\} + \log(\log x)\cdot\frac{d}{dx}(\cos x)$$

$$= \cos x \cdot \frac{1}{\log x} \cdot \frac{1}{x} + \log(\log x).(-\sin x)$$

$$= \frac{\cos x}{x \log x} - \sin x.\log(\log x)$$

$$\Rightarrow \frac{dy}{dx} = y\left\{\frac{\cos x}{x\log x} - \sin x.\log(\log x)\right\}$$

$$= (\log x)^{\cos x}\left\{\frac{\cos x}{x\log x} - \sin x.\log(\log x)\right\}$$

## TOPIC 6

# DERIVATIVES OF FUNCTIONS IN PARAMETRIC FORMS

Sometimes the relation between two variables is neither explicit nor implicit, but some link of a third variable with each of the two variables, separately, establishes a relation between the first two variables. In such a situation, we say that the relation between them is expressed via a third variable. The third variable is called the parameter. More precisely, a relation expressed between two variables $x$ and $y$ in the form $x = f(t)$, $y = g(t)$ is said to be parametric form with $t$ as a parameter.

In order to find derivative of a function in such form, we have by chain rule,

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}, \text{provided } \frac{dx}{dt} \neq 0.$$

**Example 3.4:** Find $\frac{dy}{dx}$ **for each of the following:**

**(A)** $x = \sin t$, $y = \cos 2t$

**(B)** $x = \cos\theta - \cos 2\theta$, $y = \sin\theta - \sin 2\theta$ **[NCERT]**

**Ans.** (A) Given $x = \sin t$, $y = \cos 2t$.

Differentiating w.r.t. $t$, we have

$$\frac{dx}{dt} = \cos t \text{ and } \frac{dy}{dt} = -2\sin 2t$$

Therefore, $\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}$

$$= \frac{-2\sin 2t}{\cos t}$$

$$= \frac{-4\sin t\cos t}{\cos t}$$

$$= -4\sin t$$

(B) Given $x = \cos\theta - \cos 2\theta$, $y = \sin\theta - \sin 2\theta$

Differentiating w.r.t. $\theta$, we get

$$\frac{dx}{d\theta} = -\sin\theta + 2\sin 2\theta$$

and $\frac{dy}{dt} = \cos\theta - 2\cos 2\theta$

Therefore, $\frac{dy}{dx} = \frac{dy}{d\theta} \div \frac{dx}{d\theta}$

$$= \frac{\cos\theta - 2\cos 2\theta}{-\sin\theta + 2\sin 2\theta}$$

or $\frac{\cos\theta - 2\cos 2\theta}{2\sin 2\theta - \sin\theta}$

## TOPIC 7

# DIFFERENTIATION OF A FUNCTION WITH RESPECT TO ANOTHER FUNCTION

So far we have learnt to find the derivative of a function with respect to some variable, *e.g.* derivative of $f(x)$ w.r.t. $x$. Now we will discuss the process of differentiation of one function with respect to another function.

Let $u = f(x)$ and $v = g(x)$ be two differentiable functions of $x$. We need to differentiate $u$ with respect to $v$, *i.e.*, $\frac{du}{dv}$. So, we obtain $\frac{du}{dv}$, using $\frac{du}{dv} = \frac{du}{dx} \div \frac{dv}{dx}$.

**Example 3.5:** Differentiate $\sin^2 x$ w.r.t. $e^{\cos x}$. **[NCERT]**

**Ans.** Let $u = \sin^2 x$ and $v = e^{\cos x}$. We need to find $\frac{du}{dv}$.

Here, $\frac{du}{dx} = 2\sin x\cos x$ and $\frac{dv}{dx} = e^{\cos x}(-\sin x)$

So, $\frac{du}{dv} = \frac{du}{dx} \div \frac{dv}{dx}$

$$= \frac{2\sin x\cos x}{-e^{\cos x}\sin x}$$

$$= \frac{2\cos x}{-e^{\cos x}} \text{ or } -2\cos x\, e^{-\cos x}$$

# OBJECTIVE Type Questions

[ 1 mark ]

## Multiple Choice Questions

**1.** If $y = \frac{1}{1+x^{b-a}+x^{c-a}} + \frac{1}{1+x^{c-b}+x^{a-b}} + \frac{1}{1+x^{a-c}+x^{b-c}}$, then $\frac{dy}{dx} =$

(a) $x^{a+b+c}$ (b) $x^{abc}$

(c) $\frac{1}{x^a+x^b x^c}$ (d) 0

**[Delhi Gov. 2022]**

**Ans.** *(d) 0*

**Explanation:** $y = \frac{x^a}{x^a+x^b+x^c} + \frac{x^b}{x^a+x^b+x^c} + \frac{x^c}{x^a+x^b+x^c}$

$\Rightarrow \quad = \frac{x^a+x^b+x^c}{x^a+x^b+x^c}$

Thus, $y = 1$

$\Rightarrow \quad \frac{dy}{dx} = 0.$

**2.** The derivative of $\sin^{-1}(2x\sqrt{1-x^2})$ w.r.t. $\sin^{-1}x$, $\frac{1}{\sqrt{2}} < x < 1$, is:

(a) 2 (b) $\frac{\pi}{2} - 2$

(c) $\frac{\pi}{2}$ (d) $-2$

**[CBSE Term-1 SQP 2021]**

**Ans.** *(a) 2*

let $u = \sin^{-1}(2x\sqrt{1-x^2})$

and $v = \sin^{-1}x, \frac{1}{\sqrt{2}} < x < 1$

$\Rightarrow \quad \sin v = x$ ...(1)

Using (1), we get :

$u = \sin^{-1}(2\sin v \cos v)$

$\Rightarrow \quad u = 2v$

Differentiating with respect to $v$, we get:

$$\frac{du}{dv} = 2$$

**[CBSE Marking Scheme Term-1 SQP 2021]**

**Explanation:** Let $u = \sin^{-1}(2x\sqrt{1-x^2})$

Put, $x = \sin\theta \Rightarrow \theta = \sin^{-1}x$, we get

$u = \sin^{-1}(2\sin\theta\sqrt{1-=\sin^2\theta})$

$= \sin^{-1}(\sin 2\theta)$

$= 2\theta$

$= 2\sin^{-1}x$

$\therefore \quad \frac{du}{dx} = 2\times\frac{1}{\sqrt{1-x^2}} = \frac{2}{\sqrt{1-x^2}}$ ...(i)

Also, let $v = \sin^{-1}x$

$\Rightarrow \quad \frac{dv}{dx} = \frac{1}{\sqrt{1-x^2}}$ ...(ii)

Now, $\frac{du}{dv} = \frac{du/dx}{dv/dx} = 2$

[Using (i) and (ii)]

**3.** The derivative of $x^x$ w.r.t. $x$ is:

(a) $x^{x-1}$ (b) $x^x \log x$

(c) $x^x(1+\log x)$ (d) $x^x(1-\log x)$

**4.** If $y = 10^{10^x}$, then $\frac{dy}{dx}$ is:

(a) $10^{10^x}(\log 10)$

(b) $10^{10^x}(\log 10)^2$

(c) $10^{10^x}10^x(\log 10)^2$

(d) $10^{10^x}10^x(\log 10)$ **[DIKSHA]**

**Ans.** *(c) $10^{10^x}10^x(\log 10)^2$*

**Explanation:**

Here, $y = 10^{10^x}$

$\Rightarrow \quad \log y = 10^x \log 10$

$\therefore \quad \frac{1}{y}\frac{dy}{dx} = 10^x(\log 10)^2 + 10^x \times 0$

$\Rightarrow \quad \frac{dy}{dx} = y\,10^x(\log 10)^2$

$= 10^{10^x}10^x(\log 10)^2$

**⚠ Caution**

➥ *Take the logarithm on both sides for getting $\frac{dy}{dx}$.*

**5.** If $x = a(t + \sin t)$, $y = a(1 - \cos t)$, then $\frac{dy}{dx}$ is:

(a) $\tan t$ (b) $-\cot t$

(c) $\tan\frac{t}{2}$ (d) $\cot\frac{t}{2}$

**Ans.** *(c)* $\tan\frac{t}{2}$

**Explanation:** Let $x = a(t + \sin t)$, $y = a(1 - \cos t)$. Then,

$$\frac{dx}{dt} = a(1 + \cos t)$$

and $$\frac{dy}{dt} = a(\sin t)$$

$$\Rightarrow \frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$= \frac{a\sin t}{a(1+\cos t)}$$

$$= \frac{2\sin\left(\frac{t}{2}\right)\cos\left(\frac{t}{2}\right)}{2\cos^2\left(\frac{t}{2}\right)}$$

$$= \tan\frac{t}{2}$$

**6.** If $y = \log\left(\frac{1-x^2}{1+x^2}\right)$, then $\frac{dy}{dx}$ is equal to:

(a) $\frac{4x^3}{1-x^4}$ (b) $\frac{-4x}{1-x^4}$

(c) $\frac{1}{4-x^4}$ (d) $\frac{-4x^3}{1-x^4}$

**[NCERT Exemplar]**

**7.** If $\log\left(\frac{x^2-y^2}{x^2+y^2}\right) = a$, then $\frac{dy}{dx} =$

(a) $\frac{y}{x}$ (b) $\frac{-y}{x}$

(c) $\frac{x}{y}$ (d) $\frac{-x}{y}$

**[Delhi Gov. 2022]**

**Ans.** *(a)* $\frac{y}{x}$

**Explanation:** We have,

$$\log\left(\frac{x^2-y^2}{x^2+y^2}\right) = a$$

or $$\frac{x^2-y^2}{x^2+y^2} = e^a$$

Differentiating both sides w.r.t. $x$, we get

$$\frac{(x^2+y^2)\frac{d}{dx}(x^2-y^2)-(x^2-y^2)\frac{d}{dx}(x^2+y^2)}{(x^2+y^2)^2} = 0$$

$$\Rightarrow (x^2+y^2)\left(2x-2y\frac{dy}{dx}\right) - (x^2-y^2)\left(2x+2y\frac{dy}{dx}\right) = 0$$

$$\Rightarrow (2x^3 + 2xy^2 - 2x^3 + 2xy^2) + (-2x^2y - 2y^3 - 2x^2y + 2y^3)\frac{dy}{dx} = 0$$

$$\Rightarrow 4xy^2 - 4x^2y\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{4xy^2}{4x^2y} = \frac{y}{x}$$

**8.** If $5^x + 5^y = 5^{x+y}$, then $\frac{dy}{dx} =$

(a) $5^{x-y}$ (b) $5^{y-x}$

(c) $-5^{x-y}$ (d) $-5^{y-x}$

**[Delhi Gov. 2022]**

**Ans.** *(d)* $-5^{y-x}$

**Explanation:** We have,

$$5^x + 5^y = 5^{x+y}$$

Dividing both sides by $5^{x+y}$, we get

$$5^{-y} + 5^{-x} = 1$$

Differentiating w.r.t. $x$, we get

$$5^{-y}\log 5\left(-\frac{dy}{dx}\right) + 5^{-x}\log 5\,(-1) = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{-5^{-x}}{5^{-y}} = -5^{y-x}$$

**9.** If $x = \sin^3 t$, $y = \cos^3 t$ then $\frac{dy}{dx} =$

(a) $\tan t$ (b) $\cot t$

(c) $-\tan t$ (d) $-\cot t$

**[Delhi Gov. 2022]**

**Ans.** *(d)* $-\cot t$

**Explanation:** We have,

$$x = \sin^3 t,$$

$$y = \cos^3 t$$

Differentiating them w.r.t. $t$, we get

$$\frac{dx}{dt} = 3\sin^2 t \cos t$$

and $$\frac{dy}{dt} = -3\cos^2 t \sin t$$

Now, $$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

$$= \frac{-3\cos^2 t \sin t}{3\sin^2 t \cos t}$$

$$= -\frac{\cos t}{\sin t} = -\cot t$$

**10. If $(x^2 + y^2)^2 = xy$, then $\frac{dy}{dx}$ is:**

(a) $\frac{x + 4x(x^2 + y^2)}{4y(x^2 + y^2) - x}$ (b) $\frac{y - 4x(x^2 + y^2)}{x + 4(x^2 + y^2)}$

(c) $\frac{y - 4x(x^2 + y^2)}{4y(x^2 + y^2) - x}$ (d) $\frac{4y(x^2 + y^2) - x}{y - 4x(x^2 + y^2)}$

**[CBSE Term- 1 2021]**

**Ans.** *(c)* $\frac{y - 4x(x^2 + y^2)}{4y(x^2 + y^2) - x}$

**Explanation:** $(x^2 + y^2)^2 = xy$

Differentiating both sides w.r.t. $x$, we get

$$\frac{d}{dx}(x^2 + y^2)^2 = \frac{d}{dx}(xy)$$

$$\Rightarrow 2(x^2 + y^2).\frac{d}{dx}(x^2 + y^2) = x.\frac{dy}{dx} + y.1$$

$$\Rightarrow 4x(x^2 + y^2) + 4y(x^2 + y^2)\frac{dy}{dx} = x.\frac{dy}{dx} + y$$

$$\Rightarrow [4y(x^2 + y^2) - x].\frac{dy}{dx} = y - 4x(x^2 + y^2)$$

$$\Rightarrow \frac{dy}{dx} = \frac{y - 4x(x^2 + y^2)}{4y(x^2 + y^2) - x}$$

**11. The derivative of $y = \log_x 2$ is:**

(a) $\frac{\log 2}{x(\log x)^2}$ (b) $\frac{-\log 2}{x(\log x)^2}$

(c) $\frac{-\log 2}{(\log x)^2}$ (d) $\frac{\log 2}{x}$

**12. If $y = \log(\cos e^x)$, then $\frac{dy}{dx}$ is:**

(a) $\cos e^{x-1}$ (b) $e^{-x} \cos e^x$

(c) $e^x \sin e^x$ (d) $-e^x \tan e^x$

**[CBSE Term-1 SQP 2021]**

**Ans.** *(d)* $-e^x \tan e^x$

$$y = \log(\cos e^x)$$

Differentiating w.r.t. $x$:

$\Rightarrow \quad \sin v = x \quad$ ...(1)

Using (1), we get :

$$\frac{dy}{dx} = \frac{1}{\cos(e^x)}.(-\sin e^x).e^x$$

(chain rule)

$$\Rightarrow \quad \frac{dy}{dx} = -e^x \tan e^x$$

**[CBSE Marking Scheme Term-1 SQP 2021]**

**Explanation:** $y = \log(\cos e^x)$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{\cos e^x}(-\sin e^x)e^x$$

$$= -e^x \tan e^x$$

**13. Given $2x^2 - xy - y^2 = 0$. Then the value of $\frac{dy}{dx}$ at the point $(1, -2)$ is:**

(a) $-2$ (b) $-\frac{3}{2}$

(c) $\frac{4}{3}$ (d) $2$

**14. If $x^3 - 3x^2y + y^3 = 2021 + xy$, then $\frac{dy}{dx}$ =**

(a) $\frac{3x^2 - 6xy - y}{3x^2 + 3y^2 - x}$ (b) $\frac{3x^2 - 6xy - y}{3y^2 - 3x^2 - x}$

(c) $\frac{6xy + y - 3x^2}{3y^2 - 3x^2 - x}$ (d) $\frac{3x^2 - 6xy - y}{3x^2 + 3y^2 + x}$

**[Delhi Gov. 2022]**

**Ans.** *(c)* $\frac{6xy + y - 3x^2}{3y^2 - 3x^2 - x}$

**Explanation:** We have,

$$x^3 - 3x^2y + y^3 = 2021 + xy$$

Differentiating w.r.t. $x$, we get

$$3x^2 - 3\left[x^2\frac{dy}{dx} + y\frac{d}{dx}(x^2)\right] + 3y^2\,\frac{dy}{dx}$$

$$= 0 + \left[x\frac{dy}{dx} + y(1)\right]$$

$$\Rightarrow 3x^2 - 3\left[x^2\frac{dy}{dx} + y(2x)\right] + 3y^2\,\frac{dy}{dx}$$

$$= x\frac{dy}{dx} + y$$

$$\Rightarrow \frac{dy}{dx}\,(-3x^2 + 3y^2 - x)$$

$$= y - 3x^2 + 6xy$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y - 3x^2 + 6xy}{-3x^2 + 3y^2 - x}$$

or, $$\frac{dy}{dx} = \frac{6xy + y - 3x^2}{3y^2 - 3x^2 - x}$$

**15.** If $y = \sqrt{\sin x + y}$, then $\frac{dy}{dx}$ is equal to:

(a) $\frac{\cos x}{2y-1}$ (b) $\frac{\cos x}{1-2y}$

(c) $\frac{\sin x}{1-2y}$ (d) $\frac{\sin x}{2y-1}$

**[NCERT Exemplar]**

**16.** The derivative of $\sin^{-1} x$ w.r.t. $\cos^{-1}\sqrt{1-x^2}$ is:

(a) 1 (b) 0

(c) $\cos^{-1} x$ (d) $\frac{1}{\sqrt{1-x^2}}$

**[NCERT Exemplar]**

**Ans.** *(a) 1*

**Explanation:** Let $u = \sin^{-1} x$,

$$\therefore \quad \frac{du}{dx} = \frac{1}{\sqrt{1-x^2}}$$

and $\quad v = \cos^{-1}\sqrt{1-x^2}$

$$\therefore \quad \frac{dv}{dx} = -\frac{1}{\sqrt{1-\left(\sqrt{1-x^2}\right)^2}} \times \frac{d}{dx}\left(\sqrt{1-x^2}\right)$$

$$= -\frac{1}{x} \times \frac{1}{2\sqrt{1-x^2}} \times (-2x)$$

$$= \frac{1}{\sqrt{1-x^2}}$$

$$\therefore \quad \frac{du}{dv} = \frac{du/dx}{dv/dx} = 1$$

**17.** The derivative of $\cos^{-1}(2x^2 - 1)$ w.r.t., $\cos^{-1} x$ is:

(a) 2 (b) $\frac{-1}{2\sqrt{1-x^2}}$

(c) $\frac{2}{x}$ (d) $1 - x^2$

**[NCERT Exemplar]**

**18.** If $e^x + e^y = e^{x+y}$, then $\frac{dy}{dx}$ is:

(a) $e^{y-x}$ (b) $e^{x+y}$

(c) $-e^{y-x}$ (d) $2e^{x-y}$

**[CBSE Term-1 SQP 2021]**

**Ans.** *(c)* $-e^{y-x}$

$$e^x + e^y = e^{x+y}$$

$$\Rightarrow \quad e^{-y} + e^{-x} = 1$$

Differentiating w.r.t. $x$:

$$\Rightarrow \quad -e^{-y}\frac{dy}{dx} - e^{-x} = 0 \Rightarrow \frac{dy}{dx} = -e^{y-x}$$

**[CBSE Marking Scheme Term-1 SQP 2021]**

**Explanation:** $e^x + e^y = e^{x+y}$

Dividing both sides by $e^{x+y}$, we get

$$e^{-y} + e^{-x} = 1$$

Differentiating both sides w.r.t. $x$, we get

$$e^{-y}\left(-\frac{dy}{dx}\right) + e^{-x}(-1) = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{e^{-x}}{e^{-y}} = -e^{-x}.e^{y}$$

$$= -e^{y-x}.$$

**19.** Kids love fast food such as burgers, hot dogs and pizzas. They are more fascinated by the toppings of various sauces. Soham was about to take a bite on his favourite hot dog when he observed that the sauce had been spread nicely and evenly all over and it reminded him of trigonometric and logarithmic functions he had studied in mathematics.

If $\sin(x + y) = \log(x + y)$, then $\frac{dy}{dx}$ is

(a) – 2 (b) – 1

(c) 1 (d) 2

**Ans.** *(b)* – 1

**Explanation:** Given: $\sin(x + y) = \log(x + y)$

Differentiating both sides w.r.t. $x$, we have

$$\Rightarrow \quad \cos(x+y)\left[1+\frac{dy}{dx}\right] = \frac{1}{(x+y)}\left[1+\frac{dy}{dx}\right]$$

$$\Rightarrow \left\{\cos(x+y) - \frac{1}{(x+y)}\right\}\frac{dy}{dx} = \frac{1}{(x+y)} - \cos(x+y)$$

$$\Rightarrow \quad \frac{dy}{dx} = -1$$

## VERY SHORT ANSWER Type Questions (VSA)

[ 1 mark ]

**20.** Find the derivative of the function $y = x\tan^{-1}x$.

**21.** If $y = \tan^{-1}x + \cot^{-1}x$, $x \in R$, then find $\frac{dy}{dx}$. **[CBSE 2020]**

**22.** If $\sin(xy) = k$ and $xy \neq \frac{n\pi}{2}, n \in Z$, then find $\frac{dy}{dx}$.

**Ans.** Given, $\sin(xy) = k$

On differentiating with respect to $x$, we get

$$\cos(xy)\frac{d}{dx}(xy) = 0$$

$$\cos(xy)\left[x\frac{dy}{dx} + 1.y\right] = 0$$

Since, $xy \neq \frac{n\pi}{2}$, therefore $\cos(xy) \neq 0$

So, $$x\frac{dy}{dx} + y = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{-y}{x}$$

**23.** If $f(x) = \sqrt{x^2 + 2x + 3}$, then find $f'(x)$.

**24.** If $f(x) = \sin 2x - \cos 2x$, then find $f'\left(\frac{\pi}{6}\right)$.

**Ans.** We have, $f(x) = \sin 2x - \cos 2x$

$\therefore \quad f'(x) = 2\cos 2x + 2\sin 2x$

Now, $$f'\left(\frac{\pi}{6}\right) = 2\left[\cos\frac{\pi}{3} + \sin\frac{\pi}{3}\right]$$

$$= 2\left(\frac{1}{2} + \frac{\sqrt{3}}{2}\right)$$

$$= 1 + \sqrt{3}$$

**25.** Find $\frac{dy}{dx}$ if $y = \log(\sin e^x)$.

**26.** Find the derivative of $y = (\tan x)^x$.

**Ans.** Given $y = (\tan x)^x$

On taking log both sides, we get

$$\log y = x\log(\tan x)$$

On differentiating with respect to $x$, we get

$$\frac{1}{y}\frac{dy}{dx} = x \times \frac{1}{\tan x}\frac{d}{dx}(\tan x) + 1 \times \log\tan x$$

$$\frac{dy}{dx} = y\left[\frac{x}{\tan x} \times \sec^2 x + \log\tan x\right]$$

$$= (\tan x)^x\left[\frac{x\sec^2 x}{\tan x} + \log\tan x\right]$$

**27.** If $y = |(x + 1)^2|$, then find $\frac{dy}{dx}$.

## SHORT ANSWER Type-I Questions (SA-I)

[ 2 marks ]

**28.** Differentiate w.r.t. $x$ : $\log\left(x+\sqrt{x^2+a}\right)$ **[NCERT Exemplar]**

**29.** Find $\frac{dy}{dx}$ at $x = 1, y = \frac{\pi}{4}$, if

$\sin^2 y + \cos xy = k.$ **[CBSE 2017]**

**Ans.** Given: $\sin^2 y + \cos(xy) = k$

On differentiating both sides with respect to $x$, we get

$$\frac{d}{dx}(\sin^2 y) + \frac{d}{dx}\{\cos(xy)\} = \frac{d}{dx}(k)$$

$$2\sin y\frac{d}{dx}(\sin y) - \sin(xy)\frac{d}{dx}(xy) = 0$$

$$2\sin y\cos y\frac{dy}{dx} - \sin(xy)\left[1.y + x\frac{dy}{dx}\right] = 0$$

$$\sin 2y\frac{dy}{dx} - x\sin(xy)\frac{dy}{dx} = y\sin(xy)$$

At $x = 1, y = \frac{\pi}{4}$,

$$\frac{dy}{dx}\left(\sin 2\times\frac{\pi}{4} - 1\sin\left(1\cdot\frac{\pi}{4}\right)\right) = \frac{\pi}{4}\sin\left(1\cdot\frac{\pi}{4}\right)$$

$$\frac{dy}{dx}\left(1 - \frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}\times\frac{1}{\sqrt{2}}$$

$$\therefore \frac{dy}{dx} = \frac{\pi}{4\sqrt{2}\left(\frac{\sqrt{2}-1}{\sqrt{2}}\right)} = \frac{\pi}{4(\sqrt{2}-1)}$$

**30.** If $y = e^x \log(1 + x^2)$, find $\frac{dy}{dx}$.

**31.** Differentiate $\tan^{-1}\left(\frac{1-\cos x}{\sin x}\right)$ with respect to $x$.

**Ans.** Let $y = \tan^{-1}\left(\frac{1-\cos x}{\sin x}\right)$

$$= \tan^{-1}\left(\frac{2\sin^2\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}}\right)$$

$$= \tan^{-1}\left(\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}\right) = \tan^{-1}\left(\tan\frac{x}{2}\right)$$

$$= \frac{x}{2}$$

Now on differentiating both sides with respect to $x$, we get

$$\frac{dy}{dx} = \frac{1}{2}$$

**32.** Find the differential of $\cos^2 x$ with respect to $e^{\sin x}$.

**Ans.** Let $u = \cos^2 x$ and $v = e^{\sin x}$

On differentiating both functions with respect to $x$, we get

$$\frac{du}{dx} = 2\cos x.(-\sin x)$$

and $$\frac{dv}{dx} = e^{\sin x}.\cos x$$

$$\therefore \quad \frac{du}{dv} = \frac{du/dx}{dv/dx}$$

$$= \frac{-2\sin x\cos x}{e^{\sin x}\cos x}$$

$$= -2\sin x\, e^{-\sin x}$$

**33.** Differentiate $\tan^{-1}\left(\frac{\cos x - \sin x}{\cos x + \sin x}\right)$ with respect to $x$. **[CBSE 2018]**

**34.** If $x = a\sec\theta$, $y = b\tan\theta$, then find $\frac{dy}{dx}$ at $\theta = \frac{\pi}{3}$. **[CBSE 2020]**

**Ans.** Given: $x = a\sec\theta$ and $y = b\tan\theta$

On differentiating both parametric functions with respect to $\theta$, we get

$$\frac{dx}{d\theta} = a\sec\theta\tan\theta$$

and $$\frac{dy}{d\theta} = b\sec^2\theta$$

Now $$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$$

$$= \frac{b\sec^2\theta}{a\sec\theta\tan\theta}$$

$$= \frac{b}{a}\frac{\sec\theta}{\tan\theta}$$

$$= \frac{b}{a} \frac{1}{\cos\theta \frac{\sin\theta}{\cos\theta}}$$

$$= \frac{b}{a\sin\theta}$$

At $\theta = \frac{\pi}{3}$,

$$\frac{dy}{dx} = \frac{b}{a\sin\frac{\pi}{3}} = \frac{2b}{a\sqrt{3}}$$

**35.** If $y=\sin^{-1}(6x\sqrt{1-9x^2})$, $-\frac{1}{3\sqrt{2}} < x < \frac{1}{3\sqrt{2}}$, then find $\frac{dy}{dx}$. **[CBSE 2017]**

**Ans.** Given, $y = \sin^{-1}(6x\sqrt{1-9x^2})$

$\Rightarrow$ $y = \sin^{-1}\left(2.3x\sqrt{1-(3x)^2}\right)$

Put $3x = \sin\theta$, then

$y = \sin^{-1}(2\sin\theta\sqrt{1-\sin^2\theta})$

$\Rightarrow$ $y = \sin^{-1}(2\sin\theta.\cos\theta)$

$\Rightarrow$ $y = \sin^{-1}(\sin 2\theta) \Rightarrow y = 2\theta$

$\Rightarrow$ $y = 2\sin^{-1}(3x)$

On differentiating with respect to $x$, we get

$$\frac{dy}{dx} = \frac{2}{\sqrt{1-9x^2}}(3)$$

$$\Rightarrow \frac{dy}{dx} = \frac{6}{\sqrt{1-9x^2}}$$

## SHORT ANSWER Type-II Questions (SA-II)

[ 3 marks ]

**36.** If $y = \sqrt{\cos x + \sqrt{\cos x + \sqrt{\cos x + ......\infty}}}$, then find $\frac{dy}{dx}$.

**37.** If $y = e^{x^2\cos x} + (\cos x)^x$, then find $\frac{dy}{dx}$. **[CBSE 2020]**

**Ans.** Given: $y = e^{x^2\cos x} + (\cos x)^x$

$$y = e^{x^2\cos x} + e^{x(\log\cos x)}$$

On differentiating with respect to $x$, we get

$$\frac{dy}{dx} = \frac{d}{dx}(e^{x^2\cos x}) + \frac{d}{dx}(e^{x(\log\cos x)})$$

$$= e^{x^2\cos x}\left[\frac{d}{dx}(x^2\cos x)\right] + e^{x(\log\cos x)}\frac{d}{dx}[x(\log\cos x)]$$

$$= e^{x^2\cos x}[2x\cos x - x^2\sin x] + e^{x(\log\cos x)}\left[1.\log(\cos x) + x\frac{1}{\cos x}(-\sin x)\right]$$

$$= e^{x^2\cos x}[2x\cos x - x^2\sin x] + e^{x(\log\cos x)}[\log\cos x - x\tan x]$$

**38.** Differentiate w.r.t. $x$:
$(x+1)^2(x+2)^3(x+3)^4$ **[NCERT Exemplar]**

**39.** If $(\cos x)^y = (\cos y)^x$, then find $\frac{dy}{dx}$. **[CBSE 2012]**

**Ans.** Given, $(\cos x)^y = (\cos y)^x$

On taking log both sides, we get

$$\log(\cos x)^y = \log(\cos y)^x$$

$$\Rightarrow y\log(\cos x) = x\log(\cos y)$$

On differentiating with respect to $x$, we get

$$\Rightarrow y\cdot\frac{d}{dx}\log(\cos x) + \log\cos x\cdot\frac{d}{dx}(y) = x\cdot\frac{d}{dx}\log(\cos y) + \log(\cos y)\frac{d}{dx}(x)$$

$$\Rightarrow y\cdot\frac{1}{\cos x}\frac{d}{dx}(\cos x) + \log(\cos x)\frac{dy}{dx} = x\cdot\frac{1}{\cos y}\frac{d}{dx}(\cos y) + \log\cos y.1$$

$$\Rightarrow y\cdot\frac{1}{\cos x}(-\sin x) + \log(\cos x)\cdot\frac{dy}{dx} = x\frac{1}{\cos y}(-\sin y)\cdot\frac{dy}{dx} + \log(\cos y).1$$

$$\Rightarrow -y\tan x + \log(\cos x)\frac{dy}{dx}$$

$$= -x\tan y\frac{dy}{dx} + \log(\cos y)$$

$$\Rightarrow [x\tan y + \log(\cos x)]\frac{dy}{dx}$$

$$= \log(\cos y) + y\tan x$$

$$\therefore \quad \frac{dy}{dx} = \frac{\log(\cos y) + y\tan x}{x\tan y + \log(\cos x)}$$

**40. Differentiate w.r.t. $x$: If $\sqrt{1-x^2} + \sqrt{1-y^2} = a(x-y)$, prove that $\frac{dy}{dx} = \sqrt{\frac{1-y^2}{1-x^2}}$**

**[NCERT Exemplar]**

**41. If $x = a(2\theta - \sin 2\theta)$ and $y = a(1 - \cos 2\theta)$, find $\frac{dy}{dx}$ when $\theta = \frac{\pi}{3}$. [CBSE 2018]**

**Ans.** We have $x = a(2\theta - \sin 2\theta)$ and $y = a(1 - \cos 2\theta)$

Clearly, $$\frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)} \quad \text{...(i)}$$

Here, $\frac{dx}{d\theta} = a(2 - \cos 2\theta.2) = 2a(1 - \cos 2\theta)$

and $\frac{dy}{d\theta} = a(0 + \sin 2\theta.2) = 2a\sin 2\theta$

From Eq. (i), we get

$$\frac{dy}{dx} = \frac{2a\sin 2\theta}{2a(1-\cos 2\theta)}$$

$$= \frac{\sin 2\theta}{1-\cos 2\theta}$$

$$= \frac{2\sin\theta\cos\theta}{2\sin^2\theta}$$

$[\because \cos 2\theta = 1 - 2\sin^2\theta]$

$$= \cot\theta$$

Now, $\left(\frac{dy}{dx}\right)_{\theta=\frac{\pi}{3}} = \cot\frac{\pi}{3} = \frac{1}{\sqrt{3}}$

**42. If $x = 3\sin t - \sin 3t$, $y = 3\cos t - \cos 3t$, then find $\frac{dy}{dx}$ at $t = \frac{\pi}{3}$. [NCERT Exemplar]**

**43. If $y = x\tan x + \sec x$, then find the value of $\frac{dy}{dx}$ at $x = \frac{\pi}{4}$.**

**44. Differentiate $\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$ w.r.t. $\tan^{-1}x$ when $x \neq 0$. [NCERT Exemplar]**

**Ans.** Let $u = \tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$ and $v = \tan^{-1}x$.

Now, $u = \tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$

Put $x = \tan\theta$

$$\Rightarrow \quad u = \tan^{-1}\left(\frac{\sqrt{1+\tan^2\theta}-1}{\tan\theta}\right)$$

$$= \tan^{-1}\left(\frac{\sec\theta - 1}{\tan\theta}\right)$$

$$= \tan^{-1}\left(\frac{1-\cos\theta}{\sin\theta}\right)$$

$$= \tan^{-1}\left(\frac{2\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\right)$$

$$= \tan^{-1}\left(\tan\frac{\theta}{2}\right)$$

$$= \frac{\theta}{2} = \frac{1}{2}\tan^{-1}x$$

$$\therefore \quad \frac{du}{dx} = \frac{1}{2}\times\frac{1}{1+x^2}$$

and $$\frac{dv}{dx} = \frac{d}{dx}(\tan^{-1}x)$$

$$= \frac{1}{1+x^2}$$

Now, $$\frac{du}{dv} = \frac{du/dx}{dv/dx} = \frac{1}{2}$$

**Caution**

- *The function with respect to which we need to differentiate should be taken as v and other function as u.*

**45. If $y = (\sin x)^{(\sin x)^{(\sin x)^{\dots\infty}}}$, then prove that**

$$\frac{dy}{dx} = \frac{y^2\cot x}{1 - y\log(\sin x)}$$

# LONG ANSWER Type Questions (LA)

[ 4 & 5 marks ]

**46. Find the derivative of the following functions:**

**(A)** $y = (\sin x - \cos x)^{(\sin x - \cos x)}$

**(B)** $y = \log(\sec x + \tan x)$

**Ans.** (A) Given: $y = (\sin x - \cos x)^{(\sin x - \cos x)}$

On taking log both sides, we get

$$\log y = (\sin x - \cos x)\log(\sin x - \cos x)$$

On differentiating both sides w.r.t. $x$, we get

$$\frac{1}{y}\frac{dy}{dx} = (\sin x - \cos x) \times \frac{1}{(\sin x - \cos x)} \times (\cos x + \sin x) + (\cos x + \sin x)\log(\sin x - \cos x)$$

$$= (\cos x + \sin x) + (\cos x + \sin x)\log(\sin x - \cos x)$$

$$\Rightarrow \frac{dy}{dx} = y(\cos x + \sin x)[1 + \log(\sin x - \cos x)]$$

$$\Rightarrow \frac{dy}{dx} = (\sin x - \cos x)^{(\sin x - \cos x)}.(\cos x + \sin x)[1 + \log(\sin x - \cos x)]$$

(B) Given: $y = \log(\sec x + \tan x)$

On differentiating w.r.t. $x$, we get

$$\frac{dy}{dx} = \frac{1}{(\sec x + \tan x)}\frac{d}{dx}(\sec x + \tan x)$$

$$= \frac{\sec x \tan x + \sec^2 x}{(\sec x + \tan x)}$$

$$= \frac{\sec x(\tan x + \sec x)}{(\sec x + \tan x)}$$

$$= \sec x$$

**47.** If $y = \sin^{-1}\left(\frac{a + b\cos x}{b + a\cos x}\right), b > a$, **then find its derivative.**

**48.** If $y = \tan^{-1}\left\{\frac{\sqrt{1+x^2} + \sqrt{1-x^2}}{\sqrt{1+x^2} - \sqrt{1-x^2}}\right\}, -1 < x < 1$, $x \neq 0$, **find** $\frac{dy}{dx}$.

**Ans.** Given: $y = \tan^{-1}\left\{\frac{\sqrt{1+x^2} + \sqrt{1-x^2}}{\sqrt{1+x^2} - \sqrt{1-x^2}}\right\}$

Put $x^2 = \cos 2\theta$, then

$$y = \tan^{-1}\left\{\frac{\sqrt{1+\cos 2\theta} + \sqrt{1-\cos 2\theta}}{\sqrt{1+\cos 2\theta} - \sqrt{1-\cos 2\theta}}\right\}$$

$$= \tan^{-1}\left\{\frac{\sqrt{2\cos^2\theta} + \sqrt{2\sin^2\theta}}{\sqrt{2\cos^2\theta} - \sqrt{2\sin^2\theta}}\right\}$$

$$= \tan^{-1}\left\{\frac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta}\right\}$$

$$= \tan^{-1}\left\{\frac{1 + \tan\theta}{1 - \tan\theta}\right\}$$

$$= \tan^{-1}\left\{\tan\left(\frac{\pi}{4} + \theta\right)\right\}$$

$$= \frac{\pi}{4} + \theta$$

$$\left[\begin{array}{l} \because -1 < x < 1 \Rightarrow 0 < x^2 < 1 \\ \Rightarrow 0 < \cos 2\theta < 1 \Rightarrow 0 < 2\theta < \frac{\pi}{2} \\ \Rightarrow 0 < \theta < \frac{\pi}{4} \Rightarrow \frac{\pi}{4} < \frac{\pi}{4} + \theta < \frac{\pi}{2} \end{array}\right]$$

$$\therefore y = \frac{\pi}{4} + \frac{1}{2}\cos^{-1}x^2$$

On differentiating with respect to $x$, we get

$$\frac{dy}{dx} = 0 + \frac{1}{2}\left(-\frac{1}{\sqrt{1-x^4}}\right)\frac{d}{dx}(x^2)$$

$$= -\frac{2x}{2\sqrt{1-x^4}}$$

$$= \frac{-x}{\sqrt{1-x^4}}$$

**49.** **Find the derivative of the following functions:**

(A) $\log(x^2 + y^2) = 2\tan^{-1}\left(\frac{y}{x}\right)$

(B) $y = (\cos x)^{\sin x}$

**50.** **Find the derivative of the following functions.**

(A) $y = (\sqrt{x})^{(\sqrt{x})^{(\sqrt{x})\ldots\infty}}$

(B) $x = \frac{e^t + e^{-t}}{2}$ and $y = \frac{e^t - e^{-t}}{2}$

# SECOND ORDER DERIVATIVE 4

## TOPIC 1

## SECOND ORDER DERIVATIVE OF PARAMETRIC FUNCTIONS

Let $y = f(x)$ be a function. Then,

$$\frac{dy}{dx} = f'(x) \quad \text{...(i)}$$

If $f'(x)$ is differentiable, we may differentiate (i) again with respect to $x$, then the left hand side becomes $\frac{d}{dx}\left(\frac{dy}{dx}\right)$, which is called the second order derivative of $y$ w.r.t. $x$ and is denoted by $\frac{d^2y}{dx^2}$. The second order derivative of $f(x)$ is denoted by $f''(x)$ or $y''$ or $y_2$.

If a function, represented by $x = f(t)$ and $y = g(t)$, defines $y$ as a twice differentiable function of $x$, then at any point where $\frac{dx}{dt} \neq 0$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$$

$$= \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$= \frac{d}{dx}\left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right)$$

$$= \frac{d}{dt}\left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right) \cdot \frac{dt}{dx}$$

**Example 4.1:** Find $\frac{d^2y}{dx^2}$, if $y = x^3 \log x$. **[NCERT]**

**Ans.** Given, $y = x^3 \log x$, we have

$$\frac{dy}{dx} = 3x^2 \log x + x^3 \cdot \frac{1}{x}$$

$$= x^2(1 + 3\log x)$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$= \frac{d}{dx}[x^2(1 + 3\log x)]$$

$$= 2x.(1 + 3\log x) + x^2 \cdot \frac{3}{x}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = 2x.(1 + 3\log x) + 3x, \text{ or } x(5 + 6\log x)$$

**Example 4.2:** If $y = 500e^{7x} + 600e^{-7x}$, then show that $\frac{d^2y}{dx^2} = 49y$. **[NCERT]**

**Ans.** Given, $y = 500e^{7x} + 600e^{-7x}$, we have

$$\frac{dy}{dx} = 500e^{7x}(7) + 600e^{-7x}(-7)$$

$$= 3500\,e^{7x} - 4200\,e^{-7x}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$= \frac{d}{dx}[3500e^{7x} - 4200e^{-7x}]$$

$$= 3500\,e^{7x}(7) - 4200\,e^{-7x}(-7)$$

$$= 24500\,e^{7x} + 29400\,e^{-7x}$$

$$= 49\,(500\,e^{7x} + 600\,e^{-7x})$$

$$= 49\,y$$

Hence, proved.

**Example 4.3:** If $x = a\cos\theta + b\sin\theta$ and $y = a\sin\theta - b\cos\theta$, then show that

$$y^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = 0. \quad \textbf{[Delhi 2015]}$$

**Ans.** Given, $x = a\cos\theta + b\sin\theta$ ...(i)

and $y = a\sin\theta - b\cos\theta$ ...(ii)

Here, $\theta$ is the parameter.

On differentiating Eqs. (i) and (ii), respectively w.r.t. $\theta$, we get

$$\frac{dx}{d\theta} = \frac{d}{d\theta}(a\cos\theta + b\sin\theta)$$

$$\Rightarrow \quad \frac{dx}{d\theta} = -a\sin\theta + b\cos\theta$$

and

$$\frac{dy}{d\theta} = \frac{d}{d\theta}(a\sin\theta - b\cos\theta)$$

$$\Rightarrow \quad \frac{dy}{d\theta} = a\cos\theta + b\sin\theta$$

Now,

$$\frac{dy}{dx} = \frac{(dy/d\theta)}{(dx/d\theta)}$$

$$= \frac{a\cos\theta + b\sin\theta}{-(a\sin\theta - b\cos\theta)} = -\frac{x}{y}$$

[from Eqs. (i) and (ii)]

Again, differentiating both sides w.r.t. $x$, we get

$$\frac{d^2y}{dx^2} = -\left[\frac{y\frac{d}{dx}(x) - x\frac{dy}{dx}}{y^2}\right]$$

$$\Rightarrow \quad y^2\frac{d^2y}{dx^2} = -y + x\frac{dy}{dx}$$

$$\Rightarrow y^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = 0$$

Hence, proved.

**Example 4.4:** Find $\frac{d^2y}{dx^2}$ at $\theta = \frac{\pi}{2}$, when $x = a(\theta + \sin\theta), y = a(1 - \cos\theta)$.

**Ans.** Given, $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$, we have

$$\frac{dx}{d\theta} = a(1 + \cos\theta)$$

$$= 2a\cos^2\frac{\theta}{2};$$

and $$\frac{dy}{d\theta} = a(\sin\theta)$$

$$= 2a\sin\frac{\theta}{2}\cos\frac{\theta}{2}$$

Now, $$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$$

$$= \frac{2a\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2a\cos^2\frac{\theta}{2}}$$

$$= \tan\frac{\theta}{2}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$= \frac{d}{d\theta}\left(\tan\frac{\theta}{2}\right)\cdot\frac{d\theta}{dx}$$

$$= \frac{1}{2}\sec^2\frac{\theta}{2}\cdot\frac{1}{2a\cos^2\frac{\theta}{2}}$$

$$= \frac{1}{4a}\sec^4\frac{\theta}{2}$$

Thus, $$\left[\frac{d^2y}{dx^2}\right]_{\theta=\frac{\pi}{2}} = \left[\frac{1}{4a}\sec^4\frac{\theta}{2}\right]_{\theta=\frac{\pi}{2}}$$

$$= \frac{1}{4a}\sec^4\frac{\pi}{4}$$

$$= \frac{1}{4a}\cdot(\sqrt{2})^4 = \frac{1}{a}$$

**Example 4.5:** If $y = \sin^{-1}x$, then show that $(1 - x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = 0$.

**Ans.** We have, $y = \sin^{-1}x$

On differentiating both sides w.r.t. $x$, we get

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$

$$\Rightarrow \quad \sqrt{1-x^2}\frac{dy}{dx} = 1$$

On squaring both sides, we get

$$(1-x^2)\left(\frac{dy}{dx}\right)^2 = 1$$

Again, differentiating both sides w.r.t. $x$, we get

$$\frac{d}{dx}\left[(1-x^2)\left(\frac{dy}{dx}\right)^2\right] = \frac{d}{dx}(1)$$

$$\Rightarrow (1-x^2)\frac{d}{dx}\left(\frac{dy}{dx}\right)^2 + \left(\frac{dy}{dx}\right)^2\frac{d}{dx}(1-x^2) = 0$$

$$\Rightarrow \quad (1-x^2)2\frac{dy}{dx}\cdot\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2(0-2x) = 0$$

$$\Rightarrow \quad 2\frac{dy}{dx}\left[(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx}\right] = 0$$

$$\Rightarrow \quad (1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = 0$$

Hence, proved.

## OBJECTIVE Type Questions

[ 1 mark ]

### Multiple Choice Questions

**1.** If $y = 5\cos x - 3\sin x$, then $\frac{d^2y}{dx^2}$ is equal to:

(a) $-y$ (b) $y$

(c) $25y$ (d) $9y$

**[CBSE Term-1 SQP 2021]**

**Ans.** *(a)* $-y$

$$y = 5\cos x - 3\sin x$$

$$\Rightarrow \quad \frac{dy}{dx} = -5\sin x - 3\cos x$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = -5\cos x + 3\sin x = -y$$

**[CBSE Marking Scheme Term-1 SQP 2021]**

**Explanation:**

$$y = 5\cos x - 3\sin x \quad ...(i)$$

$$\therefore \quad \frac{dy}{dx} = -5\sin x - 3\cos x$$

$$\therefore \quad \frac{d^2y}{dx^2} = -5\cos x + 3\sin x$$

$$= -(5\cos x - 3\sin x) = -y$$

[using (i)]

**2.** **The second order derivative of the function $y = \log x^3$ is:**

(a) $-\frac{3}{x^2}$ (b) $\frac{3}{x^2}$

(c) $\frac{2}{x^6}$ (d) $\frac{1}{x^3}$

**3.** **If $y = e^{5x}$, then $\frac{d^2y}{dx^2}$ is equal to**

(a) $\frac{e^{5x}}{25}$ (b) $25e^{5x}$

(c) $5e^{5x}$ (d) $\frac{e^{5x}}{5}$

**4.** **If $x = a\sec\theta$, $y = b\tan\theta$, then $\frac{d^2y}{dx^2}$ at $\theta = \frac{\pi}{6}$ is:**

(a) $\frac{-3\sqrt{3}b}{a^2}$ (b) $\frac{-2\sqrt{3}b}{a}$

(c) $\frac{-3\sqrt{3}b}{a}$ (d) $\frac{-b}{3\sqrt{3}a^2}$

**[CBSE Term-1 SQP 2021]**

**Ans.** *(a)* $\frac{-3\sqrt{3}b}{a^2}$

$$x = a\sec\theta$$

$$\Rightarrow \quad \frac{dx}{d\theta} = a\tan\theta\sec\theta$$

$$y = b\tan\theta$$

$$\Rightarrow \quad \frac{dy}{d\theta} = b\sec^2\theta$$

$$\therefore \quad \frac{dy}{dx} = \frac{b}{a}\operatorname{cosec}\theta$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{-b}{a}\operatorname{cosec}\theta.\cot\theta.\frac{d\theta}{dx}$$

$$= \frac{-b}{a^2}\cot^3\theta$$

$$\therefore \quad \left.\frac{d^2y}{dx^2}\right]\theta = \frac{\pi}{6} = \frac{-3\sqrt{3}b}{a^2}$$

**[CBSE Marking Scheme Term-1 SQP 2021]**

**Explanation:** $x = a\sec\theta$; $y = b\tan\theta$

Differentiating w.r.t. $\theta$, we get

$$\frac{dx}{d\theta} = a\sec\theta\tan\theta;\ \frac{dy}{d\theta} = b\sec^2\theta \quad ...(i)$$

$$\therefore \quad \frac{dx}{dy} = \frac{dy/d\theta}{dx/d\theta}$$

$$= \frac{b\sec^2\theta}{a\sec\theta\tan\theta}$$

$$= \frac{b}{a}\operatorname{cosec}\theta$$

So, $$\frac{d^2y}{dx^2} = \frac{b}{a}(-\operatorname{cosec}\theta\cot\theta).\frac{d\theta}{dx}$$

$$= -\frac{b}{a}\operatorname{cosec}\cot\theta \times \frac{1}{a\sec\theta\tan\theta}$$

[From (i)]

$$\therefore \quad \left(\frac{d^2y}{dx^2}\right)_{\theta=\frac{\pi}{6}} = -\frac{b}{a^2}\,\frac{\operatorname{cosec}\frac{\pi}{6}\cot\frac{\pi}{6}}{\sec\frac{\pi}{6}\tan\frac{\pi}{6}}$$

$$= -\frac{b}{a^2}\,\frac{2\times\sqrt{3}}{\frac{2}{\sqrt{3}}\times\frac{1}{\sqrt{3}}}$$

$$= \frac{-3\sqrt{3}b}{a^2}$$

**5.** **If $y = x^5 + e^x$, then $\frac{d^2y}{dx^2}$ is:**

(a) $20x^3 + 4e^{2x}$ (b) $20x^4 + 2e^{2x}$

(c) $20x^3 + e^{2x}$ (d) None of these

**6.** **If $x = \log_e y$, then $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} =$**

(a) $y$ (b) $2y$

(c) $-2y$ (d) $-y$

**[Delhi Gov. 2022]**

**Ans.** *(d)* $-y$

**Explanation:** We have,

$$x = \log_e y$$

or, $$e^x = y$$

Differentiating twice w.r.t. x, we get

$$e^x = \frac{dy}{dx}$$

and $$e^x = \frac{d^2y}{dx^2}$$

Now, $$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} = e^x - 2e^x$$

$$= -e^x = -y.$$

**7.** If $y = x \cos x$, then $\frac{d^2y}{dx^2}$ is:

(a) $\cos x - 2x \sin x$ (b) $x \cos x - 2 \sin x$

(c) $-x \cos x - 2 \sin x$ (d) $x \cos x + 2 \sin x$

**Ans.** *(c)* $-x \cos x - 2 \sin x$

**Explanation:** Given: $y = x \cos x$

On differentiating twice with respect to x, we get

$$\frac{dy}{dx} = 1.\cos x - x \sin x$$

and $$\frac{d^2y}{dx^2} = -\sin x - [1. \sin x + x \cos x]$$

$$= -x \cos x - 2 \sin x$$

**8.** If $y = \log_e\left(\frac{x^2}{e^2}\right)$, then $\frac{d^2y}{dx^2}$ is equal to:

(a) $\frac{-1}{x}$ (b) $-\frac{1}{x^2}$

(c) $\frac{2}{x^2}$ (d) $-\frac{2}{x^2}$

**[CBSE 2020]**

**9.** If $y = ax^{n+1} + bx^{-n}$, then $x^2\frac{d^2y}{dx^2}$ is:

(a) $ny$ (b) $n^2y$

(c) $n(n-1)y$ (d) $n(n+1)y$

**Ans.** *(d)* $n(n + 1)y$

**Explanation:** Given: $y = ax^{n+1} + bx^{-n}$.

Differentiating twice w.r.t. x, we get

$$\frac{dy}{dx} = a(n+1)x^n - nbx^{-n-1}$$

and, $$\frac{d^2y}{dx^2} = an(n+1)x^{n-1} + bn(n+1)x^{-n-2}$$

$$\Rightarrow x^2\frac{d^2y}{dx^2} = an(n+1)x^{n+1} + bn(n+1)x^{-n}$$

$$= n(n+1)[ax^{n+1} + bx^{-n}], \text{ or } n(n+1)y$$

**10.** If $x = t^2$ and $y = t^3$, then $\frac{d^2y}{dx^2}$ is equal to:

(a) $\frac{3}{2}$ (b) $\frac{3}{4t}$

(c) $\frac{3}{2t}$ (d) $\frac{3}{4}$

**[NCERT Exemplar]**

**11.** If $\tan y = x$, then when $x = 1$, value of $4\frac{d^2y}{dx^2}$ is:

(a) 2 (b) −2

(b) 1 (d) −1

**[Delhi Gov. 2022]**

**Ans.** *(b)* −2

**Explanation:** We have,

$$\tan y = x$$

or $$y = \tan^{-1} x$$

Differentiating twice w.r.t. x, we get

$$\frac{dy}{dx} = \frac{1}{1+x^2}$$

$$= (1 + x^2)^{-1}$$

and $$\frac{d^2y}{dx^2} = -1(1 + x^2)^{-2}(2x)$$

$$= \frac{-2x}{(1+x^2)^2}$$

So, $$4\frac{d^2y}{dx^2} = \frac{-8x}{(1+x^2)^2}$$

$$\Rightarrow \left[4\frac{d^2y}{dx^2}\right]_{x=1} = \frac{-8(1)}{[1+(1)^2]^2}$$

$$= \frac{-8}{4} = -2$$

**12.** The second order derivative of $y = \log e^{x^3}$ is:

(a) $9\log e^{x^3}$ (b) $6x$

(c) 6 (d) $6x^3$

**13.** If $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, then the value of $\frac{d^2y}{dx^2}$ is:

(a) $-\frac{b^4}{a^2y^3}$ (b) $\frac{b^4}{a^2y^3}$

(c) $-\dfrac{b^2y^2}{a^2}$ (d) $\dfrac{b^2}{a^2y^3}$

**Ans.** *(a)* $-\dfrac{b^4}{a^2y^3}$

**Explanation:** We have $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$ ...(i)

On differentiating both sides w.r.t. $x$, we get

$$\frac{2x}{a^2}-\frac{2y}{b^2}\frac{dy}{dx}=1$$

$$\frac{2y}{b^2}\frac{dy}{dx}=\frac{2x}{a^2}$$

$$\frac{dy}{dx}=\frac{b^2x}{a^2y} \quad ...(ii)$$

On again differentiating with respect to $x$, we get

$$\frac{d^2y}{dx^2}=\frac{b^2}{a^2}\left[\left(\frac{y\times 1-x\dfrac{dy}{dx}}{y^2}\right)\right]$$

$$=\frac{b^2}{a^2y^2}\left[y-x\times\frac{b^2x}{a^2y}\right]$$

[∵ From Eq. (ii)]

$$=\frac{b^2}{a^2y^2}\left[\frac{a^2y^2-b^2x^2}{a^2y}\right]$$

$$=\frac{b^2}{a^4y^3}[-a^2b^2]$$

[∵ Using Eq. (i)]

$$=-\frac{b^4}{a^2y^3}$$

## VERY SHORT ANSWER Type Questions (VSA)

[ 1 mark ]

**14.** If $y=x^4+3x^2$, then find $\dfrac{d^2y}{dx^2}$.

**15.** If $\dfrac{dy}{dx}=\cos^2 x$, then find the value of $\dfrac{d^2y}{dx^2}$ at $x=\dfrac{\pi}{4}$.

**Ans.** Given: $\dfrac{dy}{dx}=\cos^2 x$

On differentiating with respect to $x$, we get

$$\frac{d^2y}{dx^2}=2\cos x\,(-\sin x)$$

$$\Rightarrow \quad \frac{d^2y}{dx^2}=-\sin 2x$$

At $x=\dfrac{\pi}{4}$,

$$\frac{d^2y}{dx^2}=-2\sin\left(2\times\frac{\pi}{4}\right)$$

$$=-2\sin\frac{\pi}{2}=-2\times 1$$

$$=-2$$

**16.** Find the second order derivative of $y=\tan^{-1}x$.

**17.** Find the second order derivative of $y=\log e^{\sin x}$.

**Ans.** Given: $y=\log e^{\sin x}$

$\Rightarrow$ $y=\sin x$

On differentiating twice with respect to $x$, we get

$$\frac{dy}{dx}=\cos x$$

and $$\frac{d^2y}{dx^2}=-\sin x$$

**18.** Find $\dfrac{d^2y}{dx^2}$, if $y=x^2+\cot x$.

**19.** Find second order derivative of the function $y=b^{10x}$.

**Ans.** Given: $y=b^{10x}$

On differentiating twice with respect to $x$, we get

$$\frac{dy}{dx}=b^{10x}\log b\times 10$$

$$=(10\log b)\,b^{10x}$$

$$\frac{d^2y}{dx^2}=(10\log b)\,b^{10x}\times\log b\times 10$$

$$=(10\log b)^2\,b^{10x}$$

**20.** Megha had a very sharp and mathematical mind. She could relate most things she saw with its appropriate mathematical function. When she visited the amusement park with her family, she could relate the roller coaster somewhat with the trigonometric function cos $(x^2)$.

Find $\frac{d^2y}{dx^2}$, if $y = \cos^2 x$.

**Ans.** We have, $y = \cos^2 x$

On differentiating twice with respect to $x$, we get

$$\frac{dy}{dx} = -2\cos x \sin x$$

$$\Rightarrow \quad \frac{dy}{dx} = -\sin 2x$$

and $$\frac{d^2y}{dx^2} = -2\cos 2x$$

## SHORT ANSWER Type-I Questions (SA-I)

[ 2 marks ]

**21.** Find the second order derivative of $y = |x^2 + 1|$.

**22.** If $y = \cos(\log x)$, find $\frac{d^2y}{dx^2}$.

**Ans.** Given: $y = \cos(\log x)$

On differentiating twice with respect to $x$, we get

$$\frac{dy}{dx} = -\sin(\log(x))\frac{d}{dx}(\log x)$$

$$\frac{dy}{dx} = -\frac{\sin(\log x)}{x}$$

and

$$\frac{d^2y}{dx^2} = -\left[\frac{x\frac{d}{dx}\sin(\log x) - \sin(\log x)\frac{d}{dx}(x)}{x^2}\right]$$

$$= -\left[\frac{x \times \cos(\log x) \times \frac{1}{x} - \sin(\log x) \times 1}{x^2}\right]$$

$$= \frac{-\cos(\log x) + \sin(\log x)}{x^2}$$

**23.** If $y = x^{3x}$, then find $\frac{d^2y}{dx^2}$.

**24.** If $x = a\cos\theta$, $y = b\sin\theta$, then find $\frac{d^2y}{dx^2}$.

[CBSE 2020]

**Ans.** We have, $x = a\cos\theta$, $y = b\sin\theta$

$$\therefore \quad \frac{dx}{d\theta} = -a\sin\theta \quad \text{...(i)}$$

$$\frac{dy}{d\theta} = b\cos\theta \quad \text{...(ii)}$$

So, $$\frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx}$$

$$= \frac{b\cos\theta}{-a\sin\theta} = -\frac{b}{a}\cot\theta$$

Now, $$\frac{d^2y}{dx^2} = \frac{-b}{a}(-\text{cosec}^2\theta)\frac{d\theta}{dx}$$

$$= \frac{b}{a}\text{cosec}^2\theta\left(-\frac{1}{a}\text{cosec}\,\theta\right)$$

[Using (i)]

$$= \frac{-b}{a^2}\text{cosec}^3\theta$$

**25.** Find the second derivative of the function $y = e^{8x}\cos 4x$.

**26.** If $y = A\sin x + B\cos x$, then prove that $\frac{d^2y}{dx^2} + y = 0$.

**27.** If $x = at^2$, $y = 2at$ then find $\frac{d^2y}{dx^2}$.

[CBSE 2020]

**Ans.** Given: $x = at^2$, $y = 2at$

$$\therefore \quad \frac{dx}{dt} = 2at,$$

$$\frac{dy}{dt} = 2a$$

$$\therefore \quad \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2a}{2at} = \frac{1}{t}$$

So
$$\frac{d^2y}{dx^2} = \frac{-1}{t^2}\cdot\frac{dt}{dx}$$
$$= \frac{-1}{t^2}\cdot\frac{1}{2at} = -\frac{1}{2at^3}$$

## SHORT ANSWER Type-II Questions (SA-II)

[ **3** marks ]

**28.** If $y = 4e^{5x} + 5e^{4x}$, then prove that $\frac{d^2y}{dx^2} - 9\frac{dy}{dx} + 20y = 0$

**29.** If $e^y(x + 1) = 1$, then show that

$$\frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2.$$ **[CBSE 2017]**

**30.** If $y = Ae^{mx} + Be^{nx}$, show that $\frac{d^2y}{dx^2} - (m+n)\frac{dy}{dx} + mny = 0.$

**Ans.** Given: $y = Ae^{mx} + Be^{nx}$

Differentiating twice with respect to $x$, we get

$$\frac{dy}{dx} = Ae^{mx}.m + Be^{nx}.n$$

and
$$\frac{d^2y}{dx^2} = m^2 Ae^{mx} + n^2Be^{nx}$$

Now,
$$\text{L.H.S.} = \frac{d^2y}{dx^2} - (m+n)\frac{dy}{dx} + mny$$

$$= m^2Ae^{mx} + n^2Be^{nx} - (m+n)(mAe^{mx} + nBe^{nx}) + mn(Ae^{mx} + Be^{nx})$$

$$= Ae^{mx}[m^2 - (m+n)m + mn] + Be^{nx}[n^2 - (m+n)n + mn]$$

$$= Ae^{mx} \times 0 + Be^{nx} \times 0 = 0 = \text{R.H.S.}$$

Hence proved.

**31.** If $y = (\sin^{-1}x)^2$, prove that

$$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - 2 = 0.$$ **[CBSE 2019]**

**Ans.** Given: $y = (\sin^{-1}x)^2$

Differentiating w.r.t. $x$, we get

$$\frac{dy}{dx} = 2\sin^{-1}x \times \frac{1}{\sqrt{1-x^2}} \quad ...(i)$$

Again, differentiating w.r.t. $x$, we get

$$\frac{d^2y}{dx^2} = \frac{2}{\sqrt{1-x^2}}\frac{d}{dx}(\sin^{-1}x) + 2\sin^{-1}x\frac{d}{dx}\left(\frac{1}{\sqrt{1-x^2}}\right)$$

$$= \frac{2}{\sqrt{1-x^2}} \times \frac{1}{\sqrt{1-x^2}} + 2\sin^{-1}x \times \left(\left(-\frac{1}{2}\right)\frac{-2x}{(1-x^2)^{3/2}}\right)$$

$$\Rightarrow \quad (1-x^2)\frac{d^2y}{dx^2} = 2\left[1 + \frac{x\sin^{-1}x}{\sqrt{1-x^2}}\right]$$

$$\Rightarrow (1-x^2)\frac{d^2y}{dx^2} - 2x\frac{\sin^{-1}x}{\sqrt{1-x^2}} - 2 = 0$$

$$\Rightarrow \quad (1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - 2 = 0$$

Hence, proved.

**32.** If $y = (\tan^{-1}x)^2$, show that $(x^2 + 1)^2y_2 + 2x(x^2 + 1)y_1 = 2.$

**33.** If $y = x^3 \log\left(\frac{1}{x}\right)$, then prove that

$$x\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 3x^2 = 0.$$

**34.** If $y = Ae^{-mt}\sin(nt + c)$, prove that

$$\frac{d^2y}{dt^2} - 2m\frac{dy}{dt^2} + (m^2 + n^2)y = 0.$$

**35.** If $y = 2\cos(\log x) + 3\sin(\log x)$, prove that

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0.$$ **[CBSE 2016]**

**Ans.** We have, $y = 2\cos(\log x) + 3\sin(\log x)$

Differentiating with respect to $x$, we get

$$\frac{dy}{dx} = -2\sin(\log x) \times \frac{1}{x} + 3\cos(\log x) \times \frac{1}{x}$$

$$\Rightarrow x\frac{dy}{dx} = -2\sin(\log x) + 3\cos(\log x)$$

Again differentiating with respect to $x$, we get

$$x\frac{d^2y}{dx^2}+\frac{dy}{dx} = -2\cos(\log x)\times\frac{1}{x}$$

$$-3\sin(\log x)\times\frac{1}{x}$$

$$\Rightarrow \quad x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx} = -[2\cos(\log x)$$

$$+ 3\sin(\log x)]$$

$$\Rightarrow \quad x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx} = -y$$

$$\Rightarrow \quad x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+y = 0$$

Hence, proved.

**36.** Ⓐ**If $x^m.y^n = (x+y)^{m+n}$, prove that $\frac{d^2y}{dx^2} = 0$.**

**[CBSE 2017]**

## LONG ANSWER Type Questions (LA)

[ **4** & **5** marks ]

**37. If $(x - a)^2 + (y - b)^2 = c^2$, then prove that $\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}$ is a constant and independent of $a$ and $b$.**

**Ans.** Given: $(x-a)^2+(y-b)^2 = c^2$ ...(i)

On differentiating twice with respect to $x$, we get

$$2(x-a)+2(y-b)\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = -\left(\frac{x-a}{y-b}\right) \quad ...(ii)$$

and
$$\frac{d^2y}{dx^2} = -\left[\frac{(y-b)(1)-(x-a)\frac{dy}{dx}}{(y-b^2)}\right]$$

$$= -\left[\frac{(y-b)-(x-a)\left\{-\left(\frac{x-a}{y-b}\right)\right\}}{(y-b)^2}\right]$$

[From eq. (ii)]

$$= -\left[\frac{(y-b)+\frac{(x-a)^2}{(y-b)}}{(y-b)^2}\right]$$

$$= -\left[\frac{(y-b)^2+(x-a)^2}{(y-b)^3}\right]$$

$$= -\left[\frac{c^2}{(y-b)^3}\right] \quad \text{[From Eq. (i)]}$$

Now, L.H.S. $= \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}$

$$= \frac{\left[1+\left(\frac{x-a}{y-b}\right)^2\right]^{3/2}}{-\frac{c^2}{(y-b)^3}}$$

$$= \frac{\left(\frac{c^2}{(y-b)^2}\right)^{3/2}}{-\frac{c^2}{(y-b)^3}}$$

$$= -\frac{c^3}{(y-b)^3}\times\frac{(y-b)^3}{c^2}$$

$$= -c,$$

which is a constant and independent of $a$ and $b$.

Hence proved.

**38. If $y = 3\cos(\log x) + 4\sin(\log x)$, show that $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$.**

**Ans.** We have, $y = 3\cos(\log x) + 4\sin(\log x)$ ...(i)

On differentiating with respect to $x$, we get

$$\frac{dy}{dx} = -3\sin(\log x)\frac{d}{dx}(\log x)$$

$$+ 4\cos(\log x)\frac{d}{dx}(\log x)$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{-3\sin(\log x)}{x}+\frac{4\cos(\log x)}{x}$$

$$\Rightarrow \quad x\frac{dy}{dx} = -3\sin(\log x) + 4\cos(\log x)$$

Again differentiating with respect to $x$, we get

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx} = -3\cos(\log x)\frac{d}{dx}(\log x) - 4\sin(\log x)\frac{d}{dx}(\log x)$$

$$= \frac{-3\cos(\log x)}{x} - \frac{4\sin(\log x)}{x}$$

$$\Rightarrow x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} = -[3\cos(\log x) + 4\sin(\log x)]$$

$$= -y \qquad \text{[From eq. (i)]}$$

$$\Rightarrow x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0.$$

Hence, proved.

**39.** **If $x = \sin t$ and $y = \sin pt$, prove that**

$$(1 - x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + p^2y = 0$$

**[CBSE 2019, NCERT Exemplar]**

**40.** **If $y = \log(x - \sqrt{x^2 + m^2})$, prove that**

$$(x^2 + m^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} = 0.$$

**Ans.** Given, $y = \log(x - \sqrt{x^2 + m^2})$

On differentiating with respect to $x$, we get

$$\frac{dy}{dx} = \frac{1}{(x - \sqrt{x^2 + m^2})}\frac{d}{dx}(x - \sqrt{x^2 + m^2})$$

$$= \frac{1}{(x - \sqrt{x^2 + m^2})}\left(1 - \frac{2x}{2\sqrt{x^2 + m^2}}\right)$$

$$= \frac{(\sqrt{x^2 + m^2} - x)}{(x - \sqrt{x^2 + m^2})(\sqrt{x^2 + m^2})}$$

$$= -\frac{1}{\sqrt{x^2 + m^2}}$$

$$\Rightarrow \sqrt{x^2 + m^2}\frac{dy}{dx} = -1 \Rightarrow (x^2 + m^2)\left(\frac{dy}{dx}\right)^2 = 1$$

Again differentiating with respect to $x$, we get

$$(x^2 + m^2) \times 2\left(\frac{dy}{dx}\right)\left(\frac{d^2y}{dx^2}\right) + \left(\frac{dy}{dx}\right)^2(2x) = 0$$

$$2\frac{dy}{dx}\left[(x^2 + m^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx}\right] = 0$$

$$\Rightarrow (x^2 + m^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} = 0$$

Hence, proved.

$$\Rightarrow \quad x\frac{dy}{dx} = -3\sin(\log x) + 4\cos(\log x)$$

Again differentiating with respect to $x$, we get

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx} = -3\cos(\log x)\frac{d}{dx}(\log x) - 4\sin(\log x)\frac{d}{dx}(\log x)$$

$$= \frac{-3\cos(\log x)}{x} - \frac{4\sin(\log x)}{x}$$

$$\Rightarrow x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} = -[3\cos(\log x) + 4\sin(\log x)]$$

$$= -y \qquad \text{[From eq. (i)]}$$

$$\Rightarrow x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0.$$

Hence, proved.

**39. ⓐIf $x = \sin t$ and $y = \sin pt$, prove that**

$$(1 - x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + p^2y = 0$$

**[CBSE 2019, NCERT Exemplar]**

**40. If $y = \log(x - \sqrt{x^2 + m^2})$, prove that**

$$(x^2 + m^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} = 0.$$

**Ans.** Given, $y = \log(x - \sqrt{x^2 + m^2})$

On differentiating with respect to $x$, we get

$$\frac{dy}{dx} = \frac{1}{(x - \sqrt{x^2 + m^2})}\frac{d}{dx}(x - \sqrt{x^2 + m^2})$$

$$= \frac{1}{(x - \sqrt{x^2 + m^2})}\left(1 - \frac{2x}{2\sqrt{x^2 + m^2}}\right)$$

$$= \frac{(\sqrt{x^2 + m^2} - x)}{(x - \sqrt{x^2 + m^2})(\sqrt{x^2 + m^2})}$$

$$= -\frac{1}{\sqrt{x^2 + m^2}}$$

$$\Rightarrow \sqrt{x^2 + m^2}\frac{dy}{dx} = -1 \Rightarrow (x^2 + m^2)\left(\frac{dy}{dx}\right)^2 = 1$$

Again differentiating with respect to $x$, we get

$$(x^2 + m^2) \times 2\left(\frac{dy}{dx}\right)\left(\frac{d^2y}{dx^2}\right) + \left(\frac{dy}{dx}\right)^2(2x) = 0$$

$$2\frac{dy}{dx}\left[(x^2 + m^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx}\right] = 0$$

$$\Rightarrow (x^2 + m^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} = 0$$

Hence, proved.

# TOPPER'S CORNER

## VERY SHORT ANSWER Type Questions (VSA)

[ 1 mark ]

**1. Determine the value of '*k*' for which the following function *f*(*x*) is continuous at *x* = 3:**

$$f(x) = \begin{cases} \dfrac{(x+3)^2 - 36}{x-3}, & x \neq 3 \\ k, & x = 3 \end{cases}$$

**Ans.**

$$f(x) = \begin{cases} \dfrac{(x+3)^2 - 36}{x-3} & ; x \neq 0 \\ k & ; x = 3 \end{cases}$$

$f(x)$ is continuous at $x = 3$

$$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^+} f(x) = f(3)$$

$$\lim_{x \to 3} \frac{(x+3)^2 - (6)^2}{x-3} = k$$

$$\lim_{x \to 3} \frac{(x+3-6)(x+3+6)}{(x-3)} = k$$

$$\lim_{x \to 3} \frac{(x-3)(x+9)}{(x-3)} = k$$

$$\lim_{x \to 3} (x+9) = k$$

$$12 = k$$

$$k = 12$$

**[CBSE Topper 2017]**

## SHORT ANSWER Type-I Questions (SA-I)

[ 2 marks ]

**2. Differentiate $\tan^{-1}\left(\frac{1+\cos x}{\sin x}\right)$ with respect to *x*.**

**Ans.**

$$\tan^{-1}\left(\frac{1+\cos x}{\sin x}\right)$$

$$y = \tan^{-1}\left(\frac{2\cos^2 x/2}{2\sin x/2 \cos x/2}\right)$$

$$y = \tan^{-1}\left(\frac{\cos x/2}{\sin x/2}\right)$$

$$y = \tan^{-1}(\cot x/2)$$

$$y = \tan^{-1}\left[\tan\left(\pi/2 - x/2\right)\right]$$

$$y = \frac{\pi}{2} - \frac{x}{2}$$

$$y = \frac{\pi}{2} - \frac{x}{2}$$

Differentiating w.r.t. $x$

$$\frac{dy}{dx} = 0 - \frac{1}{2}$$

$$\frac{dy}{dx} = \frac{-1}{2} \quad \text{Ans}$$

[CBSE Topper 2018]

## SHORT ANSWER Type-II Questions (SA-II)

[ 3 marks ]

3. If $x = a(2\theta - \sin 2\theta)$ and $y = a(1 - \cos 2\theta)$, find $\frac{dy}{dx}$ when $\theta = \frac{\pi}{3}$.

**Ans.**

$$x = a(2\theta - \sin 2\theta)$$

$$\frac{dx}{d\theta} = a(2 - 2\cos 2\theta)$$

$$\frac{dx}{d\theta} = 2a(1 - \cos 2\theta)$$

$$y = a(1 - \cos 2\theta)$$

$$\frac{dy}{d\theta} = a(0 + 2\sin 2\theta)$$

$$\frac{dy}{d\theta} = 2a\sin 2\theta$$

$$\frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx} = \frac{2a\sin 2\theta}{2a(1 - \cos 2\theta)} = \frac{2\sin\theta\cos\theta}{2\sin^2\theta}$$

$$\frac{dy}{dx} = \cot\theta$$

$$\left(\frac{dy}{dx}\right)_{(\theta = \pi/3)} = \cot\frac{\pi}{3}$$

$$= \cot(60^\circ)$$

$$= \boxed{\frac{1}{\sqrt{3}}} \quad \text{Ans}$$

[CBSE Topper 2018]

**4.** If $x = \cos t + \log\left(\tan\frac{t}{2}\right)$, $y = \sin t$, then find the values of $\frac{d^2y}{dt^2}$ and $\frac{d^2y}{dx^2}$ at $t = \frac{\pi}{4}$.

**Ans.**

$y = \sin t$ $\qquad$ $x = \cos t + \log(\tan(t/2))$

Diff. w.r.t $t$ $\qquad$ Diff. w.r.t $t$

$\frac{dy}{dt} = \cos t$ — (1) $\qquad$ $\frac{dx}{dt} = -\sin t + \frac{1}{\tan t/2} \cdot \sec^2 t/2 \cdot \frac{1}{2}$

Diff w.r.t $t$ $\qquad$ $= -\sin t + \frac{\cos t/2}{2\sin t/2 \cos^2 t/2}$

$\frac{d^2y}{dt^2} = -\sin t$ $\qquad$ $= -\sin t + \frac{1}{\sin t}$ $[\because 2\sin A\cos A = \sin 2A]$

$\left.\frac{d^2y}{dt^2}\right]_{t=\pi/4} = -\sin\frac{\pi}{4}$ $\qquad$ $\frac{dx}{dt} = -\sin t + \frac{1}{\sin t}$ $\Rightarrow \frac{dx}{dt} = \frac{\cos^2 t}{\sin t}$ (2)

$\left.\frac{d^2y}{dt^2}\right]_{t=\pi/4} = -\frac{1}{\sqrt{2}}$

P.T.O

Q.4 contd.

Divide eqn (1) by (2)

$\frac{dy/dt}{dx/dt} = \frac{\cos t}{-\sin t + \frac{1}{\sin t}}$

$\frac{dy}{dx} = \frac{\cos t . \sin t}{1 - \sin^2 t} = \frac{\sin t}{\cos t} = \tan t$

Diff. w.r.t $x$.

$\frac{d^2y}{dx^2} = \frac{d(\tan t)}{dt} \frac{dt}{dx}$

$= \sec^2 t . \frac{1}{dx/dt} = \frac{\sec^2 t}{-\sin t + \frac{1}{\sin t}}$

$= \frac{\sec^2 t . \sin t}{1 - \sin^2 t} = \frac{\sin t}{\cos^2 t . \cos^2 t} = \frac{\sin t}{\cos^4 t}$

$= \sin t . \sec^4 t$

$\left.\frac{d^2y}{dx^2}\right]_{t=\pi/4} = \frac{1}{\sqrt{2}}(\sqrt{2})^4 = \frac{4}{\sqrt{2}} = 2\sqrt{2}$

$\therefore \left.\frac{d^2y}{dx^2}\right]_{t=\pi/4} = 2\sqrt{2}$

[CBSE Topper 2019]

**5.** Differentiate $\tan^{-1}\left(\frac{\sqrt{1+x^2} - \sqrt{1-x^2}}{\sqrt{1+x^2} + \sqrt{1-x^2}}\right)$ with respect to $\cos^{-1} x^2$.

**Ans.**

$\tan^{-1}\left(\frac{1 - \sqrt{\frac{1-x^2}{1+x^2}}}{1 + \sqrt{\frac{1-x^2}{1+x^2}}}\right) = \tan^{-1}\left(\frac{1 - \sqrt{\frac{1-\cos 2\theta}{1+\cos 2\theta}}}{1 + \sqrt{\frac{1-\cos 2\theta}{1+\cos 2\theta}}}\right)$

$1 - \cos 2\theta = 2\sin^2\theta$

$1 + \cos 2\theta = 2\cos^2\theta$