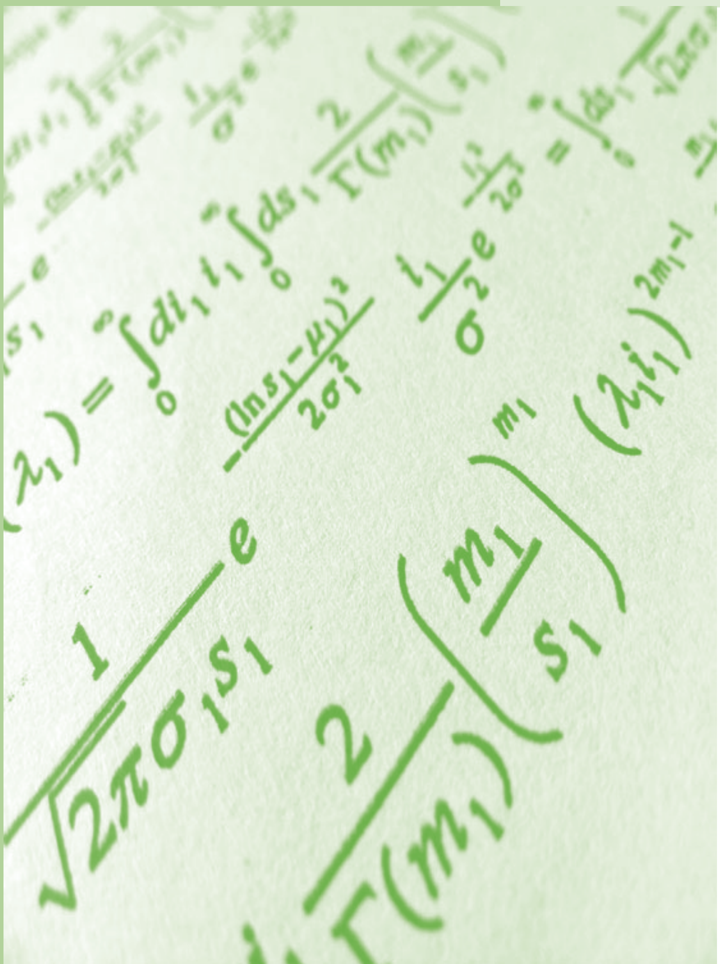
Chapter 13

Geometry



REMEMBER

Before beginning this chapter, you should be able to:

- Know types, and properties of quadrilaterals
- Understand Pythagoras theorem and its converse
- Construct polygons, triangles, quadrilaterals, circles

KEY IDEAS

After completing this chapter, you would be able to:

- Learn about symmetry, similarity of geometrical figures in general
- Study criteria for similarity of triangles, right angle theorem, results on areas of similar triangle, Pythagorean, Apollonius, Thales, vertical angle bisector theorems
- Understand properties of chords or arcs and prove theorems based on tangents
- Learn construction of geometrical figures
- Study equation of locus

INTRODUCTION

In this chapter, we shall learn about symmetry, similarity of geometrical figures in general, triangles in particular. We shall understand similarity through size transformation. Further we shall learn about concurrent lines, geometric centres in triangles, basic concepts of circles and related theorems. We shall also focus on some constructions related to polygons and circles. Finally we shall discuss the concept of locus.

SYMMETRY

Let us examine the following figures (Fig. 13.1) drawn on a rectangular piece of paper.

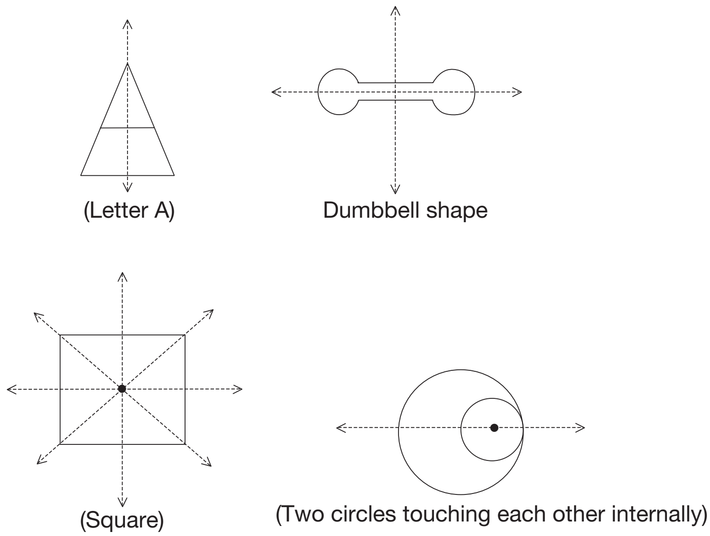


Figure 13.1

What do you infer? We can observe that when these figures are folded about the dotted lines, the two parts on either side of the dotted lines coincide. This property of geometrical figures is called symmetry.

In this chapter, we shall discuss two basic types of symmetry—**line symmetry and point symmetry**. Then we shall see how to obtain the image of a point, a line segment and an angle about a line.

Line Symmetry

Trace a geometrical figure on a rectangular piece of paper as shown below:

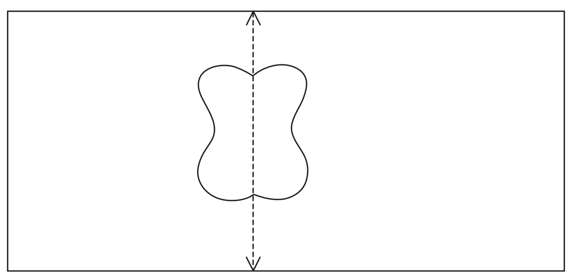


Figure 13.2

Now fold the paper along the dotted line. You will find that the two parts of the figure on either sides of the line coincide. Thus the line divides the figure into two identical parts. In this case, we say that the figure is symmetrical about the dotted line or Line Symmetric. Also, the dotted line is called the Line of Symmetry or The Axis of Symmetry.

So, a geometrical figure is said to be line symmetric or symmetrical about a line if there exists at least one line in the figure such that the parts of the figure on either sides of the line coincide when it is folded about the line.

Example: Consider the following figure (Fig. 13.3).

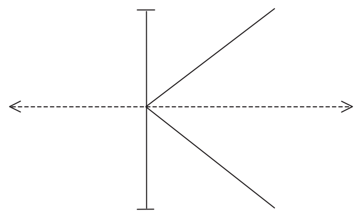


Figure 13.3

The above figure is symmetrical about the dotted line. Also, there is only one line of symmetry for the figure.

Example: Observe the figure given below (Fig. 13.4).

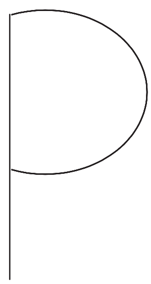


Figure 13.4

There is no line in the figure about which the figure is symmetric.

Example: Consider a rectangle as shown in the (Fig. 13.5).

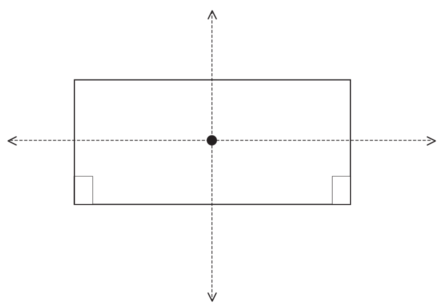


Figure 13.5

The rectangle is symmetrical about the two dotted lines. So, a rectangle has two lines of symmetry.

Example: Consider a figure in which 4 equilateral triangles are placed, one on each side of a square as shown below.

The figure is symmetrical about the dotted lines. It has 4 lines of symmetry.

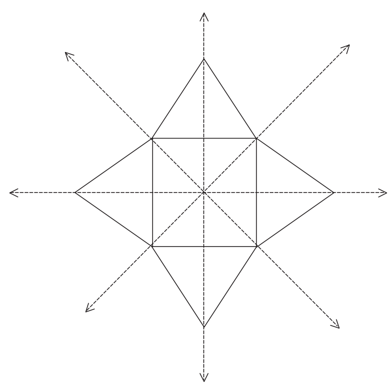


Figure 13.6

Example: A circle has an infinite number of lines of symmetry, some of which are shown in the Fig. 13.7.

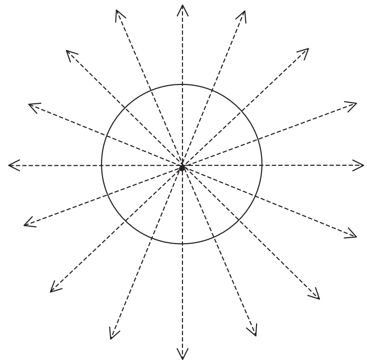


Figure 13.7

From the above illustrations, we observe the following:

1. A geometrical figure may not have a line of symmetry, i.e., a geometric figure may not be line symmetric.
2. A geometrical figure may have more than one line of symmetry, i.e., a geometrical figure may be symmetrical about more than one line.

Point Symmetry

Trace the letter N on a rectangular piece of paper as shown below. Let P be the mid-point of the inclined line in the figure. Now draw a line segment through the point P touching the two vertical strokes of N. We find that the point P divides the line segment into two equal parts. Thus, every line segment drawn through the point P and touching the vertical strokes of N is bisected at the point P. Also, when we rotate the letter N about the point P through an angle of 180°, we find that it coincides exactly with the initial position. This property of geometrical figures is called *Point Symmetry*. In this case, we say that the point P is the point of symmetry of the figure.

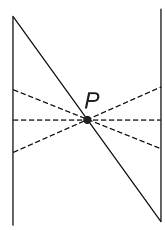


Figure 13.8

So, a geometrical figure is said to have symmetry about a point P if every line segment through the point P touching the boundary of the figure is bisected at the point P.

(Or)

A geometrical figure is said to have point symmetry if the figure does not change when rotated through an angle of 180°, about the point P.

Here, point P is called the centre of symmetry.

Illustrations

Example: Consider a rectangle $ABCD$. Let P be the point of intersection of the diagonals AC and BD. Then, the rectangle $ABCD$ has the point symmetry about the point P.

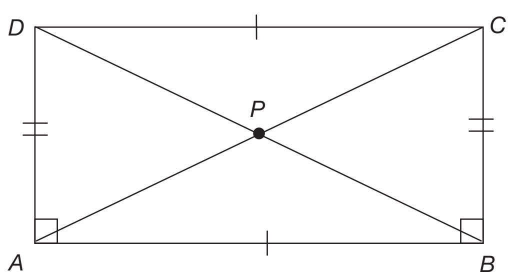


Figure 13.9

Example: The letter 'S' has a point of symmetry about the turning point in S.

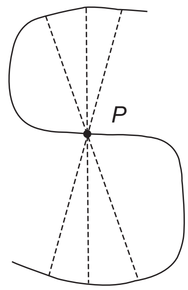


Figure 13.10

Example: A circle has a point of symmetry. This point is the centre of the circle.

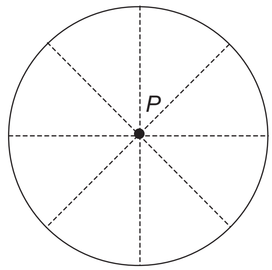


Figure 13.11

Image of a Point About a Line

Draw a line l and mark a point A on a rectangular piece of paper as shown in the Fig. 13.12.

Draw AM perpendicular to l and produce it to B such that $AM = MB$. Then the point B is said to be the image of the point A about the line l.

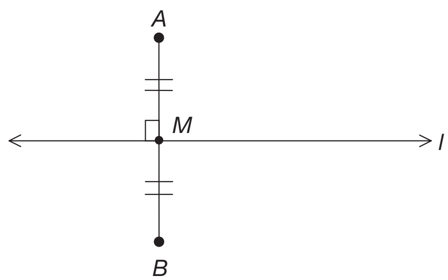


Figure 13.12

Image of a Line Segment about a Line

Draw a line l and a line segment PQ on a rectangular piece of paper as shown below.

Draw perpendiculars PL and QM from the points P and Q respectively to the line $l$and produce them to P' and Q' respectively, such that $PL = LP'$ and $QM = MQ'$. Join the points P', Q'. Then the line segment $P'Q'$ is called the image of the line segment PQ about the line l.

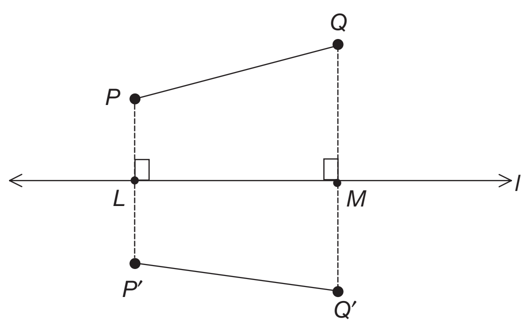


Figure 13.13

Image of an Angle About a Line

Draw a line l and an angle PQR on a piece of paper as shown below.

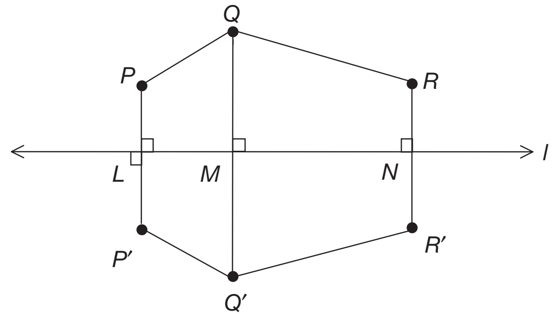


Figure 13.14

Draw perpendiculars PL, QM and RN from the points P, Q and R respectively to the line l and produce them to P', Q' and R' respectively, such that $PL = LP'$, $QM = MQ'$ and $RN = NR'$. Join the points P', Q' and Q', R'. Then, the angle $P'Q'R'$ is called the image of the angle PQR about the line l.

EXAMPLE 13.1

Determine the line of symmetry of a triangle ABC in which $\angle A = 40°$, $\angle B = 70°$ and $\angle C = 70°$.

SOLUTION

The line which bisects $\angle A$ is the line of symmetry of the given triangle ABC.

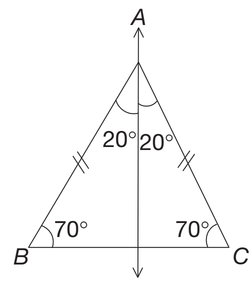


Figure 13.15

EXAMPLE 13.2

Determine the point of symmetry of a regular hexagon.

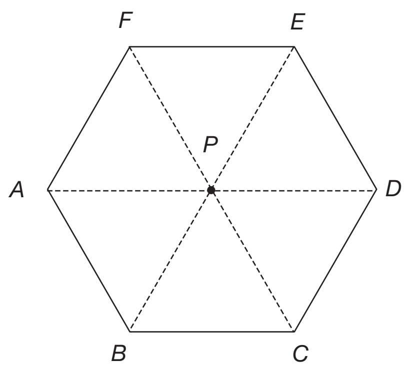


Figure 13.16

SOLUTION

The point of intersection of the diagonals of regular hexagon is the required point of symmetry.

EXAMPLE 13.3

Complete adjacent figure so that X-axis and Y-axis are the lines of symmetry of the completed figure.

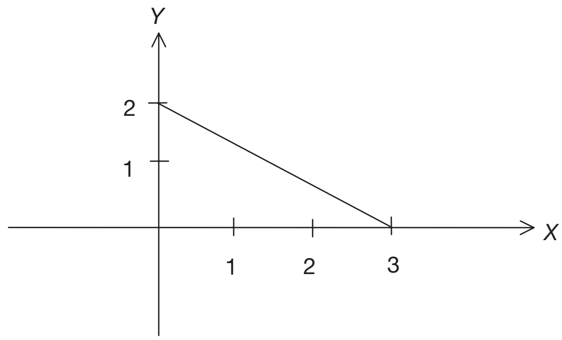


Figure 13.17

SOLUTION

Required figure is a rhombus of side $\sqrt{13}$ units.

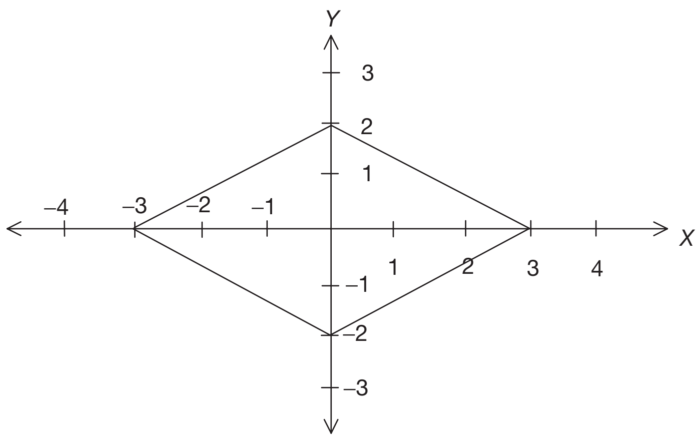
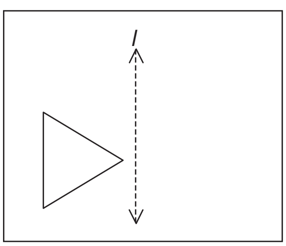


Figure 13.18

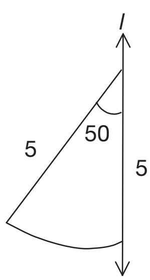
EXAMPLE 13.4

Determine the images of the following figure about the given line.

(a)

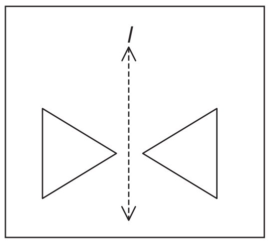


(b)

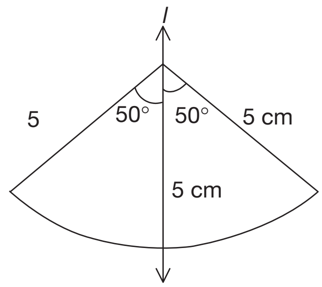


SOLUTION

(a)



(b)



SIMILARITY

Similarity of Geometrical Figures

Two geometrical figures of same shape, but not necessarily of same size are said to be similar.

Examples:

1. Any two circles are similar.
2. Any two squares are similar.
3. Any two equilateral triangles are similar.

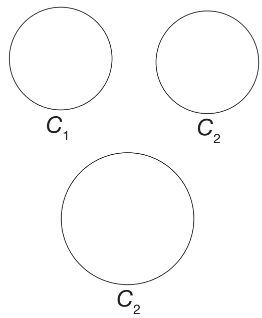


Figure 13.19

Two circles C_1 and C_2 are similar, since their shapes are the same.

C_1 and C_2 may have an equal area.

While considering the circles C_1 and C_3 they may not be of equal area, but their shapes are the same.

C_1 and C_2 are of the same shape and of same size.

C_1 and C_3 are of the same shape but of different sizes.

In both the cases, they are of the same shape.

Hence C_1, C_2 and C_3 are similar circles.

Similarity as a Size Transformation

Enlargement

Consider triangle ABC.

Construct triangle PQR whose sides are twice the corresponding sides of triangle ABC, as shown in the following figure.

Construction: First draw the triangle ABC. Take a point D outside the triangle. Join DA, DB and DC.

Now, extend DA, DB and DC to the points P, Q and R respectively such that $DP = 2DA$, $DQ = 2DB$ and $DR = 2DC$.

PQR is the enlarged image of ABC.

On verification, we find that $PQ = 2AB$

$QR = 2BC$ and $RP = 2AC$.

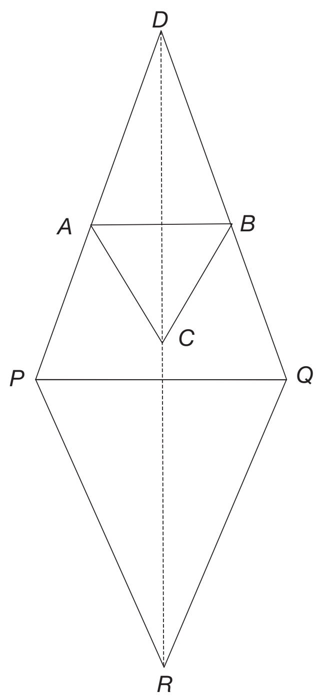


Figure 13.20

It can be defined that triangle ABC has been enlarged by a scale factor of 2 about the centre of the enlargement D to give image PQR.

Reduction

Consider square $ABCD$. One can draw a square $PQRS$ whose side is half the length of the side of $ABCD$.

Construction: Draw square $ABCD$.

Take a point E outside the square.

Join EA, EB, EC and ED. Mark the points P, Q, R and S on $\overline{EA}$, $\overline{EB}$, $\overline{EC}$, and $\overline{ED}$ such that

$$EP = \frac{1}{2}EA,\ EQ = \frac{1}{2}EB,\ ER = \frac{1}{2}EC \text{ and } ES = \frac{1}{2}ED$$

Square $PQRS$ is the reduced image of square $ABCD$.

We find that,

$$PQ = \frac{1}{2}AB,\ QR = \frac{1}{2}BC,$$

$$RS = \frac{1}{2}CD \text{ and } SP = \frac{1}{2}AD$$

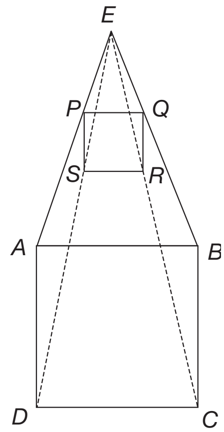


Figure 13.21

$ABCD$ has been reduced by a scale factor $\frac{1}{2}$ about the centre of reduction E to give the image $PQRS$.

Size transformation is the process which a geometrical figure is enlarged or reduced by a scale k, such that the image formed is similar to the given figure.

Properties of Size Transformation

1. The shape of the given figure remains the same.
2. If k is the scale factor of a given size transformation and $k > 1 \Rightarrow$ the image is enlarged and $k < 1 \Rightarrow$ the image is reduced.

   And $k = 1 \Rightarrow$ the image is identical to the original figure.
3. Each side of the given geometrical figure = k (The corresponding side of the given figure).
4. Area of the image = k_2 (Area of the given geometrical figure).
5. Volume of the image, if it is a 3-dimensional figure, is equal to k^3 (Volume of the original figure).

Model

The model of a plane figure and the given figure are similar to each other. If the model of a plane figure is drawn to the scale 1 : x, then scale factor, $k = \frac{1}{x}$.

1. Length of the model = k (Length of the original figure).
2. Area of the model = k^2 (Area of the original figure).
3. Volume of the model = k^3 (Volume of the original figure).

Map

If the map of a plane figure, is drawn to the scale 1 : x, then, scale factor, $k = \dfrac{1}{x}$.

1. Length in the map = k (Original length).
2. Area in the map = k^2 (Original area).

The discussion on similarity can be extended further as follows:

Two polygons are said to be similar to each other if

1. their corresponding angles are equal and
2. the lengths of their corresponding sides are proportional.

Note '~' is the symbol used for 'is similar to'.

If quadrilateral $ABCD$ is similar to quadrilateral $PQRS$ we denote this as ▱ $ABCD$ ~ ▱ $PQRS$. The relation 'is similar to' satisfies the following properties.

1. It is reflexive as every figure is similar to itself.
2. It is symmetric as A is similar to B then B is also similar to A.
3. It is transitive as, if A is similar to B and B is similar to C, then A is similar to C.

∴ The relation is 'similar to' is an equivalence relation.

Criteria for Similarity of Triangles

Similar Triangles

In two triangles, if either the corresponding angles are equal or the ratio of corresponding sides are equal, then the two triangles are similar to each other.

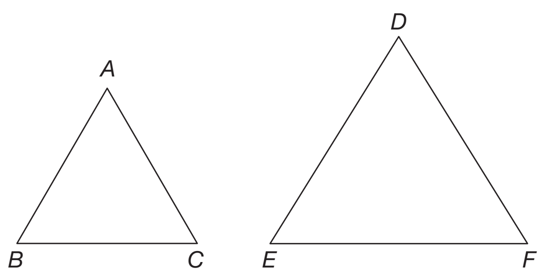


Figure 13.22

In ΔABC and ΔDEF if

1. $\angle A = \angle D$, $\angle B = \angle E$ and $\angle C = \angle F$ or
2. $\dfrac{AB}{DE} = \dfrac{BC}{EF} = \dfrac{AC}{DF}$, then $\Delta ABC \sim \Delta DEF$ or ΔABC is similar to ΔDEF.

Three Similarity Axioms for Triangles

1. **AA – Axiom or AAA – Axiom:** In two triangles, if corresponding angles are equal then the triangles are similar to each other.

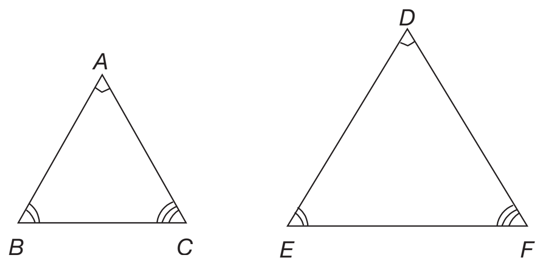


Figure 13.23

In ΔABC and ΔDEF, $\angle A = \angle D$, $\angle B = \angle E$ and $\angle C = \angle F$.

$\therefore\ \Delta ABC \sim \Delta DEF$.

Note In two triangles, if two pairs of corresponding angles are equal then the triangles are similar to each other, because the third pair corresponding angles will also be equal.

2. **SSS – Axiom:** In two triangles, if the corresponding sides are proportional, then the two triangles are similar to each other.

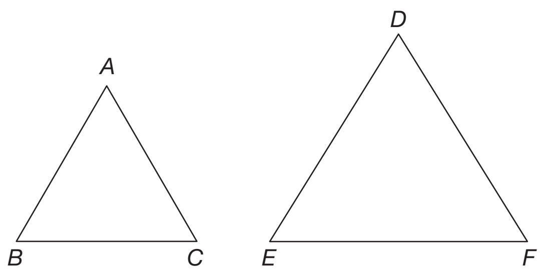


Figure 13.24

In ΔABC and ΔDEF, if $\dfrac{AB}{DE} = \dfrac{BC}{EF} = \dfrac{CA}{FD}$

$\Rightarrow \quad \Delta ABC \sim \Delta DEF$.

3. **SAS – Axiom:** In two triangles, if one pair of corresponding sides are proportional and the included angles are equal then the two triangles are similar.

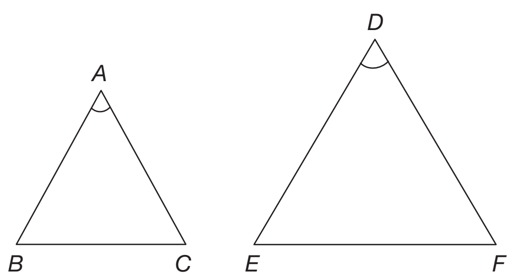


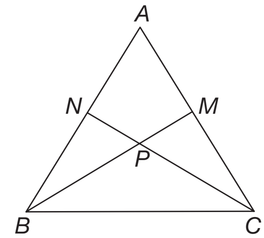
Figure 13.25

In ΔABC and ΔDEF,

If $\dfrac{AB}{DE} = \dfrac{AC}{DF}$ and $\angle A = \angle D$, then $\Delta ABC \sim \Delta DEF$.

EXAMPLE 13.5

In the given figure (not to scale), $AM : MC = 3 : 4$, $BP : PM = 3 : 2$ and $BN = 12$ cm. Find AN.



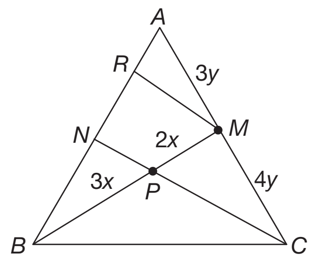
(a) 10 cm **(b)** 12 cm **(c)** 14 cm **(d)** 16 cm

SOLUTION

Given $AM : MC = 3 : 4$

$BP : PM = 3 : 2$ and $BN = 12$ cm

Draw MR parallel to CN which meets AB at the point R



Consider ΔBMR

$PN \parallel MR$ (Construction)

By BPT,

$$\frac{BN}{NR} = \frac{BP}{PM} \quad \Rightarrow \quad \frac{12}{NR} = \frac{3}{2}$$

$NR = 8$ cm

Consider ΔANC, $RM \parallel NC$ (By construction)

By BPT,

$$\frac{AR}{RN} = \frac{AM}{MC} \quad \Rightarrow \quad \frac{AR}{8} = \frac{3}{4} \quad \Rightarrow \quad AR = 6 \text{ cm}$$

$\therefore AN = AR + RN = 6 + 8 = 14$ cm.

Right Angle Theorem

In a right angled triangle, if a perpendicular is drawn from the vertex of the right angle to the hypotenuse, then the two right triangles formed on either side of the perpendicular are similar to each other and similar to the given triangle.

In the figure, ΔABC is right angled at A. $\overline{AD}$ is the perpendicular drawn from A to BC.

Let $\angle ABD = \theta$

Then, $\angle BAD = 90° - \theta$

$\angle DAC = \angle BAC - \angle BAD = 90° - (90° - \theta)$

$\Rightarrow \quad \angle DAC = \theta$

$\therefore \angle DCA = 90° - \theta$

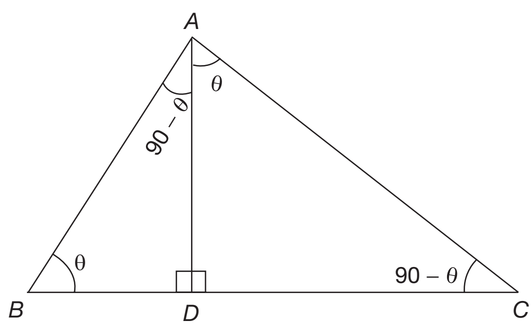


Figure 13.26

In triangles BAD, DAC, and ABC, three corresponding angles are equal. Therefore, the triangle ABC is similar to triangle DAC or triangle DBA.

In triangles DBA and DAC,

$$\because \frac{BD}{AD} = \frac{AD}{DC}$$

$$\Rightarrow \quad AD^2 = BD \times DC$$

AD is the mean proportional of BD and DC.

$$\Rightarrow \quad AD^2 = BD \cdot DC.$$

Results on Areas of Similar Triangles The ratio of areas of the two similar triangles is equal to the ratio of the squares of any two corresponding sides of the triangles.

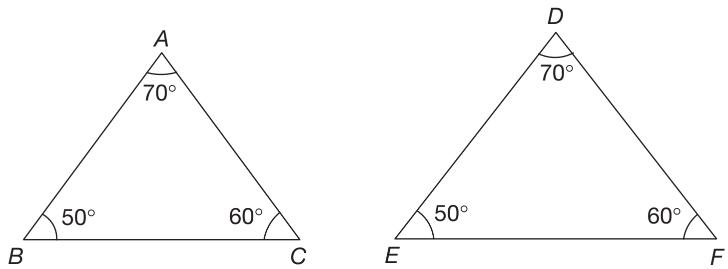


Figure 13.27

$$\Delta ABC \sim \Delta DEF \quad \Rightarrow \quad \frac{\text{Area of } \Delta ABC}{\text{Area of } \Delta DEF} = \frac{AB^2}{DE^2} = \frac{BC^2}{EF^2} = \frac{CA^2}{FD^2}.$$

1. The ratio of areas of two similar triangles is equal to the ratio of the squares of their corresponding altitudes.

   In the following figures, $\Delta ABC \sim \Delta DEF$ and AX, DY are the altitudes.

   Then, $\dfrac{\text{Area of } \Delta ABC}{\text{Area of } \Delta DEF} = \dfrac{AX^2}{DY^2}$.

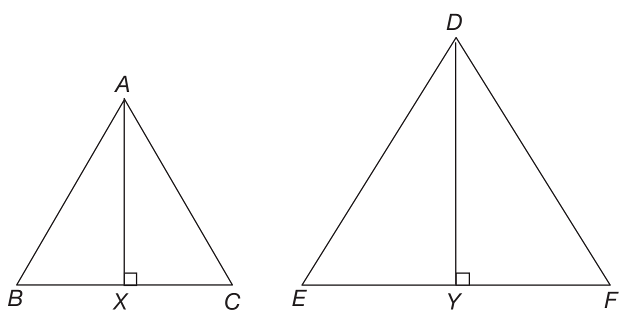


Figure 13.28

2. The ratio of areas of two similar triangles is equal to the ratio of the squares on their corresponding medians (see Fig. 13.29).

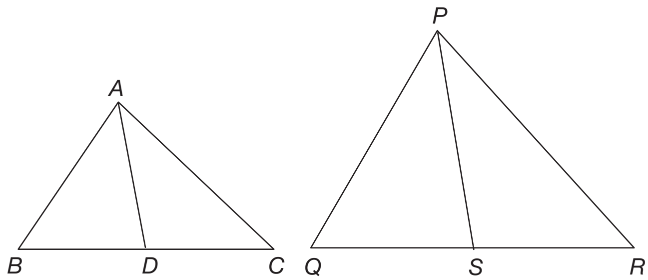


Figure 13.29

In the above figure, $\Delta ABC \sim \Delta PQR$ and AD and PS are medians.

Then, $\dfrac{\text{Area of } \Delta ABC}{\text{Area of } \Delta PQR} = \dfrac{AD^2}{PS^2}$.

3. The ratio of areas of two similar triangles is equal to the ratio of the squares of their corresponding angle bisector segments.

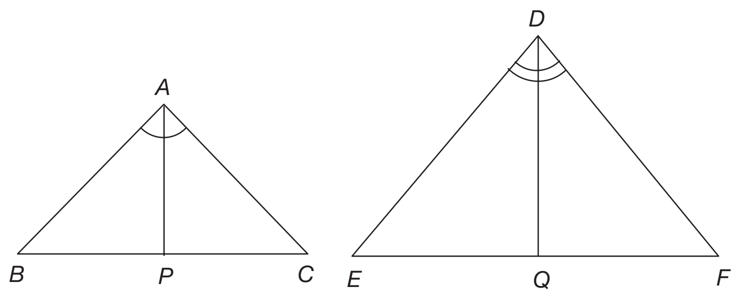


Figure 13.30

In the figure, $\Delta ABC \sim \Delta DEF$ and AP, DQ are bisectors of $\angle A$ and $\angle D$ respectively, then

$\dfrac{\text{Area of } \Delta ABC}{\text{Area of } \Delta DEF} = \dfrac{AP^2}{DQ^2}$.

Pythagoras Theorem

In a right angled triangle, the square of the hypotenuse is equal to the sum of the squares of the other two sides.

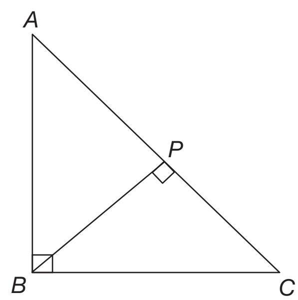


Figure 13.31

To prove: $AC^2 = AB^2 + BC^2$

Construction: Draw BP perpendicular to AC.

Proof: In triangles APB and ABC,

$\angle APB = \angle ABC$ (right angles)

$\angle A = \angle A$ (common)

$\therefore$ Triangle APB is similar to triangle ABC.

$\Rightarrow \quad \dfrac{AP}{AB} = \dfrac{AB}{AC}$

$\Rightarrow \quad AB^2 = (AP)(AC)$

Similarly, $BC^2 = (PC)(AC)$

$\therefore\ AB^2 + BC^2 = (AP)(AC) + (PC)(AC)$

$AB^2 + BC^2 = (AC)(AP + PC)$

$AB^2 + BC^2 = (AC)(AC)$

$\Rightarrow \quad AC^2 = AB^2 + BC^2$

Hence proved.

The results in an obtuse triangle and an acute triangle are as follows:

In ΔABC, $\angle ABC$ is obtuse and AD is drawn perpendicular to BC, then

$AC^2 = AB^2 + BC^2 + 2BC \cdot BD$

ΔABC is an acute angled triangle, acute angle at B and AD is drawn perpendicular to BC, then $AC^2 = AB^2 + BC^2 - 2BC \cdot BD$.

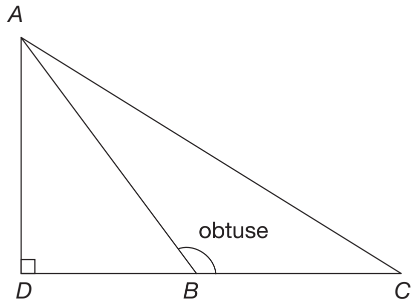


Figure 13.32

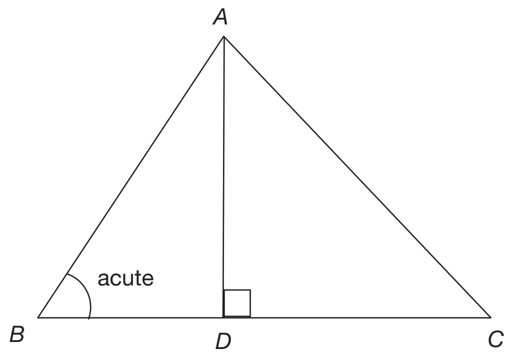


Figure 13.33

Appolonius Theorem

In a triangle, the sum of the squares of two sides of a triangle is equal to twice the sum of the square of the median which bisects the third side and the square of half the third side.

Given: $n\ \Delta ABC$, AD is the median.

RTP: $AB^2 + AC^2 = 2(AD^2 + DC^2)$ or $2(AD^2 + BD^2)$

Construction: Draw AE perpendicular to BC.

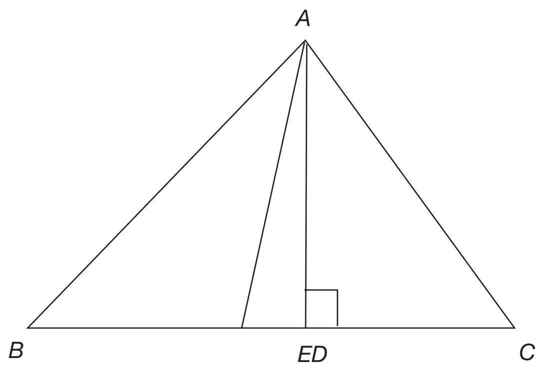


Figure 13.34

Case 1: If $\angle ADB = \angle ADC = 90°$

According to Pythagoras theorem,

In ΔABD, $AB^2 = BD^2 + AD^2$ (1)

In ΔADC, $AC^2 = CD^2 + AD^2$ (2)

Adding Eqs. (1) and (2), $AB^2 + AC^2 = BD^2 + CD^2 + 2AD^2 = 2BD^2 + 2AD^2$. [$\because CD = BD$] or $2[CD^2 + AD^2]$

Case 2: If $\angle ADC$ is acute and $\angle ADB$ is obtuse.

In triangle ADB,

$AB^2 = AD^2 + BD^2 + 2 \times BD \times DE$

In triangle ADC,

$AC^2 = AD^2 + DC^2 - 2 \times CD \times DE$

But $BD = CD$

$\therefore\ AB^2 + AC^2 = 2AD^2 + 2BD^2$ or $2(AD^2 + CD^2)$

Hence proved.

Basic Proportionality Theorem In a triangle, if a line is drawn parallel to one side of a triangle, then it divides the other two sides in the same ratio.

Given: In $\Delta ABC, \overline{DE} \parallel \overline{BC}$

RTP: $\dfrac{AD}{DB} = \dfrac{AE}{EC}$

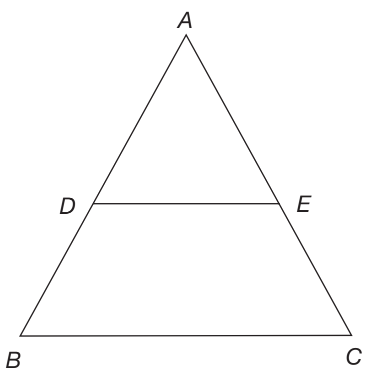


Figure 13.35

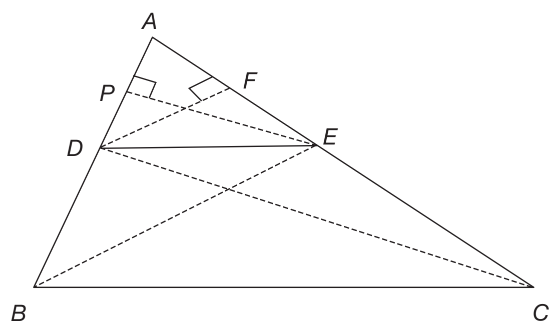


Figure 13.36

Construction: Draw $EP \perp AB$ and $DF \perp AC$. Join $\overline{DC}$ and $\overline{BE}$.

Proof: $$\frac{\text{Area of triangle } ADE}{\text{Area of triangle } BDE} = \frac{\frac{1}{2} \times AD \times PE}{\frac{1}{2} \times BD \times PE} = \frac{AD}{BD}$$

and $$\frac{\text{Area of triangle } ADE}{\text{Area of triangle } CDE} = \frac{\frac{1}{2} \times AE \times DF}{\frac{1}{2} \times EC \times DF} = \frac{AE}{EC}$$

But area so of triangles BDE and CDE are equal. (Two triangles lying on the same base and between the same parallel lines are equal in area).

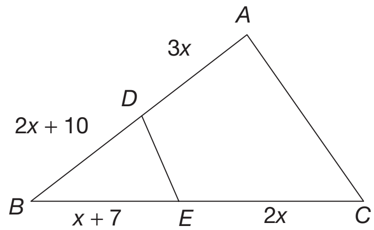
$\therefore \dfrac{AD}{DB} = \dfrac{AE}{EC}$.

Note From the above result we can prove that $\frac{AB}{AD} = \frac{AC}{AE}$ and $\frac{AB}{BD} = \frac{AC}{CE}$.

EXAMPLE 13.6

In the given figure, $\overline{DE} \parallel \overline{AC}$. Find the value of x.

(a) 1 **(b)** 2 **(c)** 3 **(d)** 4



SOLUTION

In ΔABC,

$DE \parallel AC$

$\therefore \frac{BD}{DA} = \frac{BE}{EC}$ (Basic proportionality theorem)

$\Rightarrow \quad \frac{2x+10}{3x} = \frac{x+7}{2x}$

$\Rightarrow \quad 2x(2x+10) = 3x(x+7)$

$\Rightarrow \quad 4x^2 + 20x = 3x^2 + 21x$

$\Rightarrow \quad x^2 - x = 0$

$\Rightarrow \quad x(x-1) = 0$

$\Rightarrow \quad x - 1 = 0$

$\Rightarrow \quad x = 1$

$\therefore$ Hence, option (a) is the correct answer.

Converse of Basic Proportionality Theorem

If a line divides two sides of a triangle in the same ratio then that line is parallel to the third side.

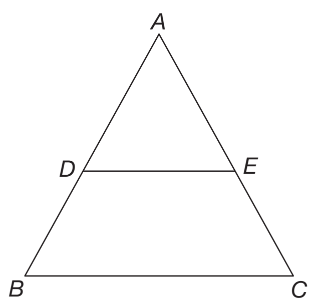


Figure 13.37

In the figure given, $\frac{AD}{DB} = \frac{AE}{EC} \quad \Rightarrow \quad \overline{DE} \parallel \overline{BC}$.

Vertical Angle Bisector Theorem

The bisector of the vertical angle of a triangle divides the base in the ratio of the other two sides.

Given: In ΔABC, AD is the bisector of $\angle A$.

RTP : $\dfrac{BD}{DC} = \dfrac{AB}{AC}$

Construction: Draw CP parallel to AD to meet BA produced at P.

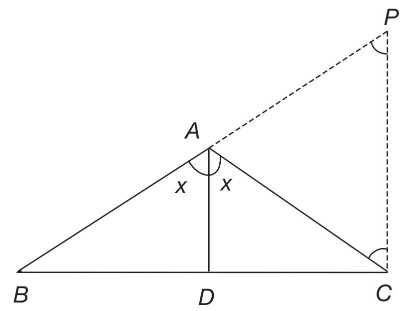


Figure 13.38

Proof: $\angle DAC = \angle ACP$ (alternate angles and $\overline{AD} \parallel \overline{CP}$)

$\angle BAD = \angle APC$ (corresponding angles)

But $\angle BAD = \angle DAC$ (given)

$\therefore\ \angle ACP = \angle APC$

In triangle APC,

$AC = AP$ (sides opposite to equal angles are equal)

In triangle BCP,

$\dfrac{BD}{DC} = \dfrac{BA}{AP}$ (by basic proportionality theorem)

$\Rightarrow\ \dfrac{BD}{DC} = \dfrac{BA}{AC}$ $(\because AP = AC)$

Hence proved.

Converse of Vertical Angle Bisector Theorem

If a line that passes through a vertex of a triangle, divides the base in the ratio of the other two sides, then it bisects the angle.

In the adjacent figure, AD divides BC in the ratio $\dfrac{BD}{DC}$ and if $\dfrac{BD}{DC} = \dfrac{AB}{AC}$, then AD is the bisector of $\angle A$.

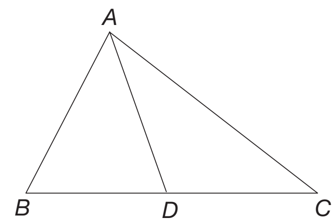
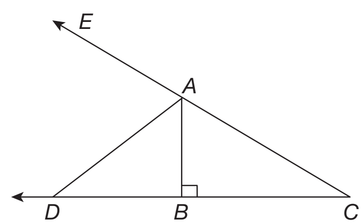


Figure 13.39

EXAMPLE 13.7

In the figure above (not to scale), $\overline{AB} \perp \overline{CD}$ and AD is the bisector of $\angle BAE$. $AB = 3$ cm and $AC = 5$ cm. Find CD.

(a) 6 cm **(b)** 8 cm **(c)** 10 cm **(d)** None of these



SOLUTION

Let BD be x cm

Given $AB = 3$ cm, $AC = 5$ cm and $\angle ABC = 90°$

$\therefore$ BC = 4 cm

And also given, AD is the bisector of $\angle BAE$.

$\therefore$ By vertical angle bisector theorem.

$\dfrac{AB}{AC} = \dfrac{BD}{CD} \Rightarrow \dfrac{3}{5} = \dfrac{x}{4+x}$

$\Rightarrow\ 12 + 3x = 5x \Rightarrow x = 6$ cm.

$\therefore\ CD = 4 + 6 = 10$ cm.

Concurrency—Geometric Centres of a Triangle

Let us recall that if three or more lines pass through a fixed point, then those lines are said to be concurrent and that fixed point is called the point of concurrence. In this context, we recall different concurrent lines and their points of concurrence associated with a triangle, also called geometric centres of a triangle.

Circum-centre

The locus of the point equidistant from the end points of the line segment is the perpendicular bisector of the line segment. The three perpendicular bisectors of the three sides of a triangle are concurrent and the point of their concurrence is called the circum-centre of the triangle and is usually denoted by *S*. The circum-centre is equidistant from all the vertices of the triangle. The circum-centre of the triangle is the locus of the point in the plane of the triangle, equidistant from the vertices of the triangle.

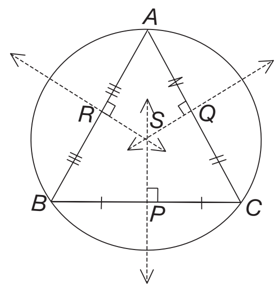


Figure 13.40

In-centre

The angle bisectors of the triangle are concurrent and the point of concurrence is called the in-centre and is usually denoted by *I*. *I* is equidistant from the sides of the triangle. The in-centre of the triangle is the locus of the point, in the plane of the triangle, equidistant from the sides of the triangle.

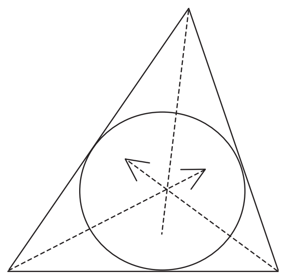


Figure 13.41

Ortho-centre

The altitudes of the triangle are concurrent and the point of concurrence of the altitudes of a triangle is called ortho-centre and is usually denoted by *O*.

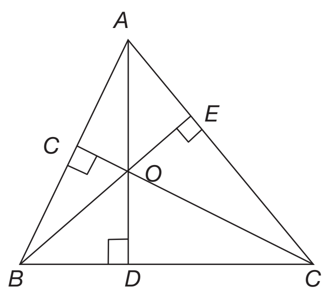


Figure 13.42

Centroid

The medians of a triangle are concurrent and the point of concurrence of the medians of a triangle is called the centroid and it is usually denoted by *G*. The centroid divides each of the medians in the ratio 2 : 1, starting from vertex, i.e., in the figure given below, $AG : GD = BG : GE = CG : GF = 2 : 1$.

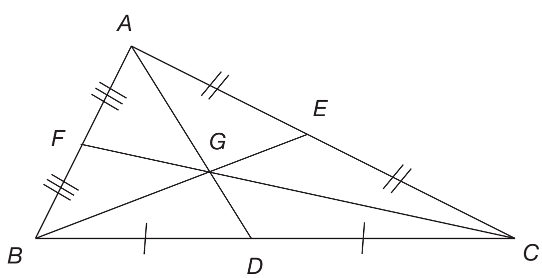


Figure 13.43

Some Important Points

1. In an equilateral triangle, the centroid, the ortho-centre, the circum-centre and the in-centre all coincide.
2. In an isosceles triangle, the centroid, the ortho-centre, the circum-centre and the in-centre all lie on the median to the base.
3. In a right-angled triangle the length of the median drawn to the hypotenuse is equal to half of the hypotenuse. The median is also equal to the circum-radius. The midpoint of the hypotenuse is the circum-centre.
4. In an obtuse-angled triangle, the circum-centre and ortho-centre lie outside the triangle and for an acute angled triangle the circum-centre and the ortho-centre lie inside the triangle.
5. For all triangles, the centroid and the in-centre lie inside the triangle.

CIRCLES

A circle is a set of points in a plane which are at a fixed distance from a fixed point.

The fixed point is the centre of the circle and the fixed distance is the radius of the circle.

In the Fig. 13.44, O is the centre of the circle and OC is a radius of the circle.

AB is a diameter of the circle. OA and OB are also the radii of the circle.

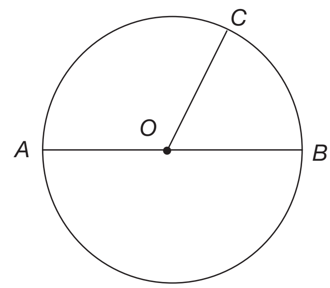


Figure 13.44

The diameter is twice the radius.

The centre of the circle is generally denoted by O, diameter by d and radius by r.

$d = 2r$.

The perimeter of the circular line is called the circumference of the circle.

The circumference of the circle is π times the diameter.

In the Fig. 13.45 with centre O; A, B and C are three points in the plane in which the circle lies. The points O and A are in the interior of the circle. The point B is located on the circumference of the circle.

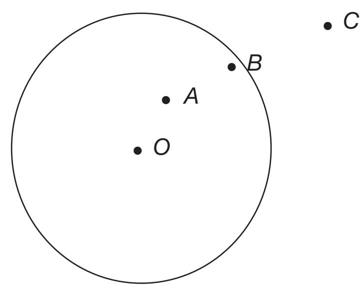


Figure 13.45

Hence, B belongs to the circle.

C is located in the exterior of the circle.

If $OB = r$, B is a point on the circumference of the circle.

As $OA < r$, A is a point in the interior of the circle.

As $OC > r$, C is a point in the exterior of the circle.

Chord

The line segment joining any two points on the circumference of a circle is a chord of the circle.

In the Fig. 13.46, $\overline{PQ}$ and $\overline{AB}$ are the chords.

AB passes through centre O, hence, it is a diameter of the circle. A diameter is the longest chord of the circle. It divides the circle into two equal parts.

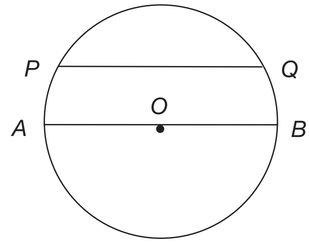


Figure 13.46

Theorem 1

One and only one circle exists through three non-collinear points.

Given: P, Q and R are three non-collinear points.

RTP: One and only one circle passes through the points P, Q and R.

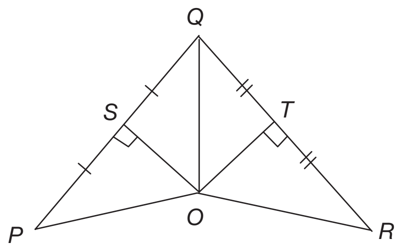


Figure 13.47

Construction:

1. Join PQ and RQ and draw the perpendicular bisectors of PQ and RQ.

   Let them meet at the point O.

   Join $\overline{OP}$, $\overline{OQ}$, and $\overline{OR}$.

   In triangles OPS and OQS, $PS = SQ$

   $\angle OSP = \angle OSQ$ (right angles)

   $OS = OS$ (common)

   $\therefore$ $\Delta OSP \cong OSQ$. (SAS Congruence Property)

   $\therefore$ $OQ = OP$

   Similarly, it can be proved that $OQ = OR$.

   $\therefore$ $OP = OQ = OR$. Therefore, the circle with O as the centre and passing through P, also passes through Q and R.

2. The perpendicular bisectors of PQ and QR intersect at only one point and that point is O. A circle passing through P, Q and R has to have this point as the centre. Thus, there can only be one circle passing through P, Q and R.

Properties of Chords and Related Theorems

Theorem 2

The perpendicular bisector of a chord of a circle passes through the centre of the circle.

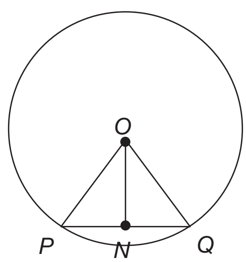


Figure 13.48

Given: PQ is a chord of a circle with centre O. N is the midpoint of chord PQ.

RTP: ON is perpendicular to PQ.

Construction: Join OP and OQ.

Proof: In triangles OPN and OQN,

$OP = OQ$ (radius of the circle)

$PN = QN$ (given)

ON is common.

By SSS Congruence Property,

$\Delta OPN \cong \Delta OQN$.

$\therefore \angle ONP = \angle ONQ$

But,

$\angle ONP + \angle ONQ = 180°$ (straight line)

$\Rightarrow \angle ONP = \angle ONQ = 90°$

$\therefore$ ON is perpendicular to chord PQ.

Note The converse of the above theorem is also true, i.e., the diameter which is perpendicular to a chord of a circle bisects the chord.

Theorem 3

Two equal chords of a circle are equidistant from the centre of the circle.

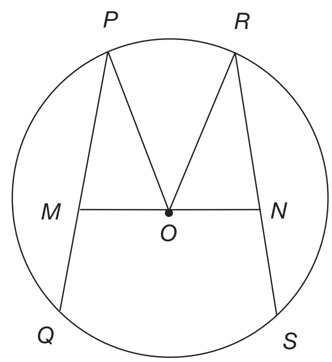


Figure 13.49

Given: In a circle with centre O, chord PQ = chord RS.

$OM \perp PQ$ and $ON \perp RS$.

RTP: $OM = ON$

Construction: Join OP and OR.

Proof: Since PQ is a chord of the circle and OM is perpendicular to PQ, OM bisects PQ.

(theorem 1).

Similarly, ON bisects RS.

$PQ = RS$ (given)

$\frac{1}{2}PQ = \frac{1}{2}RS$

$\Rightarrow \quad PM = RN$

$\angle OMP = \angle ONR$ (Right angles)

$OP = OR$ (radii of the circle)

In triangles OMP and ONR,

$OP = OR$,

$PM = RN$ and $\angle OMP = \angle ONR$

$\therefore \Delta OMP \cong \Delta ONR$ (SAS axiom)

$\therefore OM = ON$

Hence proved.

The converse of the above theorem, is also true, i.e., two chords which are equidistant from the centre of a circle are equal in length.

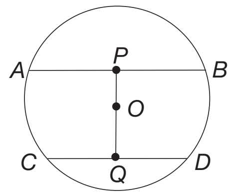


Figure 13.50

Note Longer chords are closer to the centre and shorter chords are farther from the centre.

In the given figure, AB and CD are two chords of the circle with centre at O and $AB > CD$. OP is perpendicular to chord AB and OQ is perpendicular to chord CD. By observation, we can say that $OQ > OP$.

EXAMPLE 13.8

AB and CD are two equal and parallel chords of lengths 24 cm each, in a circle of radius 13 cm. What is the distance between the chords?

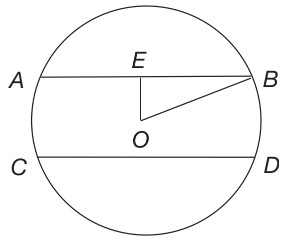


Figure 13.51

SOLUTION

In the given circle with centre O, AB and CD are the two chords each of length 24 cm and are equidistant from the centre of the circle. Therefore the distance between AB and CD = 5 + 5 = 10 cm.

Angles Subtended by Equal Chords at the Centre

Theorem 4

Equal chords subtend equal angles at the centre of the circle.

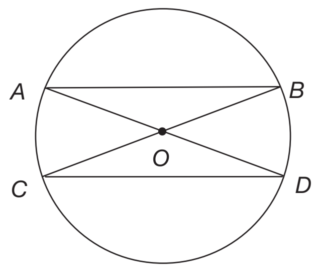


Figure 13.52

Given: AB and CD are equal chords of a circle with centre O. Join OA, OB, OC and OD.

RTP: $\angle AOB = \angle COD$.

Proof: In triangles ABO and CDO,

$OA = OC$, (radii of the same circle)

$OB = OD$

$AB = CD$ (given)

By SSS (side-side-side congruence property), triangles ABO and CDO are congruent. Hence, the corresponding angles are equal.

$\angle AOB = \angle COD$

Hence proved.

The converse of the theorem is also true, i.e., chords of a circle subtending equal angles at the centre are equal.

Angles Subtended by an Arc

Property 1

Angles subtended by an arc at any point on the rest of the circle are equal.

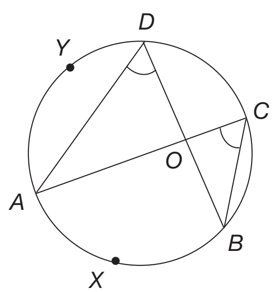


Figure 13.53

In the given circle, AXB is an arc of the circle. The angles subtended by the arc AXB at C and D are equal, i.e., $\angle ADB = \angle ACB$.

Property 2

Angle subtended by an arc at the centre of a circle is double the angle subtended by it at any point on the remaining part of the circle.

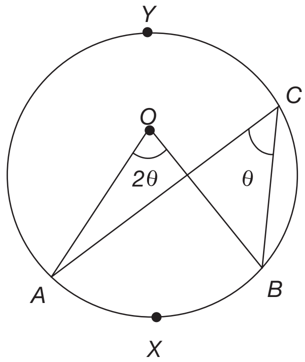


Figure 13.54

In the above figure, AXB is an arc of the circle. $\angle ACB$ is subtended by the arc AXB at the point C (a point on the remaining part of the circle), i.e., arc AYB. If O is the centre of the circle,

$$\angle AOB = 2\ \angle ACB.$$

EXAMPLE 13.9

In the following figure, AB is an arc of the circle. C and D are the points on the circle.

If $\angle ACB = 30°$, find $\angle ADB$.

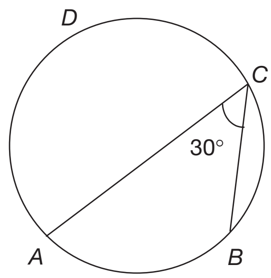


Figure 13.55

SOLUTION

Angles made by an arc in the same segment are equal. Angles made by the arc in the segment $ADCB$ are equal.

$\therefore\ \angle ADB = \angle ACB = 30°.$

EXAMPLE 13.10

In the following figure, O is the centre of the circle.

AB is an arc of the circle, such that $\angle AOB = 80°$. Find $\angle ACB$.

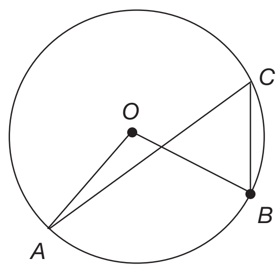


Figure 13.56

SOLUTION

The angle made by an arc at the centre of a circle is twice the angle made by the arc at any point on the remaining part of the circle.

$\angle AOB = 2\angle ACB$

$\Rightarrow\quad 2\angle ACB = 80°$

$\Rightarrow\quad \angle ACB = 40°.$

EXAMPLE 13.11

In the following figure, O is the centre of the circle. AB and CD are equal chords.

If $\angle AOB = 100°$, find $\angle CED$.

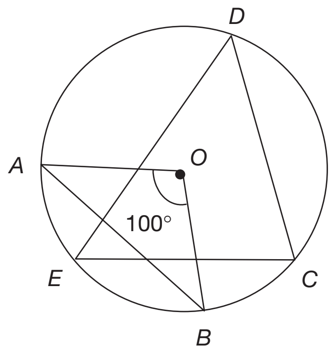


Figure 13.57

SOLUTION

Equal chords subtend equal angles at the centre of the circle.

$\angle AOB = 100°$

$\Rightarrow \quad \angle DOC = 100°$

(Angle subtended by an arc at the centre of the circle is twice the angle subtended by it anywhere in the remaining part of the circle)

$\therefore \angle DEC = \frac{1}{2}(100°) = 50°.$

Cyclic Quadrilateral

If all the four vertices of a quadrilateral lie on a circle, it is called a cyclic quadrilateral.

In the given figure, the four vertices, A, B, C and D of the quadrilateral $ABCD$ lie on the circle. Hence $ABCD$ is a cyclic quadrilateral.

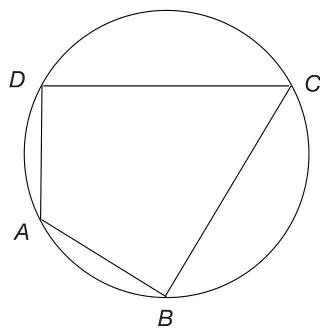


Figure 13.58

Theorem 5

The opposite angles of a cyclic quadrilateral are supplementary.

Given: $ABCD$ is a cyclic quadrilateral.

RTP:

$\angle A + \angle C = 180°$ or

$\angle B + \angle D = 180°$

Construction: Join OA and OC, where O is the centre of the circle.

$\angle AOC = 2\angle ADC$

Angle subtended by arc ABC is double the angle subtended by arc ABC at any point on the remaining part of the circle and D is such a point.

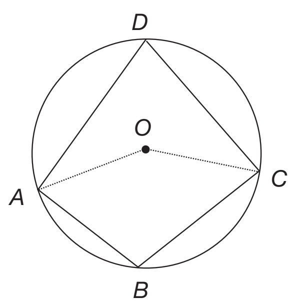


Figure 13.59

Similarly,

Reflex $\angle AOC = 2\angle ABC$

$\angle AOC +$ Reflex $\angle AOC = 360°$

$2\angle ADC + 2\angle ABC = 360°$

$\Rightarrow \quad \angle ADC + \angle ABC = 180°$

$\Rightarrow \quad \angle B + \angle D = 180°$

Similarly, it can be proved that $\angle A + \angle C = 180°$.

Theorem 6

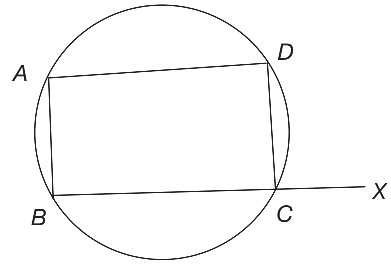


Figure 13.60

The exterior angle of a cyclic quadrilateral is equal to the interior opposite angle.

Given: $ABCD$ is a cyclic quadrilateral.

Construction: Extend BC to X.

RTP: $\angle DCX = \angle BAD$.

Proof:

$\angle BAD + \angle BCD = 180°$ (1)

(The opposite angles of a cyclic quadrilateral are supplementary).

$\angle BCD + \angle DCX = 180°$ (2)

(Angle of a straight line)

From Eqs. (1) and (2), we get

$\angle BAD + \angle BCD = \angle BCD + \angle DCX$

$\Rightarrow \quad \angle DCX = \angle BAD.$

Hence proved.

We are already familiar with what a circle is and studied some of its properties. Now, we shall study the properties of tangents and chords.

Tangents

When a line and a circle are drawn in the same plane we have the following cases.

1. The line and the circle may not intersect at all, as shown in the Fig. 13.61 (a) below. This means the line and the circle do not meet.
2. The line may intersect the circle at two points as shown in Fig. 13.61 (b).
3. The line may touch the circle at only one point as shown in Fig. 13.61 (c).

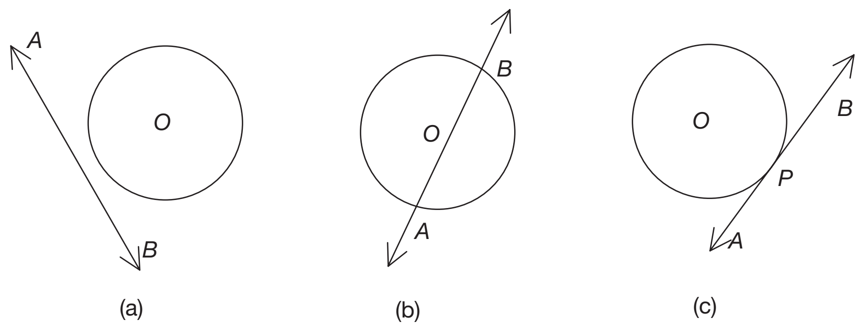


Figure 13.61

(i) If a line meets a circle at two points, then the line is called a secant of the circle.
(ii) When a line touches the circle at only one point or a line meets a circle at only one point, then the line is called a tangent to the circle at that point and that point is called point of contact or the point of tangency.
(iii) At a point on a circle, only one tangent can be drawn.
(iv) From any given external point, two tangents can be drawn to a circle.
(v) From any point inside a circle, no tangent can be drawn to the circle.

Theorem 7

The tangent at any point on a circle is perpendicular to the radius through the point of contact.

Given: PQ is a tangent to a circle with centre O. Point of contact of the tangent and the circle is A.

RTP: OA is perpendicular to PQ.

Construction: Take another point B on PQ and join OB.

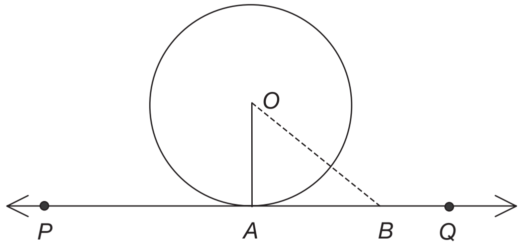


Figure 13.62

Proof: Since B is a point other than A, B may lie inside or outside the circle.

Case 1: If point B lies inside the circle but on the line PQ, then PQ cannot be a tangent to the circle. Hence B does not lie inside the circle.

Case 2: If B lies outside the circle,
$OB > OA$, i.e., among all the segment joining a point on the line to the point O, OA is the shortest.
But, among all the line segments joining point O to a point on PQ, the shortest is the perpendicular line.
$\therefore$ OA is perpendicular to PQ.

Converse of the Theorem

A line drawn through the end point of a radius of a circle and perpendicular to it is a tangent to the circle.

Theorem 8

Two tangents drawn to a circle from an external print are equal in length.

Given PQ and PT are two tangents drawn from point P to circle with centre O.

RTP: $PQ = PT$

Construction: Join OP, OQ and OT.

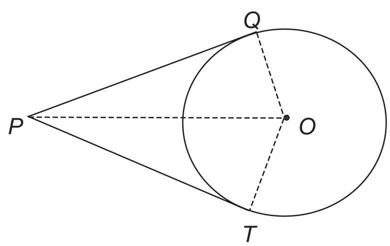


Figure 13.63

Proof: In triangles OPQ and OPT,

$OP = OP$ (Common)

$\angle OQP = \angle OTP = 90°$

$OQ = OT$ (Radius)

The tangent is perpendicular to the radius at the point of tangency. By the RHS congruence axiom, $\Delta OPQ \cong \Delta OPT$

$\Rightarrow \quad PQ = PT.$

Note We also note that $\angle OPQ = \angle OPT$, i.e., the centre of a circle lies on the bisector of the angle formed by the two tangents.

Chords

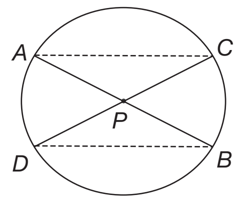
A line joining any two points on a circle is a chord of the circle. If AB is a chord of a circle and P is a point on it, P is said to divide AB internally into two segments AP and PB. Similarly, if Q is a point on the line AB, outside the circle, Q is said to divide AB externally into two segments AQ and QB.

Theorem 9

If two chords of a circle intersect each other, then the products of the lengths of their segments are equal.

Case 1: Let the two chords intersect internally.

Given: AB and CD are two chords intersecting at point P in the circle.



RTP: $(PA)\ (PB) = (PC)\ (PD)$

Construction: Join AC and BD.

Proof: In triangles APC and PDB,

$\angle APC = \angle DPB$ (Vertically opposite angels)

$\angle CAP = \angle CDB$ (Angles made by arc BC in the same segment)

ΔAPC is similar to ΔDPB. (If two angles of one triangle are equal to the corresponding angles of another triangle, the two triangles are similar or the AA Similarity Property)

$$\therefore\ \frac{PA}{PD} = \frac{PC}{PB} \quad \Rightarrow \quad (PA)(PB) = (PC)(PD)$$

Case 2: Let the two chords intersect externally.

Given: Two chords BA and CD intersect at point P which lies outside the circle.

RTP: $(PA)(PB) = (PC)(PD)$

Construction: Join AC and BD.

Proof: In triangles PAC and PDB,

$\angle PAC = \angle PDB$ and $\angle PCA = \angle PBD$ (An external angle of a cyclic quadrilateral is equal to the interior angle at the opposite vertex.

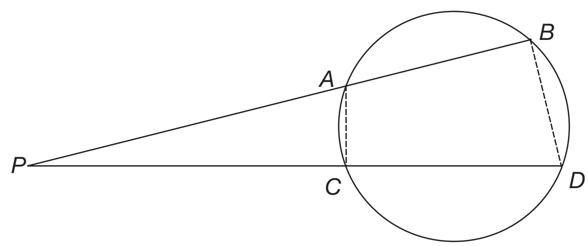


Figure 13.64

$$\frac{PA}{PD} = \frac{PC}{PB} = \text{(The } AA \text{ Similarity Property)}$$

$$\Rightarrow \quad (PA)(PB) = (PC)(PD).$$

Notes

1. The converse of the above theorem is also true, i.e., if two line segments AB and CD intersect at P and $(PA)(PB) = (PC)(PD)$, then the four points are concyclic.
2. If one of the secants (say PCD) is rotated around P so that it becomes a tangent, i.e., points C and D say at T, coincide. We get the following result.

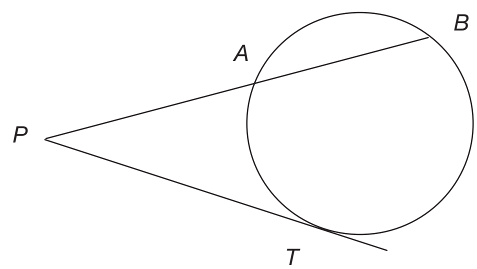


Figure 13.65

If PAB is a secant to a circle intersecting the circle at A and B and PT is the tangent drawn from P to the circle, then $PA \cdot PB = PT^2$

P is any point out side the circle with centre O and PAB is these cant drawn from P and PT is the tangent. Then $(PA)(PB) = PT^2$.

Alternate Segment and Its Angles

AB is a chord in a circle with centre O. A tangent is drawn to the circle at A. Chord AB makes two angles with the tangents $\angle BAY$ and $\angle BAX$. Chord AB divides the circle into two segments ACB and ADB. The segments ACB and ADB are called alternate segments to angles $\angle BAY$ and $\angle BAX$ respectively.

Angles made by the tangent with the chord are $\angle BAY$ and $\angle BAX$. ACB and ADB are alternate segments to those angles respectively.

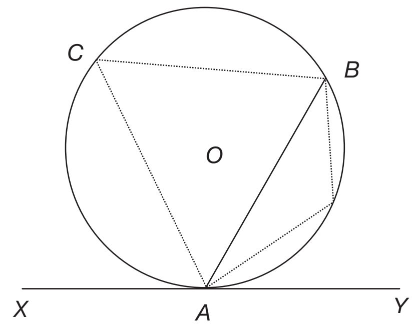


Figure 13.66

Theorem 10 Alternate Segment Theorem

If a line touches the circle at a point and if a chord is drawn from the point of contact then the angles formed between the chord and the tangent are equal to the angles in the alternate segments.

Given: XY is a tangent to the given circle with centre O at the point A, which lies in between X and Y. AB is a chord. C and D are points on the circle either side of line AB.

RTP: $\angle BAY = \angle ACB$ and $\angle BAX = \angle ADB$.

Construction: Draw the diameter AOP and join PB.

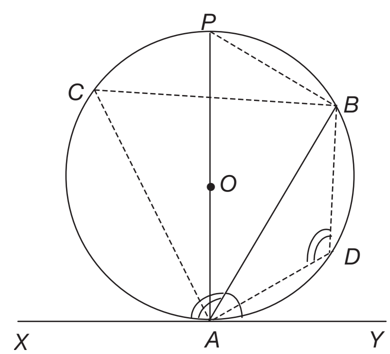


Figure 13.67

Proof: $\angle ACB = \angle APB$ (Angles in the same segment)

$\angle ABP = 90°$ (Angle in a semi-circle)

In the triangle ABP,

$$\angle APB + \angle BAP = 90° \quad (1)$$

$\angle PAY = 90°$ (the radius makes a right angle with the tangent at the point of tangency).

$$\Rightarrow \quad \angle BAP + \angle BAY = 90° \quad (2)$$

From the Eqs. (1) and (2),

$\angle APB = \angle BAY$

$\Rightarrow \quad \angle ACB = \angle BAY$. ($\because \angle APB = \angle ACB$)

Similarly, it can be proved that

$\angle BAX = \angle ADB$.

Converse of Alternate Segment Theorem

A line is drawn through the end point of a chord of a circle such that the angle formed between the line and the chord is equal to the angle subtended by the chord in the alternate segment. Then, the line is tangent to the circle at the point.

Common Tangents to Circles

When two circles are drawn on the same plane with radii r_1 and r_2, with their centres d units apart, then we have the following possibilities.

1. The two circles are concentric, then $d = 0$. The points C_1 and C_2 coincide.

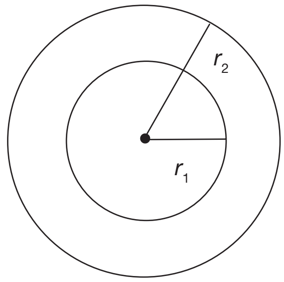


Figure 13.68

2. The two circles are such that one lies in side the other, then $|r_1 - r_2| > d$.

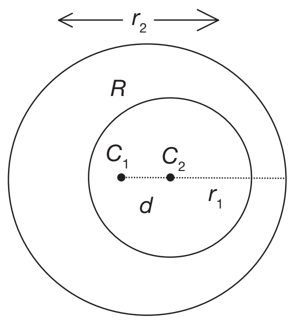


Figure 13.69

3. The two circles may touch each other internally, then $d = |r_1 - r_2|$.

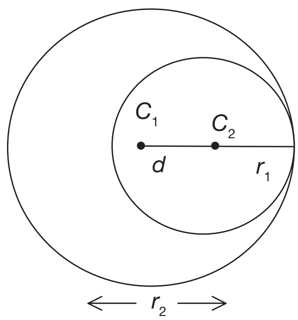


Figure 13.70

4. The two circles intersect at two points, in which case, $|r_1 - r_2| < d < r_1 + r_2$.

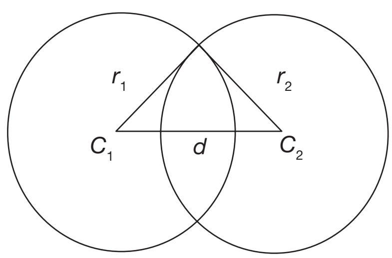


Figure 13.71

5. The two circles may touch each other externally, then $d = r_1 + r_2$.

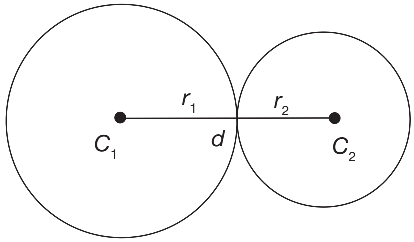


Figure 13.72

6. The two circles do not meet each other, then $d > r_1 + r_2$.

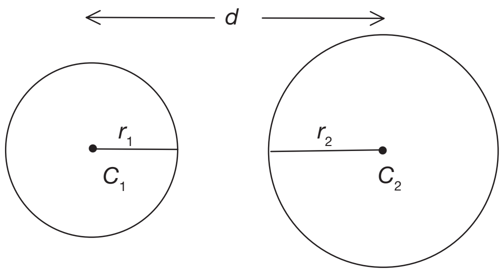


Figure 13.73

Common Tangent

If the same line is tangent to two circles drawn on the same plane, then the line is called a common tangent to the circles. The distance between the point of contacts is called the length of the common tangent.

In the figure, PQ is a common tangent to the circles, C_1 and C_2. The length of PQ is the length of the common tangent.

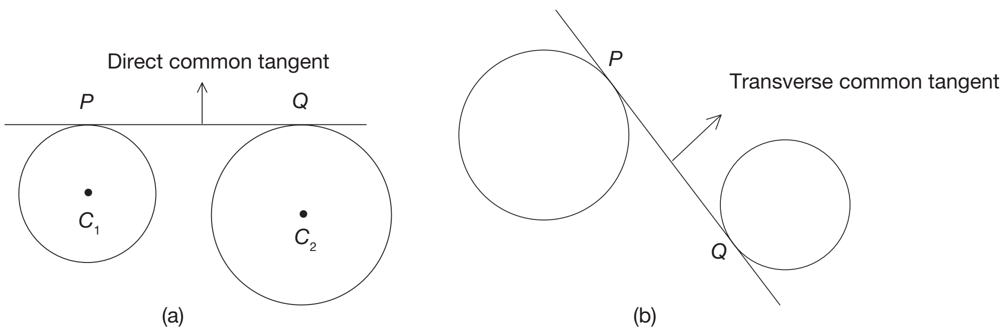


Figure 13.74

In Fig. 13.74(a), we observe that both the circles lie on the same side of PQ. In this case, PQ is a ***direct*** common tangent and in Fig. 13.74(b), we notice that the two circles lie on either side of PQ. Here PQ is a ***transverse*** common tangent.

1. The number of common tangents to the circles one lying inside the other is zero.

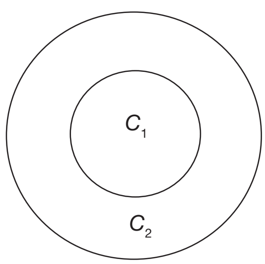


Figure 13.75

2. The number of common tangents to two circles touching internally is one.

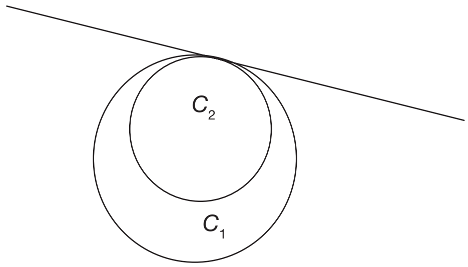


Figure 13.76

3. The number of common tangents to two intersecting circles is two, i.e., two direct common tangents.

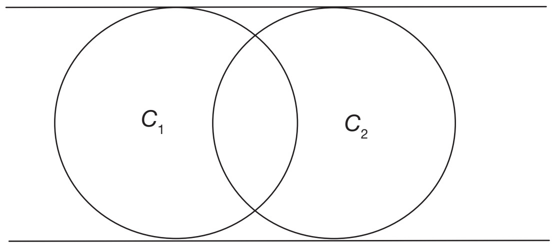


Figure 13.77

4. The number of common tangents to two circles touching externally is three, i.e., two direct common tangents and one transverse common tangent.

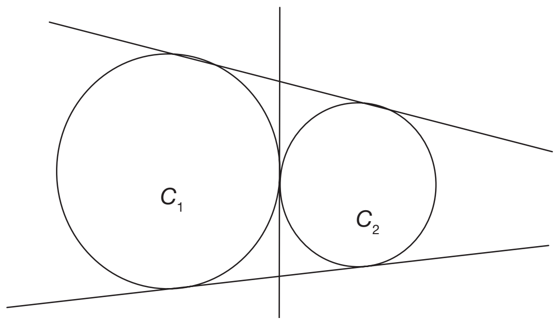


Figure 13.78

5. The number of common tangents to non-intersecting circles is four, i.e., 2 direct common tangents and 2 transverse common tangents.

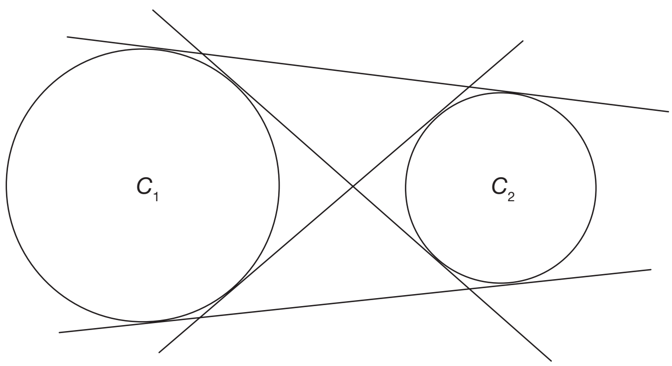


Figure 13.79

Properties of Common Tangents

1. When two circles touch each other internally or externally, then the line joining the centres is perpendicular to the tangent drawn at the point of contact of the two circles.

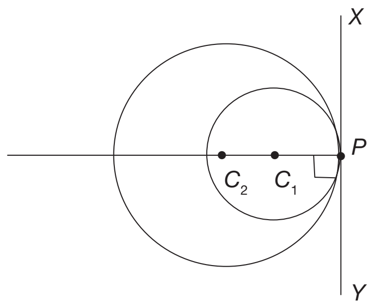


Figure 13.80

Case 1: Two circles with centres C_1 and C_2 touch each other internally at P. C_1C_2P is the line drawn through the centres and XY is the common tangent drawn at P which is common tangent to both the circles.
$\therefore$ C_1C_2 is perpendicular to XY.

Case 2: The given two circles with centres C_1 and C_2 touch each other externally at P.

C_1PC_2 is the line joining the centres of the circles and XY is the common tangent to the two circles drawn at P.

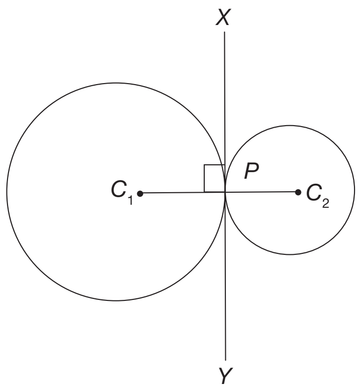


Figure 13.81

$\therefore$ C_1C_2 is perpendicular to XY.

2. The direct common tangents to two circles of equal radii are parallel to each other.

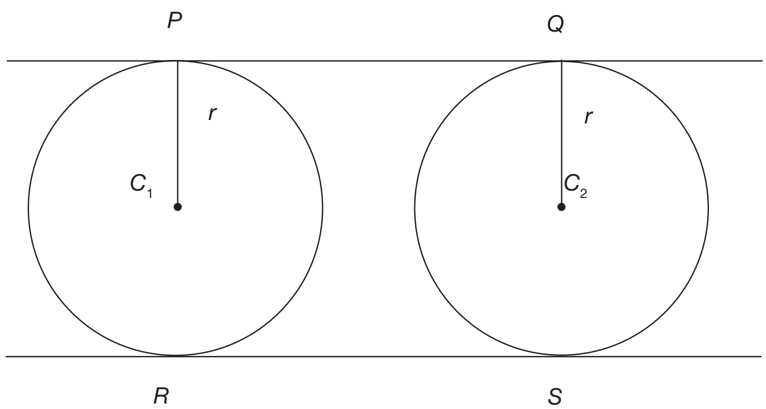


Figure 13.82

Let two circles of equal radii 'r' have centres C_1 and C_2 and PQ and RS be the direct common tangents drawn to the circles. Then PQ is parallel to RS.

EXAMPLE 13.12

In the figure, find the value of x.

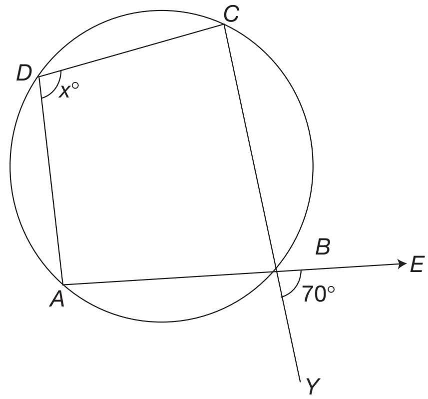


Figure 13.83

SOLUTION

In the figure, $\angle CBE$ is an exterior angle which is equal to the opposite interior angle at the opposite vertex, $\angle ADC$.

$\therefore\ \angle CBE = \angle ADC$ (1)

$\angle CBE + \angle EBY = 180°$ ($\because$ linear pair)

$\therefore\ \angle CBE = 180° - 70° = 110°$

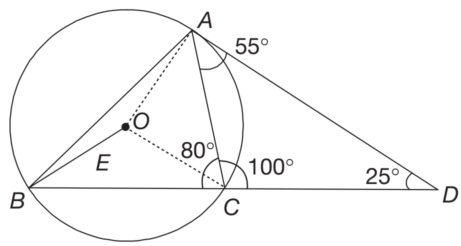
$x° = \angle ADC = \angle CBE = 110°$.

EXAMPLE 13.13

In the given figure, O is the centre of the circle and AD is a tangent to the circle at A. If $\angle CAD = 55°$ and $\angle ADC = 25°$, then find $\angle ABO$.

(a) 10° **(b)** 15° **(c)** 20° **(d)** 25°

SOLUTION



(i) Join OA and OC.

(ii) $\angle ACD = 180° - 80° = 100°$.

(iii) $\angle ACB$ and $\angle ACD$ are supplementary.

$\Rightarrow\quad \angle ACB + \angle ACD = 180°$

$\Rightarrow\quad \angle ACB = 80°$

(iv) $\angle AOB = 2\angle ACB.\quad \Rightarrow\quad \angle AOB = 2 \times 80° = 160°$

(v) Now, In ΔAOB,

$\angle O + \angle A + \angle B = 180°$

$160° + x + x = 180°$

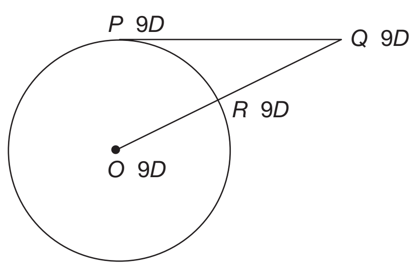
$2x = 20°$

$x = 10°$

$\therefore$ Hence, option (a) is the correct answer.

EXAMPLE 13.14

In the given figure (not to scale), PQ is a tangent segment, drawn to the circle with centre O at P and $QR = RO$. If $PQ = 3\sqrt{3}$ cm, and ORQ is a line segment, then find the radius of the circle.



(a) $\sqrt{3}$ cm **(b)** 3 cm **(c)** $2\sqrt{3}$ cm **(d)** 2 cm

SOLUTION

Let radius of the circle be x

$\therefore\ OR = RQ = x$ (given $OR = RQ$)

Join OP

$\therefore\ \overline{OP} \perp \overline{PQ}$

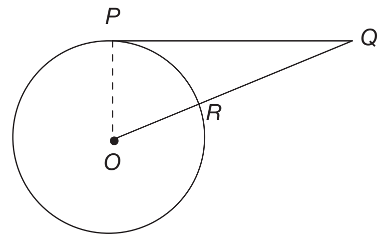
$OP^2 + PQ^2 = OQ^2$

$\Rightarrow\quad x^2 + (3\sqrt{3})^2 = (2x)^2$

$\Rightarrow\quad 3x^2 = 27 \quad \Rightarrow \quad x^2 = 9 \quad \Rightarrow \quad x = \pm 3$

But x is positive.

$\therefore\ x = 3$ cm.



Constructions Related to Circles

Construction 1: To construct a segment of a circle, on a given chord and containing a given angle.

Example: Construct a segment of a circle on a chord of length 8.5 cm, containing an angle of 65°(θ)

Step 1: Draw a line segment BC of the given length, 8.5 cm.

Step 2: Draw $\overrightarrow{BX}$ and $\overrightarrow{CY}$ such that $\angle CBX = \angle BCY = \dfrac{180 - 2\theta}{2} = 25°$.

Step 3: Mark the intersection of $\overrightarrow{BX}$ and $\overrightarrow{CY}$ as O.

Step 4: Taking O as centre and OB or OC as radius, draw BAC

Step 5: In ΔBOC, $\angle BOC = 130°$

$\Rightarrow\quad \angle BAC = (1/2)\ \angle BOC = (1/2)(130°) = 65°$

$\angle BAC = 65°$.

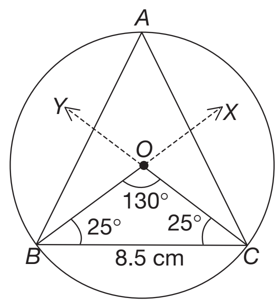


Figure 13.84

The segment bounded by $\overline{BAC}$ and $\overline{BC}$ is the required segment.

Construction 2: Construct an equilateral triangle inscribed in a circle of radius 3.5 cm.

Step 1: Draw a circle of radius 3.5 cm and mark its centre as O.

Step 2: Draw radii $\overline{OA}$, $\overline{OB}$ and $\overline{OC}$ such that $\angle AOB = \angle BOC = 120°$.

$(\therefore\ \angle COA = 120°)$

Step 3: Join AB, BC and CA which is the required equilateral ΔABC in the given circle.

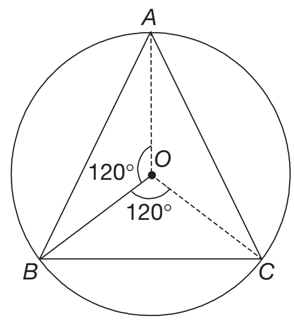


Figure 13.85

Construction 3: Construct an equilateral triangle circumscribed over a circle of radius 3 cm.

Step 1: Draw a circle of radius 3 cm with centre O.

Step 2: Draw radii $\overline{OP}$, $\overline{OQ}$ and $\overline{OR}$ such that $\angle POR = \angle ROQ = 120°$.

$(\therefore\ \angle QOP = 120°)$

Step 3: At P, Q and R draw perpendiculars to $\overline{OP}$, $\overline{OQ}$ and $\overline{OR}$ respectively to form ΔABC. ΔABC is the required circumscribing equilateral triangle.

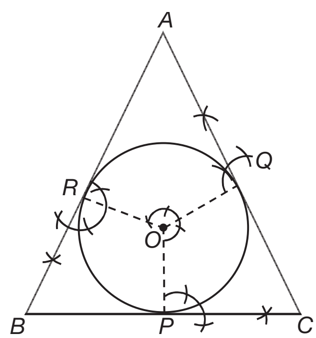


Figure 13.86

Construction 4: Draw the circum-circle of a given triangle.

Step 1: Draw ΔABC with the given measurements.

Step 2: Draw perpendicular bisectors of the two of the sides, say $\overline{AB}$ and AC to intersect at S.

Step 3: Taking S as centre, and the radius equal to AS or BS or CS, draw a circle. The circle passes through all the vertices A, B and C of the triangle.

$\therefore$ The circle drawn is the required circum-circle.

> **Note** The circum-centre is equidistant from the vertices of the triangle.

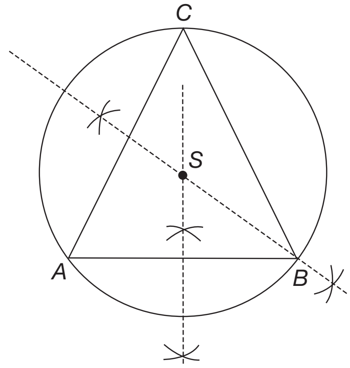


Figure 13.87

Construction 5: Construct the in circle of a given triangle ABC.

Step 1: Draw a ΔABC with the given measurements.

Step 2: Draw bisectors of two of the angles, say $\angle B$ and $\angle C$ to intersect at I.

Step 3: Draw perpendicular $\overline{IM}$ from I onto $\overline{BC}$.

Step 4: Taking I as centre and IM as the radius, draw a circle.

This circle touches all the sides of the triangle. This is the in circle of the triangle.

> **Note** The in-centre is equidistant from all the sides of the triangle.

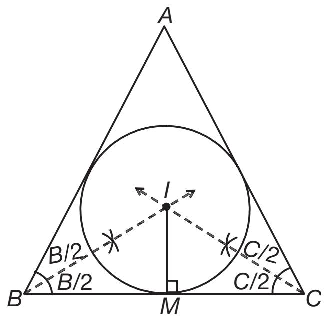


Figure 13.88

Construction 6: Construct a square inscribed in a circle of radius 3 cm.

Step 1: Draw a circle of radius 3 cm and mark centre as 'O'.

Step 2: Draw diameters $\overline{AC}$ and $\overline{BD}$ such that $\overline{AC} \perp \overline{BD}$.

Step 3: Join A, B, C and D. Quadrilateral $ABCD$ is the required square inscribed in the given circle.

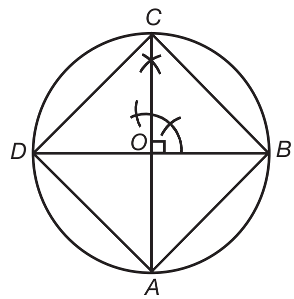


Figure 13.89

Construction 7: Construct a square circumscribed over the given circle of radius 2.5 cm.

Step 1: Draw a circle of radius 2.5 cm.

Step 2: Draw two mutually perpendicular diameters $\overline{PR}$ and $\overline{SQ}$.

Step 3: At P, Q, R and S, draw lines perpendicular to OP, OQ, OR and OS respectively to form square $ABCD$ as shown in the figure.

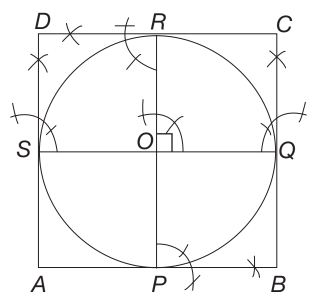


Figure 13.90

Construction 8: Construct a regular pentagon of side 4 cm. Circumscribe a circle to it.

Step 1: Draw a line segment, $AB = 4$ cm.

Step 2: Draw $\overrightarrow{BX}$ such that $\angle ABX = 108°$.

Step 2: Mark the point C on $\overrightarrow{BX}$ such that $BC = 4$ cm.

Step 4: Draw the perpendicular bisectors of $\overline{AB}$ and $\overline{BC}$ and mark their intersecting point as O.

Step 5: With O as centre and OA as radius draw a circle. And it passes through the points B and C.

Step 6: With C as centre and 4 cm as radius draw an arc which cuts the circle at the point D.

Step 7: With D as centre and 4 cm as radius draw an arc which cuts the circle at the point E.

Step 8: Join CD, DE and AE.

Step 9: $ABCDE$ is the required pentagon.

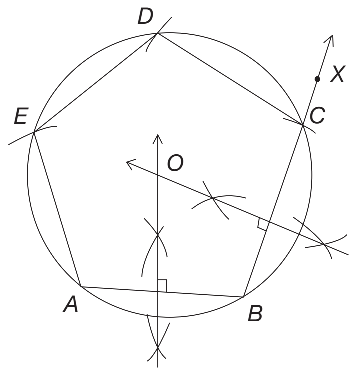


Figure 13.91

Construction 9: Construct a regular pentagon in a circle of radius 4cm.

Step 1: Construct a circle with radius 4 cm.

Step 2: Draw two radii $\overline{OA}$ and $\overline{OB}$ such that $\angle AOB = 72°$

Step 3: Join AB.

Step 4: With A as centre and AB as radius draw an arc which cuts the circle at the point E.

Step 5: With E as centre and AB as radius draw an arc which cuts the circle at the point D.

Step 6: With D as centre and AB as radius draw an arc which cuts the circle at the point C.

Step 7: Join AE, ED, DC and CB.

Step 8: $ABCDE$ is the required pentagon.

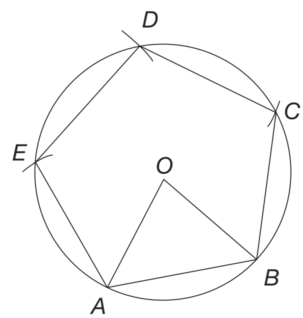


Figure 13.92

Construction 10: Construct a regular pentagon about a circle of radius 3.5 cm.

Step 1: Construct a circle with radius 3.5 cm.

Step 2: Draw radii OP, OQ, OR, OS and OT such that the angle between any two consecutive radii is 72°.

Step 3: Draw tangents to the circle at the points P, Q, R, S and T.

Step 4: Mark the intersecting points of the tangents as A, B, C, D and E.

Step 5: $ABCDE$ is the required pentagon.

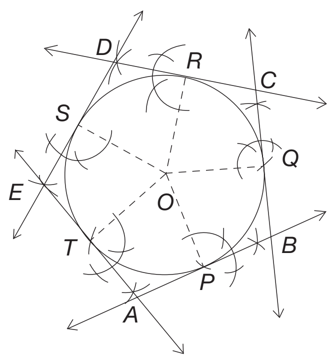


Figure 13.93

Construction 11: Construct a regular hexagon circumscribed over a circle of radius 3.5 cm.

Step 1: Draw a circle of radius 3.5 cm and mark its centre as O.

Step 2: Draw radii $\overline{OP}$, $\overline{OQ}$, $\overline{OR}$, $\overline{OS}$, $\overline{OT}$ and $\overline{OU}$ such that the angle between any two adjacent radii is 60°.

Step 3: Draw lines at P, Q, R, S, T and U perpendicular to $\overline{OP}$, $\overline{OQ}$, $\overline{OR}$, $\overline{OS}$, $\overline{OT}$ and $\overline{OU}$ respectively, to form the required circumscribed hexagon $ABCDEF$.

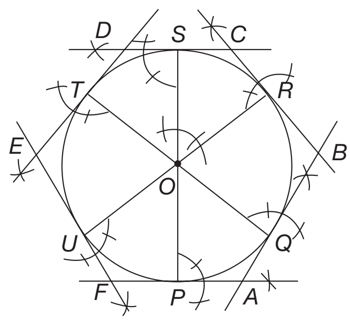


Figure 13.94

Construction 12: Inscribe a regular hexagon in a circle of radius 3 cm.

Step 1: Draw a circle of radius 3 cm taking the centre as O.

Step 2: Draw radius $\overline{OA}$. With radius, equal to OA and starting with A as the centre, mark points B, C, D, E and F one after the other.

Step 3: Join A, B, C, D, E and F. Polygon $ABCDEF$ is the required hexagon.

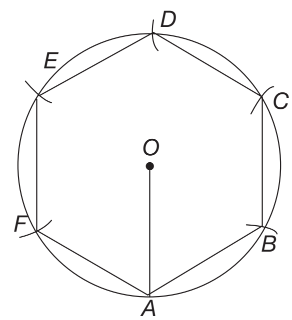


Figure 13.95

Construction 13:

1. To construct a tangent to a circle at a point on it.

When the location of the centre is known.

Step 1: Draw a circle with centre O with any radius. Let P be a point on the circle.

Step 2: Join OP.

Step 3: Construct $\angle OPY = 90°$.

Step 4: Produce YP to X and XPY is the required tangent to the given circle at point P.

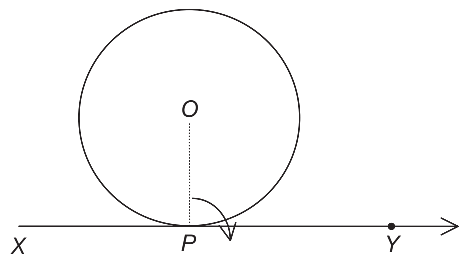


Figure 13.96

2. We have a circle, but we do not know where the centre is located.

Step 1: Draw a chord AB.

Step 2: Mark point C on major arc ACB. Join BC and AC.

Step 3: Draw $\angle BAY = \angle ACB$.

Step 4: Produce YA to X as shown in the figure. XAY is the required tangent at A.

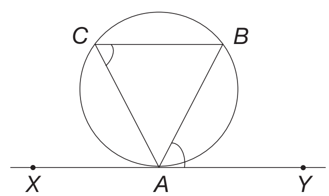


Figure 13.97

Construction 14: To construct the two tangents to a circle with centre at O, from an external point P.

Draw a circle with OP as diameter.

1. When the location of the centre is known.

Step 1: Bisect OP at A. With centre A and radius equal to OA or AP, draw a circle that intersects the given circle at two points T and Q.

Step 2: Join PT and PQ, which are the required tangents from P.

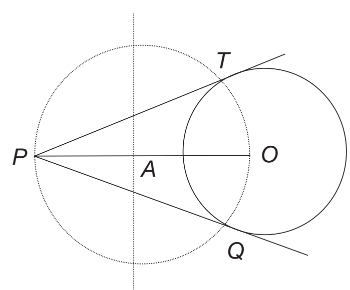


Figure 13.98

2. When the location of the centre is not known.

Step 1: Draw a circle of radius 2.5 cm. Mark a point P outside the circle.

Step 2: Draw a secant PQR through 'P' to intersect the circle at Q and R.

Step 3: Produce QP to S, such that $PS = QP$.

Step 4: With SR as diameter, construct a semi-circle.

Step 5: Construct a line perpendicular to SR at P to intersect the semi-circle at U.

Step 6: Draw arcs with PU as radius and P as the centre to intersect the given circle at T and T_1.

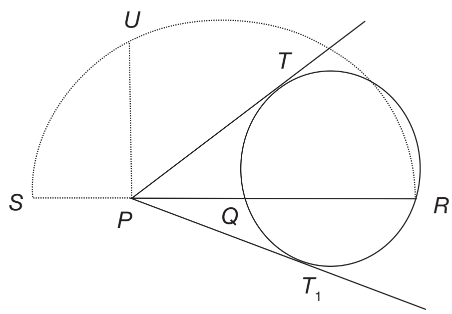


Figure 13.99

Step 7: Join PT and PT_1. These are the required tangents from P to the given circles.

Construction 15: To construct the direct common tangents to the circles of radii 3.5 cm and 1.5 cm whose centres are 6.5 cm apart.

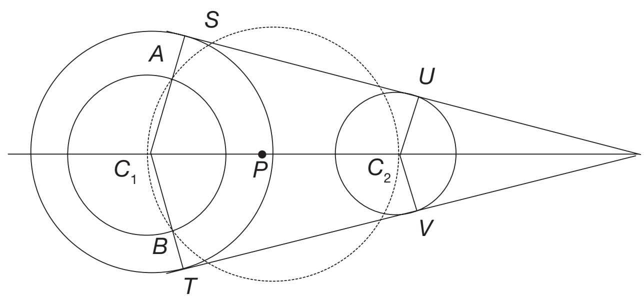


Figure 13.100

Step 1: Draw line segment, $C_1C_2 = 6.5$ cm.

Step 2: Draw circles with radii 3.5 cm and 1.5 cm respectively with C_1 and C_2 as centres.

Step 3: Bisect C_1C_2 at P and draw a circle taking P as centre and radius equal to PC_1 or PC_2.

Step 4: Draw another circle with radius equal to $(3.5 - 1.5) = 2$ cm taking the centre of larger circle C_1 intersecting the circle drawn in step 3 at A and B.

Step 5: Join C_1A and C_1B and produce C_1A and C_1B to meet the outer circle at S and T respectively.

Step 6: Draw line segments C_2U and C_2V, such that $C_2U \,||\, C_1S$ and $C_2V \,||\, C_1T$, where U, V are points on the circle with centre C_2.

Step 7: Join SU and TV. These are the required direct common tangents.

Note The length of a direct common tangent to two circles with radii r_1 and r_2 and centres d units apart,is given by $\sqrt{d^2 - (r_1 - r_2)^2}$.

Construction 16: To construct the transverse common tangents to two circles with radii 2.5 cm and 1.5 cm, with centres at a distance of 6 cm from each other.

Step 1: Draw a line segment, $C_1C_2 = 6$ cm.

Step 2: Draw circles of radii 2.5 cm and 1.5 cm respectively with C_1 and C_2 as centres.

Step 3: Bisect C_1C_2. Let M be the mid point of C_1C_2.

Step 4: Draw a circle with M as centre and C_1M or C_2M as radius.

Step 5: With C_2 as centre and radius equal to $(2.5 + 1.5)$ or 4 cm mark off 2 points A and B on the circle in step 4.

Step 6: Join C_2A and C_2B to meet the other circle with centre C_2 at S and T respectively.

Step 7: Draw lines, C_1U and C_1V such that $C_1U \parallel C_2B$ and $C_1V \parallel C_2A$, where U, V are points on the circle with centre at C_1.

Step 8: Join TU and SV. These are the required transverse common tangents.

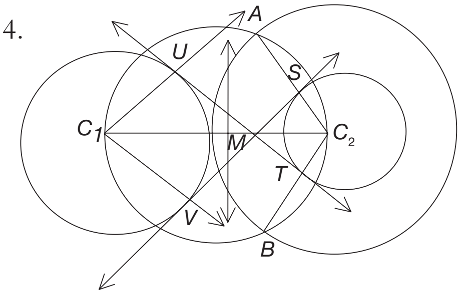


Figure 13.101

Notes

1. The length of the transverse common tangent of two circles with radii r_1 and r_2 and centres at a distance d ($d > r_1 + r_2$) is $\sqrt{d^2 - (r_1 + r_2)^2}$.
2. Transverse common tangents can be drawn to non intersecting and non touching circles only or to the circles that satisfy the condition, $d > r_1 + r_2$.

Construction 17: To construct a circle of given radius passing through two given points.

Example: Construct a circle of radius 2.5 cm passing through two points A and B 4cm apart.

Step 1: Draw a line segment, $AB = 4$ cm.

Step 2: Bisect $\overline{AB}$ by drawing a perpendicular bisector of $\overline{AB}$.

Step 3: Draw an arc with A or B as centre and with radius 2.5 cm which intersect the perpendicular bisector of $\overline{AB}$ at O.

Step 4: Draw a circle that passes through A and B with O as centre and radius 2.5 cm.

$\therefore$ This is the required circle with given radius and passing through the given points A and B.

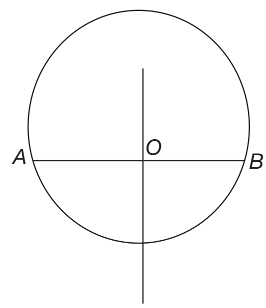


Figure 13.102

Construction 18: To draw a circle touching a given line AB at a given point P and passing through a given point Q.

Step 1: Draw a line AB and mark point P on it.

Step 2: Draw PR at P such that $PR \perp AB$, i.e., $\angle APR = 90°$.

Step 3: Join PQ and draw the perpendicular bisector of PQ to intersect PR at O

Step 4: Draw a circle, touches line AB at P and passes through the given point Q with O as centre and OP as radius.

$\therefore$ The circle drawn is the required circle.

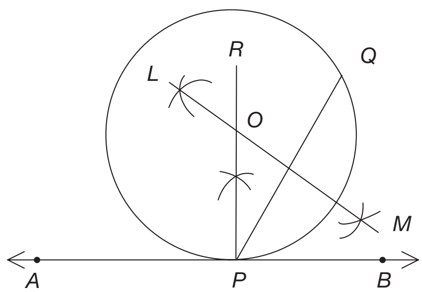


Figure 13.103

Construction 19: To construct a circle with a given radius and touching the two given intersecting lines.

Example: Construct an angle of 70° and draw a circle of radius 2.5 cm touching the arms of the angle.

Step 1: Draw an angle $\angle ABC = 70°$.

Step 2: Draw BD, the bisector of the angle ABC.

Step 3: Draw a line $\overline{PD} \parallel \overline{BC}$ at a distance of 2.5 cm (equal to the radius of the circle) to intersect the angle bisector of step 2 at O.

Step 4: Draw a circle with O as centre and radius 2.5 cm

$\therefore$ The circle drawn touches the lines at X and Y.

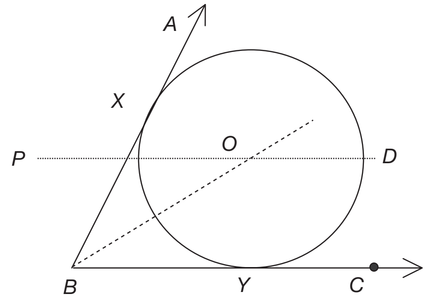


Figure 13.104

Construction 20: To construct a circle of given radius which touches a given circle and a given line.

Example: Draw a circle of radius 3 cm which touches a given line AB and a given circle of radius 2 cm with centre C.

Step 1: Let AB be the given line.

Step 2: Draw a line PQ parallel to $\overline{AB}$ at a distance 3 cm from $\overline{AB}$.

Step 3: Draw an arc with C as centre and with radius equal to $(2 + 3) = 5$ cm to cut PQ at point O.

Step 4: Draw a circle which touches the given circle at E and the given line AB at F with O as centre and radius 3 cm.

$\therefore$ The circle drawn in step 4 is the required circle.

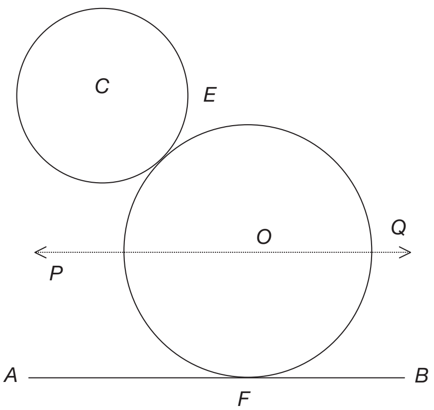


Figure 13.105

Construction 21: Construct a triangle ABC in which $BC = 6$ cm, $\angle A = 60°$ and altitude through A is 4 cm.

Step 1: Draw a line segment, $BC = 6$ cm.

Step 2: Draw $\overrightarrow{BL}$ such that $\angle CBL = 60°$.

Step 3: Draw $\overrightarrow{BM}$ such that $\angle MBL = 90°$.

Step 4: Draw a perpendicular bisector $\left(\overleftrightarrow{XY}\right)$ of $\overline{BC}$ which intersects $\overline{BC}$ at the point P.

Step 5: Mark the intersecting point of $\overleftrightarrow{XY}$ and $\overrightarrow{BM}$ as O.

Step 6: With O as centre and OB as radius draw a circle.

Step 7: Mark the point N on $\overleftrightarrow{XY}$ such that $PN = 4$cm.

Step 8: Through N, draw a line parallel to $\overrightarrow{BC}$ which intersects the circle at points A and A'.

Step 9: Join AB, AC and $A'B$, $A'C$.

Step 10: Now, ABC or $A'BC$ is required triangle.

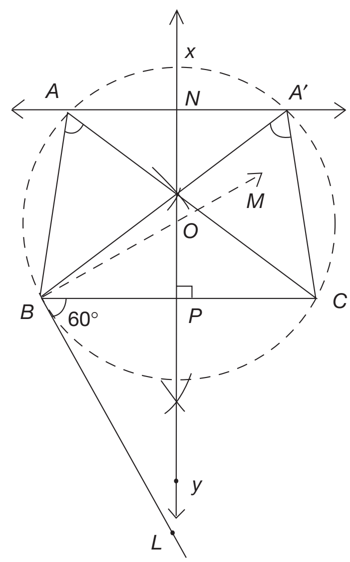


Figure 13.106

Proof: By alternate segment theorem, $\angle BAC = \angle CBL = \angle BA'C = 60°$.

Altitude through A = Altitude through A' = $PN = 4$ cm.

Construction 22: Construct a triangle ABC in which $BC = 7$ cm, $\angle A = 65°$ and median AT is 5 cm.

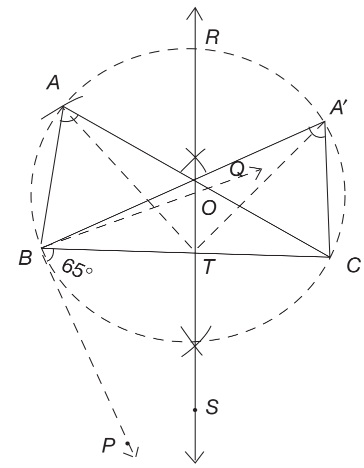


Figure 13.107

Step 1: Draw a line segment, $BC = 7$ cm.

Step 2: Draw $\overrightarrow{BP}$ such that $\angle CBP = 65°$.

Step 3: Draw $\overrightarrow{BQ}$ such that $\angle PBQ = 90°$.

Step 4: Draw a perpendicular bisector $\left(\overleftrightarrow{RS}\right)$ of $\overline{BC}$ which intersects $\overline{BC}$ at the point T.

Step 5: Mark the intersecting point of $\overleftrightarrow{RS}$ and $\overrightarrow{BQ}$ as O.

Step 6: With O as centre and OB as radius draw circle.

Step 7: With T as centre 5 cm as radius draw two arcs which intersect the circle at the points A and A'.

Step 8: Join AB and AC and A' and $A'C$.

Step 9: Now, ABC or $A'BC$ is the required triangle.

Proof: By alternate segment theorem, $\angle BAC = \angle CBP = \angle BA'C = 65°$.

Construction 23: Construct a cyclic quadrilateral $ABCD$ in which $AB = 3$ cm, $AD = 4$ cm, $AC = 6$ cm and $\angle D = 70°$.

Rough Figure

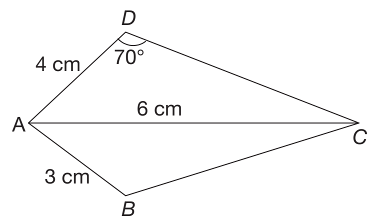


Figure 13.108

Step 1: Draw a line segment, $AC = 6$ cm.

Step 2: Draw $\overrightarrow{AP}$ such that $\angle CAP = 70°$.

Step 3: Draw $\overrightarrow{AQ}$ such that $\angle PAQ = 90°$.

Step 4: Draw a perpendicular bisectors $\left(\overleftrightarrow{RS}\right)$ of $\overline{AC}$ which intersects $\overline{AC}$ at the point T.

Step 5: Mark the intersecting point of $\overleftrightarrow{RS}$ and $\overrightarrow{AQ}$ as O.

Step 6: With O as centre and OA as radius draw a circle.

Step 7: With A as centre, 4 cm as radius draw an arc which intersects the circle atthe point D.

Step 8: With A as centre, 3 cm as radius draw an arc which intersects the circle atthe point B.

Step 9: Join AB, BC, CD and AD.

Step 10: $ABCD$ is the required cyclic quadrilateral.

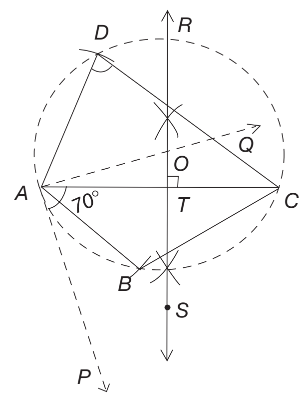


Figure 13.109

Construction 24: Find the mean proportional of the given segments p cm and q cm.

Step 1: Draw $\overleftrightarrow{XZ}$.

Step 2: Mark the point Y on $\overleftrightarrow{XZ}$ such that $XY = p$ cm and $YZ = q$ cm.

Step 3: Draw the perpendicular bisector $(\overleftrightarrow{AB})$ of $\overline{XZ}$ which meets $\overleftrightarrow{XZ}$ at the point O.

Step 4: With O as centre and OX as radius draw a semi circle. $\overleftrightarrow{XZ}$ is the diameter of the semi circle.

Step 5: Draw $\overleftrightarrow{YM}$ perpendicular to $\overleftrightarrow{XZ}$ which meets semicircle at the point M.

Step 6: Now MY, is the mean proportional of XY and YZ, i.e., p and q.

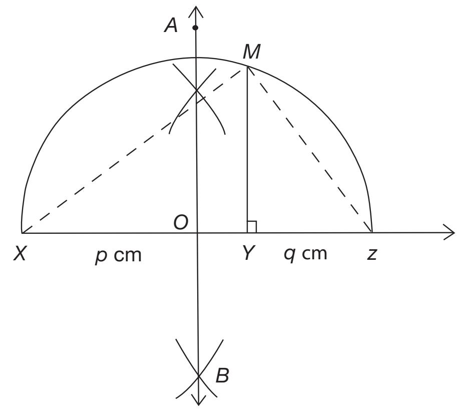


Figure 13.110

Proof:

$\Delta XYM \sim \Delta MYZ$

$$\Rightarrow \quad \frac{XY}{YM} = \frac{MY}{YZ}$$

$$\Rightarrow \quad MY = \sqrt{(XY)(YZ)} = \sqrt{pq}.$$

Uses: By using this construction, square root ($\sqrt{12}, \sqrt{15}\ \ldots$) can be found.

Construction 25: Construct a square whose area is equal to the area of the rectangle.

Step 1: Produce $\overline{AB}$ of rectangle $ABCD$ to the point E such that $BE = BC$.

Step 2: Construct mean proportional (BF) of AB and BE.

Step 3: Construct a square with side BF.

Step 4: $BFGH$ is the required square.

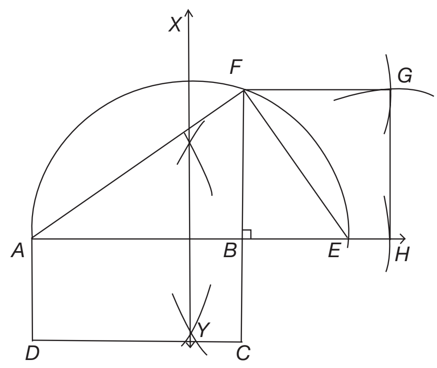


Figure 13.111

Proof: BF is the mean proportional of AB and BE.

$\Rightarrow \quad (BF)^2 = AB \times BE$

$\Rightarrow \quad (BF)^2 = AB \times BC \ (\because BC = BE)$

$\Rightarrow \quad$ Area of square $BFGH$ = Area of rectangle $ABCD$.

Construction 26: Construct a triangle similar to a given triangle ABC with its sides equal to $\left(\frac{5}{9}\right)$th of the corresponding sides of ΔABC.

Step 1: ABC is the given triangle.

Step 2: Draw $\overrightarrow{BD}$ which makes non-zero angle with $\overline{BC}$ and D is 9 cm away from the point B.

Step 3: Mark the point E on $\overrightarrow{BD}$ such that $BE = 5$ cm.

Step 4: Join $\overline{CD}$.

Step 5: Draw $\overline{EC'}$ parallel to $\overline{DC}$ which meets $\overline{BC}$ at the point C'.

Step 6: Draw $\overline{C'A'}$ parallel to $\overline{CA}$ which meets $\overline{BA}$ at the point A'.

Step 7: $A'BC'$ is the required triangle.

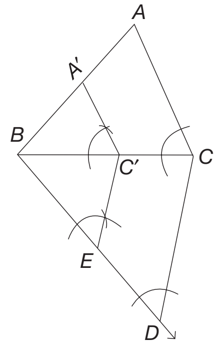


Figure 13.112

Proof: In ΔBCD, $\dfrac{BE}{BD} = \dfrac{5}{9}$ and $\overline{DC} \parallel \overline{EC'}$

$\therefore\ \Delta BEC' \sim \Delta BDC$

$\therefore\ \dfrac{BC'}{BC} = \dfrac{5}{9}$

In ΔBCA, $\dfrac{BC'}{BC} = \dfrac{5}{9}$ and $\overline{C'A'} \parallel \overline{CA}$

$\therefore\ \Delta BCA \sim \Delta BC'A'$

$\therefore\ \dfrac{BC'}{BC} = \dfrac{BA'}{BA} = \dfrac{A'C'}{AC} = \dfrac{5}{9}$

Construction 27: Construct a quadrilateral similar to a given quadrilateral $ABCD$ with its sides equal to $\left(\dfrac{6}{10}\right)$th of the corresponding sides of $ABCD$.

Step 1: $ABCD$ is the given quadrilateral.

Step 2: Join BD.

Step 3: Draw $\overrightarrow{BX}$ which makes non zero angle with $\overline{BD}$ and X is 10 cm away from the point B.

Step 4: Mark the point Y on $\overrightarrow{BX}$ such that $BY = 6$ cm.

Step 5: Join XD.

Step 6: Draw $\overline{YD'}$ parallel to $\overline{XD}$ which meets BD at the point D'.

Step 7: Draw $\overline{D'A'}$ parallel to $\overline{DA}$ which meets $\overline{AB}$ at the point A'.

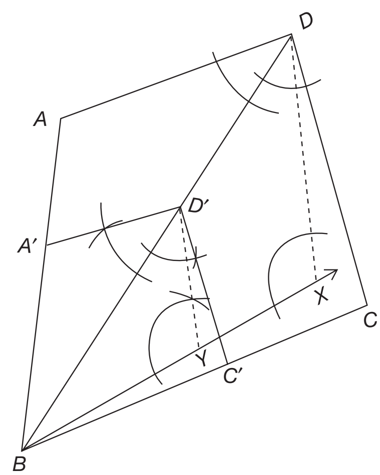


Figure 13.113

Step 8: Draw $\overline{D'C'}$ parallel to $\overline{DC}$ which meets $\overline{BC}$ at the point C'.

Step 9: $A'BC'D'$ is the required quadrilateral.

Construction 28: Construction of a pentagon similar to the given pentagon on a side of given length.

Step 1: $ABCDE$ is a given pentagon and $\overline{XY}$ is the given line segment.

Step 2: Mark the point B' on $\overline{AB}$ such that $AB' = XY$.

Step 3: Join AC and AD.

Step 4: Draw $\overline{B'C'}$ parallel to $\overline{BC}$ which meets $\overline{AC}$ at the point C'.

Step 5: Draw $\overline{C'D'}$ parallel to $\overline{CD}$ which meets $\overline{AD}$ at the point D'.

Step 6: Draw $\overline{D'E'}$ parallel to $\overline{DE}$ which meets AE at the point E'.

Step 7: $AB'C'D'E'$ is the required pentagon.

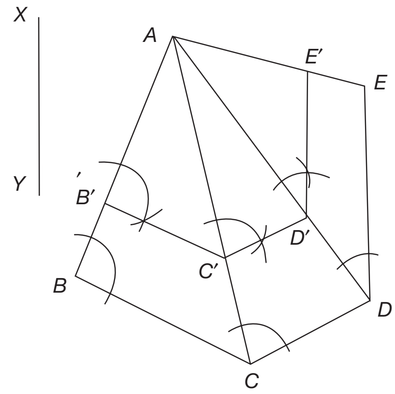


Figure 13.114

LOCUS

Mark a fixed point O on a sheet of paper. Now, start marking points P_1, P_2, P_3, P_4, … on the sheet of paper such that $OP_1 = OP_2 = OP_3 = \ldots = 4$ cm. What do we observe on joining these points by a smooth curve? We observe a pattern, which is circular in shape such that any point on the circle obtained is at a distance of 4 cm from the point O.

It can be said that whenever a set of points satisfying a certain condition are plotted, a pattern is formed. This pattern formed by all possible points satisfying the given condition is called the locus of points. In the above given example, we have a locus of points which are equidistant (4 cm) from the given point O.

The collection (set) of all points and only those points which satisfy certain given geometrical conditions is called **locus of a point**.

Alternatively, locus can be defined as the path or curve traced by a point in a plane when subjected to some geometrical conditions.

Consider the following examples:

1. The locus of the point in a plane which is at a constant distance 'r' from a fixed point 'O' is a circle with centre O and radius r units.

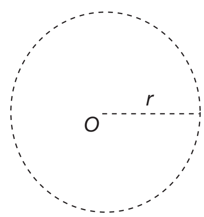


Figure 13.115

2. The locus of the point in a plane which is at a constant distance from a fixed straight line is a pair of lines, parallel to the fixed line. Let the fixed line be l. The lines m and n form the set of all points which are at a constant distance from l.

3. The locus of a point in a plane, which is equidistant from a given pair of parallel lines is a straight line, parallel to the two given lines and lying midway between them.

   In the given above, m and n are the given lines and line l is the locus.

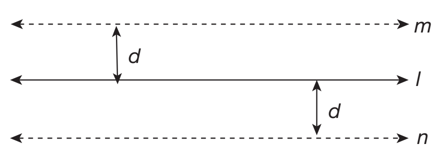


Figure 13.116

To prove that a given path or curve is the desired locus, it is necessary to prove that

(i) every point lying on the path satisfies the given geometrical conditions.

(ii) every point that satisfies the given conditions lies on the path.

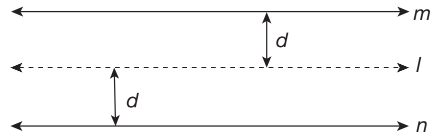


Figure 13.117

EXAMPLE 13.15

Show that the locus of a point equidistant from the end points of a line segment is the perpendicular bisector of the segment.

SOLUTION

The proof will be taken up in two steps.

Step 1: We, initially prove that any point equidistant from the end points of a line segment lies on the perpendicular bisector of the line segment.

Given: M and N are two points on a plane. A is a point in the same plane such that $AM = AN$.

RTP: A lies on the perpendicular bisector of MN.

Proof: Let M and N be the two fixed points in a plane.
Let A be a point such that $AM = AN$ and L be the mid-point of $\overline{MN}$.
If A coincides with L, then A lies on the bisector of MN.
Suppose A is different from L.
Then, in ΔMLA and ΔNLA,
$ML = NL$, $AM = AN$ and AL is a common side.
$\therefore$ By SSS congruence property, $\Delta MLA \cong \Delta NLA$.

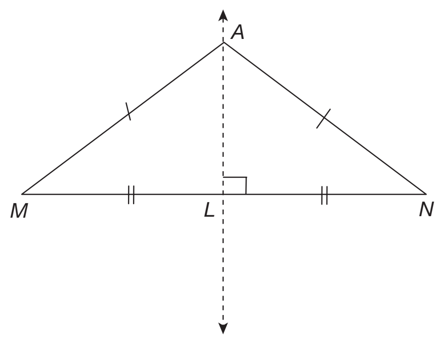


Figure 13.118

$\Rightarrow \quad \angle MLA = \angle NLA$ ($\because$ corresponding elements of congruent triangles are equal) (1)

But $\angle MLA + \angle NLA = 180°$ ($\because$ They form a straight angle)

$\Rightarrow \quad 2\angle MLA = 180°$ (using (1))

$\therefore \angle MLA = \angle NLA = 90°$

So, $\overline{AL} \perp \overline{MN}$ and hence $\overline{AL}$ is the perpendicular bisector of $\overline{MN}$.

$\therefore$ A lies on the perpendicular bisector of $\overline{MN}$.

Step 2: Now, we prove that any point on the perpendicular bisector of the line segment is equidistant from the end points of the line segment.

Given: MN is a line segment and P is a point on the perpendicular bisector. L is the mid-point of MN.

RTP: $MP = NP$.

Proof: If P coincides with L, then $MP = NP$.
Suppose P is different from L. Then, in ΔMLP and NLP,
$ML = LN$

LP is the common side and

$\angle MLP = \angle NLP = 90°$

$\therefore$ By the SAS congruence property, $\Delta MLP \cong \Delta NLP$

So, $MP = PN$ ($\because$ The corresponding elements of congruent triangles are equal)

$\therefore$ Any point on the perpendicular bisector of $\overline{MN}$ is equidistant from the points M and N.

Hence, from the steps 1 and 2 of the proof it can be said that the locus of the point equidistant from two fixed points is the perpendicular bisector of the line segment joining the two points.

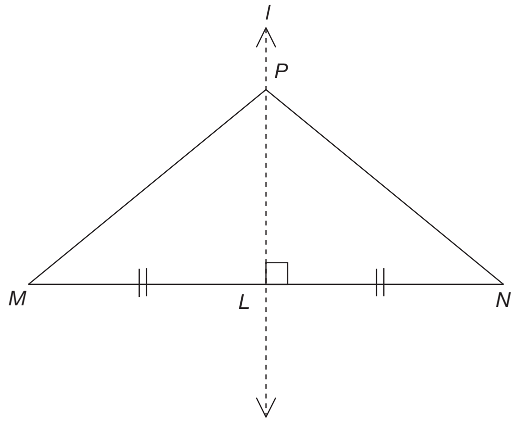


Figure 13.119

EXAMPLE 13.16

Show that the locus of a point equidistant from two intersecting lines in the plane determined by the lines is the union of the pair of lines bisecting the angles formed by given lines.

SOLUTION

Step 1: We initially prove that any point equidistant from two given intersecting lines lies on one of the lines bisecting the angles formed by given lines.

Given: $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ are two lines intersecting at O. P is the point on the plane such that $PM = PN$. Line l is the bisector of $\angle BOD$ and $\angle AOC$.

Line m is the bisector of $\angle BOC$ and $\angle AOD$.

RTP: P lies on either on line l or line m.

Proof: In ΔPOM and ΔPON,

$PM = PN$,

OP is a common side and $\angle PMO = \angle PNO = 90°$

$\therefore$ By RHS congruence property, $\Delta POM \cong \Delta PON$.

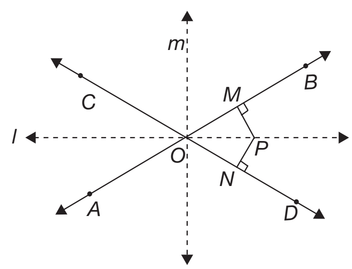


Figure 13.120

So, $\angle POM = \angle PON$, i.e., P lies on the angle bisector of $\angle BOD$.

As l is the bisector of $\angle BOD$ and $\angle AOC$, P lies on the line l.

Similarly if P lies in any of the regions of $\angle BOC$, $\angle AOC$ or $\angle AOD$, such that it is equidistant from $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$, then we can conclude that P lies on the angle bisector l or on the angle bisector m.

Step 2: We prove that any point on the bisector of one of angles formed by two intersecting lines is equidistant from the lines.

Given: Lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ intersect at O. Lines l and m are the angle bisectors.

Proof: Let l be the angle bisector of $\angle BOD$ and $\angle AOC$, and m be the angle bisector of $\angle BOC$ and $\angle AOD$.

Let P be a point on the angle bisector l, as shown in the figure.

If P coincides with O, then P is equidistant from the line $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$.

Suppose P is different from O.

Draw the perpendiculars $\overline{PM}$ and $\overline{PN}$ from the point P onto the lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ respectively.

Then in ΔPOM and ΔPON, $\angle POM = \angle PON$, $\angle PNO = \angle PMO = 90°$ and OP is a common side.

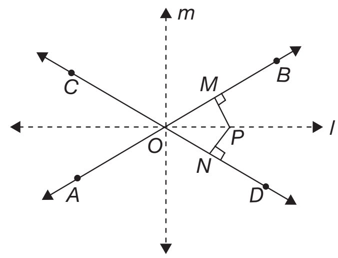


Figure 13.121

$\therefore$ By the AAS congruence property $\Delta POM \cong \Delta PON$

So, $PN = PM$ ($\because$ corresponding sides),

i.e., P is equidistant from the lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$.

Hence, from the steps 1 and 2 of the proof it can be said that the locus of the point which is equidistant from the two intersecting lines is the pair of the angle bisectors of the two pairs of vertically opposite angles formed by the lines.

Equation of a Locus

We know a locus is the set of points that satisfy a given geometrical condition. When we express the geometrical condition in the form of an algebraic equation that equation is called the equation of the locus.

Steps to Find the Equation of a Locus

1. Consider any point (x_1, y_1) on the locus.
2. Express the given geometrical condition in the form of an equation using x_1, and y_1.
3. Simplify the equation obtained in step 2.
4. Replace (x_1, y_1) by (x, y) in the simplified equation obtained in step 3, which gives the required equation of the locus.

The following formulae will be helpful in finding the equation of a locus.

1. Distance between two points $A(x_1, y_1)$ and $B(x_2, y_2)$ is $AB = AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.
2. Area of the triangle formed by joining the points (x_1, y_1), (x_2, y_2) and (x_3, y_3) is $\frac{1}{2}\begin{vmatrix} x_1 - x_2 & x_2 - x_3 \\ y_1 - y_2 & y_2 - y_3 \end{vmatrix}$, where the value of $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$.
3. Equation of the circle with centre (a, b) and radius r is given by $(x - a)^2 + (y - b)^2 = r^2$.
4. The perpendicular distance from a point $P(x_1, y_1)$ to a given line $ax + by + c = 0$ is $\left|\frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}\right|$.

EXAMPLE 13.17

Find the equation of the locus of a point that forms a triangle of area 5 units with the points $A(2, 3)$and $B(-1, 4)$.

SOLUTION

Let $P(x_1, y_1)$ be point on the locus, $(x_2, y_2) = (2, 3)$ and $(x_3, y_3) = (-1, 4)$

Given area of $\Delta PAB = 5$ sq. units.

$$\therefore \ \frac{1}{2}\begin{vmatrix} x_1 - 2 & 2-(-1) \\ y_1 - 3 & 3-4 \end{vmatrix} = 5$$

$$\begin{vmatrix} x_1 - 2 & 3 \\ y_1 - 3 & -1 \end{vmatrix} = 10$$

$-(x_1 - 2) - 3(y_1 - 3) = \pm 10$

$x_1 + 3y_1 + 11 = \pm 10$

Required equation is $x + 3y = 10 - 11$ or $x + 3y = -10 - 11$

$x + 3y + 1 = 0$ or $x + 3y + 21 = 0.$

TEST YOUR CONCEPTS

Very Short Answer Type Questions

1. There are no congruent figures which are similar. (True/False)

2. Two triangles with measures 3 cm, 4 cm, 5 cm and 0.60 cm, 0.8 cm, 1 cm are similar. (Agree/disagree)

3. A triangle is formed by joining the mid-points of the sides of a given triangle. This process is continued indefinitely. All such triangles formed are similar to one another. (True/False)

4. All the similar figures are congruent if their areas are equal. (Yes/No)

5. The ratio of corresponding sides of two similar triangles is 2 : 3, then the ratio of the perimeters of two triangles is 4 : 9. (True/False)

6. Which of the following is/are true?

   (a) All triangles are similar.

   (b) All circles are similar.

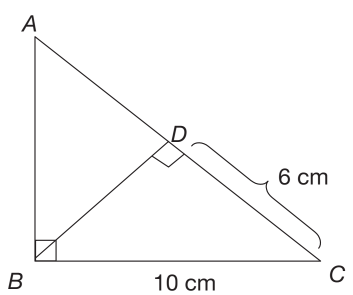
   (c) All squares are similar.

7. Two equal chords of a circle are always parallel. (True/False)

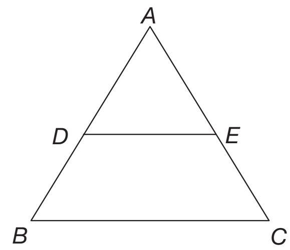
8. In a circle, chord PQ subtends an angle of 80° at the centre and chord RS subtends 100°, then which chord is longer?

9. Number of circles that pass through three collinear points is _______.

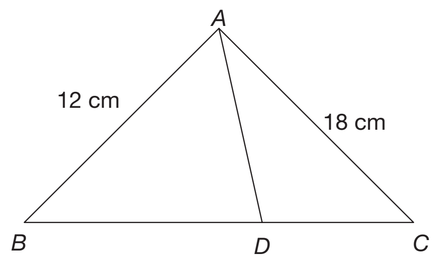
10. In the following figure, $\angle ABC = 90°$, $BC = 10$ cm, $CD = 6$ cm, then $AD =$ _______.



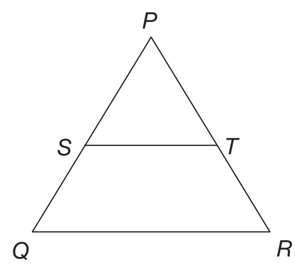
11. In the following figure, $AD = DB$ and $DE \parallel BC$, then $AE = EC$. (True/False)



12. In the following figure (not to scale), if $BC = 20$ cm and $\angle BAD = \angle CAD$, then $BD =$ _______.



13. In the following figure, $ST \parallel QR$, then $\Delta PST \sim \Delta PQR$. (Yes/No)

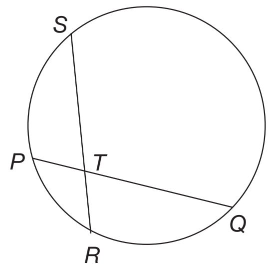


14. In the above figure, if $ST \parallel QR$ and $PS : PQ = 2 : 5$ and $TR = 15$ cm, then $PT =$ _______.

15. The number of tangents drawn from an external point to a circle is _______.

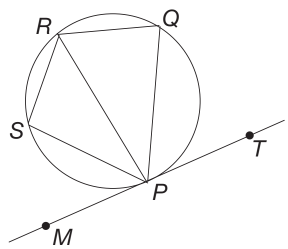
16. If two circles intersect at two distinct points, then the number of common tangents is _______.

17. In the following figure, $PT = 4$ cm, $TQ = 6$ cm and $RT = 3$ cm, then $TS =$ _______.



18. If two circles touch each other externally, then the number of transverse common tangents is _______.

19. In the following figure, to find $\angle PQR$, _______ must be given. ($\angle PRQ/\angle QPT/\angle RPT$)



20. From a point P which is at a distance of 13 cm from the centre of the circle of radius 5 cm, a tangent is drawn to the circle. The length of the tangent is _______.

21. A line drawn from the centre of a circle to a chord always bisects it. (True/False)

22. Distance between two circles with radii R and r is d units. If $d^2 = R^2 + r^2$ then the two circles _______ (intersect at one point/do not intersect/intersect at two distinct points).

23. Two circles with radii r_1 and r_2 touch externally. The length of their direct common tangent is _______.

24. In a circle, angle made by an arc in the major segment is 60°. Then the angle made by it in the minor segment is _______.

25. If a trapezium is cyclic, then its _______ are equal. (parallel sides/oblique sides)

26. In a circle, two chords PQ and RS bisect each other. Then $PRQS$ is _______.

27. Line joining the centers of two intersecting circles always bisect their common chord. (True/False)

28. The locus of the tip of a seconds hand of a watch is a _______.

29. A parallelogram has no line of symmetry. (True/False)

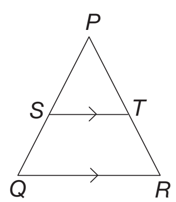
30. If the length of an enlarged rectangle is 12 cm and the scale factor is $\frac{3}{2}$, then the length of the original rectangle is _______.

Short Answer Type Questions

31. If the altitude of an equilateral triangle is 6 units, then the radius of its incircle is _________.

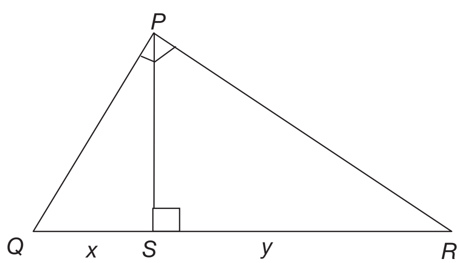
32. In a rhombus $PQRS$, PR and QS are the diagonals of the rhombus. If $PQ = 10$ cm, then find the value of $PR^2 + QS^2$.

33. In a triangle PQR, ST is parallel to QR. Show that $RT(PQ + PS) = SQ(PR + PT)$.



34. In a right angled triangle PQR, angle $Q = 90°$ and QD is the altitude. Find DR, if $PD = 17$ cm and $QD = 21$ cm.

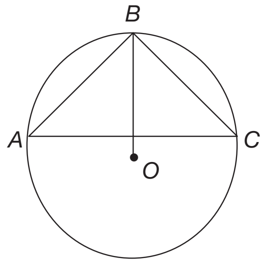
35. In the following figure, $QS = x$, $SR = y$, $\angle QPR = 90°$ and $\angle PSR = 90°$, then find $(PQ)^2 - (PR)^2$ interms of x and y.



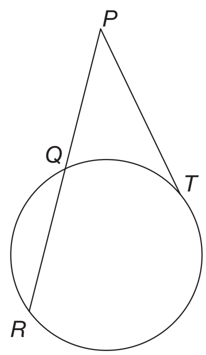
36. In the following figure, AB and BC are equidistant from the centre 'O' of the circle. Show that

(a) ABC is an isosceles triangle.

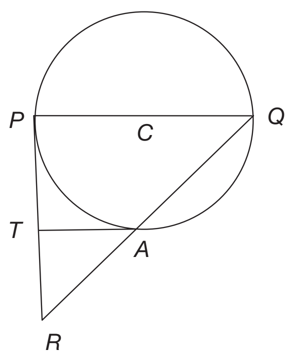
(b) OB bisects angle ABC.



37. In the following figure, PR is a secant and PT is a tangent to the circle. If $PT = 6$ cm and $QR = 5$ cm, then $PQ =$ _______ cm.



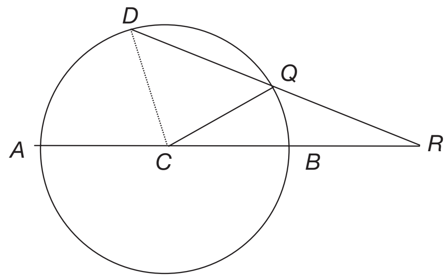
38. In the following figure, PQ is the diameter of the circle with radius 5 cm. If AT is the tangent and equal to the radius of the circle, then find the length of AR.



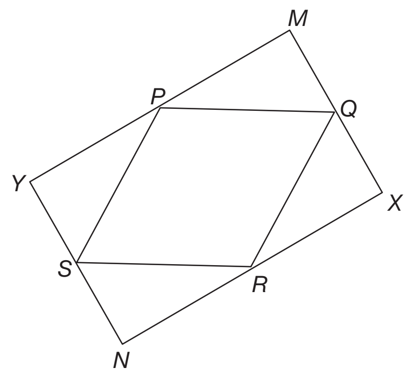
39. The distance between two buildings is 24 m. The height of the buildings are 12 m and 22 m. Find the distance between the tops.

40. Find the distance between the centres of the two circles, if their radii are 11 cm and 7 cm, and the length of the transverse common tangent is $\sqrt{301}$ cm.

41. In the figure given along side, AB is the diameter of the circle, C is the centre of the circle and CQR is an isosceles triangle, such that $CQ = QR$. Prove that $\angle DCA = 3 \cdot \angle QCR$.



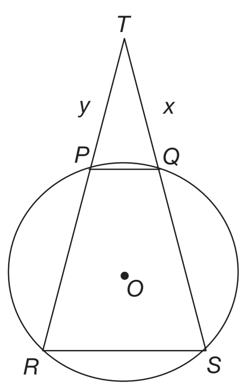
42. In the following figure, $PQRS$ is a rhombus formed by joining the mid-points of a quadrilateral $YMXN$, show that $3PQ^2 = SN^2 + NR^2 + QX^2 + XR^2 + PY^2 + YS^2$.



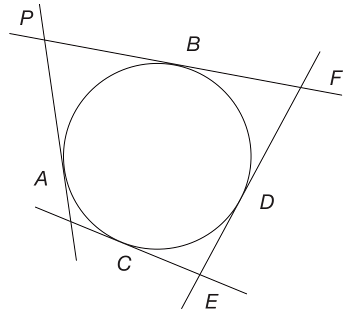
43. Find the locus of a point which is at a distance of 5 units from $(-1, -2)$.

44. In the following figure, PR and SQ are chords, of the circle with centre O, intersecting at T and $TQ = x$; $TP = y$.

Show that $(TS + TR) : (TS - TR) = (x + y) : (y - x)$.



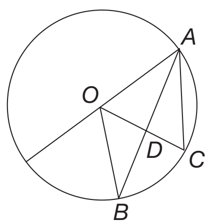
45. In the given figure, PA, PB, EC and ED are tangents to the circle. If PA = 13 cm, CE = 4.5 cm and FE = 9 cm, then find PF.



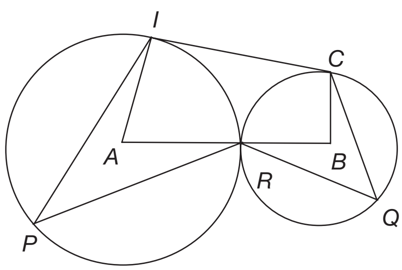
Essay Type Questions

46. EF and EH are the two chords of a circle with centre O intersecting at E. The diameter ED bisects the angle HEF. Show that the triangle FEH is an isosceles triangle.

47. In the following figure, O is the centre of the circle, AC are parallel lines. If $\angle ACO = 80°$, then find $\angle ADO$.

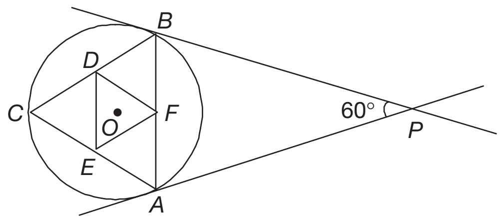


48.



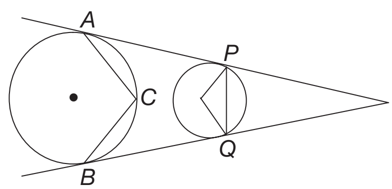
In the above figure, A and B are the centres of two circles, DC being the common tangent. If $\angle DPR$ =35°, then $\angle RQC$ =

49.



In the diagram given above, D, E and F are midpoints of BC, CA and AB. If the angle between the tangents drawn at A and B is 60°, find $\angle EFD$.

50.



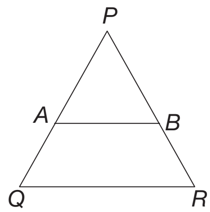
In the diagram above, A, B, P and Q are points of contacts of direct common tangents of the two circles. If $\angle ACB$ is 120°, then find the angle between the two tangents and angle made by PQ at the centre of same circle.

CONCEPT APPLICATION

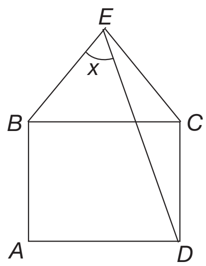
Level 1

1. In the triangle PQR, AB is parallel to QR. The ratio of the areas oftwo similar triangles PAB and PQR is 1 : 2. Then $PQ : AQ$ = ______.

(a) $\sqrt{2} : 1$ (b) $1 : \sqrt{2} - 1$
(c) $1 : (\sqrt{2} + 1)$ (d) $\sqrt{2} : \sqrt{2} - 1$



2. In the figure given below, equilateral triangle ECB surmounts square $ABCD$. Find the angle BED represented by x.



(a) 15° (b) 30°

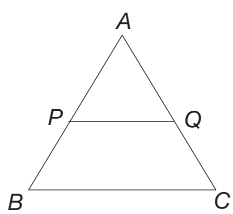
(c) 45° (d) 60°

3. In two triangles ABC and DEF, $\angle A = \angle D$. The sum of the angles A and B is equal to the sum of the angles D and E. If $BC = 6$ cm and $EF = 8$ cm, find the ratio of the areas of the triangles, ABC and DEF.

(a) 3 : 4 (b) 4 : 3

(c) 9 : 16 (d) 16 : 9

4. In the following figure, PQ is parallel to BC and $PQ : BC = 1 : 3$. If the area of the triangle ABC is 144 cm^2, then what is the area of the triangle APQ?



(a) 48 cm^2 (b) 36 cm^2

(c) 16 cm^2 (d) 9 cm^2

5. In triangle ABC, sides AB and AC are extended to D and E respectively, such that $AB = BD$ and $AC = CE$. Find DE, if $BC = 6$ cm.

(a) 3 cm (b) 6 cm

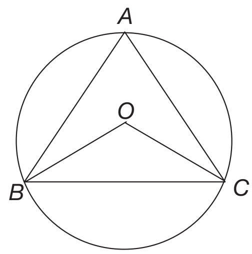
(c) 9 cm (d) 12 cm

6. A man travels on a bicycle, 10 km east from the starting point A to reach point B, then he cycles 15 km south to reach point C. Find the shortest distance between A and C.

(a) 25 km (b) 5 km

(c) $25\sqrt{13}$ km (d) $5\sqrt{13}$ km

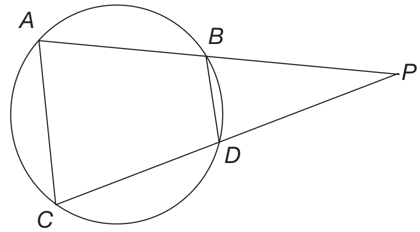
7. In the following figure, O is the centre of the circle. If $\angle BAC = 60°$, then $\angle OBC =$



(a) 120° (b) 30°

(c) 40° (d) 60°

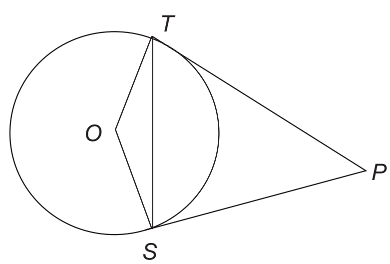
8. In the following figure (not to scale), $AB = CD$ and $\overline{AB}$ and $\overline{CD}$ are produced to meet at the point P. If $\angle BAC = 70°$, then find $\angle P$.



(a) 30° (b) 40°

(c) 45° (d) 50°

9.



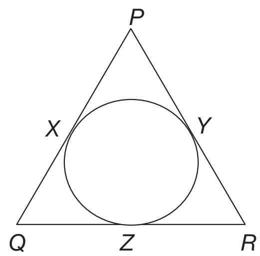
PT and PS are the tangents to the circle with centre O. If $\angle TPS = 65°$, then $\angle OTS =$

(a) 32° (b) 45°

(c) $57\frac{1}{2}°$ (d) $32\frac{1}{2}°$

10. In the following figure X, Y and Z are the points at which the incircle touches the sides of the triangle.

If $PX = 4$ cm, $QZ = 7$cm and $YR = 9$ cm, then the perimeter of triangle PQR is



(a) 20 cm (b) 46 cm

(c) 40 cm (d) 80 cm

11. The locus of the point P which is at a constant distance of 2 units from the origin and which lies in the first or the second quadrants is

(a) $y = -\sqrt{4-x^2}$ (b) $y = \sqrt{4-x^2}$

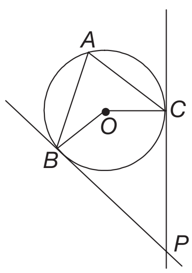
(c) $x = \sqrt{4-y^2}$ (d) $x = -\sqrt{4-y^2}$

12. If PAB is a triangle in which $\angle B = 90°$ and $A(1, 1)$ and $B(0, 1)$, then the locus of P is _______.

(a) $y = 0$ (b) $xy = 0$

(c) $x = y$ (d) $x = 0$

13. In the following figure, if the angle between two chords AB and AC is 65°, then the angle between two tangents which are drawn at B and C is _______.



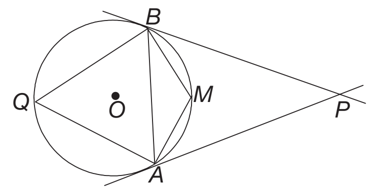
(a) 50° (b) 30°

(c) 60° (d) 40°

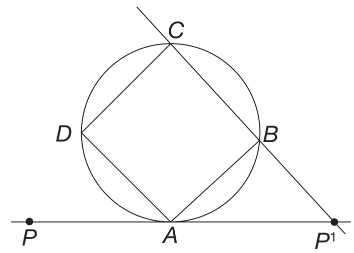
14. In the following figure, O is the centre of the circle and $\angle AMB = 120°$, Find the angle between the two tangents AP and BP.

(a) 30° (b) 45°

(c) 70° (d) 60°



15.

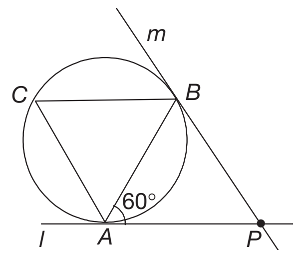


If $ABCD$ is a square inscribed in a circle and PA is a tangent, then the angle between the lines P^1A and P^1B is

(a) 30° (b) 20°

(c) 40° (d) 45°

16. In the following figure, if l and m are two tangents and AB is a chord making an angle of 60° with the tangent l, then the angle between l and m is



(a) 45° (b) 30°

(c) 60° (d) 90°

17. Find the length of a transverse common tangent of the two circles whose radii are 3.5 cm, 4.5 cm and the distance between their centres is 10 cm.

(a) 36 cm (b) 6 cm

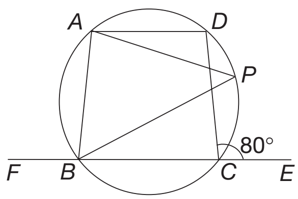
(c) 64 cm (d) 8 cm

18. If $ABCD$ is a trapezium, AC and BD are the diagonals intersecting each other at point O. Then $AC : BD =$

(a) $AB : CD$ (b) $AB + AD : DC + BC$

(c) $AO^2 : OB^2$ (d) $AO - OC : OB - OD$

19. In the following figure (not to scale), $\overline{PA}$ and $\overline{PB}$ are equal chords and $ABCD$ is a cyclic quadrilateral. If $\angle DCE = 80°$, $\angle DAP = 30°$ then find $\angle APB$.



(a) 40° (b) 80°

(c) 90° (d) 160°

20. In trapezium $KLMN$, KL and MN are parallel sides. A line is drawn, from the point A on KN, parallel to MN meeting LM at B. $KN : LM$ is equal to

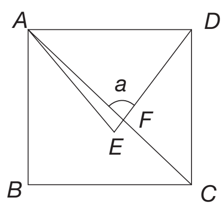
(a) $KL : NM$

(b) $(KL + KA) : (NM + BM)$

(c) $(KA - AN) : (LB - BM)$

(d) $KL^2 : MN^2$

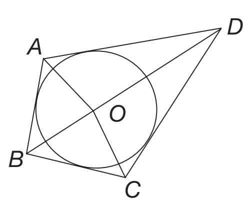
21. In the following figure, $ABCD$ is a square and AED is an equilateral triangle. Find the value of a.



(a) 30° (b) 45°

(c) 60° (d) 75°

22. A circle with centre O is inscribed in a quadrilateral $ABCD$ as shown in the figure. Which of the following statements is/are true?



(A) $\angle AOD + \angle BOC = 180°$

(B) $\angle AOB$ and $\angle COD$ are complementary

(C) OA, OB, OC and OD are the angle bisectors of $\angle A$, $\angle B$, $\angle C$ and $\angle D$ respectively.

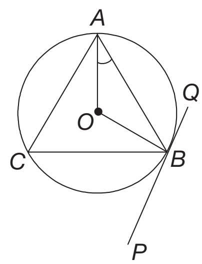
(a) Both (A) and (B)

(b) Both (B) and (C)

(c) Both (A) and (C)

(d) All the three

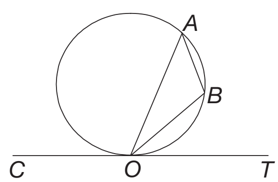
23. In the following figure, O is the centre of the circle and if $\angle OAB = 30°$, then the acute angle between AB and the tangent PQ at B is



(a) 30° (b) 60°

(c) 45° (d) 90°

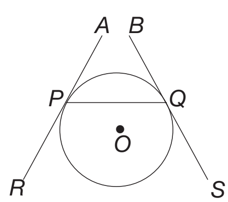
24. In the following figure, $AB = OB$ and CT is the tangent to the circle at O. If $\angle COA = 125°$, then $\angle OAB$ is



(a) 55° (b) $27\frac{1}{0}°$

(c) $82\frac{1}{2}°$ (d) 45°

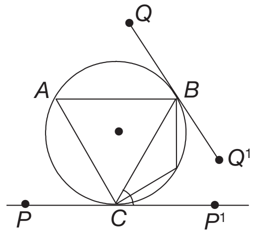
25. AR and BS are the tangents to the circle, with centre O, touching at P and Q respectively and PQ is the chord. If $\angle OQP = 25°$, then $\angle RPQ =$ ______.



(a) 100° (b) 115°

(c) 150° (d) 90°

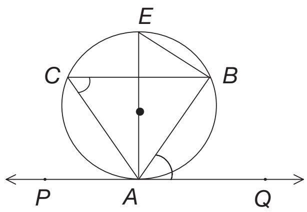
26. In the diagram, if $\angle BCP^1 = \angle ABQ = 60°$, then the triangle ABC is ______.



(a) scalene (b) equilateral

(c) right angled (d) acute angled

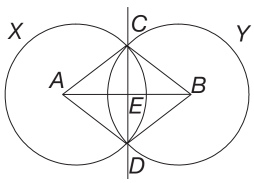
27. In the following figure, AQ is a tangent to the circle at A. If $\angle ACB = 60°$, then $\angle BAQ =$



(a) 30° (b) 60°

(c) 120° (d) 45°

28. In the following figure, two circles X and Y with centres A and B respectively intersect at C and D. The radii AC and AD of circle X are tangents to the circle Y. Radii BC and BD of circle Y are tangents to the circle X. Find $\angle AEC$.



(a) 45° (b) 60°

(c) 90° (d) Cannot be determined

29. The tangent AB touches a circle, with centre O, at the point P. If the radius of the circle is 5 cm, OB = 10 cm and $OB = AB$, then find AP.

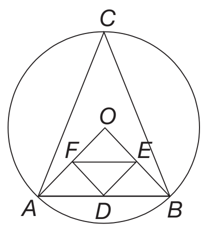
(a) $5\sqrt{5}$ cm

(b) $10\sqrt{5}$ cm

(c) $(10 - 5\sqrt{3})$ cm

(d) $\left(10 - \dfrac{5}{\sqrt{3}}\right)$ cm

30. In the following figure, O is the centre of the circle and D, E and F are mid-points of AB, BO and OA respectively. If $\angle DEF = 30°$, then find $\angle ACB$.

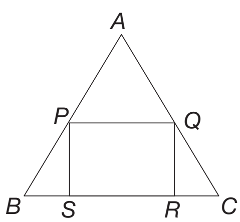


(a) 30° (b) 60°

(c) 90° (d) 120°

Level 2

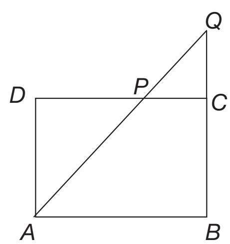
31. In the figure given below, ABC is an equilateral triangle and $PQRS$ is a square of side 6 cm. By how many cm^2 is the area of the triangle more than that of the square?



(a) $\dfrac{21}{\sqrt{3}}$ (b) 21

(c) $21\sqrt{3}$ (d) 63

32. In the following figure (not to scale) $ABCD$ is a rectangle, $BC = 24$ cm, $DP = 10$ cm and $CD = 15$ cm. Then, AQ and CQ respectively are



(a) 39 cm, 13 cm (b) 13 cm, 12 cm

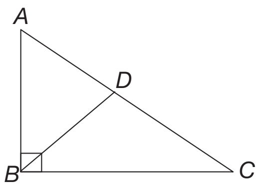
(c) 25 cm, 13 cm (d) 39 cm, 12 cm

33. At a particular time, the shadow cast by a tower is 6 m long. If the distance from top of the tower to the end of the shadow is 10 m long, determine the height of the tower.

(a) 4 m (b) 8 m

(c) 16 m (d) 12 m

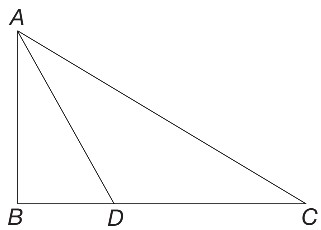
34. In the following figure, $\angle ABC = 90°$, $AD = 15$ and $DC = 20$. If BD is the bisector of $\angle ABC$, what is the perimeter of the triangle ABC?



(a) 74 (b) 84

(c) 91 (d) 105

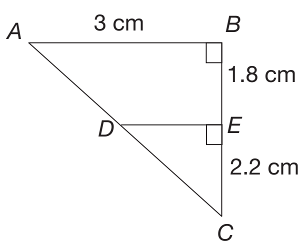
35. In the triangle ABC, $\angle ABC$ or $\angle B = 90°$. $AB : BD : DC = 3 : 1 : 3$. If $AC = 20$ cm, then what is the length of AD (in cm)?



(a) $5\sqrt{2}$ (b) $6\sqrt{3}$

(c) $4\sqrt{5}$ (d) $4\sqrt{10}$

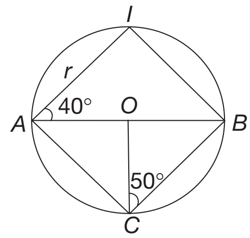
36. In the following figure, find AD.



(a) 1.8 cm (b) 2.25 cm

(c) 2.2 cm (d) 1.85 cm

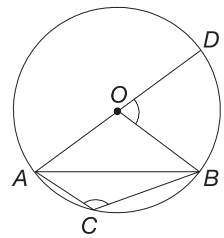
37. In the given figure, AB is a diameter, O is the centre of the circle and $\angle OCB = 50°$, then find $\angle DBC$.



(a) 80° (b) 100°

(c) 120° (d) 140°

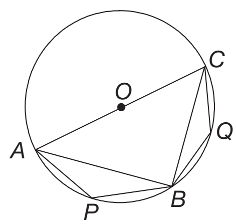
38. In the following figure, O is the centre of the circle and AD is the diameter. If $\angle ACB = 135°$, then find $\angle DOB$.



(a) 135° (b) 60°

(c) 90° (d) 45°

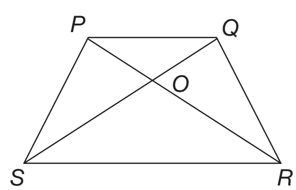
39. In the following figure, O is the centre of the circle, AC is the diameter and if $\angle APB = 120°$, then find $\angle BQC$.



(a) 30° (b) 150°

(c) 90° (d) 120°

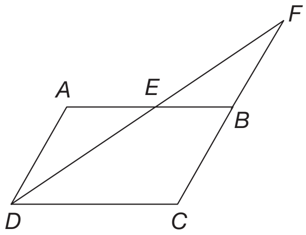
40. In the trapezium $PQRS$, PQ is parallel to RS and the ratio of the areas of the triangle POQ to triangle ROS is 225 : 900. Then $SR = ?$



(a) 30 PQ (b) 25 PQ

(c) 2 PQ (d) PQ

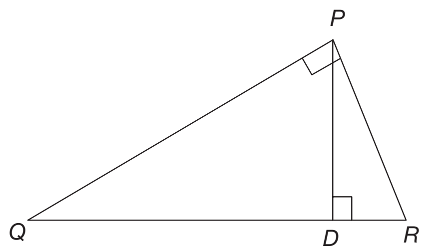
41. In the following figure, $ABCD$ is a parallelogram, CB is extended to F and the line joining D and F intersect AB at E. Then,



(a) $\frac{AD}{AE} = \frac{BF}{BE}$ (b) $\frac{AD}{AE} = \frac{CF}{CD}$

(c) $\frac{BF}{BE} = \frac{CF}{CD}$ (d) All of these

42.

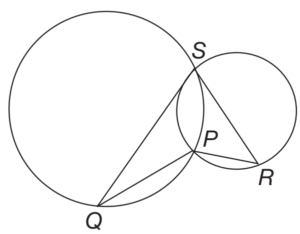


PQR is a right angled triangle, where $\angle P = 90°$. $\overline{PD}$ is perpendicular to $\overline{QR}$. $PQ : \sqrt{PR}$ =

(a) $QD : \sqrt{DR}$ (b) $\sqrt{QD} : \sqrt{DR}$

(c) $QD^2 : \sqrt{RD^2}$ (d) None of these

43. Two circles intersect at two points P and S. QR is a tangent to the two circles at Q and R. If $\angle QSR = 72°$, then $\angle QPR$ = _________.



(a) 84° (b) 96°

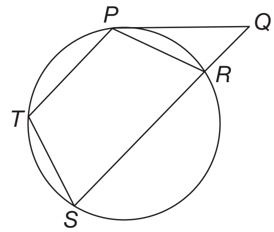
(c) 102° (d) 108°

44. An equilateral triangle CDE is constructed on a side CD of square $ABCD$. The measure of $\angle AEB$ can be _________.

(a) 150° (b) 45°

(c) 30° (d) 20°

45. In the figure above (not to scale), PQ is a tangent segment to the circle at P. If P, R, S and T are concyclic points, $\angle QPR = 40°$ and $PR = RQ$, then find $\angle PTS$.



(a) 80° (b) 100°

(c) 60° (d) 120°

46. Construct the incircle of a given triangle ABC. The following sentences are the steps involved in the above construction. Arrange them in sequencial order from first to left.

(A) Draw perpendicular $\overline{IM}$ from I onto $\overline{BC}$.

(B) Taking I as centre and IM as the radius, draw a circle.

(C) Draw a $DABC$ with the given measurements.

(D) Draw bisectors of two of the angles, say $\angle B$ and $\angle C$ to intersect at I.

(a) DCAB (b) CDAB

(c) CADB (d) DACB

47. Construct a regular pentagon in a circle of radius 4 cm. The following sentences are the steps involved in the following constructions. Arrange them in sequential order from first to last.

(A) With D as centre and AB as radius draw an arc which cuts the circle at the point C.

(B) With A as centre and AB as radius draw an arc which cuts the circle at the point E.

(C) Construct a circle with radius 4 cm.

(D) Join AB.

(E) Draw two radii $\overline{OA}$ and $\overline{OB}$ such that $\angle AOB = 72°$.

(F) Join AE, ED, DC and CB.

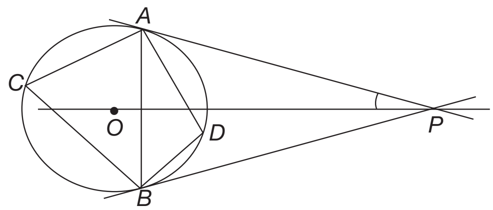
(G) With E as centre and AB as radius draw an arc which cuts the circle at the point D.

(a) CEDGBAF (b) CEBDGAF

(c) CEDBGAF (d) CEBGADF

Level 3

48. In the diagram, O is the centre of the circle and $\angle OPA = 30°$. Find $\angle ACB$ and $\angle ADB$ respectively.



(a) 120°, 60° (b) 60°, 120°

(c) 75°, 105° (d) 35°, 145°

49. Side of a square $PQRS$ is 4 cm long. $\overline{PR}$ is produced to the point M such that $PR = 2RM$. Find SM.

(a) $\sqrt{10}$ cm (b) $\sqrt{5}$ cm

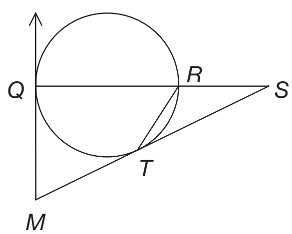
(c) $2\sqrt{5}$ (d) $2\sqrt{10}$ cm

50. ABC is an equilateral triangle of side 6 cm. If a circle of radius 1 cm is moving inside and along the sides of the triangle, then locus of the centre of the circle is an equilateral triangle of side __________.

(a) 5 cm (b) 4 cm

(c) $(6 - 2\sqrt{3})$ cm (d) $(3 + \sqrt{3})$ cm

51. In the following figure (not to scale), STM and MQ are tangents to the circle at T and Q respectively. SRQ is a straight line. $SR = TR$ and $\angle TSR = 25°$. Find $\angle QMT$.



(a) 55° (b) 60°

(c) 75° (d) 80°

52. PQ is the direct common tangent of two circles (S, 9 cm) and (R, 4 cm) which touch each other externally. Find the area of the quadrilateral $PQRS$. (in cm^2)

(a) 72 (b) 65

(c) 78 (d) 69

53. Diagonal AC of a rectangle $ABCD$ is produced to the point E such that $AC : CE = 2 : 1$. $AB = 8$ cm and $BC = 6$ cm. Find the length of DE.

(a) $2\sqrt{19}$ cm (b) 15 cm

(c) $3\sqrt{17}$ cm (d) 13 cm

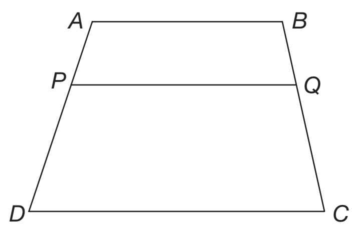
54. In ΔPQR, $PQ = 6$ cm, $PR = 9$ cm and M is a point on QR such that it divides QR in the ratio 1 : 2. $PM \perp QR$. Find QR.

(a) $\sqrt{18}$ cm (b) $3\sqrt{12}$ cm

(c) $3\sqrt{15}$ cm (d) $\sqrt{20}$ cm

55. In the following figure (not to scale), $ABCD$ is an isosceles trapezium. $\overline{AB} \parallel \overline{CD}$, $AB = 9$ cm and $CD = 12$ cm. $AP : PD = BQ : QC = 1 : 2$.

Find PQ.



(a) 11 cm (b) 10.5 cm

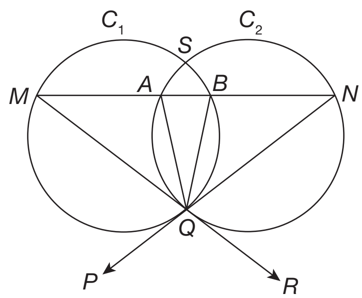
(c) 10 cm (d) 9.5 cm

56. P, Q and R are on AB, BC and AC of the equilateral triangle ABC respectively. $AP : PB = CQ : QB = 1 : 2$. G is the centroid of the triangle PQB and R is the mid-point of AC. Find $BG : GR$.

(a) 1 : 2 (b) 2 : 3

(c) 3 : 4 (d) 4 : 5

57. In the given figure (not to scale), two circles C_1 and C_2 intersect at S and Q. PQN and RQM are tangents drawn to C_1 and C_2 respectively at Q. MAB and ABN are the chords of the circles C_1 and C_2. If $\angle NQR = 85°$, then find $\angle AQB$.



(a) 5°　　(b) 10°

(c) 15°　　(d) Cannot be determined

58. Two sides of a triangle are 5 cm and 12 cm long. The measure of third side is an integer in cm. If the triangle is an obtuse triangle, then how many such triangles are possible?

(a) 9　　(b) 8

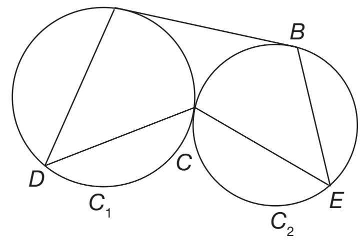
(c) 7　　(d) 6

59. In a ΔPQR, M lies on PR and between P and R such that $QR = QM = PM$. If $\angle MQR = 40°$, then find $\angle P$.

(a) 35°　　(b) 25°

(c) 45°　　(d) 55°

60. In the following figure, (not to scale), AB is the common tangent to the circles C_1 and C_2. C_1 and C_2 are touching externally at C. AD and DC are two chords of the circle C_1 and BE and CE are two chords of the circle C_2. Find the measure of $\angle ADC + \angle BEC$.



(a) 60°　　(b) 90°

(c) 120°　　(d) Cannot be determined

TEST YOUR CONCEPTS

Very Short Answer Type Questions

1. False
2. Agree
3. True
4. Yes
5. False
6. 2 and 3.
7. False
8. *RS*
9. zero
10. $\frac{32}{3}$ cm
11. True
12. 8 cm
13. Yes
14. 10 cm
15. 2
16. 2
17. 8 cm
18. 1
19. $\angle RPT$
20. 12 cm
21. False
22. intersect at two distinct points
23. $2\sqrt{r_1 r_2}$
24. 120°
25. oblique sides
26. rectangle
27. True
28. circle
29. True
30. 8 cm

Short Answer Type Questions

31. 2 units
32. 400 cm^2.
34. 26 cm (approximately).
35. $x^2 - y^2$
37. 4 cm
38. $5\sqrt{2}$
39. 26 m.
40. 25 cm
43. $x^2 + y^2 + 2x + 4y = 0$.
45. 17.5 cm

Essay Type Questions

47. 120°
48. 55°
49. 60°
50. 60°, 120°

CONCEPT APPLICATION

Level 1

1. (d)	2. (c)	3. (c)	4. (c)	5. (d)	6. (d)	7. (b)	8. (b)	9. (d)	10. (c)
11. (b)	12. (d)	13. (a)	14. (d)	15. (d)	16. (c)	17. (b)	18. (d)	19. (b)	20. (c)
21. (d)	22. (c)	23. (b)	24. (b)	25. (b)	26. (b)	27. (b)	28. (c)	29. (c)	30. (b)

Level 2

31. (c)	**32.** (d)	**33.** (b)	**34.** (b)	**35.** (d)	**36.** (b)	**37.** (b)	**38.** (c)	**39.** (b)	**40.** (c)
41. (d)	**42.** (b)	**43.** (d)	**44.** (c)	**45.** (b)	**46.** (b)	**49.** (c)			

Level 3

48. (b)	**49.** (d)	**50.** (c)	**51.** (d)	**52.** (c)	**53.** (c)	**54.** (c)	**55.** (c)	**56.** (d)	**57.** (b)
58. (d)	**59**. (a)	**60**. (b)							

CONCEPT APPLICATION

Level 1

1. The ratio of areas of two similar triangles is equal to the ratio of the squares of their corresponding sides.
2. $BC = CD = EC$ and proceed using $\angle ECD = 90° + 60° = 150°$.
3. The ratio of areas of two similar triangles is equal to the ratio of the squares of their corresponding sides.
4. The ratio of areas of two similar trianglesis equal to the ratio of the squares of their corresponding sides.
5. Apply Basic proportionality theorem.
6. Apply Pythagoras theorem.
7. $\angle BOC = 2\angle BAC$ and proceed.
8. Exterior angle of a cyclic quadrilateral is equal to its interior opposite angle.

   $\angle BAC = \angle DCA$ and proceed.
9. Radius of a circle is perpendicular to the tangent at the point of contact and tangents drawn to a circle from an external point are equal.
10. Tangents drawn to a circle from an external point are equal.
11. Recall the definition of locus.

    In the first or the second quadrant, y is positive.
12. Apply Pythagoras theorem.
13. Radius is perpendicular to the tangent at the point of contact.

    In the quadrilateral $BOCP$, $\angle BOC + \angle BPC = 180°$.
14. Recall the properties of cyclic quadrilateral and also of tangents.

    $\angle BOA + \angle BPA = 180°$ and $\angle BOA = 2\angle BQA$.
15. Apply 'Alternate segment theorem'.
16. Tangents drawn to a circle from an external point are equal.
17. Length of the transverse common tangent

    $= \sqrt{d^2 - (R+r)^2}$
18. Diagonals of a trapezium divide each other proportionally.
19. Recall the properties of cyclic quadrilateral.

    $\angle PAB = \angle PBA$ and $\angle DAB = \angle DCE$.
20. Apply BPT and use componondo–dividendo after drawing the complete figure.
21. Diagonal of a square bisects the angle at the vertices.

    $\angle FDC = 30°$ and $\angle FCD = 45°$.
22. Evaluate the solution from the options.
23. Apply 'Alternate segment theorem'.
24. Recall 'Alternate segment theorem'.
25. Radius is perpendicular to the tangent at the point of contact.
26. Apply 'Alternate segment theorem'.
27. Apply 'Alternate segment theorem'.
28. Recall the properties of a kite.
29. Apply Pythagoras theorem.
30. (i) $ADEF$ is a parallelogram.

    (ii) $\angle FAD = 30°$ and

    $\angle OAD = \angle OBA$

    (angles opposite to equal sides).

Level 2

31. (i) In triangle PBS, $\angle B = 60°$.

    $\therefore \angle P = 30°$ and $\angle S = 90°$.

    (ii) The sides of the triangle PBS, i.e., BS, SP and PB are in the ratio $1 : \sqrt{3} : 2$.

    (iii) Given $PS = 6$ cm.
32. (i) Use Pythagoras theorem to find AP.

    (ii) Triangle QAB and triangle QPC are similar.

    $\therefore \dfrac{QP}{PA} = \dfrac{QC}{CB}$.

33. (i) Apply 'Pythagoras theorem'.

(ii) The tower is perpendicular to the surface of the ground.

34. (i) Apply 'Angle bisector theorem'.

(ii) $AB : BC = AD : DC$.

(iii) Apply Pythagoras theorem, and find AB and BC.

35. (i) Apply 'Pythagoras theorem' for triangle ABC.

(ii) Let $AB = 3x$, $BD = x$ and $CD = 3x$.

(iii) First find AB and BC using $AB^2 + BC^2 = AC^2$.

(iv) Then find AD by using $AD^2 = AB^2 + BD^2$.

36. (i) Apply 'Basic proportionality theorem'.

(ii) Apply Pythagoras theorem to find AC.

(iii) Then apply basic proportionality theorem, i.e., $CE/EB = CD/DA$.

37. (i) In triangle OBC, angles opposite to equal sides are equal.

(ii) $\angle ADB = \angle ACB = 90°$ (Since angle in a semi circle is 90°).

(iii) $\angle OCB = \angle OBC$ and $\angle OAC = \angle OCA$ (Angles opposite to equal sides).

38. (i) $ACBD$ is a cyclic quadrilateral.

(ii) $\angle ACB$ and $\angle ADB$ are supplementary angles.

(iii) $\angle AOB = 2\angle ADB$.

39. (i) $APBC$ is a cyclic quadrilateral.

(ii) $\angle ABC$ is an angle in a semi circle.

(iii) $ABQC$ is a cyclic quadrilateral.

40. (i) POQ and ROS are similar triangles.

(ii) SR and PQ are proportional to the square roots of the areas of similar triangles SOR and POQ.

41. (i) Triangles FEB and FDC are similar.

(ii) Triangles AED and EFB are similar.

42. (i) Triangle PDR, QDP and QPR are similar.

(ii) Corresponding sides of similar triangles are proportional.

(iii) $\Delta PDQ \sim \Delta RDP$.

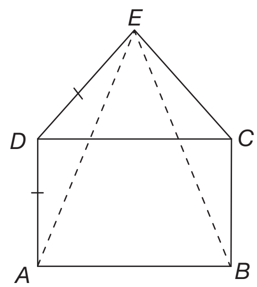
43. (i) Apply 'Alternate segment theorem'.

(ii) Join QR and join PS.

$\angle PQR = \angle PSQ$ and

$\angle PRQ = \angle PSR$ (By alternate segment theorem)

44.



$\angle ADE = \angle ADC + \angle CDE = 90° + 60°$ ($\because$ Angles in a square and equilateral triangle) $= 150°$

In ΔADE, $AD = DE$

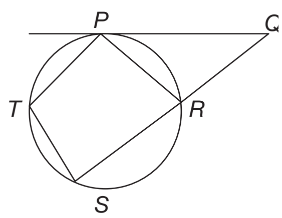
$\therefore \angle DAE = \angle DEA = \frac{1}{2}(180° - 150°)$

$= 15°$.

Similarly $\angle CEB = 15°$

$\therefore \angle AEB = 60° - (\angle DEA + \angle CEB) = 60° - (15° + 15°) = 30°$.

45.



Given $PR = RQ$ and $\angle Q = 40°$

$\Rightarrow \angle RPQ = \angle Q$

$\therefore \angle RPQ = 40°$

In ΔPQR,

$\angle PRQ = 180° - (40° + 40°) = 100°$.

We have, exterior angle of cyclic quadrilateralis equal to its interior opposite angle.

$\therefore \angle PTS = \angle PRQ$ ($\because PRST$ is a cyclic quadrilateral)

$\therefore \angle PTS = 100°$.

46. CDAB is the required sequential order.

47. CEDBGAF is the required sequential order.

Level 3

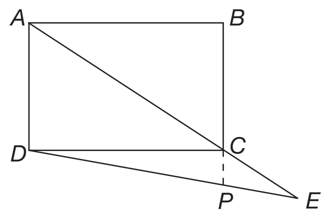
48. (i) Tangents drawn from an external point to the same circle are equal, i.e., $PA = PB$.

(ii) $\angle APO = \angle OPB$ and $\angle AOB + \angle APB = 180°$.

(iii) $\angle ACB = \frac{1}{2}\angle AOB$.

(iv) $ACBD$ is a cyclic quadrilateral.

53.



Given $AB = 8$ cm and $BC = 6$ cm

$\therefore\ AC = \sqrt{8^2 + 6^2} = 10$ cm

And also given $AC : CE = 2 : 1$

Produce BC to meet DE at the point P

As CP is parallel to AD,

$\Delta ECP \sim \Delta EAD$ (1)

$$\therefore\ \frac{CP}{AD} = \frac{CE}{AE} = \frac{CP}{6} = \frac{1}{3}$$

$\Rightarrow$ CP = 2 cm.

ΔCPD is a right triangle.

$\therefore\ DP = \sqrt{CD^2 + CP^2} = \sqrt{68} = 2\sqrt{17}$ cm

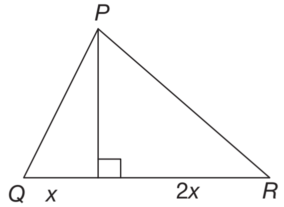
But $DP = PE = 2 : 1$ (from (1))

$PE = \sqrt{17}$ cm

$\therefore\ DE = DP + PE$

$= 2\sqrt{17} + \sqrt{17} = 3\sqrt{17}$ cm.

54.



Given $PQ = 6$ cm, PR 9 cm and $QM : MR = x : 2x$

Let $QM = x$ and $MR = 2x$.

As PM is perpendicular QR, ΔPMQ and ΔPMR are right triangles.

$\therefore\ (PM)^2 = (OQ)^2 - (QM)^2$

$(PM)^2 = (PR)^2 - (MR)^2$ (2)

From (1) and (2), we $(PQ)^2 - (QM)^2 = (PR)^2 - (MR)^2$

$(6)^2 - (x)^2 = (9)^2 - (2x)^2$

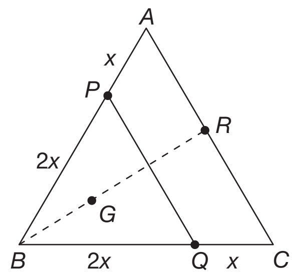
$\Rightarrow\ 3x^2 = 45 \ \Rightarrow\ x^2 = 15 \ \Rightarrow\ x = 15$ cm

$\therefore$ QR $= 3\sqrt{15}$ cm.

55. $PQ = \frac{1}{3}(CD) + \frac{2}{3}(AB)$

$= \frac{1}{3}(12) + \frac{2}{3}(9) = 10$ cm.

56.



Let $AB = BC = AC = 3x$

$\therefore\ BP = BQ = PQ = 2x$

($\because\ AP : BP = CQ : BQ = 1 : 2$)

As ΔBPQ and ΔBAC are equilateral triangle, the centroid of ΔBPQ lies on BR (where BR is median drawn on to AC)

We know that centroid divides the median in the ratio 2 : 1.

$$\therefore BG = \frac{2}{3}\frac{\left[\sqrt{3}(2x)\right]}{2} = \frac{2\sqrt{3}x}{3}$$

But $BR = \frac{\sqrt{3}(3x)}{2} = \frac{3\sqrt{3}x}{2}$

Now $GR = BR - BG$

$$= \frac{3\sqrt{3}x}{2} - \frac{2\sqrt{3}x}{3} = \frac{5\sqrt{3}x}{6}$$

Now $BG : GR = \frac{2\sqrt{3}x}{3} : \frac{5\sqrt{3}x}{6} = 4 : 5.$

57. Given $\angle NQR = 85°$

But $\angle NQR = \angle MQP$ ($\because$ Vertically opposite angles)

$\therefore \angle MQP = 85°$

By alternate segment theorem $\angle NQR = \angle QAB$

$\therefore \angle QAB = 85°$ and $\angle MQP = \angle QBA$

$\therefore \angle QBA = 85°$

In ΔAQB, $\angle AQB = 180° - (85° + 85°) = 10°$.

58. Two sides of *a* triangle 5 cm and 12 cm.

Let $a = 5$ cm and $b = 12$ cm

Let the third side be x cm

$\therefore 12 - 5 < x < 12 + 5$

$\Rightarrow \quad 7 < x < 17$

$\therefore$ Possible integer values for x are 8, 9, 10, 11, 12, 13, 14, 15 and 16.

Case 1: If b is the longest side then $b^2 > a^2 + x^2$

$\Rightarrow \quad 12^2 > 5^2 + x^2$

$\Rightarrow \quad 144 - 25 > x^2$

$\Rightarrow \quad x^2 < 119$

$\therefore x$ can be 8, 9 or 10.

Case 2: If x is the longest side, then $x^2 > a^2 + b^2$

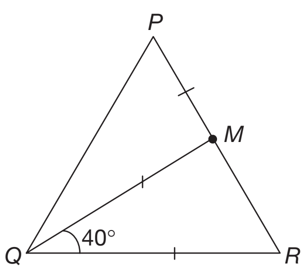
$\Rightarrow \quad x^2 > 5^2 + 12^2 \quad \Rightarrow \quad x^2 \quad \Rightarrow \quad 169$

$\therefore x$ can be 14, 15 or 16.

$\therefore$ Number of possible triangles

$= 6$ ($\because$ The measurement third side is an integer in cm).

59.



Given $QR = QM = PM$ and $\angle MQR = 40°$

In ΔQMR, $QM = QR$

$\therefore \angle QRM = \angle QMR$

Now $\angle QRM = \angle QMR$

$= \frac{1}{2}(180° - 40°) = 70°$

In ΔMPQ, $PM = MQ$

$\angle PQM = \angle MPQ \qquad (1)$

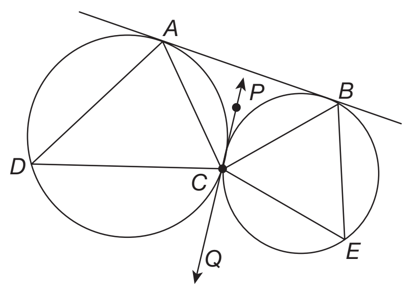
But $\angle PQM + \angle MPQ = \angle QMR$

$= 70° \qquad (2)$

From Eq. (1) and (2)

$\angle MPQ = \frac{1}{2}(70°) = 35°$.

60.



Join AC and BC

Draw a common tangent to the circles through the point C.

Let

$\angle ADC = x$ and $\angle BEC = y \qquad (1)$

By alternate segment theorem, $\angle ADC = \angle BAC$,

$\angle ADC = \angle ACP$

$\therefore \angle BAC = \angle ACP = X$

and also $\angle BEC = \angle CBA$, $\angle BEC = \angle BCP$

$\therefore \angle CBA = \angle BCP = y$

In ΔABC, $\angle BAC + \angle CBA + \angle ACB = 180°$

$x + y + (x + y) = 180° \quad \Rightarrow \quad x + y = 90° \qquad (2)$

From (1) and (2), we have

$\angle ADC + \angle BEC = 90°$.