11 Three Dimensional Geometry

"If you're a tech enthusiast, you've probably heard of the term 'Virtual Reality' (VR). As the name suggests, it creates non-physical materiality that is very similar to reality. 3D VR is created or produced digitally. 3D environments are created by using computer software and artificial intelligence with the help of three dimensional geometry to replicate the real world or to create entirely new environments.

Topic Notes

- *Basic Concepts and Line in Space*

BASIC CONCEPTS AND LINE IN SPACE 1

TOPIC 1

RECTANGULAR COORDINATE SYSTEM

Though Three dimensional Geometry was introduced in Class XI, but here we shall study it using vector algebra. In this Chapter, we shall study the direction cosines and direction ratios of a line joining two points and also discuss about the equations of lines in space under different conditions, angle between two lines and shortest distance between two skew lines. Most of the results we shall obtaining will be in vector form as they are elegant and simple. Nevertheless, we shall also translate these results in the Cartesian form which, at times, presents a more clear geometric and picture of the situation.

Three mutually perpendicular number lines (coordinate axes) having the same origin O form a rectangular coordinate system in space.

The x-axis and y-axis determine a plane, called XY plane.

Similarly, we have YZ plane and ZX plane. These planes are called coordinate planes.

Let P be any point in space.

Let its (directed) distance from the coordinate planes YZ plane; ZX plane; XY plane be x, y, z respectively.

Then, we say that P has the coordinates (x, y, z).

Any point in the XY plane is of the form $(x, y, 0)$.

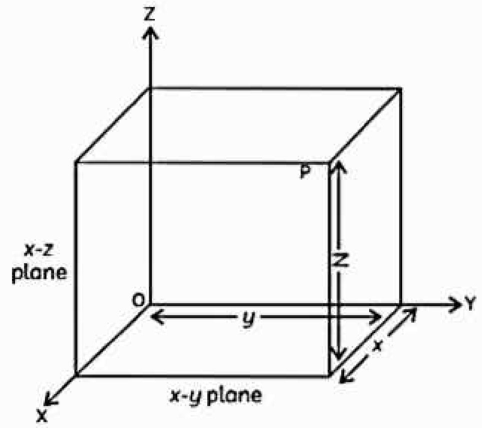
Similarly for others.

The coordinates of the origin are (0, 0, 0). All points on the YZ plane have x-coordinate 0. Hence, equation of YZ plane is $x = 0$.

Similarly, equation of ZX plane is $y = 0$; and that of XY plane is $z = 0$.

All points on x-axis have y, z-coordinates zero.

Thus, the equation of x-axis is $y = 0$ and $z = 0$ (in fact, x-axis is the intersection of the two planes $y = 0$ and $z = 0$)



Similarly, the equation of y-axis is $x = 0$ and $z = 0$; and the equation of z-axis is $x = 0$ and $y = 0$.

If $\hat{i}, \hat{j}$ and $\hat{k}$ are unit vectors along x, y and z axes respectively and P = (x, y, z), then $\vec{OP} = x\hat{i} + y\hat{j} + z\hat{k}$ and $|\vec{OP}| = \sqrt{x^2 + y^2 + z^2}$, the distance of point P from the origin.

TOPIC 2

DIRECTION COSINES AND DIRECTION RATIOS OF A LINE

In the previous chapter, we have studied about the direction cosines and direction ratios of a vector. Now, we shall define these terms in the context of a line. As we know that, vector is also defined as a directed line segment, so, direction cosines of a line L are defined as the direction cosines of a vector whose support is the given line L.

Thus, if a line L passing through the origin makes angles α, β and γ with x, y and z-axes, respectively called direction angles, then cosines of these angles, namely $\cos\alpha$, $\cos\beta$ and $\cos\gamma$ are called direction cosines of the directed line L.

If we reverse the direction of L, then the direction angles are replaced by their supplements, *i.e.*, $\pi - \alpha$, $\pi - \beta$ and $\pi - \gamma$. Thus the signs of the direction cosines are reversed.

Direction cosines of a line usually denoted by l, m and n. It is always true that $l^2 + m^2 + n^2 = 1$.

Any three numbers which are proportional to the direction cosines of a line are called the direction ratios of the line. Thus, if l, m and n are direction cosines and a, b and c are direction ratios of a line, then

$\frac{l}{a} = \frac{m}{b} = \frac{n}{c} = k$ (say), k being a constant.

i.e., $l = ak$; $m = bk$ and $n = ck$

Further, $l^2 + m^2 + n^2 = 1$ yields $k = \pm\frac{1}{\sqrt{a^2 + b^2 + c^2}}$

$\Rightarrow l = \pm\frac{a}{\sqrt{a^2 + b^2 + c^2}}$; $m = \pm\frac{b}{\sqrt{a^2 + b^2 + c^2}}$;

$n = \pm\frac{c}{\sqrt{a^2 + b^2 + c^2}}$

TOPIC 3

DIRECTION COSINES OF A LINE PASSING THROUGH TWO GIVEN POINTS

Let $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ be any two points. We know that by joining any two points, a unique line can be drawn.

The direction ratios of the line segment PQ are $x_2 - x_1$; $y_2 - y_1$; $z_2 - z_1$ or $x_1 - x_2$; $y_1 - y_2$; $z_1 - z_2$.

And the direction cosines of the line PQ are $\frac{x_2 - x_1}{PQ}$; $\frac{y_2 - y_1}{PQ}$; $\frac{z_2 - z_1}{PQ}$

i.e.,
$$\frac{x_2 - x_1}{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}};$$
$$\frac{y_2 - y_1}{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}};$$
$$\frac{z_2 - z_1}{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}}$$

Example 1.1: If a line makes angles 90°, 135° and 45° with the x, y and z axes respectively, find its direction cosines. [NCERT]

Ans. Here, the line make angles 90°, 135° and 45° with the x, y and z axes respectively.

So, $\alpha = 90°$, $\beta = 135°$ and $\gamma = 45°$.

$\Rightarrow$ Direction cosines l, m, n of the line are:

$l = \cos\alpha = \cos 90° = 0$

$m = \cos\beta = \cos 135° = \cos(180° - 45°)$

$= -\cos 45° = -\frac{1}{\sqrt{2}}$;

$n = \cos\gamma = \cos 45° = \frac{1}{\sqrt{2}}$

Thus, the required direction cosines of the line are $0, -\frac{1}{\sqrt{2}}$ and $\frac{1}{\sqrt{2}}$.

Example 1.2: Find the direction cosines of a line which make equal angles with the coordinate axes. [NCERT]

Ans. Here, $\alpha = \beta = \gamma$

$\Rightarrow \cos\alpha = \cos\beta = \cos\gamma$

$\Rightarrow l = m = n$

Since $l^2 + m^2 + n^2 = 1$,

$\Rightarrow \quad 3l^2 = 1$, or $l = \pm\frac{1}{\sqrt{3}}$

Thus, the required direction cosine of the line are $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$ or $\left(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$.

Example 1.3: If a line has the direction ratios −18, 12, − 4, then what are its direction cosines?

Ans. The required direction cosines are:

$$l = \frac{-18}{\sqrt{(-18)^2 + (12)^2 + (-4)^2}}$$
$$= \frac{-18}{\sqrt{484}} = \frac{-18}{22} = \frac{-9}{11};$$
$$m = \frac{12}{\sqrt{(-18)^2 + (12)^2 + (-4)^2}}$$
$$= \frac{12}{\sqrt{484}} = \frac{12}{22} = \frac{6}{11};$$
$$n = \frac{-4}{\sqrt{(-18)^2 + (12)^2 + (-4)^2}}$$
$$= \frac{-4}{\sqrt{484}} = \frac{-4}{22} = \frac{-2}{11}$$

Thus, required direction cosines are $\frac{-9}{11}, \frac{6}{11}$ and $\frac{-2}{11}$.

Example 1.4: Show that the points (2, 3, 4), (− 1, − 2, 1) and (5, 8, 7) are collinear. [NCERT]

Ans. Let, A(2, 3, 4), B(− 1, − 2, 1) and C(5, 8, 7) be given points.

The direction ratios of AB are: −1 − 2, −2 − 3, 1 − 4, i.e., −3, −5, −3

The direction ratios of AC are: 5 − 2, 8 − 3, 7 − 4 i.e., 3, 5, 3.

Since the directions ratios of AB and AC are proportional, So, AB and AC are either parallel or coincident lines.

Since, point A is common to both the lines, hence, the points A, B and C are collinear.

TOPIC 4

EQUATION OF A LINE IN SPACE

A line in space is uniquely determined, if

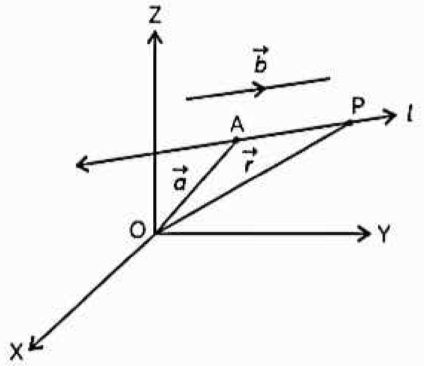
(1) it passes through a given point and has given direction, or

(2) it passes through two given points.

Equation of a Line Through a Given Point and Parallel to a Given Vector

Vector Equation

The vector equation of a line l passing through a point A with position vector $\vec{a}$ and parallel to a given vector $\vec{b}$ is $\vec{r} = \vec{a} + \lambda\vec{b}$, where $\vec{r}$ is the position vector of an arbitrary point on the line l and λ is a real number.



Special case: The vector equation of a line l passing through the origin and parallel to a given vector $\vec{b}$ is $\vec{r} = \lambda\vec{b}$.

Cartesian Equation

The Cartesian equation of a line passing through a point A(x_1, y_1, z_1) and having direction ratios a, b and c is $\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$, where (x, y, z) are coordinates of any point on the line.

If l, m and n are the direction cosines of the line, then the Cartesian equation of the line is

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}.$$

Special case: The Cartesian equation of a line l passing through the origin and having direction ratios a, b and c is $\frac{x}{a} = \frac{y}{b} = \frac{z}{c}$.

Method to Convert Vector Form of Equation of a Line into Cartesian Form

The vector equation of a line is

$$\vec{r} = \vec{a} + \lambda\vec{b} \quad ...(i)$$

where $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$; $\vec{a} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$;

$$\vec{b} = a\hat{i} + b\hat{j} + c\hat{k}$$

Substitute these values in (i) and equating the coefficients of $\hat{i}, \hat{j}$ and $\hat{k}$, we get

$$x = x_1 + \lambda a;\ y = y_1 + \lambda b;\ z = z_1 + \lambda c$$

Important

➥ *The above x, y, z are also used to take an arbitrary point on the line.*

Eliminating λ we get the desired Cartesian equation of the line as

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c} = \lambda$$

Example 1.5: Find the vector equation of the line which passes through the point (1, 2, 3) and is parallel to the vector $3\hat{i} + 2\hat{j} - 2\hat{k}$. [NCERT]

Ans. The position vector of the point (1, 2, 3) is $\hat{i} + 2\hat{j} + 3\hat{k}$.

So, the required vector equation of the line is:

$\vec{r} = \hat{i} + 2\hat{j} + 3\hat{k} + \lambda(3\hat{i} + 2\hat{j} - 2\hat{k})$, where λ is a real number.

Example 1.6: The Cartesian equation of a line is $\frac{x-5}{3} = \frac{y+4}{7} = \frac{z-6}{2}$. Write its vector form. [NCERT]

Ans. The line passes through (5, – 4, 6). So, its position vector is $\vec{a} = 5\hat{i} - 4\hat{j} + 6\hat{k}$. Vector $\vec{b}$ to which the line is parallel, is $\vec{b} = 3\hat{i} + 7\hat{j} + 2\hat{k}$

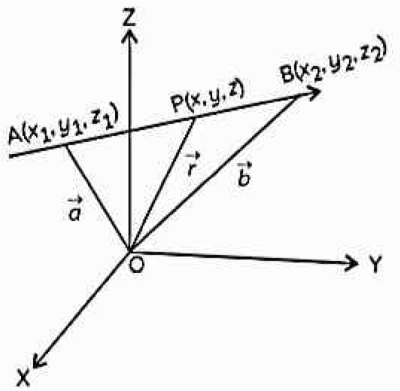
So, the vector equation of the line is:

$\vec{r} = 5\hat{i} - 4\hat{j} + 6\hat{k} + \lambda(3\hat{i} + 7\hat{j} + 2\hat{k})$, where λ is a real number.

Equation of a Line Passing Through Two Given Points

Vector Equation

Let $\vec{a}$ and $\vec{b}$ be the position vectors of two points A(x_1, y_1, z_1) and B(x_2, y_2, z_2), respectively, that are lying on a line.



Let $\vec{r}$ be the position vector of an arbitrary point P(x, y, z). P is a point on the line if and only if $\vec{AP} = \vec{r} - \vec{a}$ and $\vec{AB} = \vec{b} - \vec{a}$ are collinear vectors. Therefore, P is on the line if and only if

$$\vec{r} - \vec{a} = \lambda(\vec{b} - \vec{a})$$

or $$\vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a}), \lambda \in R$$

This is the vector equation of the line passing through two points.

Cartesian Equation

The Cartesian equation of a line passing through two points, namley, $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$, is:

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}.$$

Method to Convert Vector Form of Equation of a Line into Cartesian Form

The vector form of equation of a line is

$$\vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a}) \quad ...(i)$$

where, $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$; $\vec{a} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$;

$\vec{b} = x_2\hat{i} + y_2\hat{j} + z_2\hat{k}$

Substitute these values in (i) and equating the coefficients of $\hat{i}, \hat{j}$ and $\hat{k}$, we get

$x = x_1 + \lambda(x_2 - x_1)$; $y = y_1 + \lambda(y_2 - y_1)$; $z = z_1 + \lambda(z_2 - z_1)$

Eliminating λ, we get the desired Cartesian equation of the line as

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$$

Example 1.7: Find the vector and the Cartesian equations of the line that passes through the points (3, –2, –5) and (3, –2, 6). [NCERT]

Ans. Let $\vec{a}$ and $\vec{b}$ be the position vectors of two points A(3, –2, –5) and B(3, –2, 6) respectively. Then,

$\vec{a} = 3\hat{i} - 2\hat{j} - 5\hat{k}$; $\vec{b} = 3\hat{i} - 2\hat{j} + 6\hat{k}$

So, $\vec{b} - \vec{a} = (3 - 3)\hat{i} + (-2 + 2)\hat{j} + (6 + 5)\hat{k}$,

$= 11\hat{k}$

Hence, the required vector equation of the line is:

$$\vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a})$$

$$\Rightarrow \quad \vec{r} = (3\hat{i} - 2\hat{j} - 5\hat{k}) + \lambda(11\hat{k})$$

Also, the required Cartesian equation of the line is

$$\frac{x - 3}{3 - 3} = \frac{y - (-2)}{-2 - (-2)} = \frac{z - (-5)}{6 - (-5)}$$

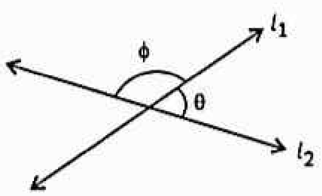
i.e., $$\frac{x - 3}{0} = \frac{y + 2}{0} = \frac{z + 5}{11}$$

TOPIC 5

ANGLE BETWEEN TWO LINES

The angle between two lines is defined as the angle between two vectors parallel to them. So, the result for angle between two vectors is also applicable for angle between two lines.

It is quite clear from the figure below that there are two angles θ and ϕ between two lines such that $0° \le \theta \le 90°$, $90° \le \phi \le 180°$ and $\theta + \phi = 180°$. [By convention, we always consider the acute angle between the two lines.]



Let the vector equations of the two lines l_1 and l_2 be

$$l_1 : \vec{r} = \vec{a_1} + \lambda\vec{b_1} \text{ and } l_2 : \vec{r} = \vec{a_2} + \mu\vec{b_2}$$

Let θ be the acute angle between the two lines. Then, the angle between the two vectors $\vec{b_1}$ and $\vec{b_2}$ is also θ. Thus, the angle between the two lines is given by

$$\cos\theta = \frac{\vec{b_1}.\vec{b_2}}{|\vec{b_1}||\vec{b_2}|}$$

Important

- *If $\vec{b_1}.\vec{b_2} = 0$, then $\vec{b_1} \perp \vec{b_2}$, i.e., the two lines are perpendicular to each other.*
- *If $\vec{b_1} = \lambda\vec{b_2}$ (where λ is a parameter), then both the lines are parallel.*

If the equations of the two lines are

$$\frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1}$$

and
$$\frac{x-x_2}{a_2} = \frac{y-y_2}{b_2} = \frac{z-z_2}{c_2}$$

then the angle θ between them is given by

$$\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}$$

Important

- *If two lines are perpendicular to each other, then $a_1a_2 + b_1b_2 + c_1c_2 = 0$.*
- *If two lines are parallel, then $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$*

Example 1.8: Find the angle between the following pair of lines:

$$\vec{r} = 2\hat{i} - 5\hat{j} + \hat{k} + \lambda(3\hat{i} + 2\hat{j} + 6\hat{k});$$

$$\vec{r} = 7\hat{i} - 6\hat{k} + \lambda(\hat{i} + 2\hat{j} + 2\hat{k})$$ **[NCERT]**

Ans. Comparing the given equations of the two lines with standard equations $\vec{r} = \vec{a_1} + \lambda\vec{b_1}$ and = + λ

$\vec{r} = \vec{a_2} + \lambda\vec{b_2}$, we have

$$\vec{a_1} = 2\hat{i} - 5\hat{j} + \hat{k};$$

$$\vec{a_2} = 7\hat{i} - 6\hat{k}$$

$$\vec{b_1} = 3\hat{i} + 2\hat{j} + 6\hat{k};$$

$$\vec{b_2} = \hat{i} + 2\hat{j} + 2\hat{k}$$

If θ be the angle between the two given lines, then

$$\cos\theta = \frac{\overline{b_1}.\overline{b_2}}{|\overline{b_1}||\overline{b_2}|} \quad ...(i)$$

Now, $\vec{b_1}.\vec{b_2} = (3\hat{i} + 2\hat{j} + 6\hat{k}).(\hat{i} + 2\hat{j} + 2\hat{k})$

$= 3 + 4 + 12 = 19$

Also, $|\overline{b_1}| = \sqrt{3^2 + 2^2 + 6^2}$

$= \sqrt{49} = 7;$

and $|\overline{b_2}| = \sqrt{1^2 + 2^2 + 2^2}$

$= \sqrt{9} = 3$

Putting these values in (i), we have

$$\cos\theta = \frac{19}{21}$$

$$\Rightarrow \quad \theta = \cos^{-1}\left(\frac{19}{21}\right)$$

Thus, the angle between the given two lines is $\cos^{-1}\left(\frac{19}{21}\right)$.

Example 1.9: Find the angle between the following pair of lines:

$$\frac{x-2}{2} = \frac{y-1}{5} = \frac{z+3}{-3} \text{ and } \frac{x+2}{-1} = \frac{y-4}{8} = \frac{z-5}{4}$$

[NCERT]

Ans. Comparing the given equations of the two lines with standard equations

$$\frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1}$$

and
$$\frac{x-x_2}{a_2} = \frac{y-y_2}{b_2} = \frac{z-z_2}{c_2},$$

we have $a_1 = 2, b_1 = 5, c_1 = -3$

And $a_2 = -1, b_2 = 8, c_2 = 4$

Let θ be the angle between the two given lines. Then

$$\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}} \quad ...(i)$$

Now, $a_1a_2 + b_1b_2 + c_1c_2 = -2 + 40 - 12 = 26$

$\sqrt{a_1^2+b_1^2+c_1^2} = \sqrt{2^2 + 5^2 + (-3)^2}$

$= \sqrt{4 + 25 + 9}$

$= \sqrt{38}$

$\sqrt{a_2^2+b_2^2+c_2^2} = \sqrt{(-1)^2 + 8^2 + 4^2}$

$= \sqrt{1 + 64 + 16}$

$= \sqrt{81} = 9$

Putting these values in (i), we have

$$\cos\theta = \frac{26}{9\sqrt{38}}$$

$$\Rightarrow \quad \theta = \cos^{-1}\left(\frac{26}{9\sqrt{38}}\right)$$

Thus, the angle between the given two lines is $\cos^{-1}\left(\frac{26}{9\sqrt{38}}\right)$.

Example 1.10: Show that the lines:

$\frac{x-5}{7} = \frac{y+2}{-5} = \frac{z}{1}$ and $\frac{x}{1} = \frac{y}{2} = \frac{z}{3}$ are perpendicular to each other. **[NCERT]**

Ans. Comparing the given equations of the two lines with standard equations

$$\frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1}$$

and $\dfrac{x-x_2}{a_2}=\dfrac{y-y_2}{b_2}=\dfrac{z-z_2}{c_2}$,

we have

$a_1 = 7, b_1 = -5, c_1 = 1$ and $a_2 = 1, b_2 = 2, c_2 = 3$.

The two lines are perpendicular to each other, when $a_1a_2 + b_1b_2 + c_1c_2 = 0$

Here, $a_1a_2 + b_1b_2 + c_1c_2 = (7 \times 1) + (-5 \times 2) + (1 \times 3) = 0$

Hence, the two given lines are perpendicular to each other.

Example 1.11: Find the value of p so that the lines $\dfrac{1-x}{3}=\dfrac{7y-14}{2p}=\dfrac{z-3}{2}$ and $\dfrac{7-7x}{3p}=\dfrac{y-5}{1}=\dfrac{6-z}{5}$ are at right angles. [NCERT]

Ans. The given lines can be written as

$$\frac{x-1}{-3}=\frac{y-2}{2p/7}=\frac{z-3}{2}$$

and $$\frac{x-1}{-3p/7}=\frac{y-5}{1}=\frac{z-6}{-5}$$

Comparing the given equations of the two lines with standard equations

$$\frac{x-x_1}{a_1}=\frac{y-y_1}{b_1}=\frac{z-z_1}{c_1}$$

and $$\frac{x-x_2}{a_2}=\frac{y-y_2}{b_2}=\frac{z-z_2}{c_2}$$

we have

$a_1 = -3, b_1 = \dfrac{2p}{7}, c_1 = 2$ and $a_2 = -\dfrac{3p}{7}, b_2 = 1,$ $c_2 = -5$

The two lines are at right angles, when

$a_1a_2 + b_1b_2 + c_1c_2 = 0$

$$\Rightarrow \left[\left(-3\times -\frac{3p}{7}\right)+\left(\frac{2p}{7}\times 1\right)+2\times(-5)\right]=0$$

$$\Rightarrow \quad \frac{9p}{7}+\frac{2p}{7}-10=0$$

$$\Rightarrow \quad p=\frac{70}{11}$$

Hence, the two given lines are at right angles when $p=\dfrac{70}{11}$.

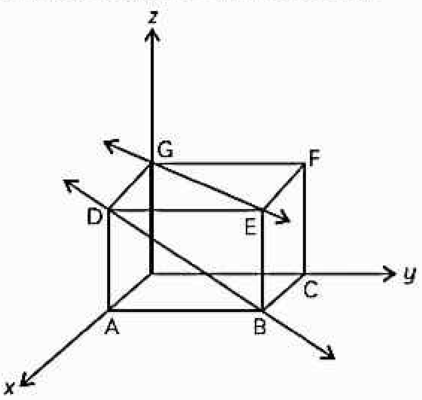
TOPIC 6

SHORTEST DISTANCE BETWEEN TWO LINES

When two lines in space intersect, then we say that the shortest distance (S.D.) between them is zero.

Also, when two lines in space are parallel, then the shortest distance (S.D.) between them is the perpendicular distance, *i.e.*, the length of the perpendicular drawn from a point on one line on to the other line.

Further, in a space, there are lines which are neither intersecting nor parallel. In fact, such lines are non coplanar and are called skew lines. In the following figure, the lines GE and DB are skew lines.



Distance between Two Skew Lines

To find the shortest distance between two skew lines, take a point on one line and join this point to a point on the other line so that the line segment so obtained is perpendicular to both the lines. The length of this line segment is called the shortest distance between the two skew lines.

Vector Equation

If two skew lines are represented by

$$\vec{r}=\overline{a_1}+\lambda\overline{b_1};\ \vec{r}=\overline{a_2}+\lambda\overline{b_2}$$

Then the shortest distance between them is given by

$$\text{S.D.}=\left|\frac{(\vec{b_1}\times\vec{b_2}).(\vec{a_2}-\vec{a_1})}{|\vec{b_1}\times\vec{b_2}|}\right|$$

Cartesian Equation

The shortest distance between the lines

$$l_1:\frac{x-x_1}{a_1}=\frac{y-y_1}{b_1}=\frac{z-z_1}{c_1}$$

and $$l_2:\frac{x-x_2}{a_2}=\frac{y-y_2}{b_2}=\frac{z-z_2}{c_2}$$ is

S.D.

$$= \left| \frac{\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\sqrt{(b_1c_2 - b_2c_1)^2 + (c_1a_2 - c_2a_1)^2 + (a_1b_2 - a_2b_1)^2}} \right|$$

Distance between Two Parallel Lines

If two lines l_1 and l_2 are parallel, then they are coplanar. Let the lines be

$$\vec{r} = \overline{a_1} + \lambda\bar{b}; \vec{r} = \overline{a_2} + \lambda\bar{b}$$

Then the shortest distance between them is given by

$$\text{S.D.} = \left| \frac{\vec{b} \times (\vec{a_2} - \vec{a_1})}{|\vec{b}|} \right|$$

Example 1.12: Find the shortest distance between the following pair of lines:

$$\vec{r} = \hat{i} + 2\hat{j} + \hat{k} + \lambda(\hat{i} - \hat{j} + \hat{k}); \text{ and}$$

$$\vec{r} = 2\hat{i} - \hat{j} - \hat{k} + \mu(2\hat{i} + \hat{j} + 2\hat{k}) \text{ [NCERT]}$$

Ans. Comparing the given equations of the two lines with standard equations $\vec{r} = \vec{a_1} + \lambda\vec{b_1}$ and

$\vec{r} = \vec{a_2} + \mu\vec{b_2}$,

we have

$$\vec{a_1} = \hat{i} + 2\hat{j} + \hat{k};$$

$$\vec{a_2} = 2\hat{i} - \hat{j} - \hat{k}$$

$$\vec{b_1} = \hat{i} - \hat{j} + \hat{k};$$

$$\vec{b_2} = 2\hat{i} + \hat{j} + 2\hat{k}$$

Shortest distance between two lines is given by

$$\text{S.D.} = \left| \frac{(\vec{b_1} \times \vec{b_2}).(\vec{a_2} - \vec{a_1})}{|\vec{b_1} \times \vec{b_2}|} \right| \quad \text{...(i)}$$

Here, $\vec{a_2} - \vec{a_1} = (2\hat{i} - \hat{j} - \hat{k}) - (\hat{i} + 2\hat{j} + \hat{k})$

$$= \hat{i} - 3\hat{j} - 2\hat{k}$$

And $\overline{b_1} \times \overline{b_2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 1 \\ 2 & 1 & 2 \end{vmatrix}$

$$= (-2-1)\hat{i} - (2-2)\hat{j} + (1+2)\hat{k}$$

$$= -3\hat{i} + 3\hat{k}$$

and $|\vec{b_1} \times \vec{b_2}| = \sqrt{(-3)^2 + (3)^2} = \sqrt{18} = 3\sqrt{2}$

$\therefore (\vec{b_1} \times \vec{b_2}).(\vec{a_2} - \vec{a_1}) = (-3\hat{i} + 3\hat{k}).(\hat{i} - 3\hat{j} - 2\hat{k})$

$$= -3 - 6 = -9$$

Substituting these values in (i), we have

$$\text{S.D.} = \left| \frac{-9}{3\sqrt{2}} \right| = \frac{3\sqrt{2}}{2} \text{ units}$$

Example 1.13: Find the shortest distance between the following pair of lines:

$$\frac{x+1}{7} = \frac{y+1}{-6} = \frac{z+1}{1} \text{ and } \frac{x-3}{1} = \frac{y-5}{-2} = \frac{z-7}{1}$$

[NCERT]

Ans. Comparing the given equations of the two lines with standard cartesian form of equations, we have

$x_1 = -1, y_1 = -1, z_1 = -1$ and $x_2 = 3, y_2 = 5, z_2 = 7$

$a_1 = 7, b_1 = -6, c_1 = 1$ and $a_2 = 1, b_2 = -2, c_2 = 1$

The shortest distance between the lines is

S.D.

$$= \left| \frac{\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\sqrt{(b_1c_2 - b_2c_1)^2 + (c_1a_2 - c_2a_1)^2 + (a_1b_2 - a_2b_1)^2}} \right| \quad \text{...(i)}$$

Now, $\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}$

$$= \begin{vmatrix} 3-(-1) & 5-(-1) & 7-(-1) \\ 7 & 6 & 1 \\ 1 & 2 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 4 & 6 & 8 \\ 7 & -6 & 1 \\ 1 & -2 & 1 \end{vmatrix}$$

$$= 4(-6+2) - 6(7-1) + 8(-14+6)$$

$$= -16 - 36 - 64 = -116$$

and

$$\sqrt{(b_1c_2 - b_2c_1)^2 + (c_1a_2 - c_2a_1)^2 + (a_1b_2 - a_2b_1)^2}$$

$$= \sqrt{(-4)^2 + (-6)^2 + (-8)^2} = \sqrt{116}$$

Substituting these values in (i), we have

$$\text{S.D.} = \left| \frac{-116}{\sqrt{116}} \right| = \sqrt{116}, \text{ or } 2\sqrt{29} \text{ units}$$

Example 1.14: Find the shortest distance between the following pair of lines:

$$\vec{r} = \hat{i} + 2\hat{j} + 3\hat{k} + \lambda(2\hat{i} + 3\hat{j} + 4\hat{k}); \text{ and}$$

$$\vec{r} = 2\hat{i} + 4\hat{j} + 5\hat{k} + \mu(4\hat{i} + 6\hat{j} + 8\hat{k})$$

Ans. Comparing the given equations of the two lines with standard equations $\vec{r} = \vec{a_1} + \lambda\vec{b_1}$ and $\vec{r} = \vec{a_2} + \mu\vec{b_2}$, we have

$$\vec{a_1} = \hat{i} + 2\hat{j} + 3\hat{k};$$

$$\vec{a_2} = 2\hat{i} + 4\hat{j} + 5\hat{k}$$

$$\vec{b_1} = 2\hat{i} + 3\hat{j} + 4\hat{k}$$

$$\vec{b_2} = 4\hat{i} + 6\hat{j} + 8\hat{k}$$

Since $\vec{b_2} = 2\vec{b_1}$, the two vectors $\vec{b_1}$ and $\vec{b_2}$ are parallel. So lines are parallel.

Let $\vec{b_2} = 2\vec{b_1} = 4\hat{i} + 6\hat{j} + 8\hat{k} = \vec{b}$ (say)

∴ Shortest distance between these lines is given by

$$\text{S.D.} = \left|\frac{\vec{b} \times (\vec{a_2} - \vec{a_1})}{|\vec{b}|}\right| \qquad ...(i)$$

Here, $\vec{a_2} - \vec{a_1} = (2\hat{i} + 4\hat{j} + 5\hat{k}) - (\hat{i} + 2\hat{j} + 3\hat{k})$

$= \hat{i} + 2\hat{j} + 2\hat{k}$

$$\Rightarrow \vec{b} \times (\vec{a_2} - \vec{a_1}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & 6 & 8 \\ 1 & 2 & 2 \end{vmatrix}$$

$= \hat{i}(12-16) - \hat{j}(8-8) + \hat{k}(8-6)$

$= -4\hat{i} + 2\hat{k}$

Also, $|\vec{b}| = \sqrt{4^2 + 6^2 + 8^2}$

$= \sqrt{16 + 36 + 64}$

$= \sqrt{116}$

Substituting these values in (i), we have

$$\text{S.D.} = \left|\frac{-4\hat{i} + 2\hat{k}}{\sqrt{116}}\right|$$

$$= \frac{\sqrt{(-4)^2 + 2^2}}{\sqrt{116}}$$

$$= \frac{\sqrt{20}}{\sqrt{116}}$$

$$= \sqrt{\frac{5}{29}} \text{ units}$$

OBJECTIVE Type Questions

[**1** mark]

Multiple Choice Questions

1. The distance of the point (6, 5, 8) from y-axis is:

(a) 5 units **(b) 6 units**

(c) 8 units **(d) 10 units**

Ans. *(d) 10 units*

Explanation: The distance of the point (x, y, z) from the y-axis is $\sqrt{x^2 + z^2}$.

Thus, the distance of the point (6, 5, 8) from the y-axis is $\sqrt{6^2 + 8^2}$, *i.e.*, 10 units.

2. A directed line makes angles of 60° and 135° with the axes of x and y respectively. The angle that this line makes with z-axis is:

(a) 15° or 165° **(b) 30° or 150°**

(c) 45° or 135° **(d) 60° or 120°**

3. The d.c's of the line making equal angles with the positive directions of the coordinate axes are:

(a) 1, 1, 1 **(b) $\frac{1}{2}, \frac{1}{2}, \frac{1}{2}$**

(c) $\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}$ **(d) $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$**

Ans. *(d)* $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$

Explanation: In this case $\alpha = \beta = \gamma$

So, $3\cos^2\alpha = 1 \Rightarrow \cos\alpha = \frac{1}{\sqrt{3}}$

4. Distance of the point (α, β, γ) from y-axis is:

(a) β **(b) $|\beta|$**

(c) $|\alpha| + |\gamma|$ **(d) $\sqrt{\alpha^2 + \gamma^2}$**

Ans. *(d)* $\sqrt{\alpha^2 + \gamma^2}$

Explanation: Coordinates of any point on the y-axis are $(0, \beta, 0)$.

∴ Required distance

$$= \sqrt{(\alpha-0)^2+(\beta-\beta)^2+(\gamma-0)^2}$$

$$= \sqrt{\alpha^2+\gamma^2}$$

5. If the directions cosines of a line are k, k, k then:

(a) $k > 0$ (b) $0 < k < 1$

(c) $k = 1$ (d) $k = \frac{1}{\sqrt{3}}$ or $-\frac{1}{\sqrt{3}}$

Ans. *(d)* $k = \frac{1}{\sqrt{3}}$ *or* $-\frac{1}{\sqrt{3}}$

Explanation: Since direction cosines of the line are k, k, k.

∴ $l = k, m = k, n = k$

We know,

$$l^2 + m^2 + n^2 = 1$$

$$\Rightarrow k^2 + k^2 + k^2 = 1$$

$$\Rightarrow 3k^2 = 1$$

$$\Rightarrow k^2 = \frac{1}{3}$$

$$\Rightarrow k = \pm\frac{1}{\sqrt{3}}$$

6. The length of the perpendicular drawn from the point (4, – 7, 3) on the y-axis is:

(a) 3 units (b) 4 units

(c) 5 units (d) 7 units [CBSE 2020]

Ans. *(c) 5 units*

Explanation: Let A(4, – 7, 3) be the given point and P be a point on y-axis.

∵ PA ⊥ y-axis

∴ P = (0, – 7, 0)

Now, $PA = \sqrt{(4-0)^2+(3-0)^2}$

$$= \sqrt{4^2+3^2} = \sqrt{16+9}$$

$$= \sqrt{25}$$

= 5 units

7. If the Cartesian equation of a line is $\frac{x-5}{3} = \frac{y+4}{7} = \frac{z-6}{12}$, then its vector equation is:

(a) $\vec{r} = (3\hat{i} + 7\hat{j} + 12\hat{k}) + \lambda(5\hat{i} + 4\hat{j} + 6\hat{k})$

(b) $\vec{r} = (3\hat{i} + 7\hat{j} + 12\hat{k}) + \lambda(5\hat{i} - 4\hat{j} + 6\hat{k})$

(c) $\vec{r} = (5\hat{i} - 4\hat{j} + 6\hat{k}) + \lambda(3\hat{i} + 7\hat{j} + 12\hat{k})$

(d) $\vec{r} = (5\hat{i} + 4\hat{j} + 6\hat{k}) + \lambda(3\hat{i} + 7\hat{j} + 12\hat{k})$

8. The direction cosines of line joining the points (– 2, 4, – 5) and (1, 2, 3) are:

(a) $\frac{-3}{\sqrt{77}}, \frac{-2}{\sqrt{77}}, \frac{8}{\sqrt{77}}$ (b) $\frac{3}{\sqrt{77}}, \frac{-2}{\sqrt{77}}, \frac{8}{\sqrt{77}}$

(c) $\frac{3}{\sqrt{77}}, \frac{-2}{\sqrt{77}}, \frac{-8}{\sqrt{77}}$ (d) $\frac{3}{\sqrt{77}}, \frac{2}{\sqrt{77}}, \frac{8}{\sqrt{77}}$

Ans. *(b)* $\frac{3}{\sqrt{77}}, \frac{-2}{\sqrt{77}}, \frac{8}{\sqrt{77}}$

Explanation: We know that the direction cosines of a line joining the points (x_1, y_1, z_1) and (x_2, y_2, z_2) are

$$\frac{x_2-x_1}{\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}};$$

$$\frac{y_2-y_1}{\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}};$$

$$\frac{z_2-z_1}{\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}}$$

Thus, the required direction cosines are

$$\frac{1+2}{\sqrt{(1+2)^2+(2-4)^2+(3+5)^2}};$$

$$\frac{2-4}{\sqrt{(1+2)^2+(2-4)^2+(3+5)^2}};$$

$$\frac{3+5}{\sqrt{(1+2)^2+(2-4)^2+(3+5)^2}}$$

i.e., $\frac{3}{\sqrt{77}}, \frac{-2}{\sqrt{77}}, \frac{8}{\sqrt{77}}$

9. The angle between the lines $2x = 3y = -z$ and $6x = -y = -4z$, is:

(a) 0° (b) 45°

(c) 30° (d) 90°

Ans. *(d) 90°*

Explanation: The given equations of the two lines can be rewritten as

$$\frac{x}{1/2} = \frac{y}{1/3} = \frac{z}{-1}; \frac{x}{1/6} = \frac{y}{-1} = \frac{z}{-1/4}$$

Since, $a_1a_2 + b_1b_2 + c_1c_2$

$$= \left[\left(\frac{1}{2}\right)\left(\frac{1}{6}\right)+\left(\frac{1}{3}\right)(-1)+(-1)\left(-\frac{1}{4}\right)\right] = 0.$$

∴ The angles between the lines is 90°.

10. The lines $\frac{-x+1}{2} = \frac{y-2}{-4} = \frac{-z+3}{6}$ and $\frac{x}{1} = \frac{y}{2} = \frac{z}{3}$ are:

(a) parallel (b) perpendicular

(c) intersecting (d) coincident

11. A line makes the same angle θ with each of the x and z-axes. If the angle β which it makes with y-axis is such that $\sin^2\beta = 3\sin^2\theta$ then find $\cos^2\theta$.

(a) $\frac{2}{5}$ (b) $\frac{1}{5}$

(c) $\frac{3}{5}$ (d) $\frac{2}{5}$

[DIKSHA]

Ans. *(c)* $\frac{3}{5}$

Explanation: If a line makes the angle α, β, γ with x, y, z axes respectively, then

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

$$\Rightarrow \cos^2\theta + \cos^2\beta + \cos^2\theta = 1$$

$$\Rightarrow 2\cos^2\theta + \cos^2\beta = 1$$

$$\Rightarrow 2\cos^2\theta = 1 - \cos^2\beta = \sin^2\beta$$

$$\Rightarrow 2\cos^2\theta = 3\sin^2\theta$$

$$[\because \sin^2\beta = 3\sin^2\theta \text{ (Given)}]$$

$$\Rightarrow 2\cos^2\theta = 3 - 3\cos^2\theta$$

$$\Rightarrow 5\cos^2\theta = 3$$

$$\Rightarrow \cos^2\theta = \frac{3}{5}$$

12. Let P(2, –1, 4) and Q(4, 3, 2) be two points and R be on PQ such that 3PQ = 5QR. Then the co-ordinates of R are:

(a) $\left(\frac{14}{5}, \frac{3}{5}, \frac{16}{5}\right)$ (b) $\left(\frac{16}{5}, \frac{7}{5}, \frac{14}{5}\right)$

(c) (6, –7, 18) (d) $\left(-1, \frac{7}{2}, \frac{-13}{2}\right)$

[DIKSHA]

13. The xy plane divides the line joining the points (1, 2, 3) and (3, 2, 1) in the ratio

(a) 1 : 2 internally (b) 2 : 1 externally

(c) 3 : 1 internally (d) 3 : 1 externally

Ans. *(d) 3 : 1 externally*

Explanation: The coordinates of the point that divides the line joining the points (1, 2, 3) and (3, 2, 1) in the ratio $k : 1$ is $\left(\frac{3k+1}{k+1}, \frac{2k+2}{k+1}, \frac{k+3}{k+1}\right)$

This point lies on xy plane So, z-coordinate of any point lying on this plane must be zero. *i.e.,*

$$\frac{k+3}{k+1} = 0, \text{ i.e., } k = -3.$$

14. The d.c.'s of a line that is perpendicular to each of the two lines with d.r's <1, – 2, – 2> and <0, 2, 1> are:

(a) $\frac{2}{3}, -\frac{1}{3}, \frac{2}{3}$ (b) $-\frac{2}{3}, \frac{1}{3}, \frac{2}{3}$

(c) $\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}$ (d) $-\frac{2}{3}, -\frac{1}{3}, -\frac{2}{3}$

15. Every point in space is characterized by coordinates, unique to its position. If we consider the coordinates of the hot air balloon to be (1, 2, 3), then its distance from the coordinate axes can be calculated using the knowledge of three dimensional geometry.

The shortest distances of the point (1, 2, 3) from x, y, z-axes are

(a) 1, 2, 3 (b) $\sqrt{5}, \sqrt{13}, \sqrt{10}$

(c) $\sqrt{10}, \sqrt{13}, \sqrt{5}$ (d) $\sqrt{13}, \sqrt{10}, \sqrt{5}$

Ans. *(d)* $\sqrt{13}, \sqrt{10}, \sqrt{5}$

Explanation: The shortest distances of the point (x, y, z) from the x, y, z-axes are $\sqrt{y^2+z^2}$, $\sqrt{x^2+z^2}, \sqrt{x^2+y^2}$ respectively.

Thus, the shortest distances of the point (1, 2, 3) from the x, y, z-axes are $\sqrt{2^2+3^2}, \sqrt{1^2+3^2}, \sqrt{1^2+2^2}$ respectively, i.e., $\sqrt{13}, \sqrt{10}, \sqrt{5}$

CASE BASED Questions (CBQs)

[4 & 5 marks]

Read the following passages and answer the questions that follow:

16. *A cricket match is organized between the Clubs A and B for which a team from each* *club is chosen. Remaining players of Club A and Club B are respectively sitting on the lines represented by the equations* $\frac{x-3}{1} = \frac{y-2}{2} = \frac{z+4}{2}$ *and* $\frac{x-5}{3} = \frac{y+2}{2} = \frac{z}{6}$ *and to cheer the team of their own clubs.*

(A) The vector equation of the line on which players of Club A are seated, is:

(a) $\vec{r} = (\hat{i} + 2\hat{j} + 2\hat{k}) + \lambda + (3\hat{i} + 2\hat{j} - 4\hat{k})$

(b) $\vec{r} = (3\hat{i} + 2\hat{j} - 4\hat{k}) + \lambda(\hat{i} + 2\hat{j} + 2\hat{k})$

(c) $\vec{r} = (\hat{i} + 2\hat{j} + 2\hat{k}) + \lambda(-3\hat{i} - 2\hat{j} + 4\hat{k})$

(d) $\vec{r} = (3\hat{i} - 2\hat{j} + 4\hat{k}) + \lambda(\hat{i} + 2\hat{j} + 2\hat{k})$

(B) The direction cosines of the line on which players of club B are seated, are:

(a) 3, 2, 6 (b) $\frac{3}{7}, \frac{2}{7}, \frac{6}{7}$

(c) 1, 2, 2 (d) $\frac{1}{3}, \frac{2}{3}, \frac{2}{3}$

(C) If the line on which players of club A are seated, is perpendicular to the line, whose cartesian equation is $\frac{x-3}{3} = \frac{y-7}{4} = \frac{kz-7}{1}$, then the value of k is:

(a) $\frac{7}{4}$ (b) $\frac{1}{6}$

(c) -1 (d) $-\frac{2}{11}$

(D) Which of the following is a player of club A?

(a) (4, 4, – 2) (b) (1, 2, 3)

(c) (0, 1, – 1) (d) (– 2, 1, 4)

(E) The angle between the lines on which players of clubs A and B are seated, is:

(a) $\sin^{-1}\left(\frac{19}{21}\right)$ (b) $\sin^{-1}\left(\frac{17}{19}\right)$

(c) $\cos^{-1}\left(\frac{19}{21}\right)$ (d) $\cos^{-1}\left(\frac{17}{19}\right)$

Ans. (A) *(b)* $\vec{r} = (3\hat{i} + 2\hat{j} - 4\hat{k}) + \lambda(\hat{i} + 2\hat{j} + 2\hat{k})$

Explanation: The vector equation of a line is given by

$$\vec{r} = \vec{a} + \lambda\vec{b}$$

From the given cartesian form of equation of line we have

$$\vec{a} = 3\hat{i} + 2\hat{j} - 4\hat{k}$$

and $$\vec{b} = \hat{i} + 2\hat{j} + 2\hat{k}$$

∴ Vector equation of given line is:

$$\vec{r} = (3\hat{i} + 2\hat{j} - 4\hat{k}) + \lambda(\hat{i} + 2\hat{j} + 2\hat{k})$$

(B) *(b)* $\frac{3}{7}, \frac{2}{7}, \frac{6}{7}$

Explanation: Equation of line B is

$$\frac{x-5}{3} = \frac{y+2}{2} = \frac{z}{6}$$

So, direction ratios of this line are <3, 2, 6>

So, direction cosines of this line are $\frac{3}{\sqrt{3^2+2^2+6^2}}$, $\frac{2}{\sqrt{3^2+2^2+6^2}}$, $\frac{6}{\sqrt{3^2+2^2+6^2}}$ *i.e.*, $\frac{3}{7}, \frac{2}{7}, \frac{6}{7}$

(C) *(d)* $-\frac{2}{11}$

Explanation: Given equation of line can also be written as

$$\frac{x-3}{3} = \frac{y-7}{4} = \frac{y-\frac{7}{k}}{\frac{1}{k}}$$

So, direction ratios of this line are <3, 4, $\frac{1}{k}$>.

Also, direction ratios of line A are <1, 2, 2>.

Since, the two lines are perpendicular,

$\therefore a_1 a_2 + b_1 b_2 + c_1 c_2 = 0$

$\Rightarrow 3 \times 1 + 4 \times 2 + \frac{1}{k} \times 2 = 0$

$\Rightarrow \quad \frac{2}{k} = -11$

or, $\quad k = -\frac{2}{11}$

17. *Consider the lines* $L_1 : \frac{x-1}{2} = \frac{y}{-1} = \frac{z+3}{1}$;

$L_2: \frac{x-4}{1} = \frac{y+3}{1} = \frac{z+3}{2}$

(A) Find the points of intersection of the lines L_1 and L_2.

(B) Find the angle between the two lines.

Ans. (B) Angle between the two lines is given as

$$\cos\theta = \frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}$$

Here,

$(a_1, b_1, c_1) = (2, -1, 1)$

and

$(a_2, b_2, c_2) = (1, 1, 2)$

$$\therefore \quad \cos\theta = \frac{2\times1+(-1)\times1+1\times2}{\sqrt{2^2+(1)^2+1^2}\sqrt{1^2+1^2+2^2}}$$

$$= \frac{2-1+2}{\sqrt{6}\sqrt{6}} = \frac{3}{6} = \frac{1}{2}$$

$$= \cos\frac{\pi}{3}$$

$$\Rightarrow \quad \theta = \frac{\pi}{3}$$

Hence, the acute angle between the two lines is $\frac{\pi}{3}$.

VERY SHORT ANSWER Type Questions (VSA)

[1 mark]

18. Given that P(3, 2, 4), Q(5, 4, -6) and R(9, 8, -10) are collinear. Find the ratio in which Q divides PR.

Ans. Let Q divides PR in the ratio of k : 1. Then, the coordinates of Q are

$$\left(\frac{9k+3}{k+1}, \frac{8k+2}{k+1}, \frac{-10k+4}{k+1}\right)$$

But according to the question,

coordinates of Q are (5, 4, – 6)

$$\therefore \frac{9k+3}{k+1} = 5, \frac{8k+2}{k+1} = 4, \frac{-10k+4}{k+1} = -6$$

Solving either of the equations, we get

$$9k + 3 = 5k + 5$$

$$\Rightarrow \quad 4k = 2$$

$$\Rightarrow \quad k = \frac{1}{2}$$

Hence, the required ratio is 1 : 2.

19. If a line makes angles 90°, 60° and θ with *x*, *y* and *z*-axes respectively, where θ is acute, then find θ. **[CBSE 2015]**

Ans. We know, $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

Here, $\alpha = 90°$, $\beta = 60°$ and $\gamma = \theta$

$\therefore \cos^2 90° + \cos^2 60° + \cos^2\theta = 1$

$$\Rightarrow 0 + \left(\frac{1}{2}\right)^2 + \cos^2\theta = 1$$

$$\Rightarrow \quad \cos^2\theta = 1 - \frac{1}{4} = \frac{3}{4}$$

$$\Rightarrow \quad \cos\theta = \frac{\sqrt{3}}{2}$$

$$\therefore \quad \theta = 30°$$

20. Find the vector equation of the line which is parallel to the vector $3\hat{i}-2\hat{j}+6\hat{k}$ and which passes through the point (1, -2, 3). **[NCERT Exemplar]**

Ans. Let $\vec{a} = 3\hat{i}-2\hat{j}+6\hat{k}$ and $\vec{b} = \hat{i}-2\hat{j}+3\hat{k}$.

So, vector equation of line which is parallel to $\vec{a}$ and passes through the vector $\vec{b}$ is

$$\vec{r} = \vec{b} + \lambda\vec{a}$$

i.e. $\quad \vec{r} = (\hat{i}-2\hat{j}+3\hat{k}) + \lambda(3\hat{i}-2\hat{j}+6\hat{k})$

⚠ Caution

- *Here we will use the formula, $\vec{r} = \vec{b} + \lambda\vec{a}$, where $\vec{r}$ is the equation of the line which passes through and is parallel to $\vec{a}$.*

21. Write the vector equation of the line $\frac{x-5}{3} + \frac{y+4}{7} = \frac{z-6}{2}$. **[CBSE 2011]**

Ans. We have $\vec{a} = 5\hat{i} - 4\hat{j} + 6\hat{k}$

and $\vec{b} = 3\hat{i} + 7\hat{j} + 2\hat{k}$

So, the vector equation will be

$\vec{r} = \vec{a} + \lambda \vec{b}$

$\Rightarrow \vec{r} = 5\hat{i} - 4\hat{j} + 6\hat{k} + \lambda(3\hat{i} + 7\hat{j} + 2\hat{k})$

22. **Find the vector equation of a line which passes through the point (3, 4, 5) and is parallel to the vector $(2\hat{i} + 2\hat{j} - 3\hat{k})$.** **[CBSE 2019]**

23. If the equation of a line AB is $\frac{3-x}{-3} = \frac{y+2}{-2} = \frac{z+2}{6}$, find the d.c.'s of a line parallel to AB. **[CBSE 2012]**

Ans. The equation of line AB is

$$\frac{3-x}{-3} = \frac{y+2}{-2} = \frac{z+2}{6}$$

$$\Rightarrow \quad \frac{x-3}{3} = \frac{y+2}{2} = \frac{z+2}{6}$$

Its direction ratios are <3, – 2, 6>.

Hence, its d.c.'s are

$$\frac{3}{\sqrt{3^2+(-2)^2+6^2}}, \frac{-2}{\sqrt{3^2+(-2)^2+6^2}}, \frac{6}{\sqrt{3^2+(-2)^2+6^2}}$$

i.e., $\frac{3}{7}, \frac{-2}{7}, \frac{6}{7}$

∴ D.c's of a line parallel to AB are proportional to $\frac{3}{7}, \frac{-2}{7}, \frac{6}{7}$.

24. Find the direction ratios of a line whose direction cosines are $\frac{1}{2}, \frac{1}{\sqrt{2}}, \frac{1}{2}$.

Ans. Since, any three number proportional to the direction cosines are the direction ratios.

So, infinite sets of d.r.'s can be found of them, one could be $1, \sqrt{2}, 1$ which we get on multiplying d.c's by 2.

25. Calculate the angle between the lines through the points (4, 7, 8), (2, 3, 4) and (– 1, – 2, 1), (1, 2, 5).

Ans. The d.r.'s of line passing through points (4, 7, 8), (2, 3, 4) is <2 – 4, 3 – 7, 4 – 8>

i.e., <– 2, – 4, – 4>.

And d.r.'s of line passing through points (– 1, – 2, 1), (1, 2, 5) is <1 + 1, 2 + 2, 5 – 1> *i.e.*, <2, 4, 4>.

Since d.r's of the two lines are proportional.

Therefore, given lines are parallel to each other.

Hence, angle between them is zero.

26. If a line makes angles α, β, γ with the positive direction of coordinate axes, then write the value of $\sin^2\alpha + \sin^2\beta + \sin^2\gamma$. **[CBSE 2015]**

Ans. The direction cosines of a line with angles α, β, γ are cos α, cos β, cos γ.

We know, $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

$\Rightarrow (1 - \sin^2\alpha) + (1 - \sin^2\beta) + (1 - \sin^2\gamma) = 1$

$\Rightarrow -(\sin^2\alpha + \sin^2\beta + \sin^2\gamma) = 1 - 3$

$\Rightarrow \sin^2\alpha + \sin^2\beta + \sin^2\gamma = 2$

27. *The electric transmission wires are fitted on poles using proper accessories. It should be ensured that there is proper tension in the wires.*

If a line makes angles 90°, 135°, 45° with the *x*, *y* and *z* axes respectively, find its direction cosines. **[CBSE 2019]**

Ans. Given, the line makes an angle of 90°, 135° and 45° with *x*, *y* and *z*-axes.

∴ $l = \cos 90° = 0$

$m = \cos 135° = -\frac{1}{\sqrt{2}}$

$n = \cos 45° = \frac{1}{\sqrt{2}}$

Hence, d.c.'s of the line are $0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}$.

SHORT ANSWER Type-I Questions (SA-I)

[2 marks]

28. The *x*-coordinate of a point on the line joining the points P(2, 2, 1) and Q(5, 1, – 2) is 4. Find its *z*-coordinate. **[CBSE 2017]**

Ans. Given points are P(2, 2, 1) and Q(5, 1, – 2).

Let R be the point on line PQ which divides the line in the ratio $k : 1$. As x-coordinate of R is 4.

$\therefore$ By section formula

$$4 = \frac{5k+2}{k+1} \Rightarrow k = 2$$

Now, z-coordinate of point R,

$$z = \frac{-2k+1}{k+1} = \frac{-2\times 2+1}{2+1} = -1$$

Hence, z-coordinate of point R is -1.

29. Find the value of k so that the lines $x = -y = kz$ and $x - 2 = 2y - 1 = -z + 1$ are perpendicular to each other. [CBSE 2020]

Ans. The given lines are

$$l_1 : \frac{x-0}{1} = \frac{y-0}{-1} = \frac{z-0}{1/k}$$

$$l_2 : \frac{x-2}{1} = \frac{y-1/2}{1/2} = \frac{z-1}{-1}$$

Given, l_1 is perpendicular to l_2

$$\therefore (1)(1) + (-1)\left(\frac{1}{2}\right) + \left(\frac{1}{k}\right)(-1) = 0$$

$$\Rightarrow 1 - \frac{1}{2} - \frac{1}{k} = 0 \Rightarrow \frac{1}{2} = \frac{1}{k} \Rightarrow k = 2$$

30. A(–1, 3, 2), B(–2, 3, –1), C(–5, –4, p) and D(–2, –4, 3) are four points in space. Lines AB and CD are parallel. Find the value of p.

Ans. Equation of line AB, passing through points A(–1, 3, 2) and B(–2, 3, –1), is

$$\frac{x+1}{-2+1} = \frac{y-3}{3-3} = \frac{z-2}{-1-2}$$

i.e., $$\frac{x+1}{-1} = \frac{y-3}{0} = \frac{z-2}{-3}$$

So, direction ratios of line AB are $<-1, 0, -3>$

Similarly, equation of line CD, passing through points C(– 5, – 4, p) and D(– 2, – 4, 3), is

$$\frac{x+2}{-5+2} = \frac{y+4}{-4+4} = \frac{z-3}{p-3}$$

i.e., $$\frac{x+2}{-3} = \frac{y+4}{0} = \frac{z-3}{p-3}$$

$\therefore$ Direction ratios of line CD are $<-3, 0, p-3>$.

$\because$ Lines AB and CD are parallel, so their direction ratios must be proportional.

$$\Rightarrow \quad \frac{-1}{-3} = \frac{0}{0} = \frac{-3}{p-3}$$

$$\Rightarrow \quad \frac{1}{3} = \frac{-3}{p-3} = p - 3 = -9$$

$$\Rightarrow \quad p = -6$$

Hence, the value of p is – 6.

31. Find the angle between the lines whose direction cosines are given by the equations $l + m + n = 0$, $l^2 + m^2 - n^2 = 0$. [NCERT Exemplar]

Ans. We have,

$$l + m + n = 0 \quad ...(i)$$

or $$n = -l - m \quad ...(ii)$$

Also, $$l^2 + m^2 - n^2 = 0$$

$$\Rightarrow l^2 + m^2 - (-l - m)^2 = 0 \quad \text{[using (ii)]}$$

$$\Rightarrow l^2 + m^2 - l^2 - m^2 - 2ml = 0$$

$$\Rightarrow 2lm = 0$$

Or $$lm = 0$$

$$\Rightarrow (-m - n)m = 0 \quad \text{[from (i)]}$$

$$\Rightarrow (m + n)\, m = 0$$

$$\Rightarrow m = -n \text{ or } m = 0$$

$$\Rightarrow l = 0 \text{ or } l = -n$$

$\therefore$ Direction cosines of the lines are proportional to $0, -n, n$ and $-n, 0, n$ *i.e.* 0, –1, 1 and –1, 0, 1.

So, the vectors parallel to these given lines are

$$\vec{b}_1 = -\hat{j} + \hat{k} \text{ and } \vec{b}_2 = -\hat{i} + \hat{k}$$

Now, $$\cos\theta = \frac{|\vec{b}_1 . \vec{b}_2|}{|\vec{b}_1||\vec{b}_2|} = \frac{0+0+1}{\sqrt{2}\sqrt{2}} = \frac{1}{2}$$

$$\Rightarrow \quad \theta = \frac{\pi}{3}$$

Hence, the angle between the two lines is $\frac{\pi}{3}$.

32. Find the position vector of a point A in space such that $\overline{OA}$ is inclined at 60° to OX and at 45° to OY and $|\overline{OA}| = 10$ units. [NCERT Exemplar]

33. If a line in the space makes angles α, β and γ with the coordinate axes, then find the value of $\cos 2\alpha + \cos 2\beta + \cos 2\gamma + \sin^2\alpha + \sin^2\beta + \sin^2\gamma$.

Ans. Given: Line makes angles α, β and γ with coordinate axes.

$$\therefore \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1 \quad ...(i)$$

Now, $\cos 2\alpha + \cos 2\beta + \cos 2\gamma + \sin^2\alpha + \sin^2\beta + \sin^2\gamma$

$= (\cos^2\alpha - \sin^2\alpha) + (\cos^2\beta - \sin^2\beta) + (\cos^2\gamma - \sin^2\gamma) + \sin^2\alpha + \sin^2\beta + \sin^2\gamma$

$= \cos^2\alpha + \cos^2\beta + \cos^2\gamma$

$= 1$ [using (i)]

Hence, the value of given expression is 1.

34. **Find the direction cosines of the following line:**

$$\frac{3-x}{-1}=\frac{2y-1}{2}=\frac{z}{4}$$

[CBSE Term-2 SQP 2022]

Ans. The given line is

$$\frac{x-3}{1}=\frac{y-\frac{1}{2}}{1}=\frac{z}{4}$$

Its direction ratios are (1, 1, 4).

Its direction cosines are

$$\left(\frac{1}{3\sqrt{2}},\frac{1}{3\sqrt{2}},\frac{4}{3\sqrt{2}}\right)$$

[CBSE Marking Scheme Term-2 SQP 2022]

Explanation: The given equation of line is:

$$\frac{3-x}{-1}=\frac{2y-1}{2}=\frac{z}{4}$$

or $$\frac{-(x-3)}{-1}=\frac{2\left(y-\frac{1}{2}\right)}{2}=\frac{z}{4}$$

$\Rightarrow$ $$\frac{x-3}{1}=\frac{y-\frac{1}{2}}{1}=\frac{z}{4}$$

Then, its d.r.'s are <1, 1, 4>.

Then, dc's are : $\frac{1}{\sqrt{1^2+1^2+4^2}},\frac{1}{\sqrt{1^2+1^2+4^2}},\frac{4}{\sqrt{1^2+1^2+4^2}}$

i.e., $\frac{1}{\sqrt{18}},\frac{1}{\sqrt{18}},\frac{4}{\sqrt{18}}$

i.e., $\frac{1}{3\sqrt{2}},\frac{1}{3\sqrt{2}},\frac{4}{3\sqrt{2}}$

35. **Find the equation of a line in Cartesian form, which is parallel to $5\hat{i}+\hat{j}-7\hat{k}$ and which passes through the point (5, – 2, 4).**

Ans. Equation of a line which passes the point (x_1, y_1, z_1) and having direction rations <a, b, c> is given by:

$$\frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c}$$

Here, $(x_1, y_1, z_1) = (5, -2, 4)$

Since, reqired line is parallel to the vector $5\hat{i}+\hat{j}-7\hat{k}$, so, direction ratios of required line is proportional to $5\hat{i}+\hat{j}-7\hat{k}$, i.e., <5λ, λ, – 7λ>

∴ Required equation of line is

$$\frac{x-5}{5\lambda}=\frac{y-(-2)}{\lambda}=\frac{z-4}{-7\lambda}$$

or $$\frac{x-5}{5}=\frac{y+2}{1}=\frac{z-4}{-7}$$

36. **Prove that the lines $x = py + q$, $z = ry + s$ and $x = p'y + q'$, $z = r'y + s'$ are perpendicular if $pp' + rr' + 1 = 0$.** **[Delhi Gov. 2022]**

SHORT ANSWER Type-II Questions (SA-II)

[3 marks]

37. **Show that the lines**

$$\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4} \text{ and } \frac{x-4}{5}=\frac{y-1}{2}=z$$

intersect. Also, find their point of intersection. **[NCERT Exemplar]**

Ans. Here, lines are

$$\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4} \text{ and } \frac{x-4}{5}=\frac{y-1}{2}=z$$

Then, $x_1 = 1, y_1 = 2, z_1 = 3$ and $a_1 = 2, b_1 = 3, c_1 = 4$

Also $x_2 = 4, y_2 = 1, z_2 = 0$ and $a_2 = 5, b_2 = 2, c_2 = 1$

If two lines intersect, then the shortest distance between them should be zero.

We know, shortest distance between two lines

$$=\frac{\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\sqrt{(b_1c_2-b_2c_1)^2+(c_1a_2-c_2a_1)^2+(a_1b_2-a_2b_1)^2}}$$

$$=\frac{\begin{vmatrix} 4-1 & 1-2 & 0-3 \\ 2 & 3 & 4 \\ 5 & 2 & 1 \end{vmatrix}}{\sqrt{(3\times1-2\times4)^2+(4\times5-1\times2)^2+(2\times2-5\times3)^2}}$$

$$=\frac{\begin{vmatrix} 3 & -1 & -3 \\ 2 & 3 & 4 \\ 5 & 2 & 1 \end{vmatrix}}{\sqrt{25+324+121}}$$

$$= \frac{3(3-8)+1(2-20)-3(4-15)}{\sqrt{470}}$$

$$= \frac{-15-18+33}{\sqrt{470}} = \frac{0}{\sqrt{470}} = 0$$

$\therefore$ Two lines are intersecting.

For finding point of intersection

Let $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4} = \lambda$

$\Rightarrow$ $x = 2\lambda + 1,$

$y = 3\lambda + 2,$

$z = 4\lambda + 3$

Since the lines are intersecting, the point (x, y, z) must satisfy the second line.

$\therefore$ Putting the values of (x, y, z) in second equation of line, we get

$$\frac{2\lambda+1-4}{5} = \frac{3\lambda+2-1}{2} = \frac{4\lambda+3}{1}$$

$$\Rightarrow \frac{2\lambda-3}{5} = \frac{3\lambda+1}{2} = \frac{4\lambda+3}{1}$$

$$\Rightarrow \frac{2\lambda-3}{5} = 4\lambda + 3$$

$\Rightarrow$ $18\lambda = -18$

$\Rightarrow$ $\lambda = -1$

So,

$x = 2(-1) + 1 = -1$

$y = 3(-1) + 2 = -1$

$z = 4(-1) + 3 = -1$

Thus, the lines intersect at point $(-1, -1, -1)$.

⚠ Caution

If shortest distance between the two lines is zero, then they intersect.

38. Prove that the line through A(0, –1, –1) and B(4, 5, 1) intersects the line through C(3, 9, 4) and D(–4, 4, 4). [NCERT Exemplar]

Ans. Equation of line passes through the points A(0, –1, –1) and B(4, 5, 1) is

$$\frac{x-0}{4-0} = \frac{y+1}{5+1} = \frac{z+1}{1+1}$$

$$\Rightarrow \frac{x}{4} = \frac{y+1}{6} = \frac{z+1}{2}$$

Now, another equation of line passing through C(3, 9, 4) and D(–4, 4, 4) is

$$\frac{x-3}{-4-3} = \frac{y-9}{4-9} = \frac{z-4}{4-4}$$

$$\Rightarrow \frac{x-3}{-7} = \frac{y-9}{-5} = \frac{z-9}{0}$$

The lines will intersect, if shortest distance between them is zero.

Now, shortest distance

$$= \frac{\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\sqrt{(b_1c_2-b_2c_1)^2+(c_1a_2-c_2a_1)+(a_1b_2-a_2b_1)^2}}$$

$$= \frac{\begin{vmatrix} 3-0 & 9+1 & 4+1 \\ 4 & 6 & 2 \\ -7 & -5 & 0 \end{vmatrix}}{\sqrt{(0+10)^2+(-14-0)^2+(-20+42)^2}}$$

$$= \frac{\begin{vmatrix} 3 & 10 & 5 \\ 4 & 6 & 2 \\ -7 & -5 & 0 \end{vmatrix}}{\sqrt{100+196+484}}$$

$$= \frac{3(0+10)-10(0+14)+5(-20+42)}{\sqrt{780}}$$

$$= \frac{30-140+110}{\sqrt{780}} = 0$$

Hence, the two lines intersect each other.

39. The direction cosines of a line segment AB are $\frac{-2}{\sqrt{17}}, \frac{3}{\sqrt{17}}, \frac{-2}{\sqrt{17}}$. If AB = $\sqrt{17}$ and the coordinates of A are (3, –6, 10), then find coordinates of B. [DIKSHA]

Ans. Let the coordinates of B be (x, y, z).

So, direction ratios of AB = $<x - 3, y + 6, z - 10>$

Also, direction cosines of AB are $\frac{-2}{\sqrt{17}}, \frac{3}{\sqrt{17}}, \frac{-2}{\sqrt{17}}$

So, its direction ratios are $< -2, 3, -2>$.

$\Rightarrow <-2, 3, -2> = <x - 3, y + 6, z - 10>$

$\Rightarrow x - 3 = -2; y + 6 = 3; z - 10 = -2$

$\Rightarrow x = 1 ; y = -3 ; z = 8$

So, coordinates of B are (1, –3, 8).

40. Find the equations of the two lines through the origin which intersect the line $\frac{x-3}{2} = \frac{y-3}{1} = \frac{z}{1}$ at angles of $\frac{\pi}{3}$ each. [NCERT Exemplar]

41. Find the image of the point (2, –1, 5) in the line $\frac{x-11}{10} = \frac{y+2}{-4} = \frac{z+8}{-11}$. Also, find equation of the line joining the given point and its image. Find the length of that line segment also. [CBSE 2013]

Ans. The given line is

$$\frac{x-11}{10}=\frac{y+2}{-4}=\frac{z+8}{-11}=\lambda \text{ (say)} \quad ...(i)$$

Then, coordinates of any general point, say M, on this line are $(10\lambda + 11, -4\lambda - 2, -11\lambda - 8)$.

Let, the given point be P(2, −1, 5).

D.r's of PM are $<10\lambda + 11 - 2, -4\lambda - 2 + 1, -11\lambda - 8 - 5>$

i.e., $<10\lambda + 9, -4\lambda - 1, -11\lambda - 13>$

As given equation of line is perpendicular to PM.

$\therefore 10(10\lambda + 9) + (-4)(-4\lambda - 1) - 11(-11\lambda - 13) = 0$

$\Rightarrow 100\lambda + 90 + 16\lambda + 4 + 121\lambda + 143 = 0$

$\Rightarrow (100 + 16 + 21)\lambda + 90 + 4 + 143 = 0$

$\Rightarrow 237\lambda + 237 = 0$

$\Rightarrow \lambda = -1$

D.r.'s of PM are $<-1, 3, -2>$ or $<1, -3, 2>$

$\therefore$ Equation of line PM passing through point P(2, − 1, 5) and having direction ratios <1, − 3, 2>.

$$\frac{x-2}{1}=\frac{y+1}{-3}=\frac{z-5}{2} \quad ...(ii)$$

If $Q(\alpha, \beta, \gamma)$ is the image of P in line (i), then M is the mid-point of PQ.

Here, M ≡ (1, 2, 3)

$\therefore \frac{\alpha+2}{2}=1, \frac{\beta-1}{2}=2, \frac{x+5}{2}=3$

$\Rightarrow \alpha = 0, \beta = 5, \gamma = 1$

$\therefore$ Q(0, 5, 1) is the image of P in line (i) and

$$PQ = \sqrt{(2-0)^2+(-1-5)^2+(5-1)^2}$$

$$= \sqrt{56} = 2\sqrt{14} \text{ units.}$$

42. Find the angle between the lines whose direction cosines are given by the equations $3l + m + 5n = 0, 6mn - 2nl + 5lm = 0$.

43. Find the shortest distance between the following lines:

$$\vec{r} = (\hat{i}+\hat{j}-\hat{k}) + s(2\hat{i}+\hat{j}+\hat{k})$$
$$\vec{r} = (\hat{i}+\hat{j}+2\hat{k}) + t(4\hat{i}+2\hat{j}+2\hat{k})$$

[CBSE Term-2 SQP 2022]

Ans. Here, the lines are parallel. The shortest distance

$$= \frac{\left|(\vec{a}_2-\vec{a}_1)\times\vec{b}\right|}{|\vec{b}|} = \frac{\left|(3\hat{k})\times(2\hat{i}+\hat{j}+\hat{k})\right|}{\sqrt{4+1+1}}$$

$$(3\hat{k})\times(2\hat{i}+\hat{j}+\hat{k}) = \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 0 & 0 & 3\\ 2 & 1 & 1\end{vmatrix} = -3\hat{i}+6\hat{j}$$

Hence, the required shortest distance $= \frac{3\sqrt{5}}{\sqrt{6}}$ units

[CBSE Marking Scheme Term-2 SQP 2022]

Explanation: The given, equation of lines are

$$\vec{r} = (\hat{i}+\hat{j}-\hat{k})+s(2\hat{i}+\hat{j}+\hat{k})$$
$$\vec{r} = (\hat{i}+\hat{j}+2\hat{k})+t(4\hat{i}+2\hat{j}+2\hat{k})$$

Since, the direction ratios of the two lines are proportional, so the two lines are parallel.

The shortest distance, between the parallel lines is given by

$$d = \frac{|(\vec{a}_2-\vec{a}_1)\times\vec{b}|}{|\vec{b}|}$$

Here,

$$\vec{a}_1 = (\hat{i}+\hat{j}-\hat{k})$$
$$\vec{a}_2 = (\hat{i}+\hat{j}+2\hat{k})$$
$$\vec{b} = 2\hat{i}+\hat{j}+\hat{k}$$

$\therefore$ Distance between parallel lines

$$= \frac{[(\hat{i}+\hat{j}+2\hat{k})-(\hat{i}+\hat{j}+\hat{k})]\times(2\hat{i}+\hat{j}+\hat{k})}{\sqrt{2^2+1^2+1^2}}$$

$$= \frac{(3\hat{k})\times(2\hat{i}+\hat{j}+\hat{k})}{\sqrt{6}}$$

$$= \frac{1}{\sqrt{6}}\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 0 & 0 & 3\\ 2 & 1 & 1\end{vmatrix}$$

$$= \frac{1}{\sqrt{6}}|(-3\hat{i}+6\hat{j})|$$

$$= \frac{1}{\sqrt{6}}\sqrt{(-3)^2+(6)^2} = \frac{1}{(6)^2}\sqrt{45}$$

$$= \frac{3\sqrt{5}}{\sqrt{6}} \text{units}$$

⚠ Caution

- *Here, both the lines given are parallel so use the formula accordingly to get the answer.*

44. The vector form of equations of two lines, l_1 and l_2, are

l_1: $\vec{r} = 2\hat{j}-\hat{k}+\lambda(-2\hat{j}+\hat{k})$

l_2: $\vec{r} = \hat{i}-3\hat{j}+2\hat{k}+4(\hat{i}\ 2\hat{k})$

Show that l_1 and l_2 are skew lines.

Ans. The direction ratios of the two lines are <0, – 2, 1> and <1, 0, – 2>.

Clearly, their direction ratios are not proportional, i.e., $\frac{a_1}{a_2} \neq \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$.

So, the two lines are not parallel.

Now, shortest distance between two lines is given as

$$d = \left|\frac{(\vec{a}_2 - \vec{a}_1)(\vec{b}_1 \times \vec{b}_2)}{|\vec{b}_1 \times \vec{b}_2|}\right|$$

Here,
$$\vec{a}_1 = 2\hat{i} - \hat{k}$$
$$\vec{b}_1 = -2\hat{j} + \hat{k}$$
$$\vec{a}_2 = \hat{i} + 3\hat{j} + 2\hat{k}$$
$$\vec{b}_2 = \hat{i} - 2\hat{k}$$

So,
$$\vec{a}_2 - \vec{a}_1 = (2\hat{i} - \hat{k}) - (\hat{i} + 3\hat{j} + 2\hat{k})$$
$$= \hat{i} - 3\hat{j} - 3\hat{k}$$

Also,
$$\vec{b}_1 \times \vec{b}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & -2 & 1 \\ 1 & 0 & -2 \end{vmatrix}$$
$$= \hat{i}(4-0) - \hat{j}(0-1) + \hat{k}(0+2)$$
$$= 4\hat{i} + \hat{j} + 2\hat{k}$$

$$\Rightarrow \quad |\vec{b}_1 \times \vec{b}_2| = \sqrt{4^2 + 1^2 + 2^2} = \sqrt{21}$$

So,
$$d = \left|\frac{(\hat{i} - 3\hat{j} - 3\hat{k}).(4\hat{i} + \hat{j} + 2\hat{k})}{\sqrt{21}}\right|$$
$$= \left|\frac{1 \times 4 + (-3) \times 1 + (-3) \times 2}{\sqrt{21}}\right|$$
$$= \left|\frac{1-3-6}{\sqrt{21}}\right| = \left|\frac{-8}{\sqrt{21}}\right|$$
$$= \frac{8}{\sqrt{21}} \text{ units}$$

Since, shortest distance between the two lines is not zero, so they are not intersecting.

As, the given lines are neither parallel nor intersecting, so, they are skew lines.

Hence, proved.

LONG ANSWER Type Questions (LA)

[**4** & **5** marks]

45. **$\overline{AB} = 3\hat{i} - \hat{j} + \hat{k}$ and $\overline{CD} = -3\hat{i} + 2\hat{j} + 4\hat{k}$ are two vectors. The position vectors of the points A and C are $6\hat{i} + 7\hat{j} + 4\hat{k}$ and $-9\hat{j} + 2\hat{k}$, respectively. Find the position vector of a point P on the line AB and a point Q on the line CD such that $\overline{PQ}$ is perpendicular to $\overline{AB}$ and $\overline{CD}$ both. [NCERT Exemplar]**

Ans. Given: $\overline{AB} = 3\hat{i} - \hat{j} + \hat{k}$ and $\overline{CD} = -3\hat{i} + 2\hat{j} + 4\hat{k}$

Also, position vectors of A and C are $6\hat{i} + 7\hat{j} + 4\hat{k}$ and $-9\hat{j} + 2\hat{k}$, respectively.

Since, $\overline{PQ}$ is perpendicular to both $\overline{AB}$ and $\overline{CD}$

So, P and Q will be foot of perpendicular to both the lines through A and C, respectively.

Now, equation of line through A and parallel to the vector $\overline{AB}$ is:

$$\frac{x-6}{3} = \frac{y-7}{-1} = \frac{z-4}{1} = \lambda(\text{say}) \quad ...(i)$$

$$\Rightarrow \quad x = 3\lambda + 6, y = -\lambda + 7, z = \lambda + 4$$

So, coordinates of a general point on this line are $(3\lambda + 6, 7 - \lambda, \lambda + 4)$.

Since, point P lies on this line.

$\therefore$ Coordinates of P = $(3\lambda + 6, 7 - \lambda, \lambda + 4)$.

Similarly, equation of a line through C and parallel to the vector $\overline{CD}$ is:

$$\frac{x-0}{-3} = \frac{y+9}{2} = \frac{z-2}{4} = \mu(\text{say}) \quad ...(ii)$$

$$\Rightarrow \quad x = -3\mu, y = 2\mu - 9, z = 4\mu + 2$$

So, coordinates of a general point on this line are $(-3\mu, 2\mu - 9, 4\mu + 2)$.

Since, point Q lies on this line.

$\therefore$ Coordinates of Q = $(-3\mu, 2\mu - 9, 4\mu + 2)$

$$\therefore \overline{PQ} = (-3\mu - 3\lambda - 6, 2\mu - 9 - 7 + \lambda, 4\mu + 2 - \lambda - 4)$$
$$= (-3\mu - 3\lambda - 6, 2\mu + \lambda - 16, 4\mu - \lambda - 2)$$

Since $\overline{PQ}$ is perpendicular to $\overrightarrow{AB}$, so it is perpendicular to line (i).

$\therefore$ Sum of product of corresponding elements of their direction ratios is zero.

$$\Rightarrow 3(-3\mu - 3\lambda - 6) - (2\mu + \lambda - 16) + (4\mu - \lambda - 2) = 0$$

$$\Rightarrow \quad -7\mu - 11\lambda - 4 = 0 \quad ...(iii)$$

Similarly, $\overline{PQ}$ is perpendicular to the line (ii).

$\therefore -3(-3\mu - 3\lambda - 6) + 2(2\mu + \lambda - 16) + 4(4\mu - \lambda - 2) = 0$

$\Rightarrow \quad 29\mu + 7\lambda - 22 = 0$ ...(iv)

On solving equations (iii) and (iv), we get

$\mu = 1$ and $\lambda = -1$

$\therefore \quad P = (3, 8, 3)$

And $\quad Q = (-3, -7, 6)$

So, position vectors of points P and Q are

$3\hat{i} + 8\hat{j} + 3\hat{k}$ and $-3\hat{i} - 7\hat{j} + 6\hat{k}$, respectively.

⚠ Caution

First calculate the position vectors of two vectors which are perpendicular to the given vectors and then get the required vectors.

46. Find the vector and Cartesian equations of a line passing through (1, 2, –4) and perpendicular to the two lines

$$\frac{x-8}{3} = \frac{y+9}{-16} = \frac{z-10}{7} \text{ and}$$

$$\frac{x-15}{3} = \frac{y-29}{8} = \frac{z-5}{-5}.$$ **[CBSE 2017, 12]**

Ans. Any line which is passing through the point (1, 2, – 4) and have d.r's as a, b, c can be written as

$$\frac{x-1}{a} = \frac{y-2}{b} = \frac{z+4}{c} \quad ...(i)$$

Consider that line (i) is perpendicular to the given lines

$$\frac{x-8}{3} = \frac{y+9}{-16} = \frac{z-10}{7} \quad ...(ii)$$

and $$\frac{x-15}{3} = \frac{y-29}{8} = \frac{z-5}{-5} \quad ...(iii)$$

The d.r's of lines (ii) and (iii) are <3, – 16, 7> and <3, 8, – 5> respectively.

Since, these two lines is perpendicular to the line with d.r's a, b, c.

$\therefore \quad 3a - 16b + 7c = 0$ ...(iv)

and $\quad 3a + 8b - 5c = 0$ ...(v)

On solving equations (iv) and (v) by cross-multiplication, we get

$$\frac{a}{80-56} = \frac{b}{21+15} = \frac{c}{24+48}$$

$$\Rightarrow \quad \frac{a}{24} = \frac{b}{36} = \frac{c}{72}$$

$$\Rightarrow \quad \frac{a}{2} = \frac{b}{3} = \frac{c}{6} = \lambda \text{ (say)}$$

$\Rightarrow a = 2\lambda, b = 3\lambda, c = 6\lambda$

The equation of required line in Cartesian form is

$$\frac{x-1}{2\lambda} = \frac{y-2}{3\lambda} = \frac{z+4}{6\lambda}$$

or $$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z+4}{6}$$

and in vector form is

$$\vec{r} = (\hat{i} + 2\hat{j} - 4\hat{k}) + \lambda(2\hat{i} + 3\hat{j} + 6\hat{k})$$

47. Find the coordinates of the foot of the perpendicular and the length of the perpendicular drawn from the point P(5, 4, 2) to the line, $\vec{r} = -\hat{i} + 3\hat{j} + \hat{k} + \lambda(2\hat{i} + 3\hat{j} - \hat{k})$. Also, find the image of P in this line.

[CBSE 2012]

Ans. The given line is,

$$\vec{r} = -\hat{i} + 3\hat{j} + \hat{k} + \lambda(2\hat{i} + 3\hat{j} - \hat{k})$$

Its equation in Cartesian form is

$$\frac{x+1}{2} = \frac{y-3}{3} = \frac{z-1}{-1} = \lambda \text{ (say)} \quad ...(i)$$

Corrdinates of any general point, say Q, on line (i) are $(2\lambda - 1, 3\lambda + 3, -\lambda + 1)$.

The given point is P(5, 4, 2).

Now, d.r's of the line PQ are

$<2\lambda - 1 - 5, 3\lambda + 3 - 4, -\lambda + 1 - 2>$

i.e., $<2\lambda - 6, 3\lambda - 1, -\lambda - 1>$

Now, PQ is perpendicular to line (i).

$\therefore a_1a_2 + b_1b_2 + c_1c_2 = 0$

$\Rightarrow 2(2\lambda - 6) + 3(3\lambda - 1) + (-1)(-\lambda - 1) = 0$

$\Rightarrow 4\lambda - 12 + 9\lambda - 3 + \lambda + 1 = 0$

$\Rightarrow 14\lambda - 14 = 0$

$\Rightarrow \lambda = 1$

$\therefore$ Coordinates of Q are (1, 6, 0).

These are the coordinates of perpendicular from P on the given line.

Now, $PQ = \sqrt{(5-1)^2 + (4-6)^2 + (2-0)^2}$

$= \sqrt{4^2 + (-2)^2 + 2^2}$

$= \sqrt{16+4+4} = \sqrt{24} = 2\sqrt{6}$ units

If K(α, β, γ) be the image of P in the given line, then Q must be the mid-point of PK.

$$\therefore \frac{\alpha+5}{2} = 1, \frac{\beta+4}{2} = 6, \frac{\gamma+2}{2} = 0$$

$\Rightarrow \alpha = -3, \beta = 8, \gamma = -2$

$\therefore$ Image of P in the given line is K (–3, 8, –2).

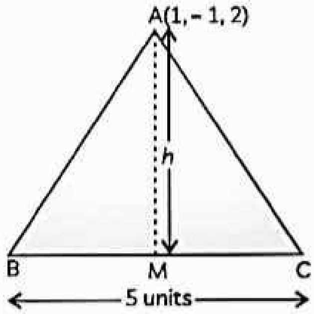
48. **Show that the lines $\frac{x-1}{3}=\frac{y-1}{-1}, \frac{z+1}{0}$ and $\frac{x-4}{2}=\frac{z+1}{3}, \frac{y}{0}$ intersect each other. Also, find their point of intersection.**

49. **Vertices B and C of ΔABC lie along the line $\frac{x+2}{2}=\frac{y-1}{1}=\frac{z-0}{4}$. Find the area of the triangle given that A has coordinates (1, – 1, 2) and the line segment BC has length 5 units.**

Ans. Let h be the height of ΔABC.

Then, h is the length of perpendicular from A(1, – 1, 2) to the line $\frac{x+2}{2}=\frac{y-1}{1}=\frac{z-0}{4}$.

Clearly, line $\frac{x+2}{2}=\frac{y-1}{1}=\frac{z-0}{4}$ passes through the point say P(– 2, 1, 0) and parallel to the vector $\vec{b}=2\hat{i}+\hat{j}+4\hat{k}$.



Let

$$\frac{x+2}{2}=\frac{y-1}{1}=\frac{z-0}{4}=\lambda$$

Then, coordinates of M are $(2\lambda-2, \lambda+1, 4\lambda)$.

Now, d.r's of AM are $(2\lambda-3, \lambda+2, 4\lambda-2)$

Since $AM \perp BC$

$\therefore a_1a_2+b_1b_2+c_1c_2=0$

$\Rightarrow 2(2\lambda-3)+1(\lambda+2)+4(4\lambda-2)=0$

[As d.r's of BC are $<2, 1, 4>$]

$\Rightarrow \quad 21\lambda = 12$

$\Rightarrow \quad \lambda = \frac{4}{7}$

Thus, the coordinates of M are $\left(-\frac{6}{7}, \frac{11}{7}, \frac{16}{7}\right)$

Now,

$$h=|AM|=\sqrt{\left(\frac{-6}{7}-1\right)^2+\left(\frac{11}{7}+1\right)^2+\left(\frac{16}{7}-2\right)^2}$$

$$=\sqrt{\frac{169}{7^2}+\frac{324}{7^2}+\frac{4}{7^2}}$$

$$=\sqrt{\frac{497}{7^2}}=\sqrt{\frac{71}{7}}$$

It is given that length of BC is 5 units.

$$\therefore \quad \text{Area of } \Delta ABC=\frac{1}{2}\times BC\times h$$

$$=\frac{1}{2}\times 5\times\sqrt{\frac{71}{7}}$$

$$=\sqrt{\frac{1775}{28}} \text{ sq. units.}$$

TOPPER'S CORNER

VERY SHORT ANSWER Type Questions (VSA)

[**1** mark]

1. If the cartesian equations of a line are $\frac{3-x}{5}=\frac{y+4}{7}=\frac{2z-6}{4}$, write the vector equation for the line.

Ans.

$\frac{x-3}{-5} = \frac{y+4}{7} = \frac{z-3}{2}$

$\therefore \vec{r} = (+3\hat{i} - 4\hat{j} + 3\hat{k}) + \lambda(-5\hat{i} + 7\hat{j} + 2\hat{k})$

[CBSE Topper 2014]

2. Find the vector equation of the line which passes through the point (3, 4, 5) and is parallel to the vector $2\hat{i}+2\hat{j}-3\hat{k}$.

Ans.

$\vec{r} = \vec{a} + \lambda\vec{b}$ where $\vec{a}$ = position vector of pt on line, $\vec{b}$ = parallel vector to line

$\vec{r} = (3\hat{i} + 4\hat{j} + 5\hat{k}) + \lambda(2\hat{i} + 2\hat{j} - 3\hat{k})$

Also, $\vec{r} = (3+2\lambda)\hat{i} + (4+2\lambda)\hat{j} + (5-3\lambda)\hat{k}$

[CBSE Topper 2019]

SHORT ANSWER Type-I Questions (SA-I)

[**2** marks]

3. The x-coordinate of a point on the line joining the points P(2, 2, 1) and Q(5, 1, –2) is 4. Find its z-coordinate.

Ans.

P (2, 2, 1) Q (5, 1, -2)

Direction ratios of the line PQ = (5-2), (1-2), (-2-1)
= 3, -1, -3

Equation of PQ

$\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$

$\frac{x-2}{3} = \frac{y-2}{-1} = \frac{z-1}{-3}$

Any point on PQ is given by $(3\lambda+2, -\lambda+2, -3\lambda+1)$

$3\lambda+2 = 4$

$3\lambda = 2$

$\lambda = 2/3$

z coordinate $= -3\lambda+1 = -3\left(\frac{2}{3}\right)+1 = -2+1 = -1$

y coordinate $= -\lambda+2 = -\frac{2}{3}+1 = \frac{1}{3}$ Hence point is $\left(4, \frac{1}{3}, -1\right)$

z coordinate = –1

[CBSE Topper 2017]