Session 4

Combination, Restricted Combinations

Combination

Each of the different groups or selections which can be made by some or all of a number of given things without reference to the order of the things in each group is called a combination.

Important Result

(1) The number of combinations of n different things taken r at a time is denoted by ^nC_r or $C(n, r)$ or $\binom{n}{r}$.

Then,

$$^nC_r = \frac{n!}{r!(n-r)!} \quad [0 \le r \le n]$$

$$= \frac{^nP_r}{r!}$$

$$= \frac{n(n-1)(n-2)\ldots(n-r+1)}{r(r-1)(r-2)\ldots 2\cdot 1}, n \in N \text{ and } r \in W$$

Proof Let the number of combinations of n different things taken r at a time be ^nC_r.

Now, each combination consists of r different things and these r things can be arranged among themselves in $r!$ ways.

Thus, for one combination of r different things, the number of arrangements is $r!$.

Hence, for ^nC_r combinations, number of arrangements is

$$r! \times {^nC_r} \quad \ldots\text{(i)}$$

But number of permutations of n different things taken r at a time is ^nP_r. ...(ii)

From Eqs. (i) and (ii), we get

$$r! \times {^nC_r} = {^nP_r} = \frac{n!}{(n-r)!}$$

$$\therefore \quad ^nC_r = \frac{n!}{r!(n-r)!}, r \in W \text{ and } n \in N$$

Note the following facts:

(i) ^nC_r is a natural number

(ii) $^nC_r = 0$, if $r > n$

(iii) $^nC_0 = {^nC_n} = 1$, $^nC_1 = n$

(iv) $^nP_r = {^nC_r}$, if $r = 0$ or 1

(v) $^nC_r = {^nC_{n-r}}$, if $r > \frac{n}{2}$

(vi) If $^nC_x = {^nC_y} \Rightarrow x = y$ or $x + y = n$

(vii) $^nC_r + {^nC_{r-1}} = {^{n+1}C_r}$ [Pascal's rule]

(viii) $^nC_r = \frac{n}{r} \cdot {^{n-1}C_{r-1}}$

(ix) $n \cdot {^{n-1}C_{r-1}} = (n-r+1) \cdot {^nC_{r-1}}$

(x) $\frac{^nC_r}{^nC_{r-1}} = \frac{n-r+1}{r}$

(xi) (a) If n is even, ^nC_r is greatest for $r = \frac{n}{2}$

(b) If n is odd, ^nC_r is greatest for $r = \frac{n-1}{2}$ or $\frac{n+1}{2}$

(xii) $^nC_0 + {^nC_1} + {^nC_2} + \ldots + {^nC_n} = 2^n$

(xiii) $^nC_0 + {^nC_2} + {^nC_4} + \ldots = {^nC_1} + {^nC_3} + {^nC_5} + \ldots = 2^{n-1}$

(xiv) $^{2n+1}C_0 + {^{2n+1}C_1} + {^{2n+1}C_2} + \ldots + {^{2n+1}C_n} = 2^{2n}$

(xv) $^nC_n + {^{n+1}C_n} + {^{n+2}C_2} + {^{n+3}C_n} + \ldots + {^{2n-1}C_n} = {^{2n}C_{n+1}}$

Example 45. If $^{15}C_{3r} = {^{15}C_{r+3}}$, find $^rC_2$.

Sol. We have, $^{15}C_{3r} = {^{15}C_{r+3}}$

$\Rightarrow \quad 3r = r + 3$

or $\quad 3r + r + 3 = 15$

$\Rightarrow \quad 2r = 3$ or $4r = 12$

$\Rightarrow \quad r = \frac{3}{2}$ or $r = 3$

but $r \in W$, so that $r \ne \frac{3}{2}$

$\therefore \quad r = 3$

Then, $\quad {^rC_2} = {^3C_2} = {^3C_1} = 3$

Example 46. If ${}^nC_9 = {}^nC_7$, find n.

Sol. We have, ${}^nC_9 = {}^nC_7 \Rightarrow n = 9 + 7$ $[\because 9 \neq 7]$

$\therefore \quad n = 16$

Example 47. *Prove that*

$$\binom{n}{r} + 2\binom{n}{r-1} + \binom{n}{r-2} = \binom{n+2}{r}$$

Sol. $\therefore \binom{n}{r} = {}^nC_r$

$$\therefore \text{ LHS} = \binom{n}{r} + 2\binom{n}{r-1} + \binom{n}{r-2}$$

$$= {}^nC_r + 2\,{}^nC_{r-1} + {}^nC_{r-2}$$

$$= ({}^nC_r + {}^nC_{r-1}) + ({}^nC_{r-1} + {}^nC_{r-2})$$

$$= {}^{n+1}C_r + {}^{n+1}C_{r-1} = {}^{n+2}C_r$$

$$= \binom{n+2}{r} = \text{RHS}$$

Example 48. If ${}^{2n}C_3 : {}^nC_3 = 11:1$, find the value of n.

Sol. We have,

$${}^{2n}C_3 : {}^nC_3 = 11:1$$

$$\Rightarrow \quad \frac{{}^{2n}C_3}{{}^nC_3} = \frac{11}{1}$$

$$\Rightarrow \quad \frac{\dfrac{2n(2n-1)(2n-2)}{1\cdot 2\cdot 3}}{\dfrac{n(n-1)(n-2)}{1\cdot 2\cdot 3}} = 11 \Rightarrow \frac{4(2n-1)}{(n-2)} = 11$$

$$\Rightarrow \quad 8n - 4 = 11n - 22 \quad \Rightarrow 3n = 18$$

$$\therefore \quad n = 6$$

Example 49. If ${}^{n+1}C_{r+1} : {}^nC_r : {}^{n-1}C_{r-1} = 11:6:3$, find the values of n and r.

Sol. Here, $\dfrac{{}^{n+1}C_{r+1}}{{}^nC_r} = \dfrac{11}{6}$

$$\Rightarrow \quad \frac{n+1}{r+1}\cdot\frac{{}^nC_r}{{}^nC_r} = \frac{11}{6} \quad \left[\because {}^nC_r = \frac{n}{r}\cdot {}^{n-1}C_{r-1}\right]$$

$$\Rightarrow \quad \frac{n+1}{r+1} = \frac{11}{6}$$

$$\Rightarrow \quad 6n + 6 = 11r + 11$$

$$\Rightarrow \quad 6n - 11r = 5 \quad \text{...(i)}$$

and $\dfrac{{}^nC_r}{{}^{n-1}C_{r-1}} = \dfrac{6}{3}$

$$\Rightarrow \quad \frac{n}{r}\cdot\frac{{}^{n-1}C_{r-1}}{{}^{n-1}C_{r-1}} = \frac{6}{3} \quad \left[\because {}^nC_r = \frac{n}{r}\cdot {}^{n-1}C_{r-1}\right]$$

$$\Rightarrow \quad \frac{n}{r} = 2$$

$$\Rightarrow \quad n = 2r \quad \text{...(ii)}$$

On solving Eqs. (i) and (ii), we get $n = 10$ and $r = 5$.

Example 50. If ${}^nC_{r-1} = 36$, ${}^nC_r = 84$ and ${}^nC_{r+1} = 126$, find r.

Sol. Here, $\dfrac{{}^nC_r}{{}^nC_{r-1}} = \dfrac{84}{36}$

$$\Rightarrow \quad \frac{n-r+1}{r} = \frac{7}{3} \quad \left[\because \frac{{}^nC_r}{{}^nC_{r-1}} = \frac{n-r+1}{r}\right]$$

$$\Rightarrow \quad 3n - 3r + 3 = 7r$$

$$\Rightarrow \quad 10r - 3n = 3 \quad \text{...(i)}$$

and $\dfrac{{}^nC_{r+1}}{{}^nC_r} = \dfrac{n-(r+1)+1}{(r+1)} = \dfrac{126}{84} \quad \left[\because \dfrac{{}^nC_r}{{}^nC_{r-1}} = \dfrac{n-r+1}{r}\right]$

$$\Rightarrow \quad \frac{n-r}{r+1} = \frac{3}{2}$$

$$\Rightarrow \quad 2n - 2r = 3r + 3$$

$$\Rightarrow \quad 5r - 2n = -3$$

or $\quad 10r - 4n = -6 \quad$ ...(ii)

On subtracting Eq. (ii) from Eq. (i), we get

$$n = 9$$

From Eq. (i), we get

$$10r - 27 = 3 \Rightarrow 10r = 30$$

$$\therefore \quad r = 3$$

Example 51. Prove that product of r consecutive positive integers is divisible by $r!$.

Sol. Let r consecutive positive integers be (m), $(m+1), (m+2), \dots, (m+r-1)$, where $m \in N$.

$$\therefore \text{ Product} = m(m+1)(m+2)\dots(m+r-1)$$

$$= \frac{(m-1)!\, m(m+1)(m+2)\dots(m+r-1)}{(m-1)!}$$

$$= \frac{(m+r-1)!}{(m-1)!} = \frac{r!\cdot(m+r-1)!}{r!\,(m-1)!}$$

$[\because {}^{m+r-1}C_r$ is natural number$]$

$$= r!\cdot {}^{m+r-1}C_r,$$

which is divisible by $r!$.

Example 52. Evaluate

$${}^{47}C_4 + \sum_{j=0}^{3} {}^{50-j}C_3 + \sum_{k=0}^{5} {}^{56-k}C_{53-k}.$$

Sol. We have, ${}^{47}C_4 + \displaystyle\sum_{j=0}^{3} {}^{50-j}C_3 + \sum_{k=0}^{5} {}^{56-k}C_{53-k}$

$$= {}^{47}C_4 + \sum_{j=0}^{3} {}^{50-j}C_3 + \sum_{k=0}^{5} {}^{56-k}C_3 \quad [\because {}^nC_r = {}^nC_{n-r}]$$

$$= {}^{47}C_4 + ({}^{50}C_3 + {}^{49}C_3 + {}^{48}C_3 + {}^{47}C_3)$$
$$+ ({}^{56}C_3 + {}^{55}C_3 + {}^{54}C_3 + {}^{53}C_3 + {}^{52}C_3 + {}^{51}C_3)$$

$$= {}^{47}C_4 + {}^{47}C_3 + {}^{48}C_3 + {}^{49}C_3 + {}^{50}C_3 + {}^{51}C_3$$
$$+ {}^{52}C_3 + {}^{53}C_3 + {}^{54}C_3 + {}^{55}C_3 + {}^{56}C_3$$

$$= ({}^{47}C_4 + {}^{47}C_3) + {}^{48}C_3 + {}^{49}C_3 + {}^{50}C_3 + {}^{51}C_3$$
$$+ {}^{52}C_3 + {}^{53}C_3 + {}^{54}C_3 + {}^{55}C_3 + {}^{56}C_3$$

$$= {}^{48}C_4 + {}^{48}C_3 + {}^{49}C_3 + {}^{50}C_3 + {}^{51}C_3 + \dots + {}^{56}C_3$$

$$= {}^{49}C_4 + {}^{49}C_3 + {}^{50}C_3 + \dots + {}^{56}C_3$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$$

$$= {}^{56}C_4 + {}^{56}C_3 = {}^{57}C_4$$

Example 53. Prove that the greatest value of ${}^{2n}C_r$ $(0 \le r \le 2n)$ is ${}^{2n}C_n$ (for $1 \le r \le n$).

Sol. We have, $\dfrac{{}^{2n}C_r}{{}^{2n}C_{r-1}} = \dfrac{2n - r + 1}{r}$ $\quad \left[\because \dfrac{{}^nC_r}{{}^nC_{r-1}} = \dfrac{n - r + 1}{r}\right]$

$$= \frac{2(n-r) + (r+1)}{r} = \frac{1 + 2(n-r) + 1}{r} > 1$$

$$\Rightarrow \quad \frac{{}^{2n}C_r}{{}^{2n}C_{r-1}} > 1 \qquad [\text{for } 1 \le r \le n]$$

$$\therefore \quad {}^{2n}C_{r-1} < {}^{2n}C_r$$

On putting $r = 1, 2, 3, \dots, n$,

then ${}^{2n}C_0 < {}^{2n}C_1, {}^{2n}C_1 < {}^{2n}C_2, \dots, {}^{2n}C_{n-1} < {}^{2n}C_n$

On combining all inequalities, we get

$$\Rightarrow \quad {}^{2n}C_0 < {}^{2n}C_1 < {}^{2n}C_2 < {}^{2n}C_3 < \dots < {}^{2n}C_{n-1} < {}^{2n}C_n$$

but ${}^{2n}C_r = {}^{2n}C_{2n-r}$, it follows that

$${}^{2n}C_{2n} < {}^{2n}C_{2n-1} < {}^{2n}C_{2n-2} < {}^{2n}C_{2n-3} < \dots < {}^{2n}C_{n+1} < {}^{2n}C_n$$

Hence, the greatest value of ${}^{2n}C_r$ is ${}^{2n}C_n$.

Example 54. Thirty six games were played in a football tournament with each team playing once against each other. How many teams were there?

Sol. Let the number of teams be n.

Then number of matches to be played is ${}^nC_2 = 36$

$$\Rightarrow \quad {}^nC_2 = \frac{9 \times 8}{1 \times 2} = {}^9C_2$$

$$\Rightarrow \quad n = 9$$

Restricted Combinations

(i) The number of selections (combinations) of r objects out of n different objects, when

(a) k particular things are always included $= {}^{n-k}C_{r-k}$.

(b) k particular things are never included $= {}^{n-k}C_r$.

(ii) The number of combinations of r things out of n different things, such that k particular things are not together in any selection $= {}^nC_r - {}^{n-k}C_{r-k}$

(iii) The number of combinations of n different objects taking r at a time when, p particular objects are always included and q particular objects are always excluded $= {}^{n-p-q}C_{r-p}$

Note

(i) The number of selections of r consecutive things out of n things in a row $= n - r + 1$.

(ii) The number of selections of r consecutive things out of n things along a circle $= \begin{cases} n, \text{ if } r < n \\ 1, \text{ if } r = n \end{cases}$.

Example 55. In how many ways can a cricket, eleven players by chosen out of a batch 15 players, if

(i) a particular is always chosen.

(ii) a particular player is never chosen?

Sol. (i) Since, particular player is always chosen. It means that $11 - 1 = 10$ players are selected out of the remaining $15 - 1 = 14$ players.

$\therefore$ Required number of ways $= {}^{14}C_{10} = {}^{14}C_4$

$$= \frac{14 \cdot 13 \cdot 12 \cdot 11}{1 \cdot 2 \cdot 3 \cdot 4} = 1001$$

(ii) Since, particular player is never chosen. It means that 11 players are selected out of the remaining $15 - 1 = 14$ players.

$\therefore$ Required number of ways $= {}^{14}C_{11} = {}^{14}C_3$

$$= \frac{14 \cdot 13 \cdot 12}{1 \cdot 2 \cdot 3} = 364$$

Example 56. How many different selections of 6 books can be made from 11 different books, if

(i) two particular books are always selected.

(ii) two particular books are never selected?

Sol. (i) Since, two particular books are always selected. It means that $6 - 2 = 4$ books are selected out of the remaining $11 - 2 = 9$ books.

$\therefore$ Required number of ways $= {}^9C_4 = \dfrac{9 \cdot 8 \cdot 7 \cdot 6}{1 \cdot 2 \cdot 3 \cdot 4} = 126.$

(ii) Since, two particular books are never selected. It means that 6 books are selected out of the remaining $11 - 2 = 9$ books.

$\therefore$ Required number of ways $= {}^9C_6$

$$= {}^9C_3 = \frac{9 \cdot 8 \cdot 7}{1 \cdot 2 \cdot 3} = 84.$$

Example 57. A person tries to form as many different parties as he can, out of his 20 friends. Each party should consist of the same number. How many friends should be invited at a time? In how many of these parties would the same friends be found?

Sol. Let the person invite r number of friends at a time. Then, the number of parties are ${}^{20}C_r$, which is maximum, when $r = 10$.

If a particular friend will be found in p parties, then p is the number of combinations out of 20 in which this particular friend must be included. Therefore, we have to select 9 more from 19 remaining friends.

Hence, $p = {}^{19}C_9$

(2) The number of ways (or combinations) of n different things selecting atleast one of them is $2^n - 1$. This can also be stated as the total number of combinations of n different things.

Proof For each things, there are two possibilities, whether it is selected or not selected.

Hence, the total number of ways is given by total possibilities of all the things which is equal to $2 \times 2 \times 2 \times \ldots \times n$ factors $= 2^n$

But, this includes one case in which nothing is selected.

Hence, the total number of ways of selecting one or more of n different things $= 2^n - 1$

Aliter Number of ways of selecting one, two, three, ..., n things from n different things are

$${}^nC_1, {}^nC_2, {}^nC_3, \ldots, {}^nC_n, \text{ respectively.}$$

Hence, the total number of ways or selecting atleast one thing is

$$ {}^nC_1 + {}^nC_2 + {}^nC_3 + \ldots + {}^nC_n$$
$$= ({}^nC_0 + {}^nC_1 + {}^nC_2 + \ldots + {}^nC_n) - {}^nC_0 = 2^n - 1$$

Example 58. Mohan has 8 friends, in how many ways he invite one or more of them to dinner?

Sol. Mohan select one or more than one of his 8 friends. So, required number of ways

$$= {}^8C_1 + {}^8C_2 + {}^8C_3 + \ldots + {}^8C_8$$
$$= 2^8 - 1 = 255.$$

Example 59. ***A question paper consists of two sections having respectively, 3 and 5 questions. The following note is given on the paper "It is not necessory to attempt all the questions one question from each section is compulsory". In how many ways can a candidate select the questions?***

Sol. Here, we have two sections A and B (say), the section A has 3 questions and section B has 5 questions and one question from each section is compulsory, according to the given direction.

$\therefore$ Number of ways selecting one or more than one question from section A is $2^3 - 1 = 7$

and number of ways selecting one or more than one question from section B is $2^5 - 1 = 31$

Hence, by the principle of multiplication, the required number of ways in which a candidate can select the questions

$$= 7 \times 31 = 217.$$

Example 60. A student is allowed to select atleast one and atmost n books from a collection of $(2n+1)$ books. If the total number of ways in which he can select books is 63, find the value of n.

Sol. Given, student select atleast one and atmost n books from a collection of $(2n+1)$ books. It means that he select one book or two books or three books or ... or n books. Hence, by the given hypothesis.

$${}^{2n+1}C_1 + {}^{2n+1}C_2 + {}^{2n+1}C_3 + \ldots + {}^{2n+1}C_n = 63 \quad \ldots\text{(i)}$$

Also, the sum of binomial coefficients, is

$${}^{2n+1}C_0 + {}^{2n+1}C_1 + \ldots + {}^{2n+1}C_n + {}^{2n+1}C_{n+1} + \ldots + {}^{2n+1}C_{n+1}$$
$$= (1+1)^{2n+1} = 2^{2n+1}$$

$$\Rightarrow {}^{2n+1}C_0 + 2({}^{2n+1}C_1 + {}^{2n+1}C_2 + \ldots + {}^{2n+1}C_n) + {}^{2n+1}C_{2n+1} = 2^{2n+1} \quad [\because {}^nC_r = C_{n-r}]$$

$$\Rightarrow 1 + 2 \times 63 + 1 = 2^{2n+1} \Rightarrow 128 = 2^{2n+1}$$

$$\Rightarrow 2^7 = 2^{2n+1} \Rightarrow 7 = 2n + 1$$

$$\therefore n = 3$$

Example 61. There are three books of Physics, four of Chemistry and five of Mathematics. How many different collections can be made such that each collection consists of

(i) one book of each subject,

(ii) atleast one book of each subject,

(iii) atleast one book of Mathematics.

Sol. (i) ${}^3C_1 \times {}^4C_1 \times {}^5C_1 = 3 \times 4 \times 5 = 60$

(ii) $(2^3 - 1) \times (2^4 - 1) \times (2^5 - 1) = 7 \times 15 \times 31 = 3255$

(iii) $(2^5 - 1) \times 2^7 = 31 \times 128 = 3968$

Exercise for Session 4

1. If $^{43}C_{r-6} = {}^{43}C_{3r+1}$, the value of r is

(a) 6 (b) 8 (c) 10 (d) 12

2. If $^{18}C_{15} + 2\,(^{18}C_{16}) + {}^{17}C_{16} + 1 = {}^{n}C_3$, the value of n is

(a) 18 (b) 20 (c) 22 (d) 24

3. If $^{20}C_{n+2} = {}^{n}C_{16}$, the value of n is

(a) 7 (b) 10 (c) 13 (d) None of these

4. If $^{47}C_4 + \sum_{r=1}^{5} {}^{52-r}C_3$ is equal to

(a) $^{47}C_6$ (b) $^{52}C_5$ (c) $^{52}C_4$ (d) None of these

5. If $^{n}C_3 + {}^{n}C_4 > {}^{n+1}C_3$ then

(a) $n > 6$ (b) $n < 6$ (c) $n > 7$ (d) $n < 7$

6. The Solution set of $^{10}C_{x-1} > 2 \cdot {}^{10}C_x$ is

(a) {1, 2, 3} (b) {4, 5, 6} (c) {8, 9, 10} (d) {9, 10, 11}

7. If $^{2n}C_2 : {}^{n}C_2 = 9 : 2$ and $^{n}C_r = 10$, then r is equal to

(a) 2 (b) 3 (c) 4 (d) 5

8. If $^{2n}C_3 : {}^{n}C_2 = 44 : 3$, for which of the following value of r, the value of $^{n}C_r$ will be 15.

(a) $r = 3$ (b) $r = 4$ (c) $r = 5$ (d) $r = 6$

9. If $^{n}C_r = {}^{n}C_{r-1}$ and $^{n}P_r = {}^{n}P_{r+1}$, the value of n is

(a) 2 (b) 3 (c) 4 (d) 5

10. If $^{n}P_r = 840$, $^{n}C_r = 35$, the value of n is

(a) 1 (b) 3 (c) 5 (d) 7

11. If $^{n}P_3 + {}^{n}C_{n-2} = 14n$, the value of n is

(a) 5 (b) 6 (c) 8 (d) 10

12. There are 12 volleyball players in all in a college, out of which a team of 9 players is to be formed. If the captain always remains the same, in how many ways can the team be formed ?

(a) 36 (b) 99 (c) 108 (d) 165

13. In how many ways a team of 11 players can be formed out of 25 players, if 6 out of them are always to be included and 5 are always to be excluded

(a) 2002 (b) 2008 (c) 2020 (d) 8002

14. A man has 10 friends. In how many ways he can invite one or more of them to a party?

(a) 10! (b) 2^{10} (c) 10! – 1 (d) $2^{10} - 1$

15. In an examination, there are three multiple choice questions and each question has four choices. Number of ways in which a student can fail to get all answers correct, is

(a) 11 (b) 12 (c) 27 (d) 63

16. In an election, the number of candidates is 1 greater than the persons to be elected . If a voter can vote in 254 ways, the number of candidates is

(a) 6 (b) 7 (c) 8 (d) 10

17. The number of groups that can be made from 5 different green balls, 4 different blue balls and 3 different red balls, if atleast one green and one blue ball is to be included

(a) 3700 (b) 3720 (c) 4340 (d) None of these

18. A person is permitted to select atleast one and atmost n coins from a collection of $(2n + 1)$ distinct coins. If the total number of ways in which he can select coins is 255, then n equals

(a) 4 (b) 8 (c) 16 (d) 32

Answers

Exercise for Session 4

1. (d)	***2.*** (b)	**3.** (d)	**4.** (c)	**5.** (a)	**6.** (c)
7. (a)	**8.** (b)	**9.** (b)	**10.** (d)	**11.** (a)	**12.** (d)
13. (a)	**14.** (d)	**15.** (d)	**16.** (c)	**17.** (b)	**18.** (a)