Session 4

Use of Complex Numbers in Binomial Theorem, Multinomial Theorem, Use of Differentiation, Use of Integration, Binomial Inside Binomial, Sum of the Series

Use of Complex Numbers in Binomial Theorem

If $\theta \in R, n \in N$ and $i = \sqrt{-1}$, then

$$(\cos\theta + i\sin\theta)^n = {}^nC_0 (\cos\theta)^{n-0} (i\sin\theta)^0 + {}^nC_1 (\cos\theta)^{n-1} (i\sin\theta)^1 + {}^nC_2 (\cos\theta)^{n-2} (i\sin\theta)^2 + {}^nC_3 (\cos\theta)^{n-3} (i\sin\theta)^3 + \dots$$

or $\cos n\theta + i\sin n\theta = \cos^n\theta + i\cdot {}^nC_1 (\cos\theta)^{n-1}\sin\theta - {}^nC_2 (\cos\theta)^{n-2}\sin^2\theta - i\cdot {}^nC_3 (\cos\theta)^{n-3}\sin^3\theta + \dots$

On comparing real and imaginary parts, we get

$$\cos n\theta = \cos^n\theta - {}^nC_2 (\cos\theta)^{n-2}\sin^2\theta - {}^nC_4 (\cos\theta)^{n-4}\sin^4\theta - \dots$$

and $\sin n\theta = {}^nC_1 (\cos\theta)^{n-1}\sin\theta - {}^nC_3 (\cos\theta)^{n-3}\sin^3\theta + {}^nC_5 (\cos\theta)^{n-5}\sin^5\theta - \dots$

Example 51. ***If*** $(1+x)^n = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + C_4 x^4 + \dots$, find the values of

(i) $C_0 - C_2 + C_4 - C_6 + \dots$

(ii) $C_1 - C_3 + C_5 - C_7 + \dots$

(iii) $C_0 + C_3 + C_6 + \dots$

Sol. $\because (1+x)^n = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + C_4 x^4 + C_5 x^5 + \dots$

Putting $x = i$, where $i = \sqrt{-1}$, then

$$(1+i)^n = C_0 + C_1 i + C_2 i^2 + C_3 i^3 + C_4 i^4 + C_5 i^5 + \dots$$
$$= (C_0 - C_2 + C_4 - \dots) + i(C_1 - C_3 + C_5 - \dots) \quad \dots\text{(i)}$$

Also, $(1+i)^n = \left[\sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)\right]^n$

$$= 2^{n/2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)^n$$
$$= 2^{n/2}\left(\cos\frac{n\pi}{4} + i\sin\frac{n\pi}{4}\right) \quad \dots\text{(ii)}$$

From Eqs. (i) and (ii), we get

$$(C_0 - C_2 + C_4 - \dots) + i(C_1 - C_3 + C_5 - \dots) = 2^{n/2}\cos\left(\frac{n\pi}{4}\right) + i\cdot 2^{n/2}\sin\left(\frac{n\pi}{4}\right)$$

On comparing real and imaginary parts, we get

$$C_0 - C_2 + C_4 - \dots = 2^{n/2}\cos\left(\frac{n\pi}{4}\right) \quad \text{[part (i)]}$$

$$C_1 - C_3 + C_5 - \dots = 2^{n/2}\sin\left(\frac{n\pi}{4}\right) \quad \text{[part (ii)]}$$

We have, $(1+x)^n = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + C_4 x^4 + C_5 x^5 + C_6 x^6 + \dots$

Putting $x = 1, \omega, \omega^2$ (cube roots of unity) and adding, we get

$$3(C_0 + C_3 + C_6 + \dots) = 2^n + (1+\omega)^n + (1+\omega^2)^n$$
$$= 2^n + (-\omega^2)^n + (-\omega)^n = 2^n + (-1)^n(\omega^{2n} + \omega^n)$$
$$= 2^n + (-1)^n\left\{e^{\frac{4\pi in}{3}} + e^{\frac{2\pi in}{3}}\right\}$$
$$= 2^n + (-1)^n \cdot e^{n\pi i} \cdot 2\cos\left(\frac{n\pi}{3}\right)$$
$$= 2^n + (-1)^n \cdot (-1)^n \cdot 2\cos\left(\frac{n\pi}{3}\right)$$
$$= 2^n + (-1)^{2n} \cdot 2\cos\left(\frac{n\pi}{3}\right) = 2^n + 2\cos\left(\frac{n\pi}{3}\right)$$

$$\therefore\ C_0 + C_3 + C_6 + \dots = \frac{1}{3}\left\{2^n + 2\cos\left(\frac{n\pi}{3}\right)\right\}$$

Example 52. ***Find the value of*** ${}^{4n}C_0 + {}^{4n}C_4 + {}^{4n}C_8 + \dots + {}^{4n}C_{4n}$.

Sol. $\because\ 4 - 0 = 8 - 4 = \dots = 4$

$\therefore$ Four roots of unity $(1)^{1/4}$ are $1, -1, i, -i$, we have

$$(1+x)^{4n} = {}^{4n}C_0 + {}^{4n}C_1 x + {}^{4n}C_2 x^2 + {}^{4n}C_3 x^3 + \dots$$

Putting $x = 1, -1, i, -i$ and then adding, we get

$$4({}^{4n}C_0 + {}^{4n}C_4 + {}^{4n}C_8 + \dots) = 2^{4n} + 0 + (1+i)^{4n} + (1-i)^{4n}$$
$$= 2^{4n} + (2i)^{2n} + (-2i)^{2n}$$

$$= 2^{4n} + 2^{2n}(-1)^n + 2^{2n}(-1)^n$$

$$= 2^{4n} + (-1)^n \cdot 2^{2n+1}$$

$$\therefore\ {}^{4n}C_0 + {}^{4n}C_4 + {}^{4n}C_8 + \ldots = 2^{4n-2} + (-1)^n \cdot 2^{2n-1}$$

Remark

If $(1+x)^n = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \ldots + C_n x^n$, then

(i) $C_0 + C_4 + C_8 + C_{12} + \ldots = \frac{1}{2}\left\{2^{n-1} + 2^{n/2}\cos\left(\frac{n\pi}{4}\right)\right\}$

(ii) $C_1 + C_5 + C_9 + C_{13} + \ldots = \frac{1}{2}\left\{2^{n-1} + 2^{n/2}\sin\left(\frac{n\pi}{4}\right)\right\}$

(iii) $C_0 + C_6 + C_{12} + \ldots = \frac{1}{3}\left\{2^{n-1}\cos\left(\frac{n\pi}{4}\right) + 3^{n/2}\cos\left(\frac{n\pi}{6}\right)\right\}$

Multinomial Theorem

If n is a positive integer and $x_1, x_2, x_3, \ldots, x_k \in C$, then

$$(x_1 + x_2 + x_3 + \ldots + x_k)^n = \sum \frac{n!}{(\alpha_1!)(\alpha_2!)(\alpha_3!)\ldots(\alpha_k!)}$$

$$x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3} \ldots x_k \alpha^k$$

where, $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_k$ are all non-negative integers such that $\alpha_1 + \alpha_2 + \alpha_3 + \ldots + \alpha_k = n$.

Remark

The coefficient of $x_1^{\alpha_1} \cdot x_2^{\alpha_2} \cdot x_3^{\alpha_3} \ldots x_k^{\alpha_k}$ in the expansion of $(x_1 + x_2 + x_3 + \ldots + x_k)^n$ is $\sum \frac{n!}{(\alpha_1!)(\alpha_2!)(\alpha_3!)\ldots(\alpha_k!)}$.

In Particular

(i) $(a+b+c)^n = \sum \frac{n!}{(\alpha!)(\beta!)(\gamma!)} a^\alpha b^\beta c^\gamma$ such that

$\alpha + \beta + \gamma = n$

(ii) $(a+b+c+d)^n = \sum \frac{n!}{(\alpha!)(\beta!)(\gamma!)(\delta!)} a^\alpha b^\beta c^\gamma d^\delta$

such that $\alpha + \beta + \gamma + \delta = n$

Example 53. ***Find the coefficient of*** $a^4 b^3 c^2 d$ ***in the expansion of*** $(a-b+c-d)^{10}$.

Sol. The coefficient of $a^4 b^3 c^2 d$ in the expansion of

$$(a-b+c-d)^{10} \text{ is } (-1)^4 \frac{10!}{4!\,3!\,2!\,1!} = 12600$$

[powers of b and d are 3 and 1 $\therefore (-1)^3(-1)$]

Example 54. ***Find the coefficient of*** $a^3 b^4 c^5$ ***in the expansion of*** $(bc+ca+ab)^6$.

Sol. In this case, write $a^3 b^4 c^5 = (ab)^x (bc)^y (ca)^z$ say

$$\therefore\quad a^3 b^4 c^5 = a^{z+x} \cdot b^{x+y} \cdot c^{y+z}$$

$$\Rightarrow\quad z + x = 3,\ x + y = 4$$

$$y + z = 5$$

On adding all, we get $2(x+y+z) = 12$

$$\therefore\quad x + y + z = 6$$

Then, $\quad x = 1, y = 3,\ z = 2$

Therefore, the coefficient of $a^3 b^4 c^5$ in the expansion of $(bc+ca+ab)^6$ or the coefficient of $(ab)^1 (bc)^3 (ca)^2$ in the expansion of $(bc+ca+ab)^6$ is $\frac{6!}{1!\,3!\,2!}$, i.e. 60.

Aliter

Coefficient of $a^3 b^4 c^5$ in the expansion of $(bc+ca+ab)^6$

= Coefficient of $a^3 b^4 c^5$ in the

expansion of $(abc)^6 \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)^6$

= Coefficient of $\left(\frac{1}{a}\right)^3 \left(\frac{1}{b}\right)^2 \left(\frac{1}{c}\right)^1$ in the expansion of

$\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)^6$ is $\frac{6!}{3!\,2!\,1!} = 60$

Number of Distinct or Dissimilar Terms in the Multinomial Expansion

Statement The number of distinct or dissimilar terms in the multinomial expansion of $(x_1 + x_2 + x_3 + \ldots + x_k)^n$ is ${}^{n+k-1}C_{k-1}$.

Proof We have, $(x_1 + x_2 + x_3 + \ldots + x_k)^n$

$$= \sum \frac{n!}{(\alpha_1!)(\alpha_2!)(\alpha_3!)\ldots(\alpha_k!)} x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3} \ldots x_k^{\alpha_k}$$

where, $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_k$ are non-negative integers such that

$$\alpha_1 + \alpha_2 + \alpha_3 + \ldots + \alpha_k = n \qquad \ldots\text{(i)}$$

Here, the number of terms in the expansion of

$$(x_1 + x_2 + x_3 + \ldots + x_k)^n$$

= The number of non-negative integral solutions of the Eq. (i)

$= {}^{n+k-1}C_{k-1}$

Example 55. ***Find the total number of distinct or dissimilar terms in the expansion of*** $(x+y+z+w)^n$, $n \in N$.

Sol. The total number of distinct or dissimilar terms in the expansion of $(x+y+z+w)^n$ is

$$= {}^{n+4-1}C_{4-1} = {}^{n+3}C_3 = \frac{(n+3)(n+2)(n+1)}{1 \cdot 2 \cdot 3}$$

$$= \frac{(n+1)(n+2)(n+3)}{6}$$

I. Aliter

We know that, $(x+y+z+w)^n = \{(x+y)+(z+w)\}^n$

$$= (x+y)^n + {}^nC_1 (x+y)^{n-1}(z+w) + {}^nC_2 (x+y)^{n-2}(z+w)^2 + \dots + {}^nC_n (z+w)^n$$

$\therefore$ Number of terms in RHS

$$= (n+1) + n\cdot 2 + (n-1)\cdot 3 + \dots + 1\cdot(n+1)$$

$$= \sum_{r=0}^{n} (n-r+1)(r+1)$$

$$= \sum_{r=0}^{n} (n+1) + nr - r^2 = (n+1)\sum_{r=0}^{n} 1 + n\sum_{r=0}^{n} r - \sum_{r=0}^{n} r^2$$

$$= (n+1)\cdot(n+1) + n\cdot\frac{n(n+1)}{2} - \frac{n(n+1)(2n+1)}{6}$$

$$= \frac{(n+1)(n+2)(n+3)}{6}$$

II. Aliter

$$(x+y+z+w)^n = \sum \frac{n!}{n_1!\, n_2!\, n_3!\, n_4!} x^{n_1} y^{n_2} z^{n_3} w^{n_4}$$

where, n_1, n_2, n_3, n_4 are non-negative integers subject to the condition $n_1 + n_2 + n_3 + n_4 = n$

Hence, number of the distinct terms

$$= \text{Coefficient of } x^n \text{ in } (x^0 + x^1 + x^2 + \dots + x^n)^4$$

$$= \text{Coefficient of } x^n \text{ in } \left(\frac{1-x^{n+1}}{1-x}\right)^4$$

$$= \text{Coefficient of } x^n \text{ in } (1-x^{n+1})^4 (1-x)^{-4}$$

$$= \text{Coefficient of } x^n \text{ in } (1-x)^{-4} \quad [\because x^{n+1} > x^n]$$

$$= {}^{n+3}C_n = {}^{n+3}C_3 = \frac{(n+3)(n+2)(n+1)}{6}$$

Greatest Coefficient in Multinomial Expansion

The greatest coefficient in the expansion of $(x_1 + x_2 + x_3 + \dots + x_k)^n$ is $\dfrac{n!}{(q!)^{k-r}((q+1)!)^r}$, where q is the quotient and r is the remainder when n is divided by k i.e.

$$k)\, n\, (q$$
$$\underline{\quad r\quad}$$

Example 56. ***Find the greatest coefficient in the expansion of*** $(a+b+c+d)^{15}$**.**

Sol. Here, $n = 15$ and $k = 4$ $\quad [\because a, b, c, d$ are four terms$]$

$$4)\,15\,(3$$
$$\underline{12}$$
$$\underline{3}$$

$\therefore \quad q = 3$ and $r = 3$

Hence, greatest coefficient $= \dfrac{15!}{(3!)^1 (4!)^3}$

Coefficient of x^r in Multinomial Expansion

If n is a positive integer and $a_1, a_2, a_3, \dots, a_k \in C$, then coefficient of x^r in the expansion of $(a_1 + a_2 x + a_3 x^2 + \dots + a_k x^{k-1})^n$, is

$$\sum \frac{n!}{(\alpha_1!)(\alpha_2!)(\alpha_3!)\dots(\alpha_k!)} a_1^{\alpha_1} a_2^{\alpha 2} a_3^{\alpha_3} \dots a_k^{\alpha_k}$$

where, $\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_k$ are non-negative integers such that $\alpha_1 + \alpha_2 + \alpha_3 + \dots + \alpha_k = n$

and $\quad \alpha_2 + 2\alpha_3 + 3\alpha_4 + \dots + (k-1)\alpha_k = r$

Example 57. Find the coefficient of x^7 in the expansion of $(1+3x-2x^3)^{10}$.

Sol. Coefficient of x^7 in the expansion of $(1+3x-2x^3)^{10}$ is

$$= \sum \frac{10!}{\alpha!\,\beta!\,\gamma!} (1)^\alpha (3)^\beta (-2)^\gamma$$

where, $\alpha + \beta + \gamma = 10$ and $\beta + 3\gamma = 7$

The possible values of α, β and γ are given below

α	β	γ
3	7	0
5	4	1
7	1	2

$\therefore$ Coefficient of x^7

$$= \frac{10!}{3!\,7!\,0!}(1)^3 (3)^7 (-2)^0 + \frac{10!}{5!\,4!\,1!}(1)^5 (3)^4 (-2)^1 + \frac{10!}{7!\,1!\,2!}(1)^7 (3)^1 (-2)^2$$

$$= 262440 - 204120 + 4320 = 62640$$

Use of Differentiation

This method applied only when the numericals occur as the product of the binomial coefficients, if

$$(1+x)^n = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \dots + C_n x^n$$

Solution Process

(i) If last term of the series leaving the plus or minus sign is m, then divide m by n. If q is the quotient and r is the remainder.

i.e. $\quad m = nq + r \quad$ or $\quad n)\, m\, (q$

$$\underline{nq}$$
$$\underline{r}$$

Then, replace x by x^q in the given series and multiplying both sides of the expression by x^r.

(ii) After this, differentiate both sides w.r.t. x and put $x=1$ or -1 or $i(i=\sqrt{-1})$, etc. According to the given series.

(iii) If product of two numericals (or square of numericals) or three numericals (or cube of numericals), then differentiate twice or thrice.

Example 58. *If*

$(1+x)^n = C_0 + C_1 x + C_2 x^2 + \ldots + C_n x^n$, ***prove that***

$C_1 + 2C_2 + 3C_3 + \ldots + nC_n = n \cdot 2^{n-1}$.

Sol. Here, last term of $C_1 + 2C_2 + 3C_3 + \ldots + nC_n$ is nC_n i.e., n and last term with positive sign.

Then, $n = n \cdot 1 + 0$ or $n)\, n\, (1$

$$\begin{array}{l} n \\ \hline 0 \\ \hline \end{array}$$

Here, $q = 1$ and $r = 0$

Then, the given series is

$$(1+x)^n = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \ldots + C_n x^n$$

Differentiating both sides w.r.t. x, we get

$$n(1+x)^{n-1} = 0 + C_1 + 2C_2 x + 3C_3 x^2 + \ldots + nC_n x^{n-1}$$

Putting $x = 1$, we get

$$n \cdot 2^{n-1} = C_1 + 2C_2 + 3C_3 + \ldots + nC_n$$

or $$C_1 + 2C_2 + 3C_3 + \ldots + nC_n = n \cdot 2^{n-1}$$

I. Aliter

$$C_1 + 2C_2 + 3C_3 + \ldots + nC_n$$

$$= n + 2 \cdot \frac{n(n-1)}{1 \cdot 2} + 3 \cdot \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} + \ldots + n \cdot 1$$

$$= n\left\{1 + (n-1) + \frac{(n-1)(n-2)}{1 \cdot 2} + \ldots + 1\right\}$$

Let $n - 1 = N$, then

$$\text{LHS} = (1+N)\left\{1 + N + \frac{N(N-1)}{1 \cdot 2} + \ldots + 1\right\}$$

$$= (1+N)\{1 + {}^N C_1 + {}^N C_2 + \ldots + {}^N C_N\}$$

$$= (1+N)\, 2^N = n \cdot 2^{n-1} = \text{RHS}$$

II. Aliter

$$\text{LHS} = C_1 + 2C_2 + 3C_3 + \ldots + nC_n = \sum_{r=1}^{n} r \cdot {}^n C_r$$

$$= \sum_{r=1}^{n} r \cdot \frac{n}{r} \cdot {}^{n-1}C_{r-1} \qquad \left[\because {}^n C_r = \frac{n}{r} \cdot {}^{n-1}C_{r-1}\right]$$

$$= n \sum_{r=1}^{n} {}^{n-1}C_{r-1}$$

$$= n({}^{n-1}C_0 + {}^{n-1}C_1 + {}^{n-1}C_2 + \ldots + {}^{n-1}C_{n-1}]$$

$$= n \cdot 2^{n-1} = \text{RHS}$$

Example 59. *If* $(1+x)^n = C_0 + C_1 x + C_2 x^2$

$+ \ldots + C_n x^n$, prove that

$C_0 + 2C_1 + 3C_2 + \ldots + (n+1)C_n = (n+2)2^{n-1}$.

Sol. Here, last term of $C_0 + 2C_1 + 3C_2 + \ldots + (n+1)C_n$ is $(n+1)C_n$ i.e., $(n+1)$ and last term with positive sign.

and $$n + 1 = n \cdot 1 + 1$$

or $$n\,)\, n+1\, (1$$

$$\begin{array}{l} -n \\ \hline 1 \\ \hline \end{array}$$

Here, $q = 1$ and $r = 1$

The given series is

$$(1+x)^n = C_0 + C_1 x + C_2 x^2 + \ldots + C_n x^n$$

Now, replacing x by x^1 and multiplying both sides by x, we get

$$x(1+x)^n = C_0 x + C_1 x^2 + C_2 x^3 + \ldots + C_n x^{n+1}$$

Differentiating both sides w.r.t. x, we get

$$x \cdot n(1+x)^{n-1} + (1+x)^n \cdot 1 = C_0 + 2C_1 x + 3C_2 x^2 + \ldots + (n+1)C_n x^n$$

Putting $x = 1$, we get

$$n(2)^{n-1} + 2^n = C_0 + 2C_1 + 3C_2 + \ldots + (n+1)C_n$$

or $$C_0 + 2C_1 + 3C_2 + \ldots + (n+1)C_n = (n+2)2^{n-1}$$

I. Aliter

$$\text{LHS} = C_0 + 2C_1 + 3C_2 + \ldots + (n+1)C_n$$

$$= C_0 + (1+1)C_1 + (1+2)C_2 + \ldots + (1+n)C_n$$

$$= (C_0 + C_1 + C_2 + \ldots + C_n) + (C_1 + 2C_2 + \ldots + nC_n)$$

[use example 58]

$$= 2^n + n \cdot 2^{n-1} = (n+2)2^{n-1} = \text{RHS}$$

II. Aliter

$$\text{LHS} = C_0 + 2C_1 + 3C_2 + \ldots + (n+1)C_n$$

$$= \sum_{r=1}^{n+1} r \cdot {}^n C_{r-1} = \sum_{r=1}^{n+1} (r-1+1) \cdot {}^n C_{r-1}$$

$$= \sum_{r=1}^{n+1} (r-1) \cdot {}^n C_{r-1} + {}^n C_{r-1}$$

$$= \sum_{r=1}^{n+1} n \cdot {}^{n-1}C_{r-2} + \sum_{r=1}^{n+1} {}^n C_{r-1}$$

$$\left[\because {}^n C_{r-1} = \frac{n}{r-1} \cdot {}^{n-1}C_{r-2}\right]$$

$$= n(0 + {}^{n-1}C_0 + {}^{n-1}C_1 + {}^{n-1}C_2 + \ldots + {}^{n-1}C_{n-1}) + ({}^n C_0 + {}^n C_1 + {}^n C_2 + \ldots + {}^n C_n)$$

$$= n \cdot 2^{n-1} + 2^n = (n+2) \cdot 2^{n-1} = \text{RHS}$$

Example 60. If $(1+x)^n = C_0 + C_1 x + C_2 x^2 + \ldots + C_n x^n$, prove that $C_0 + 3C_1 + 5C_2 + \ldots + (2n+1)C_n = (n+1)2^n$.

Sol. Here, last term of $C_0 + 3C_1 + 5C_2 + \ldots + (2n+1)C_n$ is $(2n+1)C_n$ i.e., $(2n+1)$ and last term with positive sign.

Then, $2n+1 = n \cdot 2 + 1$

or $n)\,2n+1\,(2$
$\quad -2n$
$\quad \overline{\ \ 1\ \ }$

Here, $q = 2$ and $r = 1$

The given series is

$$(1+x)^n = C_0 + C_1 x + C_2 x^2 + \ldots + C_n x^n$$

Now, replacing x by x^2, we get

$$(1+x^2)^n = C_0 + C_1 x^2 + C_2 x^4 + \ldots + C_n x^{2n}$$

On multiplying both sides by x^1, we get

$$x(1+x^2)^n = C_0 x + C_1 x^3 + C_2 x^5 + \ldots + C_n x^{2n+1}$$

On differentiating both sides w.r.t. x, we get

$$x \cdot n(1+x^2)^{n-1} \cdot 2x + (1+x^2)^n \cdot 1 = C_0 + 3C_1 x^2 + 5C_2 x^4 + \ldots + (2n+1) C_n x^{2n}$$

Putting $x = 1$, we get

$$n \cdot 2^{n-1} \cdot 2 + 2^n = C_0 + 3C_1 + 5C_2 + \ldots + (2n+1)C_n$$

or $\quad C_0 + 3C_1 + 5C_2 + \ldots + (2n+1)C_n = (n+1)2^n$

I. Aliter

$$\begin{aligned}\text{LHS} &= C_0 + 3C_1 + 5C_2 + \ldots + (2n+1)C_n \\ &= C_0 + (1+2)C_1 + (1+4)C_2 + \ldots + (1+2n)C_n \\ &= (C_0 + C_1 + C_2 + \ldots + C_n) + 2(C_1 + 2C_2 + \ldots + nC_n) \\ &= 2^n + 2 \cdot n \cdot 2^{n-1} = 2^n + n \cdot 2^n \qquad \text{[from Illusration 58]} \\ &= (n+1)2^n = \text{RHS}\end{aligned}$$

II. Aliter

$$\begin{aligned}\text{LHS} &= C_0 + 3C_1 + 5C_2 + \ldots + (2n+1)C_n \\ &= \sum_{r=0}^{n} (2r+1)\,{}^nC_r = \sum_{r=0}^{n} 2r \cdot {}^nC_r + \sum_{r=0}^{n} {}^nC_r \\ &= 2\sum_{r=0}^{n} r \cdot {}^nC_r + \sum_{r=0}^{n} {}^nC_r \\ &= 2\sum_{r=0}^{n} r \cdot \frac{n}{r} \cdot {}^{n-1}C_{r-1} + \sum_{r=0}^{n} {}^nC_r \quad \left[\because {}^nC_r = \frac{n}{r} \cdot {}^{n-1}C_{r-1}\right] \\ &= 2n \sum_{r=0}^{n} {}^{n-1}C_{r-1} + \sum_{r=0}^{n} {}^nC_r \\ &= 2n(0 + {}^{n-1}C_0 + {}^{n-1}C_1 + {}^{n-1}C_2 + \ldots + {}^{n-1}C_{n-1}) \\ &\quad + ({}^nC_0 + {}^nC_1 + {}^nC_2 + \ldots + {}^nC_n) \\ &= 2n \cdot 2^{n-1} + 2^n = (n+1) \cdot 2^n = \text{RHS}\end{aligned}$$

Example 61. If $(1+x)^n = C_0 + C_1 x + C_2 x^2 + \ldots + C_n x^n$, prove that $+ 3^2 \cdot C_3 + \ldots + n^2 \cdot C_n$ $1^2 \cdot C_1 + 2^2 \cdot C_2 = n(n+1) \cdot 2^{n-2}$.

Sol. Here, last term of $1^2 \cdot C_1 + 2^2 \cdot C_2 + 3^2 \cdot C_3 + \ldots + n^2 \cdot C_n$ is $n^2 \cdot C_n$ i.e., n^2. Linear factors of n^2 are n and n; [start always with greater factor] and last term with positive sign.

and $\quad n = n \cdot 1 + 0 \quad$ or $\quad n)\,n\,(1$
$\quad -n$
$\quad \overline{\ \ 0\ \ }$

Here, $q = 1$ and $r = 0$

Then, the given series is

$$(1+x)^n = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \ldots + C_n x^n$$

On differentiating both sides w.r.t. x, we get

$$nx(1+x)^{n-1} = C_1 + 2C_2 x + 3C_3 x^2 + \ldots + nC_n x^{n-1} \qquad \ldots\text{(i)}$$

and in last term, numerical is nC_n i.e., n and power of $(1+x)$ is $n-1$.

Then, $n = (n-1) \cdot 1 + 1 \quad$ or $\quad n-1)\,n\,(1$
$\quad n-1$
$\quad -\ +$
$\quad \overline{\ \ 1\ \ }$

Here, $q = 1$ and $r = 1$

Now, multiplying both sides by x in Eq. (i), then

$$nx(1+x)^{n-1} = C_1 x + 2C_2 x^2 + 3C_3 x^3 + \ldots + nC_n x^n$$

Differentiating on both sides w.r.t. x, we get

$$n\{x \cdot (n-1)(1+x)^{n-2} + (1+x)^{n-1} \cdot 1\} = C_1 \cdot 1 + 2^2 C_2 x + 3^2 C_3 x^2 + \ldots + n^2 C_n x^{n-1}$$

Putting $x = 1$, we get

$$n\{1 \cdot (n-1) \cdot 2^{n-2} + 2^{n-1}\} = 1^2 \cdot C_1 + 2^2 \cdot C_2 + 3^2 \cdot C_3 + \ldots + n^2 \cdot C_n$$

or $1^2 \cdot C_1 + 2^2 \cdot C_2 + 3^2 \cdot C_3 + \ldots + n^2 \cdot C_n = n(n+1)2^{n-2}$

Aliter

$$\begin{aligned}\text{LHS} &= 1^2 \cdot C_1 + 2^2 \cdot C_2 + 3^2 \cdot C_3 + \ldots + n^2 \cdot C_n \\ &= \sum_{r=1}^{n} r^2 \,.\, {}^nC_r = \sum_{r=1}^{n} r^2 \cdot \frac{n}{r} \cdot {}^{n-1}C_{r-1} \\ &\qquad \left[\because {}^nC_r = \frac{n}{r} \cdot {}^{n-1}C_{r-1}\right] \\ &= n\sum_{r=1}^{n} r \cdot {}^{n-1}C_{r-1} = n\sum_{r=1}^{n} \{(r-1)+1\} \cdot {}^{n-1}C_{r-1} \\ &= n\sum_{r=1}^{n} (r-1) \cdot {}^{n-1}C_{r-1} + n\sum_{r=1}^{n} {}^{n-1}C_{r-1} \\ &= n\sum_{r=1}^{n} (n-1) \cdot {}^{n-2}C_{r-2} + n\sum_{r=1}^{n} {}^{n-1}C_{r-1}\end{aligned}$$

$$= n(n-1)\sum_{r=1}^{n} {}^{n-2}C_{r-2} + n\sum_{r=1}^{n} {}^{n-1}C_{r-1}$$

$$= n(n-1)(0 + {}^{n-2}C_0 + {}^{n-2}C_1 + {}^{n-2}C_2 + \ldots + {}^{n-2}C_{n-2}) + n({}^{n-1}C_0 + {}^{n-1}C_1 + {}^{n-1}C_2 + \ldots + {}^{n-1}C_{n-1})$$

$$= n(n-1)\cdot 2^{n-2} + n\cdot 2^{n-1} = n(n+1)2^{n-2} = \text{RHS}$$

Example 62. *If* $(1+x)^n = C_0 + C_1 x + C_2 x^2 + \ldots + C_n x^n$, ***prove that*** $(1\cdot 2)C_2 + (2\cdot 3)C_3 + \ldots + \{(n-1)\cdot n\}C_n = n(n-1)2^{n-2}$.

Sol. Here, last term of

$(1\cdot 2)C_2 + (2\cdot 3)C_3 + \ldots + \{(n-1)\cdot n\}C_n$ is $(n-1)nC_n$

i.e. $(n-1)n$

[start with greater factor here greater factor is n] and last term with positive sign, then $n = n\cdot 1 + 0$

or $\quad n)\, n\, (1$

$\quad -n$

$\quad 0$

Here, $q = 1$ and $r = 0$

The given series is

$$(1+x)^n = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \ldots + C_n x^n$$

Differentiating on both sides w.r.t. x, we get

$$n(1+x)^{n-1} = 0 + C_1 + 2C_2 x + 3C_3 x^2 + \ldots + nC_n x^{n-1}$$

Again, differentiating on both sides w.r.t. x, we get

$$n(n-1)(1+x)^{n-2} = 0 + 0 + (1\cdot 2)C_2 + (2\cdot 3)C_3 x + \ldots + \{(n-1)\cdot n\}C_n x^{n-2}$$

Putting $x = 1$, we get

$$n(n-1)(1+1)^{n-2} = (1\cdot 2)C_2 + (2\cdot 3)C_3 + \ldots + \{(n-1)n\}\cdot C_n$$

or $(1\cdot 2)C_2 + (2\cdot 3)C_3 + \ldots + \{(n-1)n\}\cdot C_n = n(n-1)2^{n-2}$

I. Aliter

$$\text{LHS} = (1\cdot 2)C_2 + (2\cdot 3)C_3 + (3\cdot 4)C_4 + \ldots + \{(n-1)n\}\cdot C_n$$

$$= (1\cdot 2)\frac{n(n-1)}{1\cdot 2} + (2\cdot 3)\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3} + (3\cdot 4)\frac{n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\cdot 4} + \ldots + (n-1)n\cdot 1$$

$$= n(n-1)\left\{1 + \frac{(n-2)}{1} + \frac{(n-2)(n-3)}{1\cdot 2} + \ldots + 1\right\}$$

Now, in bracket, let $n-2 = N$, then

$$= n(n-1)\left\{1 + \frac{N}{1} + \frac{N(N-1)}{2!} + \ldots + 1\right\}$$

$$= n(n-1)\{{}^N C_0 + {}^N C_1 + \ldots + {}^N C_N\}$$

$$= n(n-1)2^N = n(n-1)2^{n-2} = \text{RHS}$$

II. Aliter

$$\text{LHS} = (1\cdot 2)C_2 + (2\cdot 3)C_3 + \ldots + \{(n-1)\cdot n\}C_n$$

$$= \sum_{r=2}^{n} (r-1)\cdot r\cdot {}^nC_r$$

$$= \sum_{r=2}^{n} (r-1)\cdot r\cdot \frac{n}{r}\cdot\frac{n-1}{(r-1)}\cdot {}^{n-2}C_{r-2}$$

$$= (n-1)n\sum_{r=2}^{n} {}^{n-2}C_{r-2}$$

$$= (n-1)n({}^{n-2}C_0 + {}^{n-2}C_1 + {}^{n-2}C_2 + \ldots + {}^{n-2}C_{n-2})$$

$$= (n-1)n\cdot 2^{n-2} = \text{RHS}$$

Example 63. If $(1+x)^n = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \ldots + C_n x^n$, prove that $C_0 - 2C_1 + 3C_2 - 4C_3 + \ldots + (-1)^n (n+1)C_n = 0$.

Sol. Numerical value of last term of

$C_0 - 2C_1 + 3C_2 - 4C_3 + \ldots + (-1)^n (n+1)C_n$ is $(n+1)C_n$ i.e., $(n+1)$, then

$n + 1 = n\cdot 1 + 1 \quad$ or $\quad n)\, n+1\, (1$

$\quad -n$

$\quad 1$

Here, $q = 1$ and $r = 1$

The given series is

$$(1+x)^n = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \ldots + C_n x^n$$

On multiplying both sides by x, we get

$$x(1+x)^n = C_0 x + C_1 x^2 + C_2 x^3 + C_3 x^4 + \ldots + C_n x^{n+1}$$

On differentiating both sides w.r.t. x, we get

$$x\cdot n(1+x)^{n-1} + (1+x)^n\cdot 1 = C_0 + 2C_1 x + 3C_2 x^2 + 4C_3 x^3 + \ldots + (n+1)C_n x^n$$

Putting $x = -1$, we get

$$0 = C_0 - 2C_1 + 3C_2 - 4C_3 + \ldots + (-1)^n (n+1)C_n$$

or $\quad C_0 - 2C_1 + 3C_2 - 4C_3 + \ldots + (-1)^n (n+1)C_n = 0$

I. Aliter

$$\text{LHS} = C_0 - 2C_1 + 3C_2 - 4C_3 + \ldots + (-1)^n (n+1)C_n$$

$$= C_0 - (C_1 + C_1) + (C_2 + 2C_2) - (C_3 + 3C_3) + \ldots + (-1)^n\{C_n + nC_n\}$$

$$= \{C_0 - C_1 + C_2 - C_3 + \ldots + (-1)^n C_n\} + \{-C_1 + 2C_2 - 3C_3 + \ldots + (-1)^n nC_n\}$$

$$= (1-1)^n + \left\{-n + 2\cdot\frac{n(n-1)}{1\cdot 2} - 3\,\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3} + \ldots + (-1)^n\cdot n\right\}$$

$$= 0 + n\left\{-1 + (n-1) - \frac{(n-1)(n-2)}{1\cdot 2} + \ldots + (-1)^n\right\}$$

$$= 0 - n\left\{1 - (n-1) + \frac{(n-1)(n-2)}{1\cdot 2} - \ldots + (-1)^{n-1}\right\}$$

Let in bracket, put $n - 1 = N$, we get

$$\text{LHS} = 0 - n\left\{1 - N + \frac{N(N-1)}{1\cdot 2} - \ldots + (-1)^N\right\}$$

$$= 0 - n\{{}^N C_0 - {}^N C_1 + {}^N C_2 - \ldots + (-1)^N\ {}^N C_N\}$$

$$= 0 - n(1-1)^N = 0 - 0 = 0 = \text{RHS}$$

II. Aliter

$$\text{LHS} = C_0 - 2C_1 + 3C_2 - 4C_3 + \ldots + (-1)^n (n+1) C_n$$

$$= \sum_{r=0}^{n} (-1)^r (r+1)\ {}^n C_r = \sum_{r=0}^{n} (-1)^r [r \cdot {}^n C_r + {}^n C_r]$$

$$= \sum_{r=0}^{n} (-1)^r [n\ {}^{n-1}C_{r-1} + {}^n C_r] \quad \left[\because {}^n C_r = \frac{n}{r}\ {}^{n-1}C_{r-1}\right]$$

$$= n\sum_{r=0}^{n} (-1)^r \cdot {}^{n-1}C_{r-1} + \sum_{r=0}^{n} (-1)^r \cdot {}^n C_r$$

$$= -n\sum_{r=0}^{n} (-1)^{r-1} \cdot {}^{n-1}C_{r-1} + \sum_{r=0}^{n} (-1)^r \cdot {}^n C_r$$

$$= -n(1-1)^{n-1} + (1-1)^n = 0 + 0 = 0 = \text{RHS}$$

Example 64. If $(1+x)^n = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \ldots + C_n x^n$, prove that $C_1 - 2C_2 + 3C_3 - \ldots + (-1)^{n-1} n C_n = 0$.

Sol. Numerical value of last term of

$C_1 - 2C_2 + 3C_3 - \ldots + (-1)^{n-1} n C_n$ is nC_n i.e., n, then

and $\quad n = n \cdot 1 + 0 \quad$ or $\quad n)\,n\,(1$

$$\begin{array}{r} -n \\ \hline 1 \end{array}$$

Here, $q = 1$ and $r = 0$

The given series is

$$(1+x)^n = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \ldots + C_n x^n$$

On differentiating both sides w.r.t. x, we get

$$n(1+x)^{n-1} = 0 + C_1 + 2C_2 x + 3C_3 x^2 + \ldots + n C_n x^{n-1}$$

Putting $\quad x = -1$, we get

$$0 = C_1 - 2C_2 + 3C_3 - \ldots + (-1)^{n-1} n C_n$$

or $\quad C_1 - 2C_2 + 3C_3 - \ldots + (-1)^{n-1} n C_n = 0$

I. Aliter

$$\text{LHS} = C_1 - 2C_2 + 3C_3 - \ldots + (-1)^{n-1} n \cdot C_n$$

$$= n - 2\cdot\frac{n(n-1)}{1\cdot 2} + 3\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3} - \ldots + (-1)^{n-1}\cdot n \cdot 1$$

$$= n\left\{1 - \frac{(n-1)}{1} + \frac{(n-1)(n-2)}{1\cdot 2} - \ldots + (-1)^{n-1}\right\}$$

In bracket, put $n - 1 = N$, then

$$\text{LHS} = n\left\{1 - \frac{N}{1} + \frac{N(N-1)}{1\cdot 2} - \ldots + (-1)^N\right\}$$

$$= n\{{}^N C_0 - {}^N C_1 + {}^N C_2 - \ldots + (-1)^N\ {}^N C_N\}$$

$$= n(1-1)^N = 0 = \text{RHS}$$

II. Aliter

$$\text{LHS} = C_1 - 2C_2 + 3C_3 - \ldots + (-1)^{n-1} \cdot n\, C_n$$

$$= \sum_{r=1}^{n} (-1)^{r-1} \cdot r \cdot {}^n C_r$$

$$= \sum_{r=1}^{n} (-1)^{r-1} \cdot n \cdot {}^{n-1}C_{r-1} \quad \left[\because {}^n C_r = \frac{n}{r} \cdot {}^{n-1}C_{r-1}\right]$$

$$= n\sum_{r=1}^{n} (-1)^{r-1} \cdot {}^{n-1}C_{r-1}$$

$$= n(1-1)^{n-1} = 0 = \text{RHS}$$

Example 65. *If* $(1+x)^n = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \ldots + C_n x^n$, *prove that* $C_0 - 3C_1 + 5C_2 - \ldots + (-1)^n (2n+1) C_n = 0$.

Sol. The numerical value of last term of

$C_0 - 3C_1 + 5C_2 - \ldots + (-1)^n (2n+1) C_n$ is $(2n+1) C_n$

i.e. $\quad (2n+1)$

and $\quad 2n + 1 = 2 \cdot n + 1 \quad$ or $\quad n)\,2n+1\,(2$

$$\begin{array}{r} -2n \\ \hline 1 \end{array}$$

Here, $q = 2$ and $r = 1$

The given series is

$(1+x)^n = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \ldots + C_n x^n$ now,

replacing x by x^2, then we get

$$(1+x^2)^n = C_0 + C_1 x^2 + C_2 x^4 + \ldots + C_n x^{2n}$$

On multiplying both sides by x, we get

$$x(1+x^2)^n = C_0 x + C_1 x^3 + C_2 x^5 + \ldots + C_n x^{2n+1}$$

On differentiating both sides w.r.t. x, we get

$$x \cdot n(1+x^2)^{n-1} 2x + (1+x^2)^n \cdot 1 = C_0 + 3C_1 x^2 + 5C_2 x^4 + \ldots + (2n+1) C_n x^{2n}$$

Putting $x = i$ in both sides, we get

$$0 + 0 = C_0 - 3C_1 + 5C_2 - \ldots + (2n+1)(-1)^n C_n$$

or $\quad C_0 - 3C_1 + 5C_2 - \ldots + (-1)^n (2n+1) C_n = 0$

I. Aliter

$$\text{LHS} = C_0 - 3C_1 + 5C_2 - \ldots + (-1)^n (2n+1) C_n$$

$$= C_0 - (1+2) C_1 + (1+4) C_2 - \ldots + (-1)^n (1+2n) C_n$$

$$= (C_0 - C_1 + C_2 - \ldots + (-1)^n C_n) - 2(C_1 - 2C_2 + \ldots + (-1)^{n-1} n \cdot C_n)$$

$$= (1-1)^n - 2\cdot 0 \qquad \text{[from Example 64]}$$
$$= 0 = \text{RHS}$$

II. Aliter

$$\text{LHS} = C_0 - 3C_1 + 5C_2 - \dots + (-1)^n (2n+1) C_n$$
$$= \sum_{r=1}^{n} (-1)^r (2r+1)\, {}^nC_r = \sum_{r=1}^{n} (-1)^r [2r \cdot {}^nC_r + {}^nC_r]$$
$$= 2\sum_{r=1}^{n} n \cdot {}^{n-1}C_{r-1} + \sum_{r=1}^{n} (-1)^r \cdot {}^nC_r$$
$$= 2n(1-1)^{n-1} + (1-1)^n = 0 + 0 = 0 = \text{RHS}$$

Use of Integration

This method is applied only when the numericals occur as the denominator of the binomial coefficient.

Solution Process

If $(1+x)^n = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \dots + C_n x^n$, then integrate both sides between the suitable limits which gives the required series.

1. If the sum contains $C_0, C_1, C_2, \dots, C_n$ are all positive signs, then integrate between limits 0 to 1.
2. If the sum contains alternate signs (i.e., +, −), then integrate between limits −1 to 0.
3. If the sum contains odd coefficients (i.e., $C_0, C_2, C_4, \dots$), then integrate between −1 to +1.
4. If the sum contains even coefficients (i.e., $C_1, C_3, C_5, \dots$), then subtracting (2) from (1) and then dividing by 2.
5. If in denominator of binomial coefficient product of two numericals, then integrate two times first times taken limits between 0 to x and second times take suitable limits.

Example 66. If $(1+x)^n = C_0 + C_1 x + C_2 x^2 + \dots + C_n x^n$, prove that

$$C_0 + \frac{C_1}{2} + \frac{C_2}{3} + \dots + \frac{C_n}{n+1} = \frac{2^{n+1}-1}{n+1}.$$

Sol. $\because (1+x)^n = C_0 + C_1 x + C_2 x^2 + \dots + C_n x^n$ ...(i)

Integrating both sides of Eq. (i) within limits 0 to 1, then we get

$$\int_0^1 (1+x)^n\, dx = \int_0^1 (C_0 + C_1 x + C_2 x^2 + \dots + C_n x^n)\, dx$$

$$\left[\frac{(1+x)^{n+1}}{n+1}\right]_0^1$$

$$\Rightarrow \qquad = \left[C_0 x + \frac{C_1 x^2}{2} + \frac{C_2 x^3}{3} + \dots + \frac{C_n x^{n+1}}{n+1}\right]_0^1$$

$$\Rightarrow \frac{2^{n+1}-1}{n+1} = C_0 + \frac{C_1}{2} + \frac{C_2}{3} + \dots + \frac{C_n}{n+1}$$

or $\quad C_0 + \dfrac{C_1}{2} + \dfrac{C_2}{3} + \dots + \dfrac{C_n}{n+1} = \dfrac{2^{n+1}-1}{n+1}$

I. Aliter

$$\text{LHS} = C_0 + \frac{C_1}{2} + \frac{C_2}{3} + \dots + \frac{C_n}{n+1}$$
$$= 1 + \frac{n}{1\cdot 2} + \frac{n(n-1)}{1\cdot 2\cdot 3} + \dots + \frac{1}{n+1}$$
$$= \frac{1}{n+1}\left[(n+1) + \frac{(n+1)n}{1\cdot 2} + \frac{(n+1)n(n-1)}{1\cdot 2\cdot 3} + \dots + 1\right]$$

Put $n+1 = N$, then

$$\text{LHS} = \frac{1}{N}\left[N + \frac{N(N-1)}{2!} + \frac{N(N-1)(N-2)}{3!} + \dots + 1\right]$$
$$= \frac{1}{N}[{}^NC_1 + {}^NC_2 + {}^NC_3 + \dots + {}^NC_N]$$
$$= \frac{1}{N}[(1+1)^N - 1] = \frac{2^N - 1}{N} = \frac{2^{n+1}-1}{n+1} = \text{RHS}$$

II. Aliter

$$\text{LHS} = C_0 + \frac{C_1}{2} + \frac{C_2}{3} + \dots + \frac{C_n}{n+1} = \sum_{r=0}^{n} \frac{C_r}{r+1}$$
$$= \sum_{r=0}^{n} \frac{{}^nC_r}{(r+1)} = \sum_{r=0}^{n} \frac{{}^{n+1}C_{r+1}}{(n+1)} \quad \left[\because \frac{{}^{n+1}C_{r+1}}{n+1} = \frac{{}^nC_r}{r+1}\right]$$
$$= \frac{1}{(n+1)} \sum_{r=0}^{n} {}^{n+1}C_{r+1}$$
$$= \frac{1}{(n+1)} ({}^{n+1}C_1 + {}^{n+1}C_2 + {}^{n+1}C_3 + \dots + {}^{n+1}C_{n+1})$$
$$= \frac{1}{n+1}(2^{n+1}-1) = \frac{2^{n+1}-1}{n+1} = \text{RHS}$$

Example 67. If $(1+x)^n = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \dots + C_n x^n$, prove that

$$C_0 - \frac{C_1}{2} + \frac{C_2}{3} - \dots + (-1)^n \frac{C_n}{n+1} = \frac{1}{n+1}.$$

Sol. $\because (1+x)^n = C_0 + C_1 x + C_2 x^2 + \dots + C_n x^n$ ...(i)

Integrating on both sides of Eq. (i) within limits − 1 to 0, then we get

$$\int_{-1}^{0} (1+x)^n\, dx = \int_{-1}^{0} (C_0 + C_1 x + C_2 x^2 + \dots + C_n x^n)\, dx$$

$$\Rightarrow\left[\frac{(1+x)^n}{n+1}\right]_{-1}^{0}=\left[C_0x+\frac{C_1x^2}{2}+\frac{C_2x^3}{3}+\ldots+\frac{C_nx^{n+1}}{n+1}\right]_{-1}^{0}$$

$$\Rightarrow\quad \frac{1-0}{n+1}=0-\left(-C_0+\frac{C_1}{2}-\frac{C_2}{3}+\ldots+(-1)^{n+1}\frac{C_n}{n+1}\right)$$

$$\Rightarrow\quad \frac{1}{n+1}=C_0-\frac{C_1}{2}+\frac{C_2}{3}-\ldots+(-1)^{n+2}\frac{C_n}{n+1}$$

$$\Rightarrow\quad \frac{1}{n+1}=C_0-\frac{C_1}{2}+\frac{C_2}{3}-\ldots+(-1)^n\frac{C_n}{n+1}$$

$$[\because(-1)^{n+2}=(-1)^n(-1)^2=(-1)^n]$$

Hence, $C_0-\frac{C_1}{2}+\frac{C_2}{3}-\ldots+(-1)^n\frac{C_n}{n+1}=\frac{1}{n+1}$

I. Aliter

$$\text{LHS}=C_0-\frac{C_1}{2}+\frac{C_2}{3}-\ldots+(-1)^n\frac{C_n}{n+1}$$

$$=1-\frac{n}{2}+\frac{n(n-1)}{1\cdot2\cdot3}-\ldots+(-1)^n\frac{1}{n+1}=\frac{1}{(n+1)}$$

$$\left[(n+1)-\frac{(n+1)n}{1\cdot2}+\frac{(n+1)n(n-1)}{1\cdot2\cdot3}-\ldots+(-1)^n\right]$$

Put $n+1=N$, we get

$$=\frac{1}{N}\left[N-\frac{N(N-1)}{1\cdot2}+\frac{N(N-1)(N-2)}{1\cdot2\cdot3}-\ldots+(-1)^{N-1}\right]$$

$$=\frac{1}{N}[{}^NC_1-{}^NC_2+{}^NC_3-\ldots+(-1)^{N-1}]$$

$$=-\frac{1}{N}[-{}^NC_1+{}^NC_2-{}^NC_3+\ldots+(-1)^N\,{}^NC_N]$$

$$=-\frac{1}{N}[{}^NC_0-{}^NC_1+{}^NC_2-{}^NC_3+\ldots+(-1)^N\,{}^NC_N-{}^NC_0]$$

$$=-\frac{1}{N}[(1-1)^N-{}^NC_0]=-\frac{1}{N}[0-1]=\frac{1}{N}$$

$$=\frac{1}{n+1}=\text{RHS}$$

II. Aliter

$$\text{LHS}=C_0-\frac{C_1}{2}+\frac{C_2}{3}-\ldots+(-1)^n\frac{C_n}{n+1}=\sum_{r=0}^{n}\frac{(-1)^r\cdot C_r}{r+1}$$

$$=\sum_{r=0}^{n}\frac{(-1)^r\cdot{}^nC_r}{r+1}$$

$$=\sum_{r=0}^{n}(-1)^r\cdot\frac{{}^{n+1}C_{r+1}}{(n+1)}\qquad\left[\because\frac{{}^{n+1}C_{r+1}}{n+1}=\frac{{}^nC_r}{r+1}\right]$$

$$=\frac{1}{(n+1)}\sum_{r=0}^{n}(-1)^r\cdot{}^{n+1}C_{r+1}$$

$$=\frac{1}{(n+1)}({}^{n+1}C_1-{}^{n+1}C_2+{}^{n+1}C_3-\ldots+(-1)^n\cdot{}^{n+1}C_{n+1})$$

$$=\frac{1}{(n+1)}\{{}^{n+1}C_0-({}^{n+1}C_0-{}^{n+1}C_1+{}^{n+1}C_2-{}^{n+1}C_3+\ldots+(-1)^{n+1}\,{}^{n+1}C_{n+1})\}$$

$$=\frac{1}{(n+1)}[1-(1-1)^{n+1}]=\frac{1}{(n+1)}[1-0]=\frac{1}{n+1}=\text{RHS}$$

Example 68. If $(1+x)^n=C_0+C_1x+C_2x^2+C_3x^3+\ldots+C_nx^n$, prove that $\frac{C_0}{1}+\frac{C_2}{3}+\frac{C_4}{5}+\ldots=\frac{2^n}{n+1}$.

Sol. $\because\ (1+x)^n=C_0+C_1x+C_2x^2+C_3x^3+C_4x^4+\ldots+C_nx^n\quad\ldots(\text{i})$

Integrating on both sides of Eq. (i) within limits -1 to 1, then we get

$$\int_{-1}^{1}(1+x)^n\,dx=\int_{-1}^{1}(C_0+C_1x+C_2x^2+C_3x^3+C_4x^4+\ldots+C_nx^n)\,dx$$

$$=\int_{-1}^{1}(C_0+C_2x^2+C_4x^4+\ldots)dx+\int_{-1}^{1}(C_1x+C_3x^3+\ldots)dx$$

$$=2\int_{0}^{1}(C_0+C_2x^2+C_4x^4+\ldots)\,dx+0$$

[by property of definite integral]

[since, second integral contains odd function]

$$\left[\frac{(1+x)^{n+1}}{n+1}\right]_{-1}^{1}=2\left[\left(C_0x+\frac{C_2x^3}{3}+\frac{C_4x^5}{5}+\ldots\right)\right]_0^1$$

$$\Rightarrow\qquad\frac{2^{n+1}}{n+1}=2\left(C_0+\frac{C_2}{3}+\frac{C_4}{5}+\ldots\right)$$

or $C_0+\frac{C_2}{3}+\frac{C_4}{5}+\ldots=\frac{2^n}{n+1}$

I. Aliter

$$\text{LHS}=C_0+\frac{C_2}{3}+\frac{C_4}{5}+\ldots$$

$$=1+\frac{n(n-1)}{1\cdot2\cdot3}+\frac{n(n-1)(n-2)(n-3)}{1\cdot2\cdot3\cdot4\cdot5}+\ldots$$

$$=\frac{1}{(n+1)}\left\{\frac{n+1}{1}+\frac{(n+1)n(n-1)}{1\cdot2\cdot3}+\frac{(n+1)n(n-1)(n-2)(n-3)}{1\cdot2\cdot3\cdot4\cdot5}+\ldots\right\}$$

$$=\frac{1}{n+1}\{{}^{n+1}C_1+{}^{n+1}C_3+{}^{n+1}C_5+\ldots\}$$

$$=\frac{1}{(n+1)}[\text{sum of even binomial coefficients of }(1+x)^{n+1}]$$

$$=\frac{2^{n+1-1}}{n+1}=\frac{2^n}{n+1}=\text{RHS}$$

II. Aliter LHS $= \dfrac{C_0}{1} + \dfrac{C_2}{3} + \dfrac{C_4}{5} + \ldots$

***Case* I** If n is odd say $n = 2m+1, \forall\ m \in W$, then

$$\text{LHS} = \sum_{r=0}^{m} \frac{{}^{2m+1}C_{2r}}{2r+1} = \sum_{r=0}^{m} \frac{{}^{2m+2}C_{2r+1}}{(2m+1)}$$

$$\left[\because \frac{{}^{2m+1}C_{2r}}{2r+1} = \frac{{}^{2m+2}C_{2r+1}}{2m+1}\right]$$

$$= \frac{1}{(2m+1)} \cdot 2^{2m+2-1} = \frac{2^n}{n+1} = \text{RHS}$$

$$[\because n = 2m+1]$$

***Case* II** If n is even say $n = 2m, \forall\ m \in N$, then

$$\text{LHS} = \sum_{r=0}^{m} \frac{{}^{2m}C_{2r}}{2r+1} = \sum_{r=0}^{m} \frac{{}^{2m+1}C_{2r+1}}{(2m+1)}$$

$$\left[\because \frac{{}^{2m+1}C_{2r+1}}{2m+1} = \frac{{}^{2m}C_{2r}}{2r+1}\right]$$

$$= \frac{2^{2m+1-1}}{2m+1} = \frac{2^n}{n+1} = \text{RHS} \qquad [\because n = 2m]$$

Example 69. If $(1+x)^n = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \ldots + C_n x^n$, prove that $\dfrac{C_1}{2} + \dfrac{C_3}{4} + \dfrac{C_5}{6} + \ldots = \dfrac{2^n - 1}{n+1}$.

Sol. We know that, from Examples (66) and (67)

$$C_0 + \frac{C_1}{2} + \frac{C_2}{3} + \frac{C_3}{4} + \frac{C_4}{5} + \frac{C_5}{6} + \ldots = \frac{2^{n+1} - 1}{n+1} \qquad \ldots(\text{i})$$

and $$C_0 - \frac{C_1}{2} + \frac{C_2}{3} - \frac{C_3}{4} + \frac{C_4}{5} - \frac{C_5}{6} + \ldots = \frac{1}{n+1} \qquad \ldots(\text{ii})$$

On subtracting Eq. (ii) from Eq. (i), we get

$$2\left(\frac{C_1}{2} + \frac{C_3}{4} + \frac{C_5}{6} + \ldots\right) = \frac{2^{n+1} - 2}{n+1}$$

On dividing each sides by 2, we get

$$\frac{C_1}{2} + \frac{C_3}{4} + \frac{C_5}{6} + \ldots = \frac{2^n - 1}{n+1}$$

I. Aliter LHS $= \dfrac{C_1}{2} + \dfrac{C_3}{4} + \dfrac{C_5}{6} + \ldots$

$$= \frac{n}{1\cdot 2} + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3\cdot 4} + \frac{n(n-1)(n-2)(n-3)(n-4)}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6} + \ldots$$

$$= \frac{1}{n+1}\left[\frac{(n+1)n}{1\cdot 2} + \frac{(n+1)n(n-1)(n-2)}{1\cdot 2\cdot 3\cdot 4} + \frac{(n+1)n(n-1)(n-2)(n-3)(n-4)}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6} + \ldots\right]$$

Put $n + 1 = N$, then

$$\text{LHS} = \frac{1}{N}\left[\frac{N(N-1)}{2!} + \frac{N(N-1)(N-2)(N-3)}{4!} + \frac{N(N-1)(N-2)(N-3)(N-4)(N-5)}{6!} + \ldots\right]$$

$$= \frac{1}{N}[{}^NC_2 + {}^NC_4 + {}^NC_6 + \ldots]$$

$$= \frac{1}{N}[({}^NC_0 + {}^NC_2 + {}^NC_4 + {}^NC_6 + \ldots) - {}^NC_0]$$

$$= \frac{1}{N}[2^{N-1} - 1] = \frac{2^n - 1}{n+1} = \text{RHS}$$

II. Aliter

$$\text{LHS} = \frac{C_1}{2} + \frac{C_3}{4} + \frac{C_5}{6} + \ldots$$

***Case* I** If n is odd say $n = 2m+1, \forall\ m \in W$, then

$$\text{LHS} = \sum_{r=0}^{m} \frac{{}^{2m+1}C_{2r+1}}{2r+2} = \sum_{r=0}^{m} \frac{{}^{2m+2}C_{2r+2}}{(2m+2)}$$

$$\left[\because \frac{{}^{2m+2}C_{2r+2}}{2m+2} = \frac{{}^{2m+1}C_{2r+1}}{2r+2}\right]$$

$$= \frac{1}{(2m+2)}({}^{2m+2}C_2 + {}^{2m+2}C_4 + \ldots + {}^{2m+2}C_{2m+2})$$

$$= \frac{1}{(2m+2)} \cdot (2^{2m+2-1} - {}^{2m+1}C_0) = \frac{2^n - 1}{n+1} \qquad [\because 2m+1 = n]$$

$= \text{RHS}$

***Case* II** If n is even say $n = 2m, \forall\ m \in N$, then

$$\text{LHS} = \sum_{r=0}^{m-1} \frac{{}^{2m}C_{2r+1}}{(2r+2)} = \sum_{r=0}^{m-1} \frac{{}^{2m+1}C_{2r+2}}{(2m+1)}$$

$$\left[\because \frac{{}^{2m+1}C_{2r+2}}{2m+1} = \frac{{}^{2m}C_{2r+1}}{2r+2}\right]$$

$$= \frac{1}{(2m+1)} \sum_{r=0}^{m-1} {}^{2m+1}C_{2r+2}$$

$$= \frac{1}{(2m+1)}({}^{2m+1}C_2 + {}^{2m+1}C_4 + {}^{2m+1}C_6 + \ldots + {}^{2m+1}C_{2n})$$

$$= \frac{1}{(2m+1)} \cdot (2^{2m+1-1} - {}^{2m+1}C_0)$$

$$= \frac{2^n - 1}{n+1} = \text{RHS} \qquad [\because n = 2m]$$

Example 70. *If* $(1+x)^n = C_0 + C_1 x + C_2 x^2 + \ldots + C_n x^n$, ***prove that***

$$3C_0 + 3^2 \frac{C_1}{2} + \frac{3^3 C_2}{3} + \frac{3^4 C_3}{4} + \ldots + \frac{3^{n+1} C_n}{n+1} = \frac{4^{n+1} - 1}{n+1}.$$

Sol. $\because \quad (1+x)^n = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \ldots + C_n x^n \quad \ldots(\text{i})$

Integrating on both sides of Eq. (i) within limits 0 to 3, we get

$$\int_0^3 (1+x)^n\,dx = \int_0^3 (C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \ldots + C_n x^n)\,dx$$

$$\Rightarrow \left[\frac{(1+x)^{n+1}}{n+1}\right]_0^3 = \left[C_0 x + \frac{C_1 x^2}{2} + \frac{C_2 x^3}{3} + \frac{C_3 x^4}{4} + \ldots + \frac{C_n x^{n+1}}{n+1}\right]_0^3$$

$$\Rightarrow \frac{4^{n+1}-1}{n+1} = 3C_0 + \frac{3^2 C_1}{2} + \frac{3^3 C_2}{3} + \frac{3^4 C_3}{4} + \ldots + \frac{3^{n+1} C_n}{n+1}$$

Hence,

$$3C_0 + \frac{3^2 C_1}{2} + \frac{3^3 C_2}{3} + \frac{3^4 C_3}{4} + \ldots + \frac{3^{n+1} C_n}{n+1} = \frac{4^{n+1}-1}{n+1}$$

I. Aliter

$$\text{LHS} = 3C_0 + \frac{3^2 C_1}{2} + \frac{3^3 C_2}{3} + \frac{3^4 C_3}{4} + \ldots + \frac{3^{n+1} C_n}{n+1}$$

$$= 3\cdot 1 + \frac{3^2 \cdot n}{2} + \frac{3^3 \cdot n(n-1)}{1\cdot 2\cdot 3} + \frac{3^4 \cdot n(n-1)(n-2)}{1\cdot 2\cdot 3\cdot 4} + \ldots + \frac{3^{n+1}}{n+1}$$

$$= \frac{1}{(n+1)}\left[3\cdot(n+1) + \frac{3^2 (n+1)\, n}{1\cdot 2} + \frac{3^3 (n+1)\, n(n-1)}{1\cdot 2\cdot 3} + \frac{3^4 (n+1)\, n(n-1)(n-2)}{1\cdot 2\cdot 3\cdot 4} + \ldots + 3^{n+1}\right]$$

Put $n+1 = N$, then

$$\text{LHS} = \frac{1}{N}\left[3N + \frac{3^2 N(N-1)}{2!} + \frac{3^3 N(N-1)(N-2)}{3!} + \frac{3^4 N(N-1)(N-2)(N-3)}{4!} + \ldots + 3^N\right]$$

$$= \frac{1}{N}\left[{}^N C_1 (3) + {}^N C_2 (3)^2 + {}^N C_3 (3)^3 + \ldots + {}^N C_N (3)^N\right]$$

$$= \frac{1}{N}\left[{}^N C_0 + {}^N C_1 (3) + {}^N C_2 (3)^2 + {}^N C_3 (3)^3 + \ldots + {}^N C_N (3)^N - {}^N C_0\right]$$

$$= \frac{1}{N}\{(1+3)^N - 1\} = \frac{4^N - 1}{N} = \frac{4^{n+1}-1}{n+1} = \text{RHS}$$

II. Aliter

$$\text{LHS} = 3C_0 + 3^2 \frac{C_1}{2} + \frac{3^3 C_2}{3} + \frac{3^4 C_3}{4} + \ldots + \frac{3^{n+1} C_n}{n+1}$$

$$= \sum_{r=0}^{n} \frac{3^{r+1} \cdot {}^n C_r}{(r+1)} = \sum_{r=0}^{n} \frac{3^{r+1} \cdot {}^{n+1} C_{r+1}}{(n+1)} \quad \left[\because \frac{{}^{n+1}C_{r+1}}{n+1} = \frac{{}^n C_r}{r+1}\right]$$

$$= \frac{1}{(n+1)} \sum_{r=0}^{n} {}^{n+1}C_{r+1} \cdot 3^{r+1}$$

$$= \frac{1}{(n+1)}\left({}^{n+1}C_1 \cdot 3 + {}^{n+1}C_2 \cdot 3^2 + {}^{n+1}C_3 \cdot 3^3 + \ldots + {}^{n+1}C_{n+1} \cdot 3^{n+1}\right)$$

$$= \frac{1}{(n+1)}\left[(1+3)^{n+1} - {}^{n+1}C_0\right]$$

$$= \frac{4^{n+1}-1}{n+1} = \text{RHS}$$

Example 71. If $(1+x)^n = C_0 + C_1 x + C_2 x^2 + \ldots + C_n x^n$, show that $\frac{2^2}{1\cdot 2} C_0 + \frac{2^3}{2\cdot 3} C_1 + \frac{2^4}{3\cdot 4} C_2 + \ldots + \frac{2^{n+2} C_n}{(n+1)(n+2)} = \frac{3^{n+2} - 2n - 5}{(n+1)(n+2)}$.

Sol. Given,

$$(1+x)^n = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \ldots + C_n x^n \quad \ldots(\text{i})$$

Integrating both sides of Eq. (i) within limits 0 to x, we get

$$\int_0^x (1+x)^n\,dx = \int_0^x (C_0 + C_1 x + C_2 x^2 + \ldots + C_n x^n)\,dx$$

$$\Rightarrow \left[\frac{(1+x)^{n+1}}{n+1}\right]_0^x = \left[C_0 x + \frac{C_1 x^2}{2} + \frac{C_2 x^3}{3} + \ldots + \frac{C_n x^{n+1}}{n+1}\right]_0^x$$

$$\Rightarrow \frac{(1+x)^{n+1} - 1}{(n+1)} = C_0 x + \frac{C_1 x^2}{2} + \frac{C_2 x^3}{3} + \ldots + \frac{C_n x^{n+1}}{n+1} \quad \ldots(\text{ii})$$

Again, integrating both sides of Eq. (ii) within limits 0 to 2, we get

$$\int_0^2 \frac{(1+x)^{n+1} - 1}{(n+1)}\,dx = \int_0^2 \left(C_0 x + \frac{C_1 x^2}{2} + \frac{C_2 x^3}{3} + \ldots + \frac{C_n x^{n+1}}{n+1}\right) dx$$

$$\Rightarrow \frac{1}{(n+1)}\left(\frac{(1+x)^{n+2}}{n+2} - x\right)\Bigg]_0^2 = \left[\frac{C_0 x^2}{1\cdot 2} + \frac{C_1 x^3}{2\cdot 3} + \frac{C_2 x^4}{3\cdot 4} + \ldots + \frac{C_n x^{n+2}}{(n+1)(n+2)}\right]_0^2$$

$$\Rightarrow \frac{1}{(n+1)}\left\{\frac{3^{n+2}}{n+2} - 2 - \frac{1}{n+2}\right\} = \frac{2^2}{1\cdot 2} C_0 + \frac{2^3}{2\cdot 3} C_1 + \frac{2^4}{3\cdot 4} C_2 + \ldots + \frac{2^{n+2} C_n}{(n+1)(n+2)}$$

Hence, $\frac{2^2}{1\cdot 2}C_0 + \frac{2^3}{2\cdot 3}C_1 + \frac{2^4}{3\cdot 4}C_2 + \ldots + \frac{2^{n+2}C_n}{(n+1)(n+2)}$

$$= \frac{3^{n+2} - 2n - 5}{(n+1)(n+2)}$$

I. Aliter

$$\text{LHS} = \frac{2^2}{1\cdot 2}C_0 + \frac{2^3}{2\cdot 3}C_1 + \frac{2^4}{3\cdot 4}C_2 + \ldots + \frac{2^{n+2}C_n}{(n+1)(n+2)}$$

$$= \frac{2^2}{1\cdot 2}(1) + \frac{2^3}{2\cdot 3}\cdot n + \frac{2^4}{3\cdot 4}\frac{n(n-1)}{1\cdot 2} + \ldots + \frac{2^{n+2}\cdot 1}{(n+1)(n+2)}$$

$$= \frac{1}{(n+1)(n+2)}\left\{\frac{(n+2)(n+1)}{1\cdot 2}2^2 + \frac{(n+2)(n+1)n}{1\cdot 2\cdot 3}2^3 + \frac{(n+2)(n+1)n(n-1)}{1\cdot 2\cdot 3\cdot 4}2^4 + \ldots + 2^{n+2}\right\}$$

Put $n+2 = N$, then we get

$$= \frac{1}{N(N-1)}\left\{\frac{N(N-1)}{1\cdot 2}2^2 + \frac{N(N-1)(N-2)}{1\cdot 2\cdot 3}2^3 + \frac{N(N-1)(N-2)(N-3)}{1\cdot 2\cdot 3\cdot 4}2^4 + \ldots + 2^N\right\}$$

$$= \frac{1}{N(N-1)}\{{}^NC_2(2)^2 + {}^NC_3(2)^3 + {}^NC_4(2)^4 + \ldots + {}^NC_N(2)^N]$$

$$= \frac{1}{N(N-1)}\{{}^NC_0 + {}^NC_1(2) + {}^NC_2(2)^2 + {}^NC_3(2)^3 + {}^NC_4(2)^4 + \ldots + {}^NC_N(2)^N - {}^NC_0 - {}^NC_1(2)]$$

$$= \frac{1}{N(N-1)}\{(1+2)^N - 1 - 2N\}$$

$$= \frac{3^{n+2} - 1 - 2(n+2)}{(n+2)(n+1)} = \frac{3^{n+2} - 2n - 5}{(n+1)(n+2)} = \text{RHS}$$

II. Aliter

$$\text{LHS} = \frac{2^2}{1\cdot 2}\cdot C_0 + \frac{2^3}{2\cdot 3}\cdot C_1 + \frac{2^4}{3\cdot 4}\cdot C_2 + \ldots + \frac{2^{n+2}\cdot C_n}{(n+1)(n+2)}$$

$$= \sum_{r=1}^{n+1}\frac{2^{r+1}}{r(r+1)}\cdot {}^nC_{r-1}$$

$$= \sum_{r=1}^{n+1}\frac{2^{r+1}\cdot {}^{n+2}C_{r+1}}{(n+1)(n+2)} \quad \left[\because \frac{{}^{n+2}C_{r+1}}{(n+1)(n+2)} = \frac{{}^nC_{r-1}}{r(r+1)}\right]$$

$$= \frac{1}{(n+1)(n+2)}\sum_{r=1}^{n+1}{}^{n+2}C_{r+1}\cdot 2^{r+1}$$

$$= \frac{1}{(n+1)(n+2)}[{}^{n+2}C_2\cdot 2^2 + {}^{n+2}C_3\cdot 2^3 + \ldots + {}^{n+2}C_{n+2}\cdot 2^{n+2}]$$

$$= \frac{1}{(n+1)(n+2)}[(1+2)^{n+2} - {}^{n+2}C_0 - {}^{n+2}C_1\cdot 2^1]$$

$$= \frac{(3^{n+2} - 2n - 5)}{(n+1)(n+2)} = \text{RHS}$$

When Each Term in Summation Contains the Product of Two Binomial Coefficients or Square of Binomial Coefficients

Solution Process

1. **If difference of the lower suffixes of binomial coefficients in each term is same.**

   i.e. ${}^nC_0\,{}^nC_2 + {}^nC_1\cdot{}^nC_3 + {}^nC_2\cdot{}^nC_4 + \ldots$

   Here, $2-0 = 3-1 = 4-2 = \ldots = 2$

   ***Case* I If each term of series is positive, then**

   $$(1+x)^n = C_0 + C_1x + C_2x^2 + \ldots + C_nx^n \quad \ldots(\text{i})$$

   Interchanging 1 and x, we get

   $$(x+1)^n = C_0x^n + C_1x^{n-1} + C_2x^{n-2} + \ldots + C_n \quad \ldots(\text{ii})$$

   Then, multiplying Eqs. (i) and (ii) and equate the coefficients of suitable power of x on both sides.

   Replacing x by $\frac{1}{x}$ in Eq. (i), then we get

   $$\left(1+\frac{1}{x}\right)^n = C_0 + \frac{C_1}{x} + \frac{C_2}{x^2} + \ldots + \frac{C_n}{x^n} \quad \ldots(\text{iii})$$

   Then, multiplying Eqs. (i) and (iii) and equate the coefficients of suitable power of x on both sides.

Example 72. ***If*** $(1+x)^n = C_0 + C_1x + C_2x^2 + C_3x^3 + \ldots + C_nx^n$***, prove that***

$$C_0C_r + C_1C_{r+1} + C_2C_{r+2} + \ldots + C_{n-r}C_n = \frac{2n!}{(n-r)!(n+r)!}.$$

Sol. Here, differences of lower suffixes of binomial coefficients in each term is r.

i.e., $r-0 = r+1-1 = r+2-2 = \ldots = n-(n-r) = r$

Given,

$$(1+x)^n = C_0 + C_1x + C_2x^2 + \ldots + C_{n-r}x^{n-r} + \ldots + C_nx^n \quad \ldots(\text{i})$$

Now,

$$(x+1)^n = C_0x^n + C_1x^{n-1} + C_2x^{n-2} + \ldots + C_rx^{n-r} + C_{r+1}x^{n-r-1} + C_{r+2}x^{n-r-2} + \ldots + C_n \quad \ldots(\text{ii})$$

On multiplying Eqs. (i) and (ii), we get

$$(1+x)^{2n} = (C_0 + C_1x + C_2x^2 + \ldots + C_{n-r}x^{n-r} + \ldots + C_nx^n) \times (C_0x^n + C_1x^{n-1} + C_2x^{n-2} + \ldots + C_rx^{n-r} + C_{r+1}x^{n-r-1} + C_{r+2}x^{n-r-2} + \ldots + C_n) \quad \ldots(\text{iii})$$

Now, coefficient of x^{n-r} on LHS of Eq. (iii) $= {}^{2n}C_{n-r}$

$$= \frac{2n\,!}{(n-r)\,!\,(n+r)\,!}$$

and coefficient of x^{n-r} on RHS of Eq. (iii)

$$= C_0C_r + C_1C_{r+1} + C_2C_{r+2} + \ldots + C_{n-r}C_n$$

But Eq. (iii) is an identity, therefore coefficient of x^{n-r} in RHS = coefficient of x^{n-r} in LHS.

$$\Rightarrow C_0C_r + C_1C_{r+1} + C_2C_{r+2} + \ldots + C_{n-r}C_n = \frac{2n\,!}{(n-r)!\,(n+r)\,!}$$

Aliter

Given,

$$(1+x)^n = C_0 + C_1x + C_2x^2 + \ldots + C_r\,x^r + C_{r+1}x^{r+1} + C_{r+2}x^{r+2} + \ldots + C_{n-r}x^r + \ldots + C_nx^n \quad \ldots(\text{i})$$

$$\text{Now, } \left(1+\frac{1}{x}\right)^n = C_0 + \frac{C_1}{x} + \frac{C_2}{x^2} + \ldots + \frac{C_r}{x^r} + \frac{C_{r+1}}{x^{r+1}} + \frac{C_{r+2}}{x^{r+2}} + \ldots + \frac{C_{n-r}}{x^{n-r}} + \ldots + \frac{C_n}{x^n} \quad \ldots(\text{ii})$$

On multiplying Eqs. (i) and (ii), we get

$$\frac{(1+x)^{2n}}{x^n} = (C_0 + C_1x + C_2x^2 + \ldots + C_rx^r + C_{r+1}x^{r+1} + C_{r+2}\,x^{r+2} + .. + C_{n-r}x^{n-r} + .. + C_nx^n)$$
$$\times\left(C_0 + \frac{C_1}{x} + \frac{C_2}{x^2} + \ldots + \frac{C_r}{x^r} + \frac{C_{r+1}}{x^{r+1}} + \frac{C_{r+2}}{x^{r+2}} + \ldots + \frac{C_{n-r}}{x^{n-r}} + \ldots + \frac{C_n}{x^n}\right) \ldots(\text{iii})$$

Now, coefficient of $\frac{1}{x^r}$ in RHS

$$= (C_0C_r + C_1C_{r+1} + C_2C_{r+2} + \ldots + C_{n-r}C_n)$$

$\therefore$ Coefficient of $\frac{1}{x^r}$ in LHS = Coefficient of x^{n-r} in

$$(1+x)^{2n} = {}^{2n}C_{n-r} = \frac{2n\,!}{(\,n-r)\,!\,(n+r)\,!}$$

But Eq. (iii) is an identity, therefore coefficient of $\frac{1}{x^r}$ in RHS = coefficient of $\frac{1}{x^r}$ in LHS.

$$\Rightarrow \quad C_0C_r + C_1C_{r+1} + C_2C_{r+2} + \ldots + C_{n-r}C_n = \frac{2n\,!}{(n-r)!\,(n+r)\,!}$$

***Corollary* I** For $r = 0$,

$$C_0^{\,2} + C_1^{\,2} + C_2^{\,2} + \ldots + C_n^{\,2} = \frac{2n\,!}{(n!)^2}$$

***Corollary* II** For $r = 1$,

$$C_0C_1 + C_1C_2 + C_2C_3 + \ldots + C_{n-1}C_n = \frac{2n\,!}{(n-1)\,!\,(n+1)\,!}$$

***Corollary* III** For $r = 2$,

$$C_0C_2 + C_1C_3 + C_2C_4 + \ldots + C_{n-2}C_n = \frac{2n\,!}{(n-2)\,!\,(n+2)\,!}$$

Example 73. If $(1+x)^n = C_0 + C_1x + C_2x^2 + \ldots + C_nx^n$, prove that $C_0^{\,2} + C_1^{\,2} + C_2^{\,2} = \dfrac{2n!}{n!\,n!} = \dfrac{1\cdot3\cdot5\ldots(2n-1)}{n!}\cdot 2^n$.

Sol. Given, $(1+x)^n = C_0 + C_1x + C_2x^2 + \ldots + C_nx^n$ ...(i)

Now, $(x+1)^n = C_0x^n + C_1x^{n-1} + C_2x^{n-2} + \ldots + C_n$ ...(ii)

On multiplying Eqs. (i) and (ii), we get

$$(1+x)^{2n} = (C_0 + C_1x + C_2x^2 + \ldots + C_nx^n) \times (C_0x^n + C_1x^{n-1} + C_2x^{n-2} + \ldots + C_n) \quad \ldots(\text{iii})$$

Now, coefficient of x^n in RHS

$$= C_0^{\,2} + C_1^{\,2} + C_2^{\,2} + \ldots + C_n^{\,2}$$

And coefficient of x^n in LHS $= {}^{2n}C_n = \dfrac{2n\,!}{n\,!\,n\,!}$

$$= \frac{1\cdot2\cdot3\cdot4\cdot5\ldots(2n-1)\,2n}{n\,!\,n\,!} = \frac{1\cdot3\cdot5\ldots(2n-1)\,2^n\,n\,!}{n\,!\,n\,!}$$

But Eq. (iii) is an identity, therefore coefficient of x^n in RHS = coefficient of x^n in LHS.

$$\Rightarrow C_0^{\,2} + C_1^{\,2} + C_2^{\,2} + \ldots + C_n^{\,2} = \frac{2n\,!}{n\,!\,n\,!} = \frac{1\cdot3\cdot5\cdot\ldots\cdot(2n-1)}{n\,!}\cdot 2^n$$

Aliter

Given, $(1+x)^n = C_0 + C_1x + C_2x^2 + \ldots + C_nx^n$ ...(i)

$$\text{Now, } \left(1+\frac{1}{x}\right)^n = C_0 + \frac{C_1}{x} + \frac{C_2}{x^2} + \ldots + \frac{C_n}{x^n} \quad \ldots(\text{ii})$$

On multiplying Eqs. (i) and (ii), we get

$$\frac{(1+x)^{2n}}{x^n} = (\,C_0 + C_1x + C_2x^2 + \ldots + C_nx^n) \times\left(C_0 + \frac{C_1}{x} + \frac{C_2}{x^2} + \ldots + \frac{C_n}{x^n}\right) \ldots(\text{iii})$$

Now, constant term in RHS $= C_0^2 + C_1^2 + C_2^2 + \ldots + C_n^2$

Constant term in LHS = Constant term in $\dfrac{(1+x)^{2n}}{x^n}$

$$= \text{Coefficient of } x^n \text{ in } (1+x)^{2n} = {}^{2n}C_n = \frac{2n\,!}{n\,!\,n\,!}$$

$$= \frac{n\,!\,2^n\,[1\cdot3\cdot5\ldots(2n-1)]}{n\,!\,n\,!} = \frac{2^n\,[1\cdot3\cdot5\ldots(2n-1)]}{n\,!}$$

But Eq. (iii) is an identity, therefore the constant term in RHS = constant term in LHS.

$$\Rightarrow C_0^2 + C_1^2 + C_2^2 + \ldots + C_n^2 = \frac{2n\,!}{n\,!\,n\,!} = \frac{\{1\cdot 3\cdot 5\ldots(2n-1)\}}{n\,!}2^n$$

***Case* II If terms of the series alternately positive and negative, then**

$$(1-x)^n = C_0 - C_1 x + C_2 x^2 - \ldots + (-1)^n C_n x^n \quad \ldots\text{(i)}$$

$$\text{and } (x+1)^n = C_0 x^n + C_1 x^{n-1} + C_2 x^{n-2} + \ldots + C_n \quad \ldots\text{(ii)}$$

Then, multiplying Eqs. (i) and (ii) and equate the coefficient of suitable power of x on both sides.

Or

Replacing x by $\frac{1}{x}$ in Eq. (i), we get

$$\left(1-\frac{1}{x}\right)^n = C_0 - \frac{C_1}{x} + \frac{C_2}{x^2} - \ldots + (-1)^n \frac{C_n}{x^n} \quad \ldots\text{(iii)}$$

Then, multiplying Eqs. (i) and (iii) and equate the coefficient of suitable power of x on both sides.

Example 74. *Prove that*

$$({}^{2n}C_0)^2 - ({}^{2n}C_1)^2 + ({}^{2n}C_2)^2 - \ldots + ({}^{2n}C_{2n})^2 = (-1)^n \cdot {}^{2n}C_n.$$

Sol. Since, $(1-x)^{2n} = {}^{2n}C_0 - {}^{2n}C_1 x + {}^{2n}C_2 x^2 - \ldots + (-1)^{2n} \cdot {}^{2n}C_{2n} x^{2n}$

or $(1-x)^{2n} = {}^{2n}C_0 - {}^{2n}C_1 x + {}^{2n}C_2 x^2 - \ldots + {}^{2n}C_{2n} x^{2n}$ ...(i)

and $(x+1)^{2n} = {}^{2n}C_0 x^{2n} + {}^{2n}C_1 x^{2n-1} + {}^{2n}C_2 x^{2n-2} + \ldots + {}^{2n}C_{2n}$ ...(ii)

On multiplying Eqs. (i) and (ii), we get

$$(x^2-1)^{2n} = ({}^{2n}C_0 - {}^{2n}C_1 x + {}^{2n}C_2 x^2 - \ldots + {}^{2n}C_{2n} x^{2n}) \times ({}^{2n}C_0 x^{2n} + {}^{2n}C_1 x^{2n-1} + {}^{2n}C_2 x^{2n-2} + \ldots + {}^{2n}C_{2n}) \quad \ldots\text{(iii)}$$

Now, coefficient of x^{2n} in RHS

$$= ({}^{2n}C_0)^2 - ({}^{2n}C_1)^2 + ({}^{2n}C_2)^2 - \ldots + ({}^{2n}C_{2n})^2$$

Now, LHS can also be written as $(1-x^2)^{2n}$.

$\therefore$ General term in LHS, $T_{r+1} = {}^{2n}C_r(-x^2)^r$

Putting $r = n$, we get $T_{n+1} = (-1)^n \cdot {}^{2n}C_n x^{2n}$

$\Rightarrow$ Coefficient of x^{2n} in LHS $= (-1)^n \cdot {}^{2n}C_n$

But Eq. (iii) is an identity, therefore coefficient of x^{2n} in RHS = coefficient of x^{2n} in LHS

$$\Rightarrow ({}^{2n}C_0)^2 - ({}^{2n}C_1)^2 + ({}^{2n}C_2)^2 - \ldots + ({}^{2n}C_{2n})^2 = (-1)^n \cdot {}^{2n}C_n$$

Aliter

Since, $(1+x)^{2n} = {}^{2n}C_0 + {}^{2n}C_1 x + {}^{2n}C_2 x^2 + \ldots + {}^{2n}C_{2n} x^{2n}$ ...(i)

and $\left(1-\frac{1}{x}\right)^{2n} = {}^{2n}C_0 - \frac{{}^{2n}C_1}{x} + \frac{{}^{2n}C_2}{x^2} - \ldots + \frac{{}^{2n}C_{2n}}{x^{2n}}$ ...(ii)

On multiplying Eqs. (i) and (ii), we get

$$\frac{(x^2-1)^{2n}}{x^{2n}} = ({}^{2n}C_0 + {}^{2n}C_1 x + {}^{2n}C_2 x^2 + \ldots + {}^{2n}C_{2n} x^{2n}) \times \left({}^{2n}C_0 - \frac{{}^{2n}C_1}{x} + \frac{{}^{2n}C_2}{x^2} - \ldots + \frac{{}^{2n}C_{2n}}{x^{2n}}\right) \quad \ldots\text{(iii)}$$

Now, constant term in RHS

$$= ({}^{2n}C_0)^2 - ({}^{2n}C_1)^2 + ({}^{2n}C_2)^2 - \ldots + ({}^{2n}C_{2n})^2$$

Constant term in LHS = Constant term in $\frac{(x^2-1)^{2n}}{x^{2n}}$

= Coefficient of x^{2n} in $(x^2-1)^{2n}$

= Coefficient of x^{2n} in $(1-x^2)^{2n}$

$= {}^{2n}C_n(-1)^n = (-1)^n \cdot {}^{2n}C_n$

But Eq. (iii) is an identity, therefore the constant term in RHS = constant term in LHS.

$$\Rightarrow ({}^{2n}C_0)^2 - ({}^{2n}C_1)^2 + ({}^{2n}C_2)^2 - \ldots + ({}^{2n}C_{2n})^2 = (-1)^n \cdot {}^{2n}C_n$$

Example 75. If $(1+x)^n = C_0 + C_1 x + C_2 x^2 + \ldots + C_n x^n$, prove that $C_0^2 - C_1^2 + C_2^2 - \ldots + (-1)^n \cdot C_n^2 = 0$ or $(-1)^{n/2} \cdot \frac{n!}{(n/2)!\,(n/2)!}$, according as n is odd or even. Also, evaluate $C_0^2 + C_1^2 + C_2^2 - \ldots + (-1)^n \cdot C_n^2$ for $n = 10$ and $n = 11$

Sol. Since, $(1-x)^n = C_0 - C_1 x + C_2 x^2 - \ldots + (-1)^n C_n x^n$ ...(i)

and $(x+1)^n = C_0 x^n + C_1 x^{n-1} + C_2 x^{n-2} + \ldots + C_n$ ...(ii)

On multiplying Eqs. (i) and (ii), we get

$$(1-x^2)^n = \{C_0 - C_1 x + C_2 x^2 - \ldots + (-1)^n C_n x^n\} \times (C_0 x^n + C_1 x^{n-1} + C_2 x^{n-2} + \ldots + C_n) \quad \ldots\text{(iii)}$$

Now, coefficient of x^n in RHS

$$= C_0^2 - C_1^2 + C_2^2 - \ldots + (-1)^n C_n^2$$

General term in LHS $= T_{r+1} = {}^nC_r(-x^2)^r = {}^nC_r(-1)^r x^{2r}$

Putting $2r = n$, we get $r = n/2$

$\therefore$ $T_{(n/2)+1} = {}^nC_{n/2}(-1)^{n/2} x^n$

$\therefore$ Coefficient of x^n in LHS $= {}^nC_{n/2}(-1)^{n/2}$

$$= (-1)^{n/2} \cdot \frac{n\,!}{(n/2)\,!\,(n/2)\,!}$$

$$= \begin{cases} 0, & \text{if } n \text{ is odd} \\ (-1)^{n/2} \dfrac{n\,!}{(n/2)\,!\,(n/2)\,!}, & \text{if } n \text{ is even} \end{cases} \quad \left[\because \left(\frac{\text{odd}}{2}\right)! = \infty\right]$$

But Eq. (iii) is an identity, therefore coefficient of x^n in RHS = coefficient of x^n in LHS.

$$\Rightarrow \quad C_0^2 - C_1^2 + C_2^2 - \ldots + (-1)^n C_n^2 = \begin{cases} 0 & , \text{if } n \text{ is odd} \\ (-1)^{n/2} \dfrac{n!}{(n/2)!\,(n/2)!}, & \text{if } n \text{ is even} \end{cases}$$

Now, for $n = 10$,

$$C_0^2 - C_1^2 + C_2^2 - \ldots + C_{10}^2 = (-1)^{10/2} \frac{10!}{5!\,5!} = -252$$

[$\because$ 10 is even]

and from $n = 11$,

$$C_0^2 - C_1^2 + C_2^2 - \ldots - C_{11}^2 = 0 \qquad [\because 11 \text{ is odd}]$$

Aliter

Since, $\quad (1 + x)^n = C_0 + C_1 x + C_2 x^2 + \ldots + C_n x^n \qquad \ldots\text{(i)}$

Replacing x by $-\dfrac{1}{x}$, then we get

$$\left(1 - \frac{1}{x}\right)^n = C_0 - \frac{C_1}{x} + \frac{C_2}{x^2} - \ldots + (-1)^n \frac{C_n}{x^n} \qquad \ldots\text{(ii)}$$

On multiplying Eqs. (i) and (ii), we get

$$\frac{(x^2 - 1)^n}{x^n} = (C_0 + C_1 x + C_2 x^2 + \ldots + C_n x^n) \times \left(C_0 - \frac{C_1}{x} + \frac{C_2}{x^2} - \ldots + (-1)^n \frac{C_n}{x^n}\right) \qquad \ldots\text{(iii)}$$

Now, constant term in RHS

$$= C_0^1 - C_1^2 + C_2^2 - \ldots + (-1)^n C_n^2$$

$\therefore$ Constant term in LHS

$$= \text{Constant term in } \frac{(x^2 - 1)^n}{x^n}$$

$$= \text{Coefficient of } x^n \text{ in } (x^2 - 1)^n$$

$$= \text{Coefficient of } x^n \text{ in } {}^nC_{n/2} (x^2)^{n - (n/2)} (-1)^{n/2}$$

$$= (-1)^{n/2} \cdot {}^nC_{n/2}$$

$$= (-1)^{n/2} \cdot \frac{n!}{(n/2)!\,(n/2)!}$$

$$= \begin{cases} 0, & \text{if } n \text{ is odd} \\ (-1)^{n/2} \cdot \dfrac{n!}{(n/2)!\,(n/2)!}, & \text{if } n \text{ is even} \end{cases}$$

But Eq. (iii) is an identity, therefore the constant term in RHS = constant term in LHS.

$$\Rightarrow \quad C_0^2 - C_1^2 + C_2^2 - \ldots + (-1)^n C_n^2 = \begin{cases} 0, & \text{if } n \text{ is odd} \\ (-1)^{n/2} \cdot \dfrac{n!}{(n/2)!\,(n/2)!}, & \text{if } n \text{ is even} \end{cases}$$

2. If sum of the lower suffixes of binomial coefficients in each term is same.

i.e., $C_0 C_n + C_1 C_{n-1} + C_2 C_{n-2} + \ldots + C_n C_0$

Here, $0 + n = 1 + (n - 1) = 2 + (n - 2) = \ldots = n + 0 = n$

***Case* I** If each term of series is positive, then

$$(1 + x)^n = C_0 + C_1 x + C_2 x^2 + \ldots + C_n x^n \qquad \ldots\text{(i)}$$

and $\quad (1 + x)^n = C_0 + C_1 x + C_2 x^2 + \ldots + C_n x^n \qquad \ldots\text{(ii)}$

Then, multiplying Eqs. (i) and (ii) and equate the coefficient of suitable power of x on both sides.

Example 76. Prove that ${}^{m+n}C_r = {}^mC_r + {}^mC_{r-1}\,{}^nC_1 + {}^mC_{r-2}\,{}^nC_2 + \ldots + {}^nC_r$ if $r < m, r < n$ and m, n, r are positive integers.

Sol. Here, sum of lower suffixes of binomial coefficients in each term is r.

i.e. $\qquad r = r - 1 + 1 = r - 2 + 2 = \ldots = r = r$

Since,

$$(1 + x)^m = {}^mC_0 + {}^mC_1 x + \ldots + {}^mC_{r-2} x^{r-2} + {}^mC_{r-1} x^{r-1} + {}^mC_r x^r + \ldots + {}^mC_m x^m \ldots\text{(i)}$$

and $(1 + x)^n = {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + \ldots + {}^nC_r x^r + \ldots + {}^nC_n x^n \ldots\text{(ii)}$

On multiplying Eqs. (i) and (ii), we get

$$(1 + x)^{m+n} = ({}^mC_0 + {}^mC_1 x + \ldots + {}^mC_{r-2} x^{r-2} + {}^mC_{r-1} x^{r-1} + {}^mC_r x^r + \ldots + {}^mC_m x^m) \times ({}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + \ldots + {}^nC_r x^r + \ldots + {}^nC_n x^n) \ldots\text{(iii)}$$

Now, coefficient of x^r in RHS

$= {}^mC_r \cdot {}^nC_0 + {}^mC_{r-1} \cdot {}^nC_1 + {}^mC_{r-2} \cdot {}^nC_2 + \ldots + {}^mC_0 \cdot {}^nC_r$

$= {}^mC_r + {}^mC_{r-1} \cdot {}^nC_1 + {}^mC_{r-2} \cdot {}^nC_2 + \ldots + {}^nC_r$

Coefficient of x^r in LHS = ${}^{m+n}C_r$

But Eq. (iii) is an identity, therefore coefficient of x^r in LHS = coefficient of x^r in RHS.

$$\Rightarrow \quad {}^{m+n}C_r = {}^mC_r + {}^mC_{r-1} \cdot {}^nC_1 + {}^mC_{r-2} \cdot {}^nC_2 + \ldots + {}^nC_r$$

***Case* II** If terms of the series alternately positive and negative, then

$$(1 - x)^n = C_0 - C_1 x + C_2 x^2 - \ldots + (-1)^n C_n x^n \ldots\text{(i)}$$

and $(1 + x)^n = C_0 + C_1 x + C_2 x^2 + \ldots + C_n x^n \quad \ldots\text{(ii)}$

Then, multiplying Eqs. (i) and (ii) and equate the coefficient of suitable power of x on both sides.

Example 77. If $(1 + x)^n = C_0 + C_1 x + C_2 x^2 + \ldots + C_n x^n$, prove that $C_0 C_n - C_1 C_{n-1} + C_2 C_{n-2} - \ldots + (-1)^n C_n C_0 = 0$ or $(-1)^{n/2} \dfrac{n!}{(n/2)!\,(n/2)!}$, according as n is odd or even.

Sol. Given, $(1+x)^n = C_0 + C_1x + C_2x^2 + \dots + C_{n-2}x^{n-2}$

$$+ C_{n-1}x^{n-1} + C_n x^n \quad \dots(\text{i})$$

and $(1-x)^n = C_0 - C_1x + C_2x^2 - \dots + (-1)^n C_n x^n \quad \dots(\text{ii})$

On multiplying Eqs. (i) and (ii), we get

$$(1-x^2)^n = (C_0 + C_1x + C_2x^2 + \dots + C_{n-2}x^{n-2} + C_{n-1}x^{n-1} + C_n x^n) \times (C_2 - C_1x + C_2x^2 - \dots + (-1)^n C_n x^n) \quad \dots(\text{iii})$$

Now, coefficient of x^n in RHS

$$= C_0C_n - C_1C_{n-1} + C_2C_{n-2} - \dots + (-1)^n C_nC_0$$

Now, general term in LHS,

$$T_{r+1} = {}^nC_r(-x^2)^r = (-1)^r \cdot {}^nC_r\, x^{2r}$$

Putting $2r = n$, we get

$$r = n/2$$

Now, $T_{n/2+1} = (-1)^{n/2} \cdot {}^nC_{n/2}\, x^n$

$\therefore$ Coefficient of x^n in LHS $= (-1)^{n/2} \cdot {}^nC_{n/2}$

$$= (-1)^{n/2} \cdot \frac{n!}{(n/2)!\,(n/2)!}$$

$$= \begin{cases} 0 & \text{if } n \text{ is odd} \\ (-1)^{n/2} \cdot \dfrac{n!}{(n/2)!\,(n/2)!}, & \text{if } n \text{ is even} \end{cases}$$

But Eq. (iii) is an identity, therefore the coefficient of x^n in RHS = coefficient of x^n in LHS.

$$\Rightarrow C_0C_n - C_1C_{n-1} + C_2C_{n-2} - \dots + (-1)^n C_nC_0 = \begin{cases} 0, & \text{if } n \text{ is odd} \\ (-1)^{n/2} \cdot \dfrac{n!}{(n/2)!\,(n/2)!}, & \text{if } n \text{ is even} \end{cases}$$

3. **If each term is the product of two binomial coefficient divided or multiplied by an integer, then integrating or differentiating by preceeding method. Then, multiplying two series and equate the coefficient of suitable power of x on both sides.**

Example 78. If $(1+x)^n = C_0 + C_1x + C_2x^2 + C_3x^3 + \dots + C_n x^n$, prove that

$$C_1^2 + 2C_2^2 + 3C_3^2 + \dots + nC_n^2 = \frac{(2n-1)!}{((n-1)!)^2}.$$

Sol. Given, $(1+x)^n = C_0 + C_1x + C_2x^2 + C_3x^3 + \dots + C_n x^n$

Differentiating both sides w.r.t. x, we get

$$n(1+x)^{n-1} = 0 + C_1 + 2C_2x + 3C_3x^2 + \dots + nC_n x^{n-1}$$

$$\Rightarrow n(1+x)^{n-1} = C_1 + 2C_2x + 3C_3x^2 + \dots + nC_n x^{n-1} \quad \dots(\text{i})$$

and $(x+1)^n = C_0x^n + C_1x^{n-1} + C_2x^{n-2} + C_3x^{n-3} + \dots + C_n \quad \dots(\text{ii})$

On multiplying Eqs. (i) and (ii), then we get

$$n(1+x)^{2n-1} = (C_1 + 2C_2x + 3C_3x^2 + \dots + nC_n x^{n-1}) \times (C_0x^n + C_1x^{n-1} + C_2x^{n-2} + C_3x^{n-3} + \dots + C_n) \quad \dots(\text{iii})$$

Now, coefficient of x^{n-1} on RHS

$$= C_1^2 + 2C_2^2 + 3C_3^2 + \dots + nC_n^2$$

and coefficient of x^{n-1} on LHS

$$= n \cdot {}^{2n-1}C_{n-1} = n \cdot \frac{(2n-1)!}{(n-1)!\,n!}$$

$$= \frac{(2n-1)!}{(n-1)!\,(n-1)!} = \frac{(2n-1)!}{\{(n-1)!\}^2}$$

But Eq. (iii) is an identity, therefore the coefficient of x^{n-1} in RHS = coefficient of x^{n-1} in LHS.

$$\Rightarrow \quad C_1^2 + 2C_2^2 + 3C_3^2 + \dots + nC_n^2 = \frac{(2n-1)!}{\{(n-1)!\}}$$

Example 79. If $(1+x)^n = C_0 + C_1x + C_2x^2 + \dots + C_n x^n$, prove that $C_0^2 + \dfrac{C_1^2}{2} + \dfrac{C_2^2}{3} + \dots + \dfrac{C_n^2}{n+1} = \dfrac{(2n+1)!}{\{(n+1)!\}^2}$.

Sol. Given, $(1+x)^n = C_0 + C_1x + C_2x^2 + \dots + C_n x^n$

Integrating both sides w.r.t. x within limits 0 to x, then we get

$$\int_0^x (1+x)^n\, dx = \int_0^x (C_0 + C_1x + C_2x^2 + \dots + C_n x^n)\, dx$$

$$\frac{(1+x)^{n+1} - 1}{(1+n)} = C_0x + \frac{C_1x^2}{2} + \frac{C_2x^3}{3} + \dots + \frac{C_n x^{n+1}}{n+1} \quad \dots(\text{i})$$

and $(x+1)^n = C_0x^n + C_1x^{n-1} + C_2x^{n-2} + \dots + C_n \quad \dots(\text{ii})$

Multiplying Eqs. (i) and (ii), we get

$$\frac{1}{(n+1)}\{(1+x)^{2n+1} - (1+x)^n\} = \left(C_0x + \frac{C_1x^2}{2} + \frac{C_2x^3}{3} + \dots + \frac{C_n x^{n+1}}{n+1}\right) \times (C_0x^n + C_1x^{n-1} + C_2x^{n-2} + \dots + C_n) \quad \dots(\text{iii})$$

Now, coefficient of x^{n-1} in RHS of Eq. (iii)

$$= C_0^2 + \frac{C_1^2}{2} + \frac{C_2^2}{3} + \dots + \frac{C_n^2}{n+1}$$

and coefficient of x^{n+1} in LHS of Eq. (iii)

$$= \frac{1}{(n+1)}\{{}^{2n+1}C_{n+1} - 0\}$$

$$= \frac{1}{(n+1)} \cdot \frac{(2n+1)!}{(n+1)!\,n!}$$

$$= \frac{(2n+1)!}{(n+1)!\,(n+1)!} = \frac{(2n+1)!}{\{(n+1)!\}^2}$$

But Eq. (iii) is an identity, therefore coefficient of x^{n+1} in RHS of Eq. (iii) = coefficient of x^{n+1} in LHS of Eq. (iii).

$$\Rightarrow \; C_0^2 + \frac{C_1^2}{2} + \frac{C_2^2}{3} + \ldots + \frac{C_n^2}{n+1} = \frac{(2n+1)!}{\{(n+1)!\}^2}$$

Binomial Inside Binomial

The upper suffices of binomial coefficients are different but lower suffices are same.

Example 80. Evaluate $\sum_{r=0}^{n} {}^{n+r}C_n$.

Sol. $\sum_{r=0}^{n} {}^{n+r}C_n = {}^{n}C_n + {}^{n+1}C_n + {}^{n+2}C_n + \ldots + {}^{2n}C_n$

$= \text{Coefficient of } x^n \text{ in}$

$[(1+x)^n + (1+x)^{n+1} + (1+x)^{n+2} + \ldots + \ldots + (1+x)^{2n}]$

$$= \text{Coefficient of } x^n \text{ in} \left[\frac{(1+x)^n\,[(1+x)^{n+1} - 1]}{(1+x) - 1}\right]$$

$= \text{Coefficient of } x^{n+1} \text{ in } [(1+x)^{2n+1} - (1+x)^n]$

$= {}^{2n+1}C_{n+1} - 0 = {}^{2n+1}C_n$

Example 81. If $(1+x)^n = C_0 + C_1 x + C_2 x^2 + \ldots + C_n x^n$, prove that

$C_0 \cdot {}^{2n}C_n - C_1 \cdot {}^{2n-2}C_n + C_2 \cdot {}^{2n-4}C_n - \ldots = 2^n$

Sol. LHS $= C_0 \cdot {}^{2n}C_n - C_1 \cdot {}^{2n-2}C_n + C_2 \cdot {}^{2n-4}C_n - \ldots$

$= \text{Coefficient of } x^n \text{ in}$

$[C_0 (1+x)^{2n} - C_1 (1+x)^{2n-2} + C_2 (1+x)^{2n-4} - \ldots]$

$= \text{Coefficient of } x^n \text{ in}$

$[C_0 (1+x)^2]^n - C_1 [(1+x)^2]^{n-1} + C_2 [(1+x)^2]^{n-2} - \ldots]$

$= \text{Coefficient of } x^n \text{ in } [[(1+x)^2 - 1]^n]$

$= \text{Coefficient of } x^n \text{ in } (2x + x^2)^n$

$= \text{Constant term in } (2+x)^n = 2^n = \text{RHS}$

Example 82. If $(1+x)^n = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \ldots + C_n x^n$, prove that

$C_0 \cdot {}^{2n}C_n - C_1 \cdot {}^{2n-1}C_n + C_2 \cdot {}^{2n-2}C_n - C_3 \cdot {}^{2n-3}C_n + \ldots + (-1)^n C_n \cdot {}^{n}C_n = 1$

Sol. LHS $= C_0 \cdot {}^{2n}C_n - C_1 \cdot {}^{2n-1}C_n + C_2 \cdot {}^{2n-2}C_n - C_3 \cdot {}^{2n-3}C_n + \ldots + (-1)^n C_n \cdot {}^{n}C_n$

$= \text{Coefficient of } x^n \text{ in}$

$[C_0 (1+x)^{2n} - C_1 (1+x)^{2n-1} + C_2 (1+x)^{2n-2} - C_3 (1+x)^{2n-3} + \ldots + (-1)^n C_n \cdot (1+x)^n]$

$= \text{Coefficient of } x^n \text{ in}$

$(1+x)^n [C_0 (1+x)^n - C_1 (1+x)^{n-1} + C_2 (1+x)^{n-2} - C_3 (1+x)^{n-3} + \ldots + (-1)^n C_n \cdot 1]$

$= \text{Coefficient of } x^n \text{ in } (1+x)^n [((1+x) - 1)^n]$

$= \text{Coefficient of } x^n \text{ in } (1+x)^n \cdot x^n \text{d}$

$= \text{Constant term in } (1+x)^n = 1 = \text{RHS}$

Sum of the Series

***Case* I When i and j are independent.**

In this summation, three types of terms occur, when $i < j$, $i = j$ and $i > j$,

i.e., $\sum_{i=0}^{n} \sum_{j=0}^{n} a_i a_j = \sum_{i=0}^{n} \left\{ a_i \left(\sum_{j=0}^{n} a_j \right) \right\}$

$$= \sum_{i=0}^{n} a_i \sum_{j=0}^{n} a_j = \left(\sum_{i=0}^{n} a_i \right)^2 \text{ or } \left(\sum_{j=0}^{n} a_j \right)^2$$

***Corollary* I** $\sum_{i=0}^{n} \sum_{j=0}^{n} {}^{n}C_i \, {}^{n}C_j = \left(\sum_{i=0}^{n} {}^{n}C_i \right)^2$

$$= (2^n)^2 = 2^{2n}$$

Example 83. If $(1+x)^n = C_0 + C_1 x + C_2 x^2 + \ldots + C_n x^n$, find the values of the following.

(i) $\sum_{i=0}^{n} \sum_{j=0}^{n} (C_i + C_j)$

(ii) $\sum_{i=0}^{n} \sum_{j=0}^{n} (i + j) C_i C_j$

Sol. (i) $\sum_{i=0}^{n} \sum_{j=0}^{n} (C_i + C_j) = \sum_{i=0}^{n} \sum_{j=0}^{n} C_i + \sum_{i=0}^{n} \sum_{j=0}^{n} C_j$

$$= \sum_{j=0}^{n} \left(\sum_{i=0}^{n} C_i \right) + \sum_{i=0}^{n} \left(\sum_{j=0}^{n} C_j \right)$$

$$= \sum_{j=0}^{n} (2^n) + \sum_{i=0}^{n} (2^n) = (n+1) \cdot 2^n + (n+1) \cdot 2^n$$

$$= 2(n+1)\,2^n = (n+1)\,2^{n+1}$$

(ii) $\sum_{i=0}^{n} \sum_{j=0}^{n} (i+j) C_i C_j = \sum_{i=0}^{n} \sum_{j=0}^{n} i C_i C_j + \sum_{i=0}^{n} \sum_{j=0}^{n} j C_i C_j$

$$= \sum_{i=0}^{n} i C_i \left(\sum_{j=0}^{n} C_j \right) + \sum_{j=0}^{n} j C_j \left(\sum_{i=0}^{n} C_i \right)$$

$$= \sum_{i=0}^{n} i\, C_i\,(2^n) + \sum_{j=0}^{n} j\, C_j\,(2^n)$$

$$= 2^n \sum_{i=0}^{n} i\ {}^nC_i + 2^n \sum_{j=0}^{n} j\ {}^nC_j$$

$$= 2^n \sum_{i=0}^{n} i \cdot \frac{n}{i} \cdot {}^{n-1}C_{i-1} + 2^n \sum_{j=0}^{n} j \cdot \frac{n}{j} \cdot {}^{n-1}C_{j-1}$$

$$= n \cdot 2^n \sum_{i=0}^{n} {}^{n-1}C_{i-1} + n \cdot 2^n \sum_{j=0}^{n} {}^{n-1}C_{j-1}$$

$$= n \cdot 2^n \cdot 2^{n-1} + n \cdot 2^n \cdot 2^{n-1}$$

$$= n \cdot 2 \cdot 2^{2n-1} = n \cdot 2^{2n}$$

***Case* II When *i* and *j* are dependent.**

In this summation, when $i < j$ is equal to the sum of the terms when $i > j$, if a_i and a_j are symmetrical. So, in this case

$$\sum_{i=0}^{n} \sum_{j=0}^{n} a_i\, a_j = \sum_{0 \le i < j \le n}\sum a_i\, a_j + \sum_{i=j}\sum a_i a_j + \sum_{0 \le j < i \le n}\sum a_i a_j$$

$$= 2 \sum_{0 \le i < j \le n}\sum a_i\, a_j + \sum_{i=j}\sum a_i\, a_j$$

$$\Rightarrow \sum_{0 \le i < j \le n}\sum a_i\, a_j = \frac{\sum_{i=0}^{n}\sum_{j=0}^{n} a_i\, a_j - \sum_{i=j}\sum a_i\, a_j}{2}$$

When a_i and a_j are not symmetrical, we find the sum by listing all the terms.

***Corollary* I**

$$\sum_{0 \le i < j \le n}\sum {}^nC_i\ {}^nC_j = \frac{\sum_{i=0}^{n}\sum_{j=0}^{n} {}^nC_i\,{}^nC_j - \sum_{i=j}\sum {}^nC_i \cdot {}^nC_j}{2}$$

$$= \frac{(2^n)^2 - \sum_{i=0}^{n} ({}^nC_i)^2}{2} = \frac{2^{2n} - {}^{2n}C_n}{2} = 2^{2n-1} - \frac{2n!}{2(n!)^2}$$

Example 84. If $(1+x)^n = C_0 + C_1 x + C_2 x^2 + \ldots + C_n x^n$, find the values of the following.

(i) $\sum_{0 \le i < j \le n}\sum C_i$ (ii) $\sum_{0 \le i < j \le n}\sum j\, C_i$

(iii) $\sum_{i \ne j}\sum C_i\, C_j$ (iv) $\sum_{0 \le i \le j \le n}\sum C_i\, C_j$

(v) $\sum_{0 \le i < j \le n}\sum (C_i \pm C_j)^2$

(vi) $\sum_{0 \le i < j \le n}\sum (i + j)\, C_i\, C_j$

(vii) $\sum_{0 \le i < j \le n}\sum (i \cdot j)\, C_i\, C_j$

Sol. (i) $$\sum_{0 \le i < j \le n}\sum C_i = \frac{\left(\sum_{i=0}^{n}\sum_{j=0}^{n} C_i\right) - \sum_{i=j}\sum C_i}{2}$$

$$= \frac{(n+1)\sum_{i=0}^{n} C_i - \sum_{i=0}^{n} C_i}{2} = n \cdot 2^{n-1}$$

(ii) $$\sum_{0 \le i < j \le n}\sum j\ C_i = \sum_{r=0}^{n-1} {}^nC_r \{(r+1) + (r+2) + (r+3) + \ldots + n\}$$

$$= \sum_{r=0}^{n-1} {}^nC_r \cdot \frac{(n-r)(n+r+1)}{2}$$

$$= \frac{1}{2}\sum_{r=0}^{n-1} {}^nC_r\,(n^2 - r^2 + n - r)$$

$$= \frac{1}{2}(n^2+n)\sum_{r=0}^{n-1} {}^nC_r - \frac{1}{2}\sum_{r=0}^{n-1} r \cdot {}^nC_r - \frac{1}{2}\sum_{r=0}^{n-1} r^2 \cdot {}^nC_r$$

$$= \frac{1}{2}(n^2+n)(2^n - 1) - \frac{1}{2}\cdot n \cdot (2^{n-1} - 1) - \frac{1}{2}\cdot n\,[(n-1)(2^{n-2} - 1) + 2^{n-1} - 1]$$

$$= n\,(3n+1)\cdot 2^{n-3}$$

Remark

Here, j and C_i are not symmetrical.

(iii) Here, $i \ne j$ i.e., $i > j$ or $i < j$

But C_i and C_j are symmetrical.

$$\therefore \sum_{i \ne j}\sum C_i\, C_j = 2 \sum_{0 \le i < j \le n}\sum C_i\, C_j$$

$$= 2\left(\frac{2^{2n} - {}^{2n}C_n}{2}\right) \quad \text{[from corollary I]}$$

$$= 2^{2n} - {}^{2n}C_n$$

(iv) $$\sum_{0 \le i \le j \le n}\sum C_i C_j = \sum_{0 \le i < j \le n}\sum C_i C_j + \sum_{i=j}\sum C_i C_j$$

$$= \frac{1}{2}(2^{2n} - {}^{2n}C_n) + {}^{2n}C_n \quad \text{[from corollary I]}$$

$$= \frac{1}{2}(2^{2n} + {}^{2n}C_n)$$

(v) $$\sum_{0 \le i < j \le n}\sum (C_i \pm C_j)^2 = \sum_{0 \le i < j \le n}\sum (C_i^2 + C_j^2 \pm 2\, C_i C_j)$$

$$= \sum_{0 \le i < j \le n}\sum (C_i^2 + C_j^2) \pm 2 \sum_{0 \le i < j \le n}\sum C_i\, C_j$$

$$\because \sum_{0 \le i < j \le n}\sum (C_i^2 + C_j^2)$$

$$= \frac{\sum_{i=0}^{n} \sum_{j=0}^{n} (C_i^2 + C_j^2) - 2 \sum_{i=0}^{n} C_i^2}{2}$$

$$= \frac{\sum_{i=0}^{n} \left(\sum_{j=0}^{n} C_i{}^2 + \sum_{j=0}^{n} C_j^2 \right) - 2 \cdot {}^{2n}C_n}{2}$$

$$= \frac{\sum_{i=0}^{n} ((n+1) C_i{}^2 + {}^{2n}C_n) - 2 \cdot {}^{2n} C_n}{2}$$

$$= \frac{(n+1) \sum_{i=0}^{n} C_i{}^2 + {}^{2n}C_n \sum_{i=0}^{n} 1 - 2 \cdot {}^{2n}C_n}{2}$$

$$= \frac{(n+1) \cdot {}^{2n}C_n + {}^{2n}C_n \cdot (n+1) - 2 \cdot {}^{2n}C_n}{2}$$

$$= n \cdot {}^{2n}C_n$$

$$\therefore \quad \sum\sum_{0 \le i < j \le n} (C_i \pm C_j)^2 = n \cdot {}^{2n}C_n \pm (2^{2n} - {}^{2n}C_n)$$

[from corollary 1]

$$= (n \mp 1)\ {}^{2n}C_n \pm 2^{2n}; \quad \sum\sum_{0 \le i < j \le n} (i+j) C_i C_j$$

Remark

$$\sum\sum_{0 \le i < j \le n} (C_i + C_j) = n \cdot 2^n$$

(vi) $\sum\sum_{0 \le i < j \le n} (i+j) C_i C_j$

Let $\quad P = \sum\sum_{0 \le i < j \le n} (i+j)\, C_i\, C_j$ ...(i)

Replacing i by $n - i$ and j by $n - j$ in Eq. (i), then we get

$$P = \sum\sum_{0 \le i < j \le n} (n - i + n - j)\, C_{n-i}\, C_{n-j}$$

[$\because$ sum of binomial expansion does not change if we replace r by $n - r$]

$$P = \sum\sum_{0 \le i < j \le n} (2n - i - j)\, C_i\, C_j$$

[$\because {}^nC_r = {}^nC_{n-r}$] ...(ii)

On adding Eqs. (i) and (ii), we get

$$2P = 2n \sum\sum_{0 \le i < j \le n} C_i\, C_j$$

or $\quad P = n \sum\sum_{0 \le i < j \le n} C_i\, C_j = \frac{n}{2} (2^{2n} - {}^{2n}C_n)$

[from corollary I]

(vii) $\sum\sum_{0 \le i < j \le n} (i \cdot j)\, C_i\, C_j = \sum\sum_{0 \le i < j \le n} (i\ {}^nC_i)(j \cdot {}^nC_j)$

$$= n^2 \sum\sum_{0 \le i < j \le n} {}^{n-1}C_{i-1}\ {}^{n-1}C_{j-1}$$

$$= n^2 \left[\frac{2^{2(n-1)} - {}^{2n-2}C_{n-1}}{2} \right] \text{[from corollary I]}$$

$$= n^2 \left(2^{2n-3} - \frac{1}{2} \cdot {}^{2n-2}C_{n-1} \right)$$

Exercise for Session 4

1. The coefficient of $a^4 b^8 c^9 d^9$ in the expansion of $(abc + abd + acd + bcd)^{10}$ is

(a) $10!$ (b) $\frac{10!}{4!8!9!9!}$ (c) 2520 (d) None of these

2. If $(1+2x+3x^2)^{10} = a_0 + a_1 x + a_2 x^2 + \dots + a_{20} x^{20}$, then a_1 equals

(a) 210 (b) 20 (c) 10 (d) None of these

3. If $(1+x+x^2+x^3)^5 = a_0 + a_1 x + a_2 x^2 + \dots + a_{15} x^{15}$, then a_{10} equals

(a) 99 (b) 100 (c) 101 (d) 110

4. Coefficient of x^{15} in $(1+x+x^3+x^4)^n$ is

(a) $\sum_{r=0}^{5} {}^nC_{5-r} \cdot {}^nC_{3r}$ (b) $\sum_{r=0}^{5} {}^nC_{5r}$ (c) $\sum_{r=0}^{5} {}^nC_{2r}$ (d) $\sum_{r=0}^{3} {}^nC_{3-r} \cdot {}^nC_{5r}$

5. The number of terms in the expansion of $\left(x^2 + 1 + \frac{1}{x^2}\right)^n, n \in N$ is

(a) ${}^{n+2}C_2$ (b) ${}^{n+3}C_2$ (c) ${}^{2n+1}C_{2n}$ (d) ${}^{3n+1}C_{3n}$

6. If $(1+x)^{10} = a_0 + a_1 x + a_2 x^2 + \dots + a_{10} x^{10}$, then $(a_0 - a_2 + a_4 - a_6 + a_8 - a_{10})^2 + (a_1 - a_3 + a_5 - a_7 + a_9)^2$ is equal to

(a) 2^9 (b) 3^9 (c) 2^{10} (d) 3^{10}

7. If $(1+x)^n = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \dots + C_n x^n$, n being even the value of $C_0 + (C_0 + C_1) + (C_0 + C_1 + C_2) + \dots + (C_0 + C_1 + C_2 + \dots + C_{n-1})$ is equal to

(a) $n \cdot 2^n$ (b) $n \cdot 2^{n-1}$ (c) $n \cdot 2^{n-2}$ (d) $n \cdot 2^{n-3}$

8. The value of $\frac{C_0}{1\cdot 3} - \frac{C_1}{2\cdot 3} + \frac{C_2}{3\cdot 3} - \frac{C_3}{4\cdot 3} + \dots + (-1)^n \frac{C_n}{(n+1)\cdot 3}$ is

(a) $\frac{3}{n+1}$ (b) $\frac{n+1}{3}$ (c) $\frac{1}{3(n+1)}$ (d) None of these

9. The value of $\binom{50}{0}\binom{50}{1} + \binom{50}{1}\binom{50}{2} + \dots + \binom{50}{49}\binom{50}{50}$, where ${}^nC_r = \binom{n}{r}$, is

(a) $\binom{100}{50}$ (b) $\binom{100}{51}$ (c) $\binom{50}{25}$ (d) $\binom{50}{25}^2$

10. If C_r stands for ${}^4C_r$, then $C_0 C_4 - C_1 C_3 + C_2 C_2 - C_3 C_1 + C_4 C_0$ is equal to

(a) C_1 (b) C_2 (c) C_3 (d) C_4

11. The sum $\sum_{r=0}^{n} (r+1)({}^nC_r)^2$ is equal to

(a) $\frac{(n+2)(2n-1)!}{n!(n-1)!}$ (b) $\frac{(n+2)(2n+1)!}{n!(n-1)!}$ (c) $\frac{(n+2)(2n+1)!}{n!(n+1)!}$ (d) $\frac{(n+2)(2n-1)!}{n!(n+1)!}$

12. $\sum_{r=1}^{n}\left(\sum_{p=0}^{r-1} {}^nC_r \, {}^rC_p \, 2^p\right)$ is equal to

(a) $4^n - 3^n + 1$ (b) $4^n - 3^n - 1$ (c) $4^n - 3^n + 2$ (d) $4^n - 3^n$

13. $\left(\sum_{r=0}^{10} {}^{10}C_r\right)\left(\sum_{m=0}^{10} (-1)^m \frac{{}^{10}C_m}{2^m}\right)$ is equal to

(a) 1 (b) 2^5 (c) 2^{10} (d) 2^{20}

14. The value of $\sum\sum\sum\sum_{0 \le i < j < k < l \le n} 2$ is equal to

(a) $2(n+1)^3$ (b) $2 \cdot {}^{n+1}C_4$ (c) $2(n+1)^4$ (d) $2 \cdot {}^{n+2}C_3$

Answers

Exercise for Session 4

1. (c)	**2.** (b)	**3.** (c)	**4.** (a)	**5.** (a)	**6.** (c)
7. (b)	**8.** (c)	**9.** (b)	**10.** (b)	**11.** (a)	**12.** (d)
13. (a)	**14.** (b)				