JEE Type Solved Examples :
Single Option Correct Type Questions

- This section contains **10 multiple choice examples.** Each example has four choices (a), (b), (c) and (d) out of which **ONLY ONE** is correct.

● **Ex. 1** *If A is a square matrix of order 2 such that $A\begin{bmatrix}1\\-1\end{bmatrix}=\begin{bmatrix}-1\\2\end{bmatrix}$ and $A^2\begin{bmatrix}1\\-1\end{bmatrix}=\begin{bmatrix}1\\0\end{bmatrix}$. The sum of elements and product of elements of A are S and P, then $S+P$ is*

(a) -1 (b) 2 (c) 4 (d) 5

Sol. (*d*) Let $A=\begin{bmatrix}a & b\\c & d\end{bmatrix}$

From first part, $A\begin{bmatrix}1\\-1\end{bmatrix}=\begin{bmatrix}-1\\2\end{bmatrix}$ ...(i)

$\Rightarrow \begin{bmatrix}a & b\\c & d\end{bmatrix}\begin{bmatrix}1\\-1\end{bmatrix}=\begin{bmatrix}-1\\2\end{bmatrix}$

or $a-b=-1$ ...(ii)

and $c-d=2$ ...(iii)

From second part,

$$A^2\begin{bmatrix}1\\-1\end{bmatrix}=\begin{bmatrix}1\\0\end{bmatrix} \Rightarrow A\left(A\begin{bmatrix}1\\-1\end{bmatrix}\right)=\begin{bmatrix}1\\0\end{bmatrix}$$

From Eq. (i), we get

$$A\begin{bmatrix}-1\\2\end{bmatrix}=\begin{bmatrix}1\\0\end{bmatrix} \Rightarrow \begin{bmatrix}a & b\\c & d\end{bmatrix}\begin{bmatrix}-1\\2\end{bmatrix}=\begin{bmatrix}1\\0\end{bmatrix}$$

or $-a+2b=1$ ...(iv)

and $-c+2d=0$ ...(v)

From Eqs. (ii) and (iv), we get

$$a=-1,\ b=0$$

and from Eqs. (iii) and (v), we get

$$c=4,\ d=2$$

$\therefore$ $S=a+b+c+d=5$

and $P=abcd=0$

Hence, $S+P=5$

● **Ex. 2** *If P is an orthogonal matrix and $Q=PAP^T$ and $B=P^TQ^{1000}P$, then B^{-1} is, where A is involutory matrix*

(a) A (b) A^{1000} (c) I (d) None of these

Sol. (*c*) Given, P is orthogonal

$\therefore$ $P^TP=I$ ...(i)

and $Q=PAP^T$ ...(ii)

Now, $B=P^TQ^{1000}P=P^T(PAP^T)^{1000}P$ [from Eq. (ii)]

$=P^TPAP^T(PAP^T)^{999}P$

$=IAP^T\cdot PAP^T(PAP^T)^{998}P$

$=AIAP^T(PAP^T)^{998}P$

$=A^2P^TPAP^T(PAP^T)^{997}P$

$=A^2IAP^T(PAP^T)^{997}P$

$=A^3P^T(PAP^T)^{997}P$

... ...

$=A^{1000}P^TP=A^{1000}=I$ [$\because A$ is involutory]

Hence, $B^{-1}=I^{-1}=I$

● **Ex. 3** *If A is a diagonal matrix of order 3×3 is commutative with every square matrix of order 3×3 under multiplication and trace $(A)=12$, then*

(a) $|A|=64$ (b) $|A|=16$

(c) $|A|=12$ (d) $|A|=4$

Sol. (*a*) A diagonal matrix is commutative with every square matrix, if it is scalar matrix so every diagonal element is 4.

$$\therefore \quad A=\begin{bmatrix}4 & 0 & 0\\0 & 4 & 0\\0 & 0 & 4\end{bmatrix}$$

$$\Rightarrow \quad |A|=4\cdot4\cdot4=64$$

● **Ex. 4** *If $A=[a_{ij}]_{4\times4}$, such that $a_{ij}=\begin{cases}2, & \text{when } i=j\\0, & \text{when } i\neq j\end{cases}$, then $\left\{\dfrac{\det(\operatorname{adj}(\operatorname{adj}A))}{7}\right\}$ is [when $\{\cdot\}$ represents fractional part function]*

(a) $\frac{1}{7}$ (b) $\frac{2}{7}$ (c) $\frac{3}{7}$ (d) $\frac{4}{7}$

Sol. (*a*) $\because$ $A=\begin{bmatrix}2 & 0 & 0 & 0\\0 & 2 & 0 & 0\\0 & 0 & 2 & 0\\0 & 0 & 0 & 2\end{bmatrix}$

$$\therefore \quad |A|=\begin{vmatrix}2 & 0 & 0 & 0\\0 & 2 & 0 & 0\\0 & 0 & 2 & 0\\0 & 0 & 0 & 2\end{vmatrix}=2^4=16$$

$\because$ $\det(\operatorname{adj}(\operatorname{adj}A))=|\operatorname{adj}(\operatorname{adj}A)|=|A|^{3^2}=|A|^9$

$=(2^4)^9=2^{36}=(2^3)^{12}=(1+7)^{12}$

$=1+{}^{12}C_1(7)+{}^{12}C_2(7)^2+\dots$

$$\frac{\det(\operatorname{adj}(\operatorname{adj}A))}{7}=\frac{1}{7}+\text{Positive integer}$$

$$\therefore \quad \left\{\frac{\det(\operatorname{adj}(\operatorname{adj}A))}{7}\right\}=\frac{1}{7}$$

Ex. 5 *If* $A=\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}$ *and* $\det(A^n - I) = 1-\lambda^n$, $n \in N$, *then the value of* λ, *is*

(a) 1 (b) 2 (c) 3 (d) 4

Sol. (*b*) $\because$ $A=\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}$

$\therefore$ $A^2=\begin{bmatrix}2 & 2\\2 & 2\end{bmatrix}=2\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}=2A$

$\Rightarrow$ $A^3 = A^2\cdot A = 2A^2 = 2^2 A$

Similarly, $A^n = 2^{n-1}A$

$\therefore$ $A^n - I = \begin{bmatrix}2^{n-1} & 2^{n-1}\\2^{n-1} & 2^{n-1}\end{bmatrix}-\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$

$= \begin{bmatrix}2^{n-1}-1 & 2^{n-1}\\2^{n-1} & 2^{n-1}-1\end{bmatrix}$

$\Rightarrow$ $\det(A^n - I) = (2^{n-1}-1)^2 - (2^{n-1})^2$

$= 1 - 2^n = 1 - \lambda^n$ [given]

$\therefore$ $\lambda = 2$

Ex. 6 *If* $A=\begin{bmatrix}1 & 2\\2 & 1\end{bmatrix}$ *and* $f(x)=\dfrac{1+x}{1-x}$, *then* $f(A)$ *is*

(a) $\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}$ (b) $\begin{bmatrix}2 & 2\\2 & 2\end{bmatrix}$

(c) $\begin{bmatrix}-1 & -1\\-1 & -1\end{bmatrix}$ (d) None of these

Sol. (*c*) $\because$ $f(x)=\dfrac{1+x}{1-x}$

$\Rightarrow$ $(1-x)f(x) = 1+x$

$\Rightarrow$ $(I-A)f(A) = (I+A)$

$\Rightarrow$ $f(A) = (I-A)^{-1}(I+A)$

$= \left(\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}-\begin{bmatrix}1 & 2\\2 & 1\end{bmatrix}\right)^{-1}\left(\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}+\begin{bmatrix}1 & 2\\2 & 1\end{bmatrix}\right)$

$= \begin{bmatrix}0 & -2\\-2 & 0\end{bmatrix}^{-1}\begin{bmatrix}2 & 2\\2 & 2\end{bmatrix} = -\dfrac{1}{4}\begin{bmatrix}0 & 2\\2 & 0\end{bmatrix}\begin{bmatrix}2 & 2\\2 & 2\end{bmatrix}$

$= -\dfrac{1}{4}\begin{bmatrix}4 & 4\\4 & 4\end{bmatrix}=\begin{bmatrix}-1 & -1\\-1 & -1\end{bmatrix}$

Ex. 7 *The number of solutions of the matrix equation* $X^2=\begin{bmatrix}1 & 1\\2 & 3\end{bmatrix}$ *is*

(a) more than 2 (b) 2

(c) 0 (d) 1

Sol. (*a*) Let $X=\begin{bmatrix}a & b\\c & d\end{bmatrix}$

$\therefore$ $X^2=\begin{bmatrix}a^2+bc & b(a+d)\\c(a+d) & bc+d^2\end{bmatrix}=\begin{bmatrix}1 & 1\\2 & 3\end{bmatrix}$ [given]

$\Rightarrow$ $a^2+bc=1, b(a+d)=1,$

$c(a+d)=2$ and $bc+d^2=3$

$\Rightarrow$ $d^2-a^2=2$

$\Rightarrow$ $d-a=\dfrac{2}{d+a}=2b$ and $d+a=\dfrac{1}{b}$

$\therefore$ $2d=2b+\dfrac{1}{b}$ and $2a=\dfrac{1}{b}-2b$

Also, $c=2b$

Now, from $bc+d^2=3$

$\Rightarrow$ $2b^2+\left(b+\dfrac{1}{2b}\right)^2=3 \Rightarrow 3b^2+\dfrac{1}{4b^2}-2=0$

$\Rightarrow$ $12b^4-8b^2+1=0$

or $(6b^2-1)(2b^2-1)=0$

$\Rightarrow$ $b=\pm\dfrac{1}{\sqrt{6}}$ or $b=\pm\dfrac{1}{\sqrt{2}}$

Therefore, matrices are

$$\begin{bmatrix}0 & \dfrac{1}{\sqrt{2}}\\\sqrt{2} & \sqrt{2}\end{bmatrix},\begin{bmatrix}0 & -\dfrac{1}{\sqrt{2}}\\-\sqrt{2} & -\sqrt{2}\end{bmatrix},\begin{bmatrix}\dfrac{2}{\sqrt{6}} & \dfrac{1}{\sqrt{6}}\\\dfrac{2}{\sqrt{6}} & \dfrac{4}{\sqrt{6}}\end{bmatrix}\text{ and }\begin{bmatrix}-\dfrac{2}{\sqrt{6}} & -\dfrac{1}{\sqrt{6}}\\-\dfrac{2}{\sqrt{6}} & -\dfrac{4}{\sqrt{6}}\end{bmatrix}.$$

Ex. 8 *For a matrix* $A=\begin{bmatrix}1 & 2r-1\\0 & 1\end{bmatrix}$, *the value of* $\prod_{r=1}^{50}\begin{bmatrix}1 & 2r-1\\0 & 1\end{bmatrix}$ *is equal to*

(a) $\begin{bmatrix}1 & 100\\0 & 1\end{bmatrix}$ (b) $\begin{bmatrix}1 & 4950\\0 & 1\end{bmatrix}$ (c) $\begin{bmatrix}1 & 5050\\0 & 1\end{bmatrix}$ (d) $\begin{bmatrix}1 & 2500\\0 & 1\end{bmatrix}$

Sol. (*d*) $\prod_{r=1}^{50}\begin{bmatrix}1 & 2r-1\\0 & 1\end{bmatrix}=\begin{bmatrix}1 & 1+3+5+\ldots+99\\0 & 1\end{bmatrix}$

$=\begin{bmatrix}1 & (50)^2\\0 & 1\end{bmatrix}=\begin{bmatrix}1 & 2500\\0 & 1\end{bmatrix}$

Ex. 9 *If* $A_1, A_2, A_3, \ldots, A_{2n-1}$ *are n skew-symmetric matrices of same order, then* $B=\sum_{r=1}^{n}(2r-1)(A_{2r-1})^{2r-1}$ *will be*

(a) symmetric
(b) skew-symmetric
(c) neither symmetric nor skew-symmetric
(d) data not adequate

Sol. (*b*) $\because$ $B = A_1 + 3A_3^3 + 5A_5^5 + \ldots + (2n-1)(A_{2n-1})^{2n-1}$

$\therefore$ $B^T = (A_1 + 3A_3^3 + 5A_5^5 + \ldots + (2n-1)(A_{2n-1})^{2n-1})^T$

$= A_1^T + 3(A_3^T)^3 + 5(A_5^T)^5 + \ldots + (2n-1)(A_{2n-1}^T)^{2n-1}$

$= -A_1 + 3(-A_3)^3 + 5(-A_5)^5 + \ldots + (2n-1)(-A_{2n-1})^{2n-1}$

$= -(A_1 + 3A_3^3 + 5A_5^5 + \ldots + (2n-1)A_{2n-1}^{2n-1})$

$= -B$

Hence, B is skew-symmetric.

● **Ex. 10** *Elements of a matrix A of order 10×10 are defined as $a_{ij} = \omega^{i+j}$ (where ω is cube root of unity), then trace (A) of the matrix is*

(a) 0 (b) 1 (c) 3 (d) None of these

Sol. (*d*) $\operatorname{tr}(A) = \sum_{i=j=1}^{10} a_{ij} = \sum_{i=j=1}^{10} \omega^{i+j} = \sum_{i=1}^{10} \omega^{2i}$

$= \omega^2 + \omega^4 + \omega^6 + \omega^8 + \ldots + \omega^{20}$

$= (\omega^2 + \omega + 1) + (\omega^2 + \omega + 1) + (\omega^2 + \omega + 1) + \omega^{20}$

$= 0 + 0 + 0 + \omega^2 = \omega^2$

JEE Type Solved Examples :
More than One Correct Option Type Questions

- This section contains **5 multiple choice examples.** Each example has four choices (a), (b), (c) and (d) out of **which more than one** may be correct.

● **Ex. 11** *If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ (where $bc \neq 0$) satisfies the equations $x^2 + k = 0$, then*

(a) $a + d = 0$ (b) $k = -|A|$

(c) $k = |A|$ (d) None of these

Sol. (*a, c*) We have, $A^2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a^2 + bc & ab + bd \\ ac + cd & bc + d^2 \end{bmatrix}$

As A satisfies $x^2 + k = 0$, therefore

$$A^2 + kI = 0$$

$\Rightarrow$ $\begin{bmatrix} a^2 + bc + k & (a+d)b \\ (a+d)c & bc + d^2 + k \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\Rightarrow$ $a^2 + bc + k = 0, (a+d)b = 0,$

$(a+d)c = 0$ and $bc + d^2 + k = 0$

As $bc \neq 0 \Rightarrow b \neq 0, c \neq 0$

So, $a + d = 0 \Rightarrow a = -d$

Also, $k = -(a^2 + bc) = -(-ad + bc) = (ad - bc) = |A|$

● **Ex. 12** *If $A = [a_{ij}]_{n \times n}$ and f is a function, we define $f(A) = [f(a_{ij})]_{n \times n}$. Let $A = \begin{bmatrix} \frac{\pi}{2} - \theta & \theta \\ -\theta & \frac{\pi}{2} - \theta \end{bmatrix}$, then*

(a) $\sin A$ is invertible (b) $\sin A = \cos A$

(c) $\sin A$ is orthogonal (d) $\sin 2A = 2 \sin A \cos A$

Sol. (*a, c*) $\sin A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ and $\cos\theta = \begin{bmatrix} \sin\theta & \cos\theta \\ \cos\theta & \sin\theta \end{bmatrix}$

$\therefore$ $|\sin A| = \cos^2\theta + \sin^2\theta = 1 \neq 0$

Hence, $\sin A$ is invertible.

Also, $(\sin A)(\sin A)^T = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

Hence, $\sin A$ is orthogonal.

Also, $2 \sin A \cos A = 2\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} \sin\theta & \cos\theta \\ \cos\theta & \sin\theta \end{bmatrix}$

$= 2\begin{bmatrix} \sin 2\theta & 1 \\ \cos 2\theta & 0 \end{bmatrix} \neq \sin 2A$

● **Ex. 13** *Let A and B are two square idempotent matrices such that $AB \pm BA$ is a null matrix, the value of* $\det(A - B)$ *can be equal*

(a) -1 (b) 0

(c) 1 (d) 2

Sol. (*a, b, c*)

$\because (A - B)^2 = A^2 - AB - BA + B^2$

$= A + B$ $[\because AB + BA = 0$ and $A^2 = A, B^2 = B]$

$\therefore$ $|A - B|^2 = |A + B|$ ...(i)

and $(A + B)^2 = A^2 + AB + BA + B^2$

$= A + B$ $[\because AB + BA = 0$ and $A^2 = A, B^2 = B]$

$\Rightarrow$ $|A + B|^2 = |A + B|$

$\Rightarrow$ $|A + B|(|A + B| - 1) = 0$

$\therefore$ $|A + B| = 0, 1$

From Eq. (i),

$$|A-B|^2 = 0, 1 \Rightarrow |A-B| = 0, \pm 1$$

or $\quad \det(A-B) = 0, -1, 1$

● ***Ex. 14*** *If $AB = A$ and $BA = B$, then*

(a) $A^2B = A^2$ (b) $B^2A = B^2$
(c) $ABA = A$ (d) $BAB = B$

Sol. (*a, b, c, d*)

We have, $A^2B = A(AB) = A \cdot A = A^2$,

$B^2A = B(BA) = BB = B^2$,

$ABA = A(BA) = AB = A$, $BAB = B(AB) = BA = B$

● ***Ex. 15*** *If A is a square matrix of order 3 and I is an Identity matrix of order 3 such that $A^3 - 2A^2 - A + 2I = 0$, then A is equal to*

(a) I (b) $2I$ (c) $\begin{bmatrix} 2 & -1 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ (d) $\begin{bmatrix} 2 & 1 & -2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

Sol. (*a, b, d*) It is clear that $A = I$ and $A = 2I$ satisfy the given equaion $A^3 - 2A^2 - A + 2I = 0$ and the characteristic equation of the matrix in (c) is

$$\begin{vmatrix} 2-\lambda & -1 & 2 \\ 1 & -\lambda & 0 \\ 0 & 1 & -\lambda \end{vmatrix} = 0$$

$\Rightarrow \quad \lambda^3 - 2\lambda^2 + \lambda - 2 = 0,$

giving $\quad A^3 - 2A^2 + A - 2I = 0$

$\neq A^3 - 2A^2 - A + 2I = 0$

and the characteristic equation of the matrix in (d) is

$$\begin{vmatrix} 2-\lambda & 1 & -2 \\ 1 & -\lambda & 0 \\ 0 & 1 & -\lambda \end{vmatrix} = 0$$

$\Rightarrow \quad \lambda^3 - 2\lambda^2 - \lambda + 2 = 0,$

giving $\quad A^3 - 2A^2 - A + 2I = 0$

JEE Type Solved Examples : Passage Based Questions

- This section contains **2 solved passages**. Base upon each of the passage **3 multiple choice question** have to be answered. Each of these question has four choices (a), (b), (c) and (d) out of which **ONLY ONE** is correct.

Passage I

(Ex. Nos. 16 to 18)

If $A_0 = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$ and $B_0 = \begin{bmatrix} -4 & -3 & -3 \\ 1 & 0 & 1 \\ 4 & 4 & 3 \end{bmatrix}$ and

$B_n = \text{adj}(B_{n-1}), n \in N$ and I is an identity matrix of order 3.

16. det $(A_0 + A_0^2 B_0^2 + A_0^3 + A_0^4 B_0^4 + \ldots$ upto 12 terms) is equal to

(a) 1200 (b) −960
(c) 0 (d) −9600

17. $B_2 + B_3 + B_4 + \ldots + B_{50}$ is equal to

(a) B_0 (b) $7B_0$
(c) $49B_0$ (d) $49I$

18. For a variable matrix X, the equation $A_0 X = B_0$ will have

(a) unique solution
(b) infinite solution
(c) finitely many solution
(d) no solution

Sol. (Ex. Nos 16 to 18)

$\because \quad A_0 = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix} \Rightarrow |A_0| = 0$

and $\quad \text{adj}\, B_0 = \begin{bmatrix} -4 & 1 & 4 \\ -3 & 0 & 4 \\ -3 & 1 & 3 \end{bmatrix}^T = \begin{bmatrix} -4 & -3 & -3 \\ 1 & 0 & 1 \\ 4 & 4 & 3 \end{bmatrix} = B_0$

$\because \quad B_n = \text{adj}(B_{n-1}), n \in N$

$\therefore \quad B_1 = \text{adj}(B_0) = B_0$

$\Rightarrow \quad B_2 = \text{adj}(B_1) = \text{adj}(B_0) = B_0,$

Similarly $B_3 = B_0, B_4 = B_0, \ldots$

$\therefore \quad B_n = B_0 \ \forall\, n \in N$

16. (*c*) det $(A_0 + A_0^2 B_0^2 + A_0^3 + A_0^4 B_0^4 + \ldots$ upto 12 terms) = det

$\{A_0 (I + A_0 B_0^2 + A_0^2 + A_0^3 B_0^4 + \ldots \text{upto 12 terms})\}$

$= |A_0|(I + A_0 B_0^2 + A_0^2 + A_0^3 B_0^4 + \ldots$ upto 12 terms)

$= 0 \qquad [\because |A_0| = 0]$

17. (*c*) $B_2 + B_3 + B_4 + \ldots + B_{50} = B_0 + B_0 + B_0 + \ldots + B_0 = 49 B_0$

18. (*d*) $\because |A_0| = 0$

$\Rightarrow \quad A_0^{-1}$ is not possible.

Hence, system of equation $A_0 X = B_0$ has no Sol.

Passage II

(Ex. Nos. 19 to 21)

Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ satisfies $A^n = A^{n-2} + A^2 - I$ for $n \geq 3$ and consider a matrix $\underset{3 \times 3}{U}$ with its columns as U_1, U_2, U_3, such that

$$A^{50} U_1 = \begin{bmatrix} 1 \\ 25 \\ 25 \end{bmatrix}, A^{50} U_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \text{ and } A^{50} U_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

19. The value of $|A^{50}|$ equals

(a) -1 (b) 0 (c) 1 (d) 25

20. Trace of A^{50} equals

(a) 0 (b) 1 (c) 2 (d) 3

21. The value of $|U|$ equals

(a) -1 (b) 0 (c) 1 (d) 2

Sol. (Ex. Nos. 19 to 21)

$\because \quad A^n = A^{n-2} + A^2 - I \Rightarrow A^{50} = A^{48} + A^2 - I$

Further, $A^{48} = A^{46} + A^2 - I$

$A^{46} = A^{44} + A^2 - I$

$\vdots \quad \vdots \quad \vdots \quad \vdots$

$A^4 = A^2 + A^2 - I^4$

On adding all, we get

$$A^{50} = 25A^2 - 24I \quad \text{...(i)}$$

19. (*c*) $|A^{50}| = |A|^{50} = \begin{vmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix}^{50} = (-1)^{50} = 1$

20. (*d*) $\because A^2 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

$\therefore \quad A^{50} = 25A^2 - 24I = \begin{bmatrix} 25 & 0 & 0 \\ 25 & 25 & 0 \\ 25 & 0 & 25 \end{bmatrix} - \begin{bmatrix} 24 & 0 & 0 \\ 0 & 24 & 0 \\ 0 & 0 & 24 \end{bmatrix}$ [from Eq. (i)]

$= \begin{bmatrix} 1 & 0 & 0 \\ 25 & 1 & 0 \\ 25 & 0 & 1 \end{bmatrix}$ ...(ii)

Hence, trace of $A^{50} = 1 + 1 + 1 = 3$

21. (*c*) Let $U_1 = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

Given, $A^{50} U_1 = \begin{bmatrix} 1 \\ 25 \\ 25 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 25 & 1 & 0 \\ 25 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 25 \\ 25 \end{bmatrix}$ [from Eq. (ii)]

$\Rightarrow \begin{bmatrix} x \\ 25x + y \\ 25x + z \end{bmatrix} = \begin{bmatrix} 1 \\ 25 \\ 25 \end{bmatrix}$, we get $x = 1, y = 20$ and $z = 0$

$\therefore \quad U_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, similarly $U_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

and $\quad U_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \Rightarrow U = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$

$\therefore \quad |U| = 1$

JEE Type Solved Examples :
Single Integer Answer Type Questions

- This section contains **2 examples.** The answer to each example is a **single digit integer** ranging from **0** to **9** (both inclusive).

● ***Ex. 22*** *Let A be a 3×3 diagonal matrix which commutes with every 3×3 matrix. If $\det(A) = 8$, then tr A is*

Sol. (6) Let $A = \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{bmatrix}$

$$\therefore \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{bmatrix} \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix} = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix} \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{bmatrix}$$

$\Rightarrow \quad \alpha = \beta = \gamma$

$\therefore \quad A = \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha \end{bmatrix}$

$\Rightarrow \quad \det(A) = \alpha^3 = 8$ [given]

$\therefore \quad \alpha = 2$

$\therefore \quad A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

$\Rightarrow \quad \text{tr } A = 2 + 2 + 2 = 6$

● **Ex. 23** *Let A and B be two non-singular matrices such that $A \neq I$, $B^3 = I$ and $AB = BA^2$, where I is the identity matrix, the least value of k such that $A^k = I$ is*

Sol. (7) Given, $AB = BA^2 \Rightarrow B = A^{-1}BA^2 \Rightarrow B^3 = I$

$\Rightarrow (A^{-1}BAA)(A^{-1}BAA)(A^{-1}BAA) = I$

$\Rightarrow (A^{-1}BA)(BA)(BAA) = I$ $[\because A^{-1}A = I]$

$\Rightarrow A^{-1}B(AB)(AB)AA = I$

$\Rightarrow A^{-1}B(BA^2)(BA^2)AA = I$ $[\because AB = BA^2]$

$\Rightarrow A^{-1}BBA(AB)A^4 = I$

$\Rightarrow A^{-1}BBA(BA^2)A^4 = I$ $[\because AB = BA^2]$

$\Rightarrow A^{-1}BB(AB)A^6 = I$

$\Rightarrow A^{-1}BB(BA^2)A^6 = I$ $[\because AB = BA^2]$

$\Rightarrow A^{-1}B^3A^8 = I$

$\Rightarrow (A^{-1}I)A^8 = I$. $[\because B^3 = I]$

$\Rightarrow A^{-1}A^8 = I$

$\Rightarrow A^7 = I = A^k$ $[\because A^k = I]$

$\Rightarrow A^k = A^7$

$\therefore$ Least value of k is 7.

JEE Type Solved Examples : Matching Type Questions

- This section contains **2 examples.** Example 24 have three statements (A, B and C) given in **Column I** and four statements (p, q, r and s) in **Column II** and example 25 have three statements (A, B and C) given in **Column I** and five statements (p, q, r, s and t). In **Column II** any given statement in **Column I** can have correct matching with one or more statement(s) given in **Column II.**

● **Ex. 24**

	Column I		Column II
(A)	If A is a square matrix of order 3 and $\det(A) = 3$, then $\det(6A^{-1})$ is divisible by	(p)	3
(B)	If A is a square matrix of order 3 and $\det(A) = \frac{1}{4}$, then det [adj (adj (2A))] is divisible by	(q)	4
(C)	If A and B are square matrices of odd order and $(A + B)^2 = A^2 + B^2$, if det $(A) = 2$, then det(B) is divisible by	(r)	5
		(s)	6

Sol. (A) → (p, q, s); (B) →(q); (C) →(p, q, r, s)

(A) $\det(6A^{-1}) = 6^3 \det(A^{-1}) = \dfrac{216}{\det(A)} = \dfrac{216}{3} = 72$

(B) $\det[\text{adj}(\text{adj}(2A))] = [\det(2A)]^4 = [2^3 \det(A)]^4$

$= 2^{12}[\det(A)]^4$

$= 2^{12}\left(\dfrac{1}{4}\right)^4 = 2^4 = 16$

(C) $\because (A + B)^2 = A^2 + AB + BA + B^2$

$\Rightarrow A^2 + B^2 = A^2 + AB + BA + B^2$

$[\because (A + B)^2 = A^2 + B^2]$

$\Rightarrow AB + BA = O \Rightarrow AB = -BA$

$\therefore \det(AB) = \det(-BA) = -\det(BA)$

$\Rightarrow \det(A)\cdot\det(B) = -\det(B)\cdot\det(A)$

$\Rightarrow 2\det(A)\cdot\det(B) = 0 \Rightarrow 4\det(B) = 0$ $[\because \det(A) = 2]$

$\therefore \det(B) = 0$

● **Ex. 25**

	Column I		Column II
(A)	If $\begin{bmatrix} 1 & 2 & a \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}^n = \begin{bmatrix} 1 & 18 & 2007 \\ 0 & 1 & 36 \\ 0 & 0 & 1 \end{bmatrix}$, then $(n + a)$ is divisible by	(p)	4
(B)	If A is a square matrix of order 3 such that $\|A\| = a$, $B = \text{adj}(A)$ and $\|B\| = b$, then $(ab^2 + a^2b + 1)\lambda$ is divisible by, where $\frac{1}{2}\lambda = \frac{a}{b} + \frac{a^2}{b^3} + \frac{a^3}{b^5} + \ldots$ upto ∞ and $a = 3$	(q)	6
(C)	Let $A = \begin{bmatrix} a & b & c \\ p & q & r \\ 1 & 1 & 1 \end{bmatrix}$ and $B = A^2$. If $(a-b)^2 + (p-q)^2 = 25$, $(b-c)^2 + (q-r)^2 = 36$ and $(c-a)^2 + (r-p)^2 = 49$, then det $\left(\frac{B}{2}\right)$ is divisible by	(r)	10
		(s)	12
		(t)	15

Sol. (A) → (p, r); (B) →(t); (C) → (q, s)

(A) $\because A = \begin{bmatrix} 1 & 2 & a \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$

$\therefore A^2 = \begin{bmatrix} 1 & 2 & a \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & a \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 4 & 2a+8 \\ 0 & 1 & 8 \\ 0 & 0 & 1 \end{bmatrix}$

$$\Rightarrow \quad A^3 = \begin{bmatrix} 1 & 4 & 2a+8 \\ 0 & 1 & 8 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & a \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 6 & 3a+24 \\ 0 & 1 & 12 \\ 0 & 0 & 1 \end{bmatrix}$$

Similarly, we get

$$A^n = \begin{bmatrix} 1 & 2n & na + 8\sum_{r=0}^{n-1} r \\ 0 & 1 & 4n \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 18 & 2007 \\ 0 & 1 & 36 \\ 0 & 0 & 1 \end{bmatrix} \text{ [given]}$$

$\Rightarrow \; 2n = 18 \; \Rightarrow \; n = 9$

$\therefore \quad na + 8 \sum_{r=0}^{n-1} r = 2007 \; \Rightarrow \; 9a + 8 \sum_{r=0}^{8} r = 2007$

$\Rightarrow \quad 9a + 8 \cdot \left(\frac{8 \times 9}{2} \right) = 2007 \Rightarrow 9a = 2007 - 288 = 1719$

$\therefore \qquad a = 191$

Hence, $n + a = 9 + 191 = 200$

(B) $B = \text{adj } A$

$\Rightarrow \quad b = |B| = |\text{adj } A| = |A|^2 = a^2 = 9 \; \Rightarrow \; a = 3, b = 9$

and $\quad \frac{1}{2}\lambda = \frac{3}{9} + \frac{3^2}{9^3} + \frac{3^3}{9^5} + \ldots + \infty$

$$= \frac{1}{3} + \frac{1}{81} + \frac{1}{27 \times 81} + \ldots + \infty = \frac{\frac{1}{3}}{1 - \frac{1}{27}} = \frac{9}{26}$$

$\Rightarrow \qquad \lambda = \frac{9}{13}$

Now, $(ab^2 + a^2 b + 1)\lambda = (3 \times 81 + 9 \times 9 + 1) \times \frac{9}{13} = 225$

(C) $\det(A) = \begin{vmatrix} a & b & c \\ p & q & r \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} a & p & 1 \\ b & q & 1 \\ c & r & 1 \end{vmatrix} = 2 \times \frac{1}{2} \begin{vmatrix} a & p & 1 \\ b & q & 1 \\ c & r & 1 \end{vmatrix}$

$= 2 \times$ Area of the triangle with vertices $(a, p), (b, q)$ and (c, r) with sides 5, 6, 7

$= 2 \times \sqrt{s(s-a)(s-b)(s-c)} = 2 \times 6\sqrt{6} = 12\sqrt{6}$

Hence, $\det\left(\frac{B}{2}\right) = \left(\frac{1}{2}\right)^3 \det(B) = \frac{1}{8} \det(A^2)$

$$= \frac{1}{8} (\det A)^2 = \frac{1}{8} (12\sqrt{6})^2 = 108$$

JEE Type Solved Examples : Statement I and II Type Questions

- **Direction** example numbers 26 and 27 are Assertion-Reason type examples. Each of these examples contains two statements:

Statement-1 (Assertion) and **Statement-2** (Reason)

Each of these examples also has four alternative choices, **ONLY ONE** of which is the correct answer. You have to select the correct choice as given below.

(a) Statement-1 is true, Statement-2 is true; Statement-2 is correct explanation for Statement-1
(b) Statement-1 is true, Statement-2 is true; Statement-2 is not a correct explanation for Statement-1
(c) Statement-1 is true, Statement-2 is false
(d) Statement-1 is false, Statement-2 is true

● ***Ex. 26*** **Statement-1** *A is singular matrix of order $n \times n$, then adj A is singular.*

Statement-2 $|\text{adj } A| = |A|^{n-1}$

Sol. (*d*) If A is non-singular matrix of order $n \times n$, then

$$|\text{adj } A| = |A|^{n-1}$$

Hence, Statement-1 is false and Statement-2 is true.

● ***Ex. 27*** **Statement-1** *If A and B are two matrices such that $AB = B, BA = A$, then $A^2 + B^2 = A + B$.*

Statement-2 *A and B are idempotent matrices, then $A^2 = A, B^2 = B$.*

Sol. (*b*) $\because AB = B$

$\Rightarrow \quad B(AB) = B \cdot B$

$\Rightarrow \quad (BA) B = B^2$ [by associative law]

$\Rightarrow \quad AB = B^2$ [$\because BA = A$]

$\Rightarrow \quad B = B^2$ [$\because AB = B$]

and $\quad BA = A$

$\Rightarrow \quad A(BA) = A \cdot A$

$\Rightarrow \quad (AB) A = A^2$ [by associative law]

$\Rightarrow \quad BA = A^2$ [$\because AB = B$]

$\Rightarrow \quad A = A^2$ [$\because BA = A$]

Hence, $\therefore \quad A^2 + B^2 = A + B$

Here, both statements are true and Statement-2 is not a correct explanation for Statement-1.

Subjective Type Examples

- In this section, there are **12 subjective** solved examples.

Ex. 28 *If $A^n = 0$, then evaluate*

(i) $I + A + A^2 + A^3 + \ldots + A^{n-1}$

(ii) $I - A + A^2 - A^3 + \ldots + (-1)^{n-1} A^{n-1}$ for odd 'n', where I is the identity matrix having the same order of A.

Sol. (i) $A^n = 0 \Rightarrow A^n - I = -I$

$$\Rightarrow \quad A^n - I^n = -I \Rightarrow I^n - A^n = I$$

$$\Rightarrow (I - A)(I + A + A^2 + A^3 + \ldots + A^{n-1}) = I$$

$$\Rightarrow (I + A + A^2 + A^3 + \ldots + A^{n-1}) = (I - A)^{-1} I = (I - A)^{-1}$$

(ii) $A^n = 0 \Rightarrow A^n + I = I$

$$\Rightarrow \quad A^n + I^n = I$$

$$\Rightarrow \quad I^n + A^n = I$$

$$\Rightarrow (I + A)(I - A + A^2 - A^3 + \ldots + A^{n-1}) = I \quad [\because n \text{ is odd}]$$

$$\Rightarrow I - A + A^2 - A^3 + \ldots + A^{n-1} = (I + A)^{-1} I = (I + A)^{-1}$$

Ex. 29 *If A is idempotent matrix, then show that $(A + I)^n = I + (2^n - 1)A, \forall n \in N$, where I is the identity matrix having the same order of A.*

Sol. $\because$ A is idempotent matrix

$\therefore \quad A^2 = A,$

similarly $A = A^2 = A^3 = A^4 = \ldots = A^n$ ...(i)

Now, $(A + I)^n = (I + A)^n$

$$= I + {}^nC_1 A + {}^nC_2 A^2 + {}^nC_3 A^3 + \ldots + {}^nC_n A^n$$

$$= I + ({}^nC_1 + {}^nC_2 + {}^nC_3 + \ldots + {}^nC_n) A \quad \text{[from Eq. (i)]}$$

$$= I + (2^n - 1)A$$

Hence, $(A + I)^n = I + (2^n - 1)A, \forall n \in N$.

Ex. 30 *If the matrices $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ (a, b, c, d not all simultaneously zero) commute, find the value of $\dfrac{d - b}{a + c - b}$. Also, show that the matrix which commutes with A is of the form $\begin{bmatrix} \alpha - \beta & \dfrac{2\beta}{3} \\ \beta & \alpha \end{bmatrix}$.*

Sol. Given, $AB = BA$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} a + 2c & b + 2d \\ 3a + 4c & 3b + 4d \end{bmatrix} = \begin{bmatrix} a + 3b & 2a + 4b \\ c + 3d & 2c + 4d \end{bmatrix}$$

On comparing, we get

$$a + 2c = a + 3b$$

$$\Rightarrow \quad b = \frac{2c}{3} \quad \text{...(i)}$$

$$b + 2d = 2a + 4b$$

$$\Rightarrow \quad d = a + \frac{3b}{2} \quad \text{...(ii)}$$

$$\Rightarrow \quad 3a + 4c = c + 3d$$

$$d = a + c \quad \text{...(iii)}$$

and $\quad 3b + 4d = 2c + 4d$

$$\Rightarrow \quad b = \frac{2c}{3} \quad \text{...(iv)}$$

$$\Rightarrow \quad \frac{d - b}{a + c - b} = \frac{d - b}{d - b} = 1 \quad \text{[from Eq. (iii)]}$$

Now, $\quad B = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} d - c & \dfrac{2c}{3} \\ c & d \end{bmatrix}$

If $c = \beta$ and $d = \alpha$, then $B = \begin{bmatrix} \alpha - \beta & \dfrac{2\beta}{3} \\ \beta & \alpha \end{bmatrix}$

Ex. 31 *Given the matrix $A = \begin{bmatrix} -1 & 3 & 5 \\ 1 & -3 & -5 \\ -1 & 3 & 5 \end{bmatrix}$ and X be the solution set of the equation $A^x = A$, where $x \in N - \{1\}$. Evaluate $\prod \left(\dfrac{x^3 + 1}{x^3 - 1} \right)$, where the continued extends for all $x \in X$.*

Sol. $\because A^2 = \begin{bmatrix} -1 & 3 & 5 \\ 1 & -3 & -5 \\ -1 & 3 & 5 \end{bmatrix}\begin{bmatrix} -1 & 3 & 5 \\ 1 & -3 & -5 \\ -1 & 3 & 5 \end{bmatrix} = \begin{bmatrix} -1 & 3 & 5 \\ 1 & -3 & -5 \\ -1 & 3 & 5 \end{bmatrix}$

$= A$

$$\therefore \quad A^2 = A^3 = A^4 = A^5 = \ldots = A$$

but given $\quad A^x = A$

$$\Rightarrow \quad x = 2, 3, 4, 5, \ldots \quad [\because x \neq 1, \text{given}]$$

$$\therefore \quad \prod \left(\frac{x^3 + 1}{x^3 - 1} \right) = \prod \left(\frac{x + 1}{x - 1} \right) \prod \left(\frac{x^2 - x + 1}{x^2 + x + 1} \right)$$

On putting $\quad x = 2, 3, 4, 5, \ldots$

$$\prod\left(\frac{x^3+1}{x^3-1}\right)=\lim_{n\to\infty}\prod_{x=2}^{n}\left(\frac{x+1}{x-1}\right)\prod_{x=2}^{n}\left(\frac{x^2-x+1}{x^2+x+1}\right)$$

$$=\lim_{n\to\infty}\left(\frac{3\cdot4\cdot5\ldots(n-1)n(n+1)}{1\cdot2\cdot3\ldots(n-3)(n-2)(n-1)}\right)$$

$$\times\lim_{n\to\infty}\left(\frac{3\cdot7\cdots(n^2-n+1)}{7\cdot13\ldots(n^2-n+1)(n^2+n+1)}\right)$$

$$=\lim_{n\to\infty}\frac{n(n+1)}{2}\times\frac{3}{(n^2+n+1)}$$

$$=\frac{3}{2}\lim_{n\to\infty}\frac{\left(1+\frac{1}{n}\right)}{\left(1+\frac{1}{n}+\frac{1}{n^2}\right)}=\frac{3}{2}\cdot\frac{(1+0)}{(1+0+0)}=\frac{3}{2}$$

- ***Ex. 32*** *If P is a non-singular matrix, with $\operatorname{adj}(P^{-1})$ in terms of 'P', then show that $\operatorname{adj}(Q^{-1}BP^{-1})=PAQ$. Given that, $\operatorname{adj}(B)=A$ and $|P|=|Q|=1$.*

Sol. $\because \operatorname{adj}(P^{-1})=|P|(P^{-1})^{-1}=|P|P=P \qquad [\because |P|=1]$

and $\operatorname{adj}(Q^{-1}BP^{-1})=\operatorname{adj}(P^{-1})\cdot\operatorname{adj}B\cdot\operatorname{adj}(Q^{-1})$

$$=\frac{P}{|P|}A\cdot\frac{Q}{|Q|}=PAQ \qquad [\because |P|=|Q|=1]$$

- ***Ex. 33*** *Let A and B be matrices of order n. Prove that if $(I-AB)$ is invertible, $(I-BA)$ is also invertible and $(I-BA)^{-1}=I+B(I-AB)^{-1}A$, where I be the identity matrix of order n.*

Sol. Here, $I-BA=BIB^{-1}-BABB^{-1}=B(I-AB)B^{-1}$ ...(i)

Hence, $|I-BA|=|B||I-AB||B^{-1}|=|B||I-AB|\frac{1}{|B|}$

$=|I-AB|$

If $|I-AB|\neq0$, then $|I-BA|\neq0$

i.e. if $(I-AB)$ is invertible, then $(I-BA)$ is also invertible.

Now, $(I-BA)[I+B(I-AB)^{-1}A]$

$=(I-BA)+(I-BA)B(I-AB)^{-1}A$ [using Eq. (i)]

$=(I-BA)+B(I-AB)B^{-1}B(I-AB)^{-1}A$

$=(I-BA)+B(I-AB)(I-AB^{-1})A$

$=(I-BA)+BA=I$

Hence, $(I-BA)^{-1}=I+B(I-AB)^{-1}A$.

- ***Ex. 34*** *Prove that the inverse of $\begin{bmatrix}A & O\\ B & C\end{bmatrix}$ is $\begin{bmatrix}A^{-1} & O\\ -C^{-1}BA^{-1} & C^{-1}\end{bmatrix}$, where A, C are non-singular matrices and O is null matrix and find the inverse* $\begin{bmatrix}1&0&0&0\\1&1&0&0\\1&1&1&0\\1&1&1&1\end{bmatrix}$

Sol. We have, First part $\begin{bmatrix}A & O\\ B & C\end{bmatrix}\begin{bmatrix}A^{-1} & O\\ -C^{-1}BA^{-1} & C^{-1}\end{bmatrix}$

$$=\begin{bmatrix}AA^{-1} & O\\ BA^{-1}-CC^{-1}BA^{-1} & CC^{-1}\end{bmatrix}$$

$$=\begin{bmatrix}I & O\\ BA^{-1}-BA^{-1} & I\end{bmatrix}=\begin{bmatrix}I & O\\ 0 & I\end{bmatrix}$$

Hence, $\begin{bmatrix}A^{-1} & O\\ -C^{-1}BA^{-1} & C^{-1}\end{bmatrix}$ is the inverse of $\begin{bmatrix}A & O\\ B & C\end{bmatrix}$.

Second part $\begin{bmatrix}1&0&0&0\\1&1&0&0\\1&1&1&0\\1&1&1&1\end{bmatrix}=\begin{bmatrix}A & O\\ B & C\end{bmatrix}$

where $A=\begin{bmatrix}1&0\\1&1\end{bmatrix}$, $B=\begin{bmatrix}1&1\\1&1\end{bmatrix}$, $C=\begin{bmatrix}1&0\\1&1\end{bmatrix}$ and $O=\begin{bmatrix}0&0\\0&0\end{bmatrix}$

and $A^{-1}=\begin{bmatrix}1&0\\-1&1\end{bmatrix}$, $C^{-1}=\begin{bmatrix}1&0\\-1&1\end{bmatrix}$

Now, $C^{-1}BA^{-1}=\begin{bmatrix}1&0\\-1&1\end{bmatrix}\begin{bmatrix}1&1\\1&1\end{bmatrix}\begin{bmatrix}1&0\\-1&1\end{bmatrix}=\begin{bmatrix}0&1\\0&0\end{bmatrix}$

$\therefore$ Inverse of $\begin{bmatrix}1&0&0&0\\1&1&0&0\\1&1&1&0\\1&1&1&1\end{bmatrix}$ is $\begin{bmatrix}1&0&0&0\\-1&1&0&0\\0&-1&1&0\\0&0&-1&1\end{bmatrix}$

- ***Ex. 35*** *Let $A=\begin{bmatrix}3&a&-1\\2&5&c\\b&8&2\end{bmatrix}$ is symmetric and $B=\begin{bmatrix}d&3&a\\b-a&e&-2b-c\\-2&6&-f\end{bmatrix}$ is skew-symmetric, find AB. If AB is symmetric or skew-symmetric or neither of them. Justify your answer.*

Sol. $\because A$ is symmetric

$\therefore \quad c=8, b=-1$ and $a=2$ ...(i)

and B is skew-symmetric

$\therefore \quad d=e=f=0$ and $2b+c=6, a=2, b-a=-3$ ...(ii)

From Eqs. (i) and (ii), we get

$a=2, b=-1, c=8, d=0, e=0, f=0$

$$\therefore \quad A=\begin{bmatrix}3&2&-1\\2&5&8\\-1&8&2\end{bmatrix} \text{ and } B=\begin{bmatrix}0&3&2\\-3&0&-6\\-2&6&0\end{bmatrix}$$

$$\Rightarrow \quad AB=\begin{bmatrix}-4&3&-6\\-31&54&-26\\-28&9&-50\end{bmatrix}$$

which neither symmetric nor skew-symmetric.

Ex. 36 *If B, C are square matrices of order n and if $A = B + C$, $BC = CB$, $C^2 = 0$, show that for any positive integer p, $A^{p+1} = B^p[B + (p+1)C]$.*

Sol. $\because$ $A = B + C \Rightarrow A^{p+1} = (B + C)^{p+1}$

$$= {}^{p+1}C_0 B^{p+1} + {}^{p+1}C_1 B^p C + {}^{p+1}C_2 B^{p-1} C^2 + \dots + {}^{p+1}C_{p+1} C^{p+1}$$

$$= B^{p+1} + {}^{p+1}C_1 B^p C + 0 + 0 + \dots \quad [\because C^2 = 0 \Rightarrow C^2 = C^3 = \dots = 0]$$

$$= B^p[B + (p+1)C]$$

Hence, $A^{p+1} = B^p[B + (p+1)C]$

Ex. 37 *If there an three square matrices A, B, C of same order satisfying the equation $A^2 = A^{-1}$ and let $B = A^{2^n}$ and $C = A^{2^{(n-2)}}$, prove that $\det(B - C) = 0$.*

Sol. $\because$ $B = A^{2^n} = A^{2 \cdot 2^{n-1}} = (A^2)^{2^{n-1}} = (A^{-1})^{2^{n-1}}$ $[\because A^2 = A^{-1}]$

$$= (A^{2^{n-1}})^{-1} = (A^{2 \cdot 2^{n-2}})^{-1} = [(A^2)^{2^{n-2}}]^{-1}$$

$$= [(A^{-1})^{2^{n-2}}]^{-1} = ((A^{-1})^{-1})^{2^{n-2}} = A^{2^{n-2}} = C$$

$\Rightarrow$ $B - C = 0 \Rightarrow \det(B - C) = 0$

Ex. 38 *Construct an orthogonal matrix using the skew-symmetric matrix $A = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$.*

Sol. $\because$ $A = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix} \Rightarrow I - A = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$

$\Rightarrow$ $(I - A)^{-1} = \frac{1}{5}\begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$ and $(I + A) = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$

Let B be the orthogonal matrix from a skew-symmetric matrix A, then $B = (I - A)^{-1}(I + A)$

$$= \frac{1}{5}\begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} -3 & 4 \\ -4 & -3 \end{bmatrix} = \begin{bmatrix} -\frac{3}{5} & \frac{4}{5} \\ -\frac{4}{5} & -\frac{3}{5} \end{bmatrix}$$

Ex. 39 *If $A = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 4 & 1 \\ -2 & -4 & -1 \end{bmatrix}$ and X, Y are two non-zero column vectors such that $AX = \lambda X$, $AY = \mu Y$, $\lambda \neq \mu$, find angle between X and Y.*

Sol. $\because$ $AX = \lambda X \Rightarrow (A - \lambda I)X = 0$

$\because$ $X \neq 0$

$\therefore$ $\det(A - \lambda I) = 0$

$$\Rightarrow \begin{vmatrix} 3-\lambda & 2 & 2 \\ 2 & 4-\lambda & 1 \\ -2 & -4 & -1-\lambda \end{vmatrix} = 0$$

Applying $R_3 \to R_3 + R_2$, then

$$\begin{vmatrix} 3-\lambda & 2 & 2 \\ 2 & 4-\lambda & 1 \\ 0 & -\lambda & -\lambda \end{vmatrix} = 0$$

Applying $C_2 \to C_2 - C_3$, then

$$\Rightarrow \begin{vmatrix} 3-\lambda & 0 & 2 \\ 2 & 3-\lambda & 1 \\ 0 & 0 & -\lambda \end{vmatrix} = 0$$

$\Rightarrow$ $-\lambda(3-\lambda)^2 = 0$

$\Rightarrow$ $\lambda = 0, 3$

It is clear that $\lambda = 0, \mu = 3$

For $\lambda = 0$, $AX = 0 \Rightarrow \begin{bmatrix} 3 & 2 & 2 \\ 2 & 4 & 1 \\ -2 & -4 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow$ $3x + 2y + 2z = 0$ and $2x + 4y + z = 0$

$\therefore$ $\frac{x}{-6} = \frac{y}{1} = \frac{z}{8}$

So, $X = \begin{bmatrix} -6 \\ 1 \\ 8 \end{bmatrix}$

For $\mu = 3$, $(A - 3I)Y = 0$

$$\Rightarrow \begin{bmatrix} 0 & 2 & 2 \\ 2 & 1 & 1 \\ -2 & -4 & -4 \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow$ $0 \cdot \alpha + 2\beta + 2\gamma = 0$ and $2\alpha + \beta + \gamma = 0$

$\therefore$ $\frac{\alpha}{0} = \frac{\beta}{4} = \frac{\gamma}{-4}$

$\Rightarrow$ $\frac{\alpha}{0} = \frac{\beta}{-1} = \frac{\gamma}{1}$

So, $Y = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$

If θ be the angle between X and Y, then

$$\cos\theta = \frac{0 \cdot (-6) + (-1) \cdot 1 + 1 \cdot 8}{\sqrt{(0+1+1)}\sqrt{36+1+64}} = \frac{7}{\sqrt{202}}$$

$\therefore$ $\theta = \cos^{-1}\left(\frac{7}{\sqrt{202}}\right)$