9 Differential Equations

> “ *Skydiving is a method of transiting from a high point in the atmosphere to the surface of the Earth with the aid of gravity. This involves the control of speed during the descent using a parachute(s). Once the sky diver jumps from an airplane, the net force experienced by the diver can be calculated using differential equations. The adventure sports equipments are designed to support many such parameters.*

Topic Notes

- *Basic Concepts Related to Differential Equations*
- *Solutions of First Order, First Degree Differential Equations*

BASIC CONCEPTS RELATED TO DIFFERENTIAL EQUATIONS

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TOPIC 1

DIFFERENTIAL EQUATION

In chapter 5, we have learnt how to differentiate a given function f with respect to an independent variable x, *i.e.*, how to find $f'(x)$ for a given function f at each x in its domain of definition. Further, in chapter 7, we discussed how to find the integral of a given function, *i.e.*, how to find a function f whose derivative is the given function g, which may be formulated as follows:

For a given function g, find a function f such that

$$\frac{dy}{dx} = g(x), \text{ where } y = f(x) \qquad ...(i)$$

An equation of the form (i) is known as a differential equation.

These equations arise in a variety of applications, may it be in Physics, Chemistry, Biology, Anthropology, Geology, Economics, etc. Hence, an indepth study of differential equations has assumed a great importance in all modern scientific investigations.

In this chapter, we will study some basic concepts related to differential equations, general and particular solutions of differential equations, and some methods to solve differential equations.

An equation involving only one independent variable (say x), one dependent variable (say y) and derivatives of the dependent variable with respect to the independent variable is called an ordinary differential equation.

Important

In this chapter, we shall confine ourselves to ordinary differential equations. Of course, there are differential equations involving derivatives with respect to more than one indepenent variables, called partial differential equations.

Some examples of ordinary differential equations are given below:

(1) $2x\frac{dy}{dx} = y - 3$

(2) $\frac{dy}{dx} = x\log x + 7$

(3) $\frac{d^2y}{dx^2} - 3\left(\frac{dy}{dx}\right)^2 + 12y = \sin x$

Following notations for derivatives are commonly used in the study of differential equations:

$$\frac{dy}{dx} = y'; \frac{d^2y}{dx^2} = y''; \frac{d^3y}{dx^3} = y'''$$

Order of a Differential Equation

Order of a differential equation is defined as the order of the highest order derivative of the dependent variable with the respect to the independent variable, involved in the differential equation. For example,

(1) $\frac{dy}{dx} = x\log x$ is a differential equation of order 1.

(2) $\frac{d^2y}{dx^2} + 2y = \sin x$ is a differential equation of order 2.

(3) $\frac{d^3y}{dx^2} + \left(\frac{d^2y}{dx^2}\right)^2 = e^x + 2$ is a differential equation of order 3.

Degree of a Differential Equation

The highest power (positive integral index) of the highest order derivative involved in a differential equation, given that it is a polynomial equation in derivatives, is called the degree of a differential equation.

For example,

(1) The differential equation $\frac{d^3y}{dx^3} + \left(\frac{d^2y}{dx^2}\right)^2 = e^x + 2$ is of degree 1, as the highest power of the highest order derivative $\frac{d^3y}{dx^3}$ is 1.

(2) The differential equation $\left(\frac{d^3y}{dx^3}\right)^2 + \frac{dy}{dx} = x^2\sin x$ is of degree 2, as the highest power of the highest order derivative $\frac{d^3y}{dx^3}$ is 2.

(3) The differential equation $\frac{dy}{dx}+\sin\left(\frac{dy}{dx}\right)=0$ is not a polynomial equation in $\frac{dy}{dx}$ and hence its degree is not defined.

Important

➥ *To define the degree of a differential equation, the key point is that the differential equation must be a polynomial.*
➥ *The order and degree of a differential equation are always positive integers.*
➥ *While determining the degree of a differential equation, first make derivatives of all order free from radicals and fractions.*

Illustration: The degree of the differential equation $\sqrt[3]{\frac{d^2y}{dx^2}}=\sqrt[4]{\frac{dy}{dx}}$ is determined as shown below:

$\sqrt[3]{\frac{d^2y}{dx^2}}=\sqrt[4]{\frac{dy}{dx}}$ can be expressed as $\left(\frac{d^2y}{dx^2}\right)^4=\left(\frac{dy}{dx}\right)^3$

So, the order of the differential equation is 2 and its degree is 4.

Example 1.1: Find the order and degree (if defined) of the following differential equations:

(A) $\frac{d^4y}{dx^4}+\sin\left(\frac{d^3y}{dx^3}\right)=0$

(B) $\frac{dy}{dx}+5y=0$

(C) $\left(\frac{d^2y}{dx^2}\right)^2+\cos\left(\frac{dy}{dx}\right)=0$

(D) $\frac{d^2y}{dx^2}=\cos 3x+\sin 3x$

[NCERT]

Ans. (A) The differential equation $\frac{d^4y}{dx^4}+\sin\left(\frac{d^3y}{dx^3}\right)=0$ is of order 4, but its degree is not defined as it is not a polynomial equation in derivatives.

(B) The differential equation $\frac{dy}{dx}+5y=0$ is of order 1 and degree 1.

(C) The differential equation $\left(\frac{d^2y}{dx^2}\right)^2+\cos\left(\frac{dy}{dx}\right)=0$ is of order 2, but its degree is not defined as it is not a polynomial equation in derivatives.

(D) The differential equation $\frac{d^2y}{dx^2}=\cos 3x+\sin 3x$ is of order 2 and degree 1.

Example 1.2: Find the order and degree (if defined) of the following differential equations:

(A) $\frac{dy}{dx}=\frac{x^{1/2}}{y^{1/2}(1+x)^{1/2}}$

(B) $\frac{d^2y}{dx^2}=\left[1+\left(\frac{dy}{dx}\right)^2\right]^{5/3}$

(C) $\left(\frac{d^2y}{dx^2}\right)^2=\left[2+\left(\frac{dy}{dx}\right)^2\right]^{3/4}$

(D) $\frac{d^2y}{dx^2}+2\frac{dy}{dx}+y=\log\left(\frac{dy}{dx}\right)$

Ans. (A) The given differential equation is of order 1 and degree 2. Since the differential equation can be expressed as $\left(\frac{dy}{dx}\right)^2=\frac{x}{y(1+x)}$

(B) The given differential equation is of order 2 and degree 3, since the differential equation can be expressed as $\left(\frac{d^2y}{dx^2}\right)^3=\left[1+\left(\frac{dy}{dx}\right)^2\right]^5$

(C) The given differential equation is of order 2 and degree 8, since the differential equation can be expressed as $\left(\frac{d^2y}{dx^2}\right)^8=\left[2+\left(\frac{dy}{dx}\right)^2\right]^3$

(D) The given differential equation is of order 2 but degree is not defined, as the equation cannot be expressed as a polynomial equation in derivatives.

Caution

➥ *To find the degree of differential equation containing radicals or fractions, first free it from radicals and fractions by simplifying, and then check its degree.*

OBJECTIVE Type Questions

[**1** mark]

Multiple Choice Questions

1. **The order and degree of the differential equation**

$$1+\left(\frac{d^2y}{dx^2}\right)^2=\left[2+\left(\frac{dy}{dx}\right)^2\right]^{3/2},$$

respectively are:

(a) 2, 3 **(b)** 1, 2

(c) 2, 4 **(d)** 2, 2

Ans. *(c) 2, 4*

Explanation: The given differential equation can be simplified as

$$\left[1+\left(\frac{d^2y}{dx^2}\right)^2\right]^2=\left[2+\left(\frac{dy}{dx}\right)^2\right]^3$$

The highest order derivative is $\frac{d^2y}{dx^2}$ whose order is 2.

And, the degree of the highest order derivative in the equation is 4.

2. **If p and q are the order and degree of the differential equation**

$$y^2\left(\frac{d^2y}{dx^2}\right)^2+3x\frac{dy}{dx}+x^2y^2=\sin x$$

then:

(a) $p<q$ **(b)** $p=q$

(c) $2p=q$ **(d)** $p>q$

3. **The order and degree of the differential equation** $\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^{\frac{1}{4}}+x^{\frac{1}{5}}=0$, **respectively, are:**

(a) 2 and 4 **(b)** 2 and 2

(c) 2 and 3 **(d)** 3 and 3

Ans. *(a) 2 and 4*

Explanation: We have,

$$\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^{1/4}+x^{1/5}=0$$

$$\Rightarrow \left(\frac{dy}{dx}\right)^{1/4}=-\left(x^{1/5}+\frac{d^2y}{dx^2}\right)$$

$$\Rightarrow \frac{dy}{dx}=\left[-\left(x^{1/5}+\frac{d^2y}{dx^2}\right)\right]^4$$

$$\Rightarrow \frac{dy}{dx}=\left(x^{1/5}+\frac{d^2y}{dx^2}\right)^4$$

Here, the highest order derivative is $\frac{d^2y}{dx^2}$, whose order is 2.

Also, the highest power of $\frac{d^2y}{dx^2}$ will be 4.

$\therefore$ Order = 2, Degree = 4

⚠ Caution

➥ *Order of differential equation is the order of highest order derivative occurring in the differential equation.*

4. **The degree of the differential equation** $\frac{d^2y}{dx^2}+\frac{dy}{dx}=x\sin\left(\frac{dy}{dx}\right)$ **is:**

(a) 1 (b) 2

(c) 3 (d) not defined

5. **The order and degree, of the differential equation** $\frac{d^2y}{dx^2}+2\frac{dy}{dx}+y=\log\left(\frac{dy}{dx}\right)$ **respectively, is:**

(a) 2, 1 **(b)** 1, 2

(c) 2, 0 **(d)** 2, not defined

Ans. *(d) 2, not defined*

Explanation: The order of highest order derivative is $\frac{d^2y}{dx^2}$ whose order is 2.

Given differential equation is not a polynomial equation in its derivative. So, its degree is not defined.

VERY SHORT ANSWER Type Questions (VSA)

[**1** mark]

6. **What is the degree of the given differential equation:**

$$5x\left[\frac{dy}{dx}\right]^2-\frac{d^2y}{dx^2}-6y=\log x$$

Ans. For the given differential equation, the highest order derivative is $\frac{d^2y}{dx^2}$, with maximum degree 1.

Hence, the degree of differential equation is 1.

7. Find the order and degree of the differential equation $\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}=\frac{d^2y}{dx^2}$

Ans. Given, equation is $\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}=\frac{d^2y}{dx^2}$

On squaring both sides, we get,

$$\left[1+\left(\frac{dy}{dx}\right)^2\right]^3=\left(\frac{d^2y}{dx^2}\right)^2$$

Here, highest order derivative is $\frac{d^2y}{dx^2}$, whose highest power is 2.

So, order = 2, degree = 2

8. Shown below is a differential equation:

$$y=e^{\sin\left(\frac{d^3y}{dx^3}\right)^2}+\left(\frac{dy}{dx}\right)^4$$

Find the order and the degree of the given differential equation.

Ans. The highest order derivative in the given differential equation is $\frac{d^3y}{dx^3}$.

So, the order of the given differential equation is 3.

Since, the differential equation is not a polynomial equation, so its degree is not defined.

9. Write the sum of order and degree of differential equation

$$1+\left(\frac{dy}{dx}\right)^4=7\left(\frac{d^2y}{dx^2}\right)^3$$ **[CBSE 2015]**

Ans. Given, differential equation is

$$1+\left(\frac{dy}{dx}\right)^4=7\left(\frac{d^2y}{dx^2}\right)^3$$

Here, highest order derivative is $\frac{d^2y}{dx^2}$, whose highest power is 3.

So, order = 2 and degree = 3

∴ Sum of order and degree is 2 + 3 *i.e.*, 5.

10. Find the order and degree of the differential equation $y=px+\sqrt{a^2p^2+b^2}$, where $p=\frac{dy}{dx}$.

[DIKSHA]

SHORT ANSWER Type-I Questions (SA-I)

[**2** marks]

11. Write the sum of the order and the degree of the following differential equation: $\frac{d}{dx}\left(\frac{dy}{dx}\right)=5$. **[CBSE Term-2 SQP 2022]**

Ans. *Order = 2, Degree = 1, Sum = 3*

[CBSE Marking Scheme Term-2 SQP 2022]

Explanation: Given, differential equation is:

$$\frac{d}{dx}\left(\frac{dy}{dx}\right)=5$$

or $$\frac{d^2y}{dx^2}=5$$

∴ Order of D.E. = 2

Degree of D.E. = 1

∴ Required sum = 2 + 1 = 3

Concept Applied:

The order of a differential equation is defined as the highest order derviative it contains and the degree of a differential equation is defined as the power to which highest order derivative is raised.

12. Write the sum of the order and degree of the differential equation $\frac{d}{dx}\left\{\left(\frac{dy}{dx}\right)^3\right\}=0$.

[CBSE 2015]

Ans. Given, equation is $\frac{d}{dx}\left\{\left(\frac{dy}{dx}\right)^3\right\}=0$

$$\Rightarrow \quad 3\left(\frac{dy}{dx}\right)^2\frac{d^2y}{dx^2}=0$$

Clearly, the highest order derivative occuring in the differential equation is $\frac{d^2y}{dx^2}$.

So, its order is 2. Also, it is a polynomial equation in derivative and highest power raised to $\frac{d^2y}{dx^2}$ is 1, so its degree is 1.

Hence, the sum of the order and degree of the given differential equation is 2 + 1 = 3

SOLUTIONS OF FIRST ORDER, FIRST DEGREE DIFFERENTIAL EQUATIONS

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TOPIC 1

GENERAL AND PARTICULAR SOLUTIONS OF DIFFERENTIAL EQUATIONS

The general solution of a differential equation is a relation between dependent and independent variables having n arbitrary constants.

Caution

The general solution of a differential equation contains as many arbitrary constants as the order of the differential equation.

The solution obtained from the general solution by giving some particular values to these arbitrary constants is called particular solution.

Thus, the solution which contains arbitrary constants, is called a general solution of the differential equation, whereas, the solution free from arbitrary constants, *i.e.*, after particular values to the arbitrary constants, is called particular solution of the differential equation.

Illustration: $y = A\cos x + B\sin x$, where A and B are arbitrary (*i.e.*, unknown) constants, is the general solution of the differential equation $\frac{d^2y}{dx^2} + y = 0$; and $y = 2\cos x - 3\sin x$ is the particular solution of the differential equation $\frac{d^2y}{dx^2} + y = 0$.

Method to check whether the function provided is a solution of given differential equation or not

Step 1: Take the given function, say $y = f(x)$ [Here, y is dependent variable and x is independent variable.]

Step 2: Differentiate the function $y = f(x)$ with respect to the independent variable *i.e.*, x as many as times as the order of the given differential equation.

Step 3: Substitute the values of y, $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$ etc. in the L.H.S. of given differential equation.

If L.H.S. = R.H.S., then the given function is a solution of the differential equation, otherwise not.

Example 2.1: Verify that the given function (explicit or implicit) is a solution of the corresponding differential equations:

(A) $y = \sqrt{1+x^2}$; $\frac{dy}{dx}$ (or y') $= \frac{xy}{1+x^2}$

(B) $y - \cos y = x$; $(y\sin y + \cos y + x)\, y' = y$

(C) $x + y = \tan^{-1} y$; $y^2y' + y^2 + 1 = 0$ [NCERT]

Ans. (A) Here, the given solution is

$$y = \sqrt{1+x^2} \quad ...(i)$$

Differentiating both sides of (i) w.r.t. x, we get,

$$\frac{dy}{dx} = \frac{1}{2}\frac{2x}{\sqrt{1+x^2}}$$

Substituting the values of $\frac{dy}{dx}$ and y in the differential equation, we get,

$$\text{L.H.S.} = \frac{1}{2}\frac{2x}{\sqrt{1+x^2}} = \frac{x}{\sqrt{1+x^2}}$$

$$\text{R.H.S.} = \frac{x\sqrt{1+x^2}}{1+x^2} = \frac{x}{\sqrt{1+x^2}}$$

i.e., L.H.S. = R.H.S.

Hence, the given function is a solution of the given differrential equation.

(B) Here, the given solution is

$$y - \cos y = x \quad ...(i)$$

Differentiating both sides of (i) w.r.t. x, we get

$$\frac{dy}{dx} + \sin y \cdot \frac{dy}{dx} = 1$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{1+\sin y}$$

Substituting the values of $\frac{dy}{dx}$ and x in the differential equation, we get

$$\text{L.H.S.} = [y\sin y + \cos y + (y - \cos y)]\frac{1}{1+\sin y}$$

$$= \frac{y(1+\sin y)}{1+\sin y} = y = \text{R.H.S.}$$

Hence, the given function is a solution of the given differential equation.

(C) Here, the given solution is

$$x + y = \tan^{-1} y \qquad \text{...(i)}$$

Differentiating both sides of (i) w.r.t. x, we get

$$1 + \frac{dy}{dx} = \frac{1}{1+y^2}\frac{dy}{dx}$$

$$\Rightarrow \qquad \frac{dy}{dx} = -\frac{1+y^2}{y^2}$$

Substituting the value of $\frac{dy}{dx}$ in the differential equation, we get,

$$\text{L.H.S.} = y^2\left(-\frac{1+y^2}{y^2}\right) + y^2 + 1$$

$$= 0 = \text{R.H.S.}$$

Hence, the given function is a solution of the given differential equation.

TOPIC 2

DIFFERENTIAL EQUATIONS WITH VARIABLES SEPARABLE

A first order, first degree differential equation is of the form

$$\frac{dy}{dx} = F(x, y) \qquad \text{...(i)}$$

It is said to be in Variable Separable form, if the function F can be expressed as the product of the function of x and the function of y.

Solving Differential Equations of the Form $\frac{dy}{dx} = f(x)$

We have, $\frac{dy}{dx} = f(x)$

$$\Rightarrow \qquad dy = f(x)dx$$

Integrating both sides, we have

$$\int dy = \int f(x)dx + C$$

i.e., $y = g(x) + C$,

where $g(x) = \int f(x)\,dx$ and C is a constant.

Example 2.2: Find the general solution of the following differential equations:

(A) $\frac{dy}{dx} = \frac{1-\cos x}{1+\cos x}$

(B) $\frac{dy}{dx} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$ **[NCERT]**

Ans. (A) $\frac{dy}{dx} = \frac{1-\cos x}{1+\cos x}$

$$\Rightarrow \qquad \frac{dy}{dx} = \frac{2\sin^2\frac{x}{2}}{2\cos^2\frac{x}{2}}$$

$$\left[\begin{array}{l}\because 1-\cos^2\theta = 2\sin^2\theta \\ \quad 1+\cos^2\theta = 2\cos^2\theta\end{array}\right]$$

$$= \tan^2\frac{x}{2}$$

$$\Rightarrow \qquad dy = \tan^2\frac{x}{2}\cdot dx$$

Integrating both sides, we get,

$$\int dy = \int \tan^2\frac{x}{2}dx + C$$

or $$\int dy = \int\left(\sec^2\frac{x}{2} - 1\right)dx + C$$

i.e., $y = 2\tan\frac{x}{2} - x + C$, which is the required solution of the given differential equation.

(B) $$\frac{dy}{dx} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

$$\Rightarrow \qquad dy = \left(\frac{e^x - e^{-x}}{e^x + e^{-x}}\right)dx$$

Integrating both sides, we have,

$$\int dy = \int\left(\frac{e^x - e^{-x}}{e^x + e^{-x}}\right)dx + C$$

i.e., $y = \log(e^x + e^{-x}) + C$, which is the required solution of the given differential equation.

Solving Differential Equations of the Form $\frac{dy}{dx} = f(y)$

We have, $$\frac{dy}{dx} = f(y)$$

$$\Rightarrow \quad \frac{dy}{f(y)} = dx$$

Integrating both sides, we have

$$\int \frac{dy}{f(y)} = \int dx + C_1$$

which gives the required solution.

Example 2.3: Find the general solution of the following differential equations:

(A) $\frac{dy}{dx} = \sqrt{4 - y^2}, -2 < y < 2$

(B) $\frac{dy}{dx} + y = 1, y \neq 1$ **[NCERT]**

Ans. (A) $$\frac{dy}{dx} = \sqrt{4 - y^2}, -2 < y < 2$$

$$\Rightarrow \frac{dy}{\sqrt{4 - y^2}} = dx$$

Integrating both sides, we have

$$\int \frac{dy}{\sqrt{4 - y^2}} = \int dx + C$$

i.e., $\sin^{-1}\frac{y}{2} = x + C$, or $y = 2\sin(x + C)$,

which is the required solution of the given differential equation.

(B) $$\frac{dy}{dx} + y = 1, y \neq 1$$

$$\Rightarrow \quad \frac{dy}{dx} = 1 - y$$

$$\Rightarrow \quad \frac{dy}{1 - y} = dx$$

Integrating both sides, we have

$$\int \frac{dy}{1 - y} = \int dx + C$$

i.e., $-\log|1 - y| = x + C$

or $-\log|1 - y| = x + \log C$,

or $x = -\log C(1 - y)$

or $e^{-x} = C(1 - y)$

or $1 - y = \frac{1}{C}e^{-x}$

or $y = 1 - \frac{1}{C}e^{-x}$

i.e., $y = 1 + Ae^{-x}$, where $A = -\frac{1}{C}$

which is the required solution of the given differential equation.

Solving Differential Equations of the Form $\frac{dy}{dx} = PQ$, where P is a Function of x only and Q is a Function of y only

We have, $$\frac{dy}{dx} = PQ$$

$$\Rightarrow \quad \frac{dy}{Q} = P\,dx$$

Integrating both sides, we have

$$\int \frac{dy}{Q} = \int P\,dx + C$$

which gives the required solution.

Example 2.4: Find the general solution of the following differential equations:

(A) $\frac{dy}{dx} = (1 + x^2)(1 + y^2)$

(B) $y \log y\, dx - x\, dy = 0$ **[NCERT]**

Ans. (A) $$\frac{dy}{dx} = (1 + x^2)(1 + y^2)$$

$$\Rightarrow \quad \frac{dy}{1 + y^2} = (1 + x^2)\,dx$$

Integrating both sides, we get,

$$\int \frac{dy}{1 + y^2} = \int (1 + x^2)\,dx$$

$$\Rightarrow \quad \tan^{-1} y = x + \frac{x^3}{3} + C$$

which is the required solution of the given differential equation.

(B) $y \log y\, dx - x\, dy = 0$

$$\Rightarrow \quad y \log y\, dx = x\, dy$$

$$\Rightarrow \quad \frac{dy}{y \log y} = \frac{dx}{x}$$

Integrating both sides, we have

$$\int \frac{dy}{y \log y} = \int \frac{dx}{x} + C$$

$$\Rightarrow \quad \log(\log y) = \log x + \log C$$

$$\Rightarrow \quad \log(\log y) = \log(xC)$$

$$\Rightarrow \quad \log y = Cx \text{ or } y = e^{Cx}$$

which is the required solution of the given differential equation.

TOPIC 3

EQUATIONS REDUCIBLE TO VARIABLES SEPARABLE FORM

Equation of the form $\frac{dy}{dx} = f(ax + by + c)$ can be reduced to the form in which variables are separable. (How?)

Putting $ax + by + c = z$, and then differentiating it w.r.t. x, we have

$$\frac{dz}{dx} = a + b\frac{dy}{dx}$$

or $$\frac{dy}{dx} = \frac{1}{b}\left(\frac{dz}{dx} - a\right)$$

Thus, given D.E. becomes

$$\frac{1}{b}\left(\frac{dz}{dx} - a\right) = f(z)$$

i.e., $$\frac{dz}{dx} = a + b\,f(z)$$

or $$\frac{dz}{a + bf(z)} = dx$$

which can now be integrated.

Example 2.5: Find the general solution of the following differential equations:

(A) $\frac{dy}{dx} = \frac{x + y + 1}{x + y}$

(B) $\sin^{-1}\left(\frac{dy}{dx}\right) = x + y$

Ans. (A) $$\frac{dy}{dx} = \frac{x + y + 1}{x + y} \quad ...(i)$$

Put $x + y = z$ so that $1 + \frac{dy}{dx} = \frac{dz}{dx}$

$\therefore$ Equation (i) becomes

$$\frac{dz}{dx} - 1 = \frac{z + 1}{z}.$$

$$\Rightarrow \frac{dz}{dx} = \frac{z + 1}{z} + 1$$

$$\Rightarrow \frac{dz}{dx} = \frac{2z + 1}{z}$$

or $$\frac{1}{2}\left(1 - \frac{1}{2z + 1}\right)dz = dx$$

Integrating both sides, we have

$$\int \frac{1}{2}\left(1 - \frac{1}{2z + 1}\right)dz = \int dx + C$$

$$\Rightarrow \frac{1}{2}\left[z - \frac{1}{2}\log(2z + 1)\right] = x + C$$

$$\Rightarrow \frac{1}{2}(x + y) - \frac{1}{4}\log(2x + 2y + 1) = x + C$$

This solution may further be simplified as $2(y - x) = \log(2x + 2y + 1) + D$, where $D = 4C$ which is the required solution.

(B) $$\sin^{-1}\left(\frac{dy}{dx}\right) = x + y$$

$$\Rightarrow \frac{dy}{dx} = \sin(x + y) \quad ...(i)$$

Put $x + y = z$ so that $1 + \frac{dy}{dx} = \frac{dz}{dx}$

$\therefore$ Equation (i) becomes

$$\frac{dz}{dx} - 1 = \sin z$$

or $$\frac{dz}{dx} = \sin z + 1$$

$$\Rightarrow \left(\frac{1}{\sin z + 1}\right)dz = dx$$

$$\Rightarrow \left[\left(\frac{1}{1 + \sin z}\right) \times \left(\frac{1 - \sin z}{1 - \sin z}\right)\right]dz = dx$$

$$\Rightarrow \left(\frac{1 - \sin z}{1 - \sin^2 z}\right)dz = dx$$

or $$\left(\frac{1 - \sin z}{\cos^2 z}\right)dz = dx$$

Integrating both sides, we have

$$\int\left(\frac{1 - \sin z}{\cos^2 z}\right)dz = \int dx + C$$

$$\Rightarrow \int(\sec^2 z - \tan z \sec z)\,dz = \int dx + C$$

$$\Rightarrow (\tan z - \sec z) = x + C$$

or $\tan(x + y) - \sec(x + y) = x + C$

which is the required solution.

Particular Solution of Differential Equations or Initial Value Problems

We already know that solution satisfying certain values of independent and dependent variables is called a particular solution and the probem in which a particular solution is required is an initial value problem.

Example 2.6: **Solve the differential equation:**

$(x^3 + x^2 + x + 1)\dfrac{dy}{dx} = 2x^2 + x;\ y = 1$ **when** $x = 0$

[NCERT]

Ans. $(x^3 + x^2 + x + 1)\dfrac{dy}{dx} = 2x^2 + x$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{2x^2 + x}{x^3 + x^2 + x + 1}$$

$$= \frac{2x^2 + x}{(x+1)(x^2+1)}$$

$$\frac{2x^2 + x}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx + C}{x^2+1}$$

$$\Rightarrow \frac{2x^2 + x}{(x+1)(x^2+1)} = \frac{A(x^2+1) + (Bx + C)(x+1)}{(x+1)(x^2+1)}$$

$$\Rightarrow \quad 2x^2 + x = x^2(A + B) + x(B + C) + A + C$$

Comparing coefficients of x^2, x and constant terms, we get

$$A + B = 2$$
$$B + C = 1$$
$$A + C = 0$$

Solving these equations, we get

$$A = \frac{1}{2}, B = \frac{3}{2} \text{ and } C = -\frac{1}{2}$$

$$\therefore \frac{2x^2 + x}{(x+1)(x^2+1)} = \frac{1}{2}\cdot\frac{1}{x+1} + \frac{\frac{3}{2}x - \frac{1}{2}}{x^2+1}$$

$$= \frac{1}{2}\cdot\frac{1}{x+1} + \frac{3}{2}\cdot\frac{x}{x^2+1} - \frac{1}{2}\cdot\frac{1}{x^2+1}$$

Using partial fractions above, we have

$$\frac{dy}{dx} = \frac{1}{2(x+1)} + \frac{3x - 1}{2(x^2+1)}$$

$$\Rightarrow \quad dy = \left(\frac{1}{2}\frac{1}{x+1} + \frac{3}{2}\frac{x}{x^2+1} - \frac{1}{2}\frac{1}{x^2+1}\right)dx$$

Integrating both sides, we have

$$\int dy = \int\left(\frac{1}{2}\frac{1}{x+1} + \frac{3}{4}\frac{2x}{x^2+1} - \frac{1}{2}\frac{1}{x^2+1}\right)dx + C$$

$$\Rightarrow y = \frac{1}{2}\log|x+1| + \frac{3}{4}\log(x^2+1) - \frac{1}{2}\tan^{-1}x + C$$

Given that $y = 1$ when $x = 0$.

$$\therefore 1 = \frac{1}{2}\log|0+1| + \frac{3}{4}\log|(0)+1| - \frac{1}{2}\tan^{-1}(0) + C$$

$\Rightarrow \quad 1 = 0 + 0 - 0 + C$, *i.e.*, $C = 1$

Hence, the particular solution of the given D.E. is

$$y = \frac{1}{2}\log(x+1) + \frac{3}{4}\log(x^2+1) - \frac{1}{2}\tan^{-1}x + 1$$

This solution may further be simplified as

$$y = \frac{1}{4}\log(x+1)^2 + \frac{1}{4}\log(x^2+1)^3 - \frac{1}{2}\tan^{-1}x + 1$$

$$\text{or } y = \frac{1}{4}\log[(x+1)^2(x^2+1)^3] - \frac{1}{2}\tan^{-1}x + 1$$

TOPIC 4

HOMOGENEOUS DIFFERENTIAL EQUATIONS

Consider the following functions in x and y:

$$F_1(x, y) = y^2 + xy;$$

$$F_2(x, y) = y + \sqrt{x^2 + y^2};$$

$$F_3(x, y) = \frac{x^2 - y^2}{2xy};$$

$$F_4(x, y) = \sin x + \cos y;$$

$$F_5(x, y) = 4x^2y + 2xy^2;$$

$$F_6(x, y) = 6x + 7y^2$$

If we replace x and y by λx and λy respectively in the above functions for any non-zero constant λ, we get

$$F_1(\lambda x, \lambda y) = (\lambda y)^2 + (\lambda x)(\lambda y)$$
$$= \lambda^2(y^2 + xy) = \lambda^2 F_1(x, y)$$

$$F_2(\lambda x, \lambda y) = (\lambda y) + \sqrt{(\lambda x)^2 + (\lambda y)^2}$$

$$= (\lambda y) + \lambda\sqrt{x^2 + y^2}$$

$$= \lambda^1[y + \sqrt{x^2 + y^2}]$$

$$= \lambda^1 F_2(x, y)$$

$$F_3(\lambda x, \lambda y) = \frac{(\lambda x)^2 - (\lambda y)^2}{2(\lambda x)(\lambda y)}$$

$$= \frac{\lambda^2(x^2 - y^2)}{\lambda^2(2xy)}$$

$$= \lambda^0\left(\frac{x^2 - y^2}{2xy}\right) = \lambda^0 F_3(x, y)$$

$$F_4(\lambda x, \lambda y) = \sin(\lambda x) + \cos(\lambda y)$$
$$\neq \lambda^n F_4(x, y), \text{ for any } n \in N$$

$$F_5(\lambda x, \lambda y) = 4(\lambda x)^2(\lambda y) + 2(\lambda x)(\lambda y)^2$$
$$= \lambda^3(4x^2y + 2xy^2)$$
$$= \lambda^3 F_5(x, y)$$
$$F_6(\lambda x, \lambda y) = 6(\lambda x) + 7(\lambda y)^2$$
$$\neq \lambda^n F_4(x, y), \text{ for any } n \in N$$

Here, we observe that the functions F_1, F_2, F_3 and F_5 can be written in the form $F(\lambda x, \lambda y) = \lambda^n F(x, y)$, but F_4 and F_6 cannot be written in this form.

This leads to the following definition :

A function $F(x, y)$ is said to be a homogeneous function of degree n if

$$F(\lambda x, \lambda y) = \lambda^n F(x, y), \text{ for any non-zero constant } \lambda.$$

Thus, the above functions F_1, F_2, F_3 and F_5 are homogeneous functions of degree 2, 1, 0 and 3 respectively. Functions F_4 and F_6 are called non-homogeneous functions.

Important

➥ *A differential equation of the form* $\frac{dy}{dx} = F(x, y)$ *is said to be homogeneous if* $F(x, y)$ *is a homogeneous function of degree zero.*

Working Procedure to Solve Homogeneous Differential Equations

Step 1: Write the given differential equation $\frac{dy}{dx} = F(x, y)$ in the form $\frac{dy}{dx} = g\left(\frac{y}{x}\right)$.

Step 2: Make a substitution $y = vx$ so that we get $\frac{dy}{dx} = v + x\frac{dv}{dx}$

Step 3: Substitute these values in $\frac{dy}{dx} = g\left(\frac{y}{x}\right)$ and get $v + x\frac{dv}{dx} = g(v)$.

Step 4: Simplify it and get $x\frac{dv}{dx} = g(v) - v$.

Step 5: Reduce the above in the form $\frac{dv}{g(v) - v} = \frac{dx}{x}$

Step 6: Integrate both sides and get $\int \frac{dv}{g(v) - v} = \int \frac{dx}{x} + C$

Step 7: After integration, replace v by $\frac{y}{x}$ and get the required solution.

Example 2.7: Solve the differential equation: $(x^2 - y^2)dx + 2xy\,dy = 0$. [NCERT]

Ans. The given D.E. is $(x^2 - y^2)\,dx + 2xy\,dy = 0$, which can be written as

$$\frac{dy}{dx} = \frac{y^2 - x^2}{2xy} \quad ...(i)$$

Let $F(x, y) = \frac{y^2 - x^2}{2xy}$.

Then, $F(\lambda x, \lambda y) = \frac{(\lambda y)^2 - (\lambda x)^2}{2(\lambda x)(\lambda y)}$

$$= \frac{\lambda^2(y^2 - x^2)}{\lambda^2(2xy)}$$
$$= \lambda^0\left(\frac{y^2 - x^2}{2xy}\right)$$
$$= \lambda^0 F(x, y)$$

So, the given differential equation is homogeneous.

Put $y = vx$ so that $\frac{dy}{dx} = v + x\frac{dv}{dx}$

With these substitutions, equation (i) reduces to

$$v + x\frac{dv}{dx} = \frac{(vx)^2 - x^2}{2x(vx)}$$
$$\Rightarrow \quad v + x\frac{dv}{dx} = \frac{v^2 - 1}{2v}$$
$$\Rightarrow \quad x\frac{dv}{dx} = \frac{v^2 - 1}{2v} - v$$
$$\Rightarrow \quad x\frac{dv}{dx} = \frac{-v^2 - 1}{2v}$$
$$\Rightarrow \quad \frac{2v}{v^2 + 1}dv = -\frac{dx}{x}$$

On integrating both sides, we have

$$\int \frac{2v}{v^2 + 1}\,dv = -\int \frac{dx}{x}$$
$$\Rightarrow \quad \log(v^2 + 1) = -\log|x| + \log C$$
$$\Rightarrow \log[(v^2 + 1)x] = \log C,$$
$$\text{or} \quad x(v^2 + 1) = C$$

Replacing v by $\frac{y}{x}$, we get

$$x\left[\left(\frac{y}{x}\right)^2 + 1\right] = C, \text{ or } y^2 + x^2 = Cx$$

which is the required solution of the given differential equation.

Example 2.8: Show that the following differential equation is homogeneous and solve it.

$$x\frac{dy}{dx} - y + x\sin\left(\frac{y}{x}\right) = 0 \quad \text{[NCERT]}$$

Ans. The given differential equation can be written as

$$\frac{dy}{dx} = \frac{y}{x} - \sin\left(\frac{y}{x}\right) \quad ...(i)$$

Let $F(x, y) = \frac{y}{x} - \sin\left(\frac{y}{x}\right)$.

Then, $F(\lambda x, \lambda y) = \frac{\lambda y}{\lambda x} - \sin\left(\frac{\lambda y}{\lambda x}\right)$

$$= \frac{y}{x} - \sin\left(\frac{y}{x}\right) = \lambda^0 F(x, y)$$

Therefore, F(x, y) is a homogeneous function of degree zero. So, the given differential equation is a homogeneous differential equation.

Put $y = vx$

so that, $\frac{dy}{dx} = v + x\frac{dv}{dx}$

With these substitutions, equation (i) reduces to

$$v + x\frac{dv}{dx} = v - \sin v$$

$$\Rightarrow \quad x\frac{dv}{dx} = -\sin v$$

$$\Rightarrow \quad \operatorname{cosec} v\, dv = -\frac{dx}{x}$$

On integrating both sides, we have

$$\int \operatorname{cosec} v\, dv = -\int \frac{dx}{x} + C$$

$\Rightarrow \quad \log|\operatorname{cosec} v - \cot v| = -\log|x| + \log C$

$\Rightarrow \quad \log[(\operatorname{cosec} v - \cot v)x] = \log C$

$\Rightarrow \quad x(\operatorname{cosec} v - \cot v) = C,$

or $\quad x(1 - \cos v) = C \sin v$

Replacing v by $\frac{y}{x}$, we get

$$x\left(1 - \cos\frac{y}{x}\right) = C\sin\frac{y}{x}$$

which is the required solution of the given differential equation.

Example 2.9: Show that the following differential equation is homogeneous and solve it.

$$(1 + e^{\frac{x}{y}})dx + e^{\frac{x}{y}}\left(1 - \frac{x}{y}\right)dy = 0. \quad \textbf{[NCERT]}$$

Ans. The given differential equation can be written as

$$\frac{dx}{dy} = \frac{e^{x/y}\left(\frac{x}{y} - 1\right)}{1 + e^{x/y}} \quad ...(i)$$

Let $F(x, y) = \frac{e^{x/y}\left(\frac{x}{y} - 1\right)}{1 + e^{x/y}}$.

Then, $F(\lambda x, \lambda y) = \frac{e^{\lambda x/\lambda y}\left(\frac{\lambda x}{\lambda y} - 1\right)}{1 + e^{\lambda x/\lambda y}}$

$$= \frac{e^{x/y}\left(\frac{x}{y} - 1\right)}{1 + e^{x/y}} = \lambda^0 F(x, y)$$

Therefore, F(x, y) is a homogeneous function of degree zero. So, the given differential equation is a homogeneous differential equation.

Put $x = vy$ so that $\frac{dx}{dy} = v + y\frac{dv}{dy}$

With these substitutions, equation (i) reduces to

$$v + y\frac{dv}{dy} = \frac{e^v(v - 1)}{1 + e^v}$$

$$\Rightarrow \quad y\frac{dv}{dy} = \frac{e^v(v - 1)}{1 + e^v} - v$$

$$\Rightarrow \quad y\frac{dv}{dy} = \frac{-e^v - v}{1 + e^v}$$

$$\Rightarrow \quad \frac{1 + e^v}{e^v + v}dv = -\frac{dy}{y}$$

On integrating both sides, we have

$$\int \frac{1 + e^v}{e^v + v}dv = -\int\frac{dy}{y} + C$$

$\Rightarrow \quad \log|e^v + v| = -\log|y| + \log C$

$\Rightarrow \log[|e_v + v|y] = \log C$

$\Rightarrow \quad y|e^v + v| = C$

Replacing v by $\frac{x}{y}$, we get

$$y\left|e^{x/y} + \frac{x}{y}\right| = C,$$

or $ye^{x/y} + x = C$

which is the required solution of the given differential equation.

Example 2.10: Show that the following differential equation is homogeneous and solve it.

$$\left[x\sin^2\left(\frac{y}{x}\right) - y\right]dx + x\,dy = 0; \; y = \frac{\pi}{4} \text{ when } x = 1$$

[NCERT]

Ans. The given differential equation can be written as

$$\frac{dy}{dx} = \frac{y}{x} - \sin^2\left(\frac{y}{x}\right) \quad ...(i)$$

Let $F(x, y) = \frac{y}{x} - \sin^2\left(\frac{y}{x}\right)$.

Then, $F(\lambda x, \lambda y) = \frac{\lambda y}{\lambda x} - \sin^2\left(\frac{\lambda y}{\lambda x}\right)$

$$= \frac{y}{x} - \sin^2\left(\frac{y}{x}\right)$$

$$= \lambda^0 F(x, y)$$

Therefore, $F(x, y)$ is a homogeneous function of degree zero. So, the given differential equation is a homogeneous differential equation.

Put $y = vx$ so that $\frac{dy}{dx} = v + x\frac{dv}{dx}$

With these substitutions, equation (i) reduces to

$$v + x\frac{dv}{dx} = v - \sin^2 v$$

$$\Rightarrow \quad x\frac{dv}{dx} = -\sin^2 v$$

$$\Rightarrow \quad \text{cosec}^2 v\, dv = -\frac{dx}{x}$$

On integrating both sides, we have

$$\int \text{cosec}^2 v\, dv = -\int \frac{dx}{x} + C$$

$$\Rightarrow \quad -\cot v = -\log|x| + C$$

Replacing v by $\frac{y}{x}$, we get

$$\log|x| - \cot\left(\frac{y}{x}\right) = C \quad \text{...(ii)}$$

Given that $y = \frac{\pi}{4}$ when $x = 1$.

$$\therefore \quad \log|1| - \cot\left(\frac{\pi/4}{1}\right) = C$$

$$\Rightarrow \quad 0 - 1 = C$$

$$\Rightarrow \quad C = -1$$

Putting this value of C in (ii), we get

$$\log|x| - \cot\left(\frac{y}{x}\right) + 1 = 0$$

or $\cot\left(\frac{y}{x}\right) = \log|ex|$ $\quad [\because \log e = 1]$

which is the required solution of the given differential equation.

TOPIC 5

FIRST ORDER LINEAR DIFFERENTIAL EQUATIONS

A differential equation in which dependent variable and its derivatives are of degree one, is called a linear differential equation.

For example,

(1) $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - y = 0$ is a linear differential equation.

(2) $\frac{d^3y}{dx^3} - x\left(\frac{dy}{dx}\right)^2 = 0$ is not a linear differential equation, as degree of $\frac{dy}{dx}$ is two.

Further, a linear differential equation of order one is called a first order linear differential equation, provided dependent variable and its derivative do not occur as a product.

Thus, a differential equation is said to be linear differential equation of first order, if it can be put in either of the forms:

(1) $\frac{dy}{dx} + Py = Q$, where P and Q are constants or functions of x only.

(2) $\frac{dx}{dy} + Px = Q$, where P and Q are constants or functions of y only.

Some examples of these types of linear differential equations of first order are:

(1) $\frac{dy}{dx} + y = \sin x$

(2) $\frac{dy}{dx} + \frac{1}{x}y = e^x$

(3) $\frac{dx}{dy} + x = \cos y$

(4) $\frac{dx}{dy} - \frac{1}{y}x = 2y$

Working Procedure to Solve First Order Linear Differential Equations

Step 1: Express the given differential equation in the form

$$\frac{dy}{dx} + Py = Q \quad \text{...(i)}$$

Step 2: Identify P and Q.

Step 3: Determine the integrating factor (I.F.): $e^{\int P\,dx}$

Step 4: Write the general solution of the given differential equation as:

$$(\text{I.F.})\, y = \int [(\text{I.F.})\, Q]\, dx + C$$

OR

Step 1: Express the given differential equation in the form

$$\frac{dx}{dy} + Px = Q \quad ...(i)$$

Step 2: Identify P and Q.

Step 3: Determine the integrating factor (I.F.) = $e^{\int P\,dy}$

Step 4: Write the general solution of the given differential equation as:

$$(\text{I.F.})\, x = \int [(\text{I.F.})\,Q]\,dy + C$$

Example 2.11: Find the general solution of the following differential equations:

(A) $x \log x \dfrac{dy}{dx} + y = \dfrac{2}{x}\log x$

(B) $(1 + x^2)dy + 2xy\,dx = \cot x\,dx$ **[NCERT]**

Ans. (A) The given differential equation can be rewritten as

$$\frac{dy}{dx} + \frac{1}{x\log x}y = \frac{2}{x^2}$$

[Dividing both side by $x \log x$]

It is clearly a first order linear differential equation, with $P = \dfrac{1}{x\log x}$, and $Q = \dfrac{2}{x^2}$, each is a function of x only.

Here, $\text{I.F.} = e^{\int P\,dx}$

$$= e^{\int \frac{1}{x\log x}dx}$$

$$= e^{\log(\log x)} = \log x$$

Thus, the required general solution is

$$y \log x = \int \left[\frac{2}{x^2}\cdot(\log x)\right]dx + C$$

$$= \left(-\frac{2}{x}\right)(\log x) - \int\left(-\frac{2}{x}\right)\cdot\frac{1}{x}\,dx + C$$

$$= -\frac{2\log x}{x} - \frac{2}{x} + C$$

$$\Rightarrow \quad y \log x = -\frac{2\log x}{x} - \frac{2}{x} + C$$

(B) The given differential equation can be rewritten as

$$\frac{dy}{dx} + \frac{2x}{1+x^2}y = \frac{\cot x}{1+x^2}$$

[Dividing both sides by $(1 + x^2)dx$]

It is clearly a first order linear differential equation, with $P = \dfrac{2x}{1+x^2}$, and $Q = \dfrac{\cot x}{1+x^2}$, each is a function of x only.

Here, $\text{I.F.} = e^{\int P\,dx}$

$$= e^{\int \frac{2x}{1+x^2}dx}$$

$$= e^{\log(1+x^2)} = 1 + x^2$$

Thus, the required general solution is:

$$y(1 + x^2) = \int \left[\frac{\cot x}{1+x^2}\cdot(1 + x^2)\right]dx + C$$

$$= \int \cot x + C$$

$$= \log|\sin x| + C$$

$$\Rightarrow \quad y = (1 + x^2)^{-1}\log|\sin x| + C(1 + x^2)^{-1}$$

Example 2.12: Find the general solution of the following differential equations:

(A) $(x + y)\dfrac{dy}{dx} = 1$

(B) $y\,dx + (x - y^2)\,dy = 0$ **[NCERT]**

Ans. (A) $(x + y)\dfrac{dy}{dx} = 1$

$$\Rightarrow \quad x + y = \frac{dx}{dy}$$

$$\Rightarrow \quad \frac{dx}{dy} - x = y$$

It is clearly a first order linear differential equation, with P = – 1, and Q = y, each is a constant or function of y.

Here, $\text{I.F.} = e^{\int P\,dy}$

$$= e^{\int(-1)dy} = e^{-y}$$

Thus, the required general solution is:

$$xe^{-y} = \int ye^{-y}\,dy + C$$

$$= -ye^{-y} + \int e^{-y}\,dy + C$$

$$= -ye^{-y} - e^{-y} + C$$

$$= -(y + 1)e^{-y} + C$$

$$\Rightarrow \quad xe^{-y} = -(y + 1)e^{-y} + C$$

or $\quad x = -(y + 1) + Ce^{y}$

(B) $y\,dx + (x - y^2)dy = 0$

$$\Rightarrow \quad y\,dx = (y^2 - x)dy$$

Dividing both sides by $y\,dy$, we get

$$\Rightarrow \quad \frac{dx}{dy} = y - \frac{x}{y}$$

$$\Rightarrow \quad \frac{dx}{dy} + \frac{1}{y}x = y$$

It is clearly a first order linear differential equation, with $P = \dfrac{1}{y}$, and Q = y, each is a function of y.

Here, $\quad$ I.F. $= e^{\int P\,dy} = e^{\int \frac{1}{y}dy} = e^{\log y} = y$

Thus, the required general solution is:

$$xy = \int y.y\,dy + C$$

$$= \frac{y^3}{3} + C$$

$$\Rightarrow \quad x = \frac{y^2}{3} + \frac{C}{y}$$

Example 2.13: Find the particular solution of the following differential equations:

(A) $\frac{dy}{dx} + 2y\tan x = \sin x; y = 0$ **when** $x = \frac{\pi}{3}$

(B) $(1 + x^2)\frac{dy}{dx} + 2xy = \frac{1}{1+x^2}; y = 0$ **when** $x = 1$

[NCERT]

Ans. (A) The given differential equation is clearly a first order linear differential equation, with P = 2 tan x, and Q = sin x, each is a function of x only.

Here, $\quad$ I.F. $= e^{\int P\,dx}$

$$= e^{\int 2\tan x\,dx}$$

$$= e^{2\log(\sec x)}$$

$$= e^{\log \sec^2 x}$$

$$= \sec^2 x$$

Thus, the required general solution is:

$$y\sec^2 x = \int \sin x.(\sec^2 x)dx + C$$

$$= \int \sec x \tan x\,dx + C$$

$$= \sec x + C$$

$$\Rightarrow \quad y\sec^2 x = \sec x + C \qquad ...(i)$$

Given that $y = 0$ when $x = \frac{\pi}{3}$.

$$\therefore \quad 0.\sec^2\frac{\pi}{3} = \sec\frac{\pi}{3} + C$$

$$\Rightarrow \quad 0 = 2 + C$$

$$\Rightarrow \quad C = -2$$

Hence, the particular solution is:

$y\sec^2 x = \sec x - 2$, or $y = \cos x - 2\cos^2 x$.

(B) The given differential equation can be written as

$$\frac{dy}{dx} + \frac{2x}{1+x^2}y = \left(\frac{1}{1+x^2}\right)^2$$

[Dividing both sides by $(1 + x^2)$.]

It is clearly a first order linear differential equation, with $P = \frac{2x}{1+x^2}$, and $Q = \left(\frac{1}{1+x^2}\right)^2$, each is a function of x only.

Here, $\quad$ I.F. $= e^{\int P\,dx}$

$$= e^{\int \frac{2x}{1+x^2}dx}$$

$$= e^{\log(1+x^2)}$$

$$= 1 + x^2$$

Thus, the required general solution is:

$$y(1 + x^2) = \int \left(\frac{1}{1+x^2}\right)^2 \cdot (1 + x^2)dx + C$$

$$= \int \frac{1}{1+x^2}dx + C$$

$$= \tan^{-1} x + C$$

$$\Rightarrow y(1 + x^2) = \tan^{-1} x + C \qquad ...(i)$$

Given that $y = 0$ when $x = 1$.

$$\therefore \quad 0.(1 + 1) = \tan^{-1}(1) + C$$

$$\Rightarrow \quad 0 = \frac{\pi}{4} + C$$

$$\Rightarrow \quad C = -\frac{\pi}{4}$$

Hence, the particular solution is:

$$y(1 + x^2) = \tan^{-1} x - \frac{\pi}{4}$$

OBJECTIVE Type Questions

[1 mark]

Multiple Choice Questions

1. The number of arbitrary constants in the particular solution of a differential equation of second order is (are):

(a) 0 $\qquad$ **(b)** 1

(c) 2 $\qquad$ **(d)** 3 $\qquad$ **[CBSE 2020]**

Ans. *(a) 0*

Explanation: Particular solution of a differential equation of any order, does not have any arbitrary constant.

2. The number of arbitrary constants in the general solution of differential equation of fourth order is:

(a) 0 $\qquad$ **(b)** 2

(c) 3 $\qquad$ **(d)** 4 $\qquad$ **[DIKSHA]**

Ans. *(d) 4*

Explanation: The number of arbitrary constants in general solution of a differential equation of n^{th} order is n.

3. **The general solution of the differential equation $\frac{dy}{dx} = e^{x+y}$ is:**

(a) $e^x + e^{-y} = c$ (b) $e^x + e^y = c$

(c) $e^{-x} + e^y = c$ (d) $e^{-x} + e^{-y} = c$

Ans. *(a)* $e^x + e^{-y} = c$

Explanation: $\frac{dy}{dx} = e^{x+y}$ is equivalent to

$\frac{dy}{dx} = e^x e^y$, or $e^{-y}\,dy = e^x\,dx$

Integrating both sides, we have

$$\int e^{-y}\,dy = \int e^x\,dx + c'$$

$$\Rightarrow \quad -e^{-y} = e^x + c'$$

$$\Rightarrow \quad e^x + e^{-y} = c, \text{ where } c = -c'$$

4. **The integrating factor of the differential equation**

$$(1 - y^2)\frac{dx}{dy} + yx = ay\,(-1 < y < 1)$$

is:

(a) $\frac{1}{y^2 - 1}$ (b) $\frac{1}{\sqrt{y^2 - 1}}$

(c) $\frac{1}{1 - y^2}$ (d) $\frac{1}{\sqrt{1 - y^2}}$

Ans. *(d)* $\frac{1}{\sqrt{1 - y^2}}$

Explanation: The given differential equation can be rewritten as

$$\frac{dx}{dy} + \frac{y}{1 - y^2}x = \frac{ay}{1 - y^2}$$

Clearly, $P = \frac{y}{1 - y^2}, Q = \frac{ay}{1 - y^2}$

Integrating factor $= e^{\int \frac{y}{1-y^2}dy}$

$$\therefore \quad = e^{-\frac{1}{2}\log(1-y^2)}$$

$$= e^{\log\left(\frac{1}{\sqrt{1-y^2}}\right)} = \frac{1}{\sqrt{1 - y^2}}$$

5. **The solution of the differential equation $2x\frac{dy}{dx} - y = 3$ represents:**

(a) a circle (b) an ellipse

(c) a straight line (d) a parabola

Ans. *(d) a parabola*

Explanation: We have,

$$2x\frac{dy}{dx} - y = 3$$

$$\Rightarrow \quad 2x\frac{dy}{dx} = 3 + y$$

$$\Rightarrow \quad 2\frac{dy}{3 + y} = \frac{dx}{x}$$

On integrating both sides, we get

$$2\log(y + 3) = \log x + \log c$$

$$\Rightarrow \quad \log(y + 3)^2 = \log cx$$

$$\text{or} \quad (y + 3)^2 = cx$$

6. **A homogeneous differential equation of the form $\frac{dy}{dx} = h\left(\frac{x}{y}\right)$ can be solved by making the substitution:**

(a) $y = vx$ (b) $v = yx$

(c) $x = vy$ (d) $x = v$

7. **The integrating factor of the differential equation $(x + 3y^2)\frac{dy}{dx} = y$ is:**

(a) y (b) $-y$

(c) $\frac{1}{y}$ (d) $-\frac{1}{y}$ **[CBSE 2020]**

Ans. *(c)* $\frac{1}{y}$

Explanation: Given, $(x + 3y^2)\frac{dy}{dx} = y$

$$\Rightarrow \quad \frac{x + 3y^2}{y} = \frac{dx}{dy}$$

$$\Rightarrow \quad \frac{x}{y} + 3y = \frac{dx}{dy}$$

$$\Rightarrow \quad \frac{dx}{dy} - \frac{x}{y} = 3y$$

This is a linear differential equation with

$$P = -\frac{1}{y},\ Q = 3y$$

$$\therefore \quad \text{I.F.} e^{\int p\,dy} = e^{-\int \frac{dy}{y}} = e^{-\log y} = \frac{1}{y}$$

8. Ⓐ**The integrating factor of the differential equation $x\frac{dy}{dx} - y = 2x^2$ is:**

(a) e^{-x} (b) e^x

(c) $\frac{1}{x}$ (d) x

9. The general solution of the differential equation $\frac{y\,dx - x\,dy}{y} = 0$ is:

(a) $xy = c$ (b) $x = cy^2$

(c) $y = cx$ (d) $y = cx^2$

Ans. *(c)* $y = cx$

Explanation: The given differential equation can be reduced to $\frac{dy}{y} = \frac{dx}{x}$

On integrating both sides, we get

$$\int \frac{dy}{y} = \int \frac{dx}{x}$$

$$\Rightarrow \quad \log y = \log x + \log c$$

$$\Rightarrow \quad \log y = \log cx$$

$$\Rightarrow \quad y = cx$$

10. Solution of differential equation $x\,dy - y\,dx = 0$ represents:

(a) a rectangular hyperbola.

(b) parabola whose vertex is at origin.

(c) straight line passing through origin.

(d) a circle whose centre is at origin.

[NCERT Exemplar]

Ans. *(c) straight line passing through origin.*

Explanation: We have,

$$x\,dy - y\,dx = 0$$

$$\Rightarrow \quad x\,dy = y\,dx$$

$$\Rightarrow \quad \frac{dy}{y} = \frac{dx}{x}$$

Integrating both sides, we get

$$\log y = \log x + \log c$$

$$\Rightarrow \quad y = cx$$

Which is a straight line passing through origin.

⚠ Caution

➥ *Compare the given equation with the standard equations to get the answer.*

11. Ⓐ**The general solution of the differential equation $\frac{dy}{dx} = e^{\frac{x^2}{2}} + xy$ is:**

(a) $y = ce^{\frac{-x^2}{2}}$ (b) $y = ce^{\frac{x^2}{2}}$

(c) $y = (x+c)e^{\frac{x^2}{2}}$ (d) $y = (c-x)e^{\frac{x^2}{2}}$

[NCERT Exemplar]

12. The differential equation $x\frac{dy}{dx} - y = x^2$, has the general solution:

(a) $y - x^3 = 2cx$ (b) $2y - x^3 = cx$

(c) $y = x^2 + cx$ (d) $y = x^2 - cx$

[DIKSHA]

Ans. *(c)* $y = x^2 + cx$

Explanation: Given, differential equation is:

$$x\frac{dy}{dx} - y = x^2$$

$$\Rightarrow \quad \frac{dy}{dx} - \frac{y}{x} = x$$

Here, $P = \frac{-1}{x}$ and $Q = x$

$$\therefore \quad \text{I.F.} = e^{\int \frac{-1}{x}dx} = e^{-\log x} = \frac{1}{x}$$

$$\therefore \quad y \times \frac{1}{x} = \int x \times \frac{1}{x}dx + c$$

$$\Rightarrow \quad \frac{y}{x} = x + c$$

$$\Rightarrow \quad y = x^2 + cx$$

13. The general solution of the differential equation $(e^x + 1)\,y dy = (y + 1)\,e^x dx$ is:

(a) $(y + 1) = k(e^x + 1)$

(b) $y + 1 = e^x + 1 + k$

(c) $y = \log\{k(y + 1)(e^x + 1)\}$

(d) $y = \log\left(\frac{e^x+1}{y+1}\right) + k$

[NCERT Exemplar]

Ans. *(c) $y = \log\{k(y + 1)(e^x + 1)\}$*

Explanation:

Given, $(e^x + 1)\,y\,dy = (y + 1)e^x\,dx$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{e^x(1+y)}{(e^x+1)y}$$

$$\Rightarrow \quad \frac{y}{1+y}dy = \frac{e^x}{1+e^x}dx$$

$$\Rightarrow \quad \int\left(\frac{y}{1+y}\right)dy = \int\frac{e^x}{1+e^x}dx$$

$$\Rightarrow \quad \int\left(\frac{1+y-1}{1+y}\right)dy = \int\frac{e^x}{1+e^x}dx$$

$\Rightarrow \int 1.dy - \int \frac{1}{1+y} dy = \int \frac{e^x}{1+e^x} dx$

$\Rightarrow \quad y - \log(1 + y) = \log (1 + e^x) + \log k$

$\Rightarrow \quad y = \log [(1 + e^x) k(1 + y)]$

or, $\quad y = \log (k(1 + e^x)(1 + y))$

⚠ Caution

➥ *Separate the variables for solving the differential equation.*

14. Ⓐ**The general solution of a differential equation of the type $\frac{dx}{dy} + P_1x = Q_1$ is:**

(a) $y.e^{\int P_1 dy} = \int(Q_1 e^{\int P_1 dy})dy + c$

(b) $y.e^{\int P_1 dy} = \int(Q_1 e^{\int P_1 dx})dx + c$

(c) $x.e^{\int P_1 dy} = \int(Q_1 e^{\int P_1 dy})dy + c$

(d) $x.e^{\int P_1 dx} = \int(Q_1 e^{\int P_1 dx})dx + c$

15. The solution of the differential equation $\cos x \sin y\, dx + \sin x \cos y\, dy = 0$ is:

(a) $\frac{\sin x}{\sin y} = c$ (b) $\sin x \sin y = c$

(c) $\sin x + \sin y = c$ (d) $\cos x \cos y = c$

[NCERT Exemplar]

Ans. *(b)* $\sin x \sin y = c$

Explanation:

Given, $\cos x \sin y\, dx + \sin x \cos y\, dy = 0$

$\Rightarrow \quad \cos x \sin y\, dx = -\sin x \cos y\, dy$

$\Rightarrow \quad \frac{\cos x}{\sin x} dx = -\frac{\cos y}{\sin y} dy$

On integrating both sides, we get

$\Rightarrow \quad \log \sin x = -\log \sin y + \log c$

$\Rightarrow \quad \log (\sin x \sin y) = \log c$

$\Rightarrow \quad \sin x \sin y = c$

⚠ Caution

➥ *Separate the variables, and then integrate to get the answer.*

16. The solution of the differential equation $\frac{dy}{dx} + \frac{1}{x} = \frac{e^y}{x}$ is:

(a) $e^{-y} = 1 + cx$ (b) $e^y = 1 + cx$

(c) $e^{-y} = 1 - cx$ (d) $e^y + 1 = cx$

Ans. *(c)* $e^{-y} = 1 - cx$

Explanation: The given D.E. can be written as

$$\frac{dy}{dx} = \frac{1}{x}(e^y - 1)$$

$\Rightarrow \quad \frac{dy}{e^y - 1} = \frac{dx}{x}$

$\Rightarrow \quad \frac{e^{-y}}{1 - e^{-y}} dy = \frac{dx}{x}$

Integrating both sides, we get

$$\log (1 - e^{-y}) = \log x + \log c$$

$\Rightarrow \quad 1 - e^{-y} = cx$, or $e^{-y} = 1 - cx$

17. The integrating factor of the differential equation

$$(1 + y^2)\, dx - (\tan^{-1} y - x)dy = 0$$

is:

(a) $\tan^{-1} y$ (b) $e^{\tan^{-1} y}$

(c) $\frac{1}{1 + y^2}$ (d) $\frac{1}{x(1 + y^2)}$

Ans. *(b)* $e^{\tan^{-1} y}$

Explanation: The given D.E. can be written as

$$\frac{dx}{dy} + \frac{1}{1+y^2} \cdot x = \frac{\tan^{-1} y}{1+y^2}$$

Integrating factor $= e^{\int \frac{1}{1+y^2} dy} = e^{\tan^{-1} y}$

18. Which of the following differential equations has $y = x$ as the particular solution?

(a) $\frac{d^2y}{dx^2} - x^2 \frac{dy}{dx} + xy = x$

(b) $\frac{d^2y}{dx^2} + x^2 \frac{dy}{dx} + xy = x$

(c) $\frac{d^2y}{dx^2} - x^2 \frac{dy}{dx} + xy = 0$

(d) $\frac{d^2y}{dx^2} + x^2 \frac{dy}{dx} + xy = 0$

Ans. *(c)* $\frac{d^2y}{dx^2} - x^2 \frac{dy}{dx} + xy = 0$

Explanation: On differentiating the given equation $y = x$, we have

$$\frac{dy}{dx} = 1$$

On differentiating once again, we have

$$\frac{d^2y}{dx^2} = 0$$

Equation $\frac{d^2y}{dx^2} - x^2 \frac{dy}{dx} + xy = 0$ in option (c) is the only equation which is satisfied by these values of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

CASE BASED Questions (CBQs)

[4 & 5 marks]

Read the following passages and answer the questions that follow:

19. *A Veterinary doctor was examining a sick cat brought by a pet lover. When it was brought to the hospital, it was already dead. The pet lover wanted to find its time of death. He took the temperature of the cat at 11.30 pm which was 94.6°F. He took the temperature again after one hour; the temperature was lower than the first observation. It was 93.4°F. The room in which the cat was put is always at 70°F. The normal temperature of the cat is taken as 98.6°F when it was alive. The doctor estimated the time of death using Newton's law of cooling which is governed by the differential equation:* $\frac{dT}{dt} \propto (T - 70)$, *where 70°F is the room temperature and T is the temperature of an object at time t.*

Substituting the two different observations of T and t made, in the solution of the differential equation $\frac{dT}{dt} = k(T - 70)$ *where k is a constant of proportion, time of death is calculated.*

(A) State the degree of the above given differential equation.

(B) Write the solution of the differential equation $\frac{dT}{dt} = k(T - 70)$ **given that** $t = 0$ **when** $T = 72$.

Ans. (A) Given differential equation is

$$\frac{dT}{dt} = k(T - 70)$$

Here, highest order derivative is $\frac{dT}{dt}$, whose power is one.

So, the degree of given differentiatal equation is 1.

20. *The order of a differential equation is the order of the highest derivative occurring in the differential equation. The degree of a differential equation whose terms are polynomials in the derivatives is defined as the highest power (positive integral index) of the highest order derivative in it.*

(A) If p and q be the order and degree of the differential equation

$$y^2\left(\frac{d^2y}{dx^2}\right)^2 + 3x\frac{dy}{dx} + x^2y^2 = \sin x$$

then,

(a) $p < q$ (b) $p = q$

(c) $2p = q$ (d) $p > q$

(B) If m and n be the degree and order of the differential equation $(1 + y_1^2)^{2/3} = y_2$ **then,** $\frac{m+n}{m-n}$ **is equal to:**

(a) 2 (b) 5

(c) 4 (d) 3

(C) The degree of the differential equation

$$\left[1 + \left(\frac{dy}{dx}\right)^2 + \sin\left(\frac{dy}{dx}\right)\right]^{3/4} = \frac{d^2y}{dx^2} \text{ is:}$$

(a) 1 (b) 3

(c) 4 (d) Not defined

(D) The product of the order and degree of the differential equation

$$\left(\frac{d^2y}{dx^2}\right)^2 - \left(\frac{dy}{dx}\right)^3 = y^4$$

is

(a) 2 (b) 3

(c) 4 (d) 5

(E) The order of the differential equation, whose general solution of $y = A\cos x + B\sin x + Ce^x$, **A, B and C are arbitrary constants, is:**

(a) 2 (b) 3

(c) 4 (d) 1

Ans. (B) *(b) 5*

Explanation: We have,

$$(1 + y_1^2)^{2/3} = y_2$$

$$\Rightarrow (1 + y_1^2)^2 = y_2^3$$

$\therefore$ Order (n) = 2, Degree (m) = 3

So, $\frac{m+n}{m-n} = \frac{5}{1} = 5$

(D) *(c) 4*

Explanation: Here,

Order = 2, Degree = 2.

So, product is 4.

21. *Consider the differential equation* $\frac{dy}{dx} + Py = Q$, *where P and Q are functions of x (or constants).*

Here, I.F. = $e^{\int P\,dx}$ and solution of the D.E. is given by

$$y\,(I.F.) = \int Q.\,(I.F.)\,dx + c.$$

Now consider the differential equation: $\frac{dy}{dx} + \frac{y}{x} = 3x.$

Based on this, answer the following questions:

(A) What is the value of I.F.?

(B) Solve the given differential equation.

Ans. (B) Solution of the given differential equation is given by

$$y.x = \int x.3x\,dx + c$$

$$\Rightarrow y.x = \frac{3x^3}{3} + c$$

$$\Rightarrow y = x^2 + \frac{c}{x}$$

VERY SHORT ANSWER Type Questions (VSA)

[1 mark]

22. How many arbitrary constants will be there in the general solution of a diferential equation of order three?

23. Write the integrating factor of the following differential equation:

$$(1 + y^2) + (2xy - \cot y)\frac{dy}{dx} = 0.$$

[CBSE 2015]

Ans. The given differential equation can be rewritten as

$$(1 + y^2)\frac{dx}{dy} + (2xy - \cot y) = 0$$

$$\Rightarrow \frac{dx}{dy} + \frac{2y}{1+y^2}\cdot x = \frac{\cot y}{1+y^2}$$

$$\therefore \quad I.F. = e^{\int P.dy} = e^{\int \frac{2y}{1+y^2}dy}$$

$$= e^{\log|1+y^2|}$$

$$= 1 + y^2$$

24. Write the solution of the differential equation:

$$\frac{dy}{dx} = 2^{-y}$$

[CBSE 2015]

Ans. Given, $\frac{dy}{dx} = 2^{-y}$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2^y}$$

$$\Rightarrow 2^y\,dy = dx$$

Integrating both sides, we get

$$\Rightarrow \int 2^y dy = \int dx + c$$

$$\Rightarrow \frac{2^y}{\log 2} = x + c$$

$$\Rightarrow 2^y = x\log 2 + c'$$

where $c' = c\log 2$.

25. Solve the differential equation:

$$\frac{dy}{dx} = \sqrt{4 - y^2}\ (-2 < y < 2)$$

26. Solve the differential equation:

$$\frac{dy}{dx} - \frac{x}{x^2+1} = 0$$

Ans. Given, differential equation is

$$\frac{dy}{dx} - \frac{x}{x^2+1} = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{x}{x^2+1}$$

On separating the variables, we get

$$\Rightarrow dy = \frac{x}{x^2+1}dx$$

On integrating both sides, we get

$$\Rightarrow \int dy = \int \frac{x}{x^2+1}dx$$

$$\Rightarrow y = \frac{1}{2}\int \frac{2x}{x^2+1}dx$$

$$\Rightarrow y = \frac{1}{2}\log|x^2+1| + c$$

which is the required solution.

27. Find the integrating factor of the differential equation $\left(\frac{e^{-2\sqrt{x}}}{\sqrt{x}} - \frac{y}{\sqrt{x}}\right)\frac{dx}{dy} = 1.$ **[CBSE 2015]**

Ans. Given, $\left(\frac{e^{-2\sqrt{x}}}{\sqrt{x}} - \frac{y}{\sqrt{x}}\right)\frac{dx}{dy} = 1$

$$\Rightarrow \quad \frac{e^{-2\sqrt{x}}}{\sqrt{x}} - \frac{y}{\sqrt{x}} = \frac{dy}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx} + \frac{y}{\sqrt{x}} = \frac{e^{-2\sqrt{x}}}{\sqrt{x}}$$

$$\therefore \quad P = \frac{1}{\sqrt{x}}, d = \frac{e^{-2\sqrt{x}}}{\sqrt{x}}$$

$$\therefore \quad \text{I.F.} = e^{\int P dx} = e^{\int \frac{1}{\sqrt{x}} dx} = e^{2\sqrt{x}}$$

28. Solve the differential equation $\frac{dy}{dx} + 1 = e^{x+y}$.

[NCERT Exemplar]

Ans. Given, differential equation is

$$\frac{dy}{dx} + 1 = e^{x+y} \quad ...(i)$$

Let $x + y = t$

On differentiating w.r.t. x, we get

$$1 + \frac{dy}{dx} = \frac{dt}{dx} \quad ...(ii)$$

$\therefore$ Equation (i) becomes

$$\frac{dt}{dx} = e^t$$

$$\Rightarrow \quad e^{-t} dt = dx$$

On integrating both sides, we get

$$-e^{-t} = x + c$$

$$\Rightarrow \quad \frac{-1}{e^t} = x + c$$

$$\Rightarrow \quad \frac{-1}{e^{x+y}} = x + c$$

$$\Rightarrow \quad -1 = (x + c)\, e^{x+y}$$

$$\Rightarrow \quad (x + c)\, e^{x+y} + 1 = 0$$

Caution

Do the necessary substitution, wherever needed.

29. Find the solution of $\frac{dy}{dx} = 2^{y-x}$.

[NCERT Exemplar]

Ans. Given, $\frac{dy}{dx} = 2^{y-x}$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{2^y}{2^x}$$

$$\Rightarrow \quad \frac{dy}{2^y} = \frac{dx}{2^x}$$

On integrating both sides, we get

$$\int 2^{-y} dy = \int 2^{-x} dx$$

$$\Rightarrow \quad \frac{-2^{-y}}{\log 2} = \frac{-2^{-x}}{\log 2} + c$$

$$\Rightarrow \quad -2^{-y} + 2^{-x} = c \log 2$$

$$\Rightarrow \quad 2^{-x} - 2^{-y} = k, \text{ where } k = c \log 2$$

Hence, the required solution is $2^{-x} - 2^{-y} = k$.

30. Find the solution of $\frac{dy}{dx} + y = e^{-x}$.

31. For what value of 'n' is the following a homogeneous differential equation?

$$\frac{dy}{dx} = \frac{x^3 - y^n}{x^2 y + xy^2}$$ **[CBSE SQP 2020]**

Ans. In homogeneous differential equation, the degree of terms are same. So, the value of 'n' must be 3.

32. If $\sin x$ is an integrating factor of the differential equation $\frac{dy}{dx} + Py = Q$, then write the value of P. **[Delhi Gov. 2022]**

33. What is the value of the constant of integration in the particular solution of the given differential equation?

$\frac{dy}{dx} = \frac{2x}{y^2}$, if $f(-2) = 3$. **[Delhi Gov. 2022]**

Ans. Given, equation is

$$y^2 dy = 2x\, dx$$

On integrating both sides, we get

$$\int y^2 dy = 2\int x\, dx$$

$$\Rightarrow \quad \frac{y^3}{3} = \frac{2x^2}{2} + c$$

$$\Rightarrow \quad \frac{y^3}{3} = x^2 + c$$

It is given that $f(-2) = 3$.

$$\therefore \quad \frac{(3)^3}{3} = (-2)^2 + c$$

$$\Rightarrow \quad 9 = 4 + c$$

$$\Rightarrow \quad c = 5$$

Hence, the value of constant of integration *i.e.*, c is 5.

34. Find the general solution of $\frac{dy}{dx} + y \tan x = \sec x$.

SHORT ANSWER Type-I Questions (SA-I)

[**2** marks]

35. Verify that the function $x + y = \tan^{-1} y$ is a solution of the differential equation

$$y^2y' + y^2 + 1 = 0.$$

Ans. Given, function is

$$x + y = \tan^{-1} y \quad ...(i)$$

Since, order of differential equation is 1.

So, on differentating equation (i) w.r.t. x, we get

$$1 + \frac{dy}{dx} = \frac{1}{1+y^2}\frac{dy}{dx}$$

$$\Rightarrow \quad \left(1 + \frac{dy}{dx}\right)(1+y^2) = \frac{dy}{dx}$$

$$\Rightarrow \quad 1 + \frac{dy}{dx} + y^2 + y^2\frac{dy}{dx} = \frac{dy}{dx}$$

$$\Rightarrow \quad y^2\frac{dy}{dx} + y^2 + 1 = 0,$$

which is the given differential equation.

Hence, $x + y = \tan^{-1} y$ is a solution of the given differential equation.

36. Verify that the function $y = \sqrt{a^2 - x^2}$, $x \in (-a, a)$ is a solution of differential equation $x + y\frac{dy}{dx} = 0 \; (y \neq 0)$.

37. Solve the following differential equation:

$$\frac{dy}{dx} + y = \cos x - \sin x. \qquad \text{[CBSE 2019]}$$

Ans. The given equation is a linear differential equation.

On comparing it with $\frac{dy}{dx} + Py = Q$, we get

$P = 1, Q = \cos x - \sin x$

$\therefore \quad \text{I.F.} = e^{\int P.dx} = e^{\int 1.dx} = e^x$

The solution of the given differential equation is

$$y\,e^x = \int e^x(\cos x - \sin x)dx + c$$

$$\Rightarrow \quad y\,e^x = e^x \cos x + c$$

$$\Rightarrow \quad y = \cos x + c\,e^{-x}$$

38. Solve the differential equation

$$\cos\left(\frac{dy}{dx}\right) = a(a \in R) \qquad \text{[CBSE 2018]}$$

Ans. Given, differential equation is

$$\cos\left(\frac{dy}{dx}\right) = a$$

The given equation can be rewritten as

$$\frac{dy}{dx} = \cos^{-1}(a)$$

$$\Rightarrow \quad dy = \cos^{-1} a\, dx$$

Now, on integrating both sides, we get

$$\int dy = \int \cos^{-1} a\, dx$$

$$\Rightarrow \quad y = x\cos^{-1} a + c$$

which is the required solution.

39. If $\frac{dy}{dx} = e^{-2y}$ and $y = 0$ when $x = 5$, find the value of x when $y = 3$. **[NCERT Exemplar]**

Ans. Given: $\quad \frac{dy}{dx} = e^{-2y}$

$$\Rightarrow \quad \frac{dy}{e^{-2y}} = dx$$

$$\Rightarrow \quad \int e^{2y}\,dy = \int dx$$

$$\Rightarrow \quad \frac{e^{2y}}{2} = x + c \quad ...(i)$$

It is given that when $x = 5$, $y = 0$.

$$\therefore \quad \frac{e^0}{2} = 5 + c$$

$$\Rightarrow \quad \frac{1}{2} = 5 + c$$

$$\Rightarrow \quad c = \frac{1}{2} - 5 = \frac{-9}{2}$$

$\therefore$ Equation (i) becomes

$$\frac{e^{2y}}{2} = \; --$$

Or $\quad e^{2y} = 2x - 9$

When $y = 3$, then

$$e^6 = 2x - 9$$

$$\Rightarrow \quad x = \frac{e^6 + 9}{2}$$

Hence, the required value of x is $\frac{e^6+9}{2}$.

40. Find the general solution of $\frac{dy}{dx} + ay = e^{mx}$ **[NCERT Exemplar]**

41. Find the general solution of the following differential equation:

$$\frac{dy}{dx} = \frac{xy + y}{xy + x}$$

Ans. The given equation is

$$\frac{dy}{dx} = \frac{y(x+1)}{x(y+1)}$$

$$\Rightarrow \quad \frac{y+1}{y}dy = \frac{x+1}{x}dx$$

$$\Rightarrow \quad \left(1+\frac{1}{y}\right)dy = \left(1+\frac{1}{x}\right)dx$$

On integrating both sides, we get

$$\int\left(1+\frac{1}{y}\right)dy = \int\left(1+\frac{1}{x}\right)dx$$

$$\Rightarrow \quad y + \log y = x + \log x + \log c$$

$$\Rightarrow \quad y - x = \log\left[c\cdot\left(\frac{x}{y}\right)\right]$$

42. Solve the differential equation

$$\log\left(\frac{dy}{dx}\right) = 3x + 4y. \qquad \textbf{[CBSE 2017]}$$

43. Solve the differential equation $\frac{dy}{dx}+2xy = y$.

[NCERT Exemplar]

Ans. Given, differential equation is

$$\frac{dy}{dx} \quad xy = y$$

$$\Rightarrow \quad \frac{dy}{dx}+2xy-y = 0$$

$$\Rightarrow \quad \frac{dy}{dx}+(2x-1)y = 0$$

Which is a linear differential equation of the type $\frac{dy}{dx}+Py = Q$.

where, $P = (2x-1)$ and $Q = 0$

$\therefore$ $IF = e^{\int P.dx} = e^{\int (2x-1)dx}$

$$= e^{\left(\frac{2x^2}{2}-x\right)} = e^{x^2-x}$$

$\therefore$ The complete solution is:

$$y.IF = \int(Q.\ IF)dx + c.$$

$$\Rightarrow \quad y.e^{x^2-x} = \int 0.e^{x^2-x}\,dx + c$$

$$\Rightarrow \quad y.e^{x^2-x} = 0 + c$$

$$\Rightarrow \quad y = c\,e^{x-x^2}$$

44. Find the general solution of the following differential equation:

$\cos x\,(1 + \cos y)\,dx - \sin y\,(1 + \sin x)\,dy = 0$

Ans. The given equation is

$$\cos x(1 + \cos y)\,dx - \sin y\,(1 + \sin x)\,dy = 0$$

$$\Rightarrow \cos x(1 + \cos y)\,dx = \sin y\,(1 + \sin x)\,dy$$

$$\Rightarrow \quad \frac{\cos x}{1+\sin x}dx = \frac{\sin y}{1+\cos y}dy$$

Now, integrating both sides, we get

$$\int\frac{\cos x}{(1+\sin x)}dx = \int\frac{\sin y}{(1+\cos y)}dy$$

$$\Rightarrow \quad \log(1 + \sin x) = -\log(1 + \cos y) + \log c$$

$$\left[\because \frac{d}{dx}(1+\sin x) = \cos x, \frac{d}{dx}(1+\cos y) = -\sin y\right]$$

$$\Rightarrow \quad 1+\sin x = \frac{c}{1+\cos y}$$

$\Rightarrow (1 + \sin x)(1 + \cos y) = c$ is the required solution.

45. Solve the following differential equation:

$\frac{dy}{dx} = x^3 \text{cosec}\, y$ given that $y(0) = 0$.

[CBSE SQP 2020]

Ans. Here,

$$\frac{dy}{dx} = x^3\,\text{cosec}\,y$$

$$\Rightarrow \quad \frac{dy}{\text{cosec}\,y} = x^3\,dx$$

On integrating both sides, we get

$$\int \sin y\,dy = \int x^3 dx$$

$$\Rightarrow \quad -\cos y = \frac{x^4}{4}+c$$

Put $x = 0, y = 0$

Then, $$-\cos 0 = \frac{0^4}{4}+c$$

$$\Rightarrow \quad -1 = c$$

$$\Rightarrow \quad c = -1$$

Then, $$-\cos y = \frac{x^4}{4}-1$$

$$\Rightarrow \quad \cos y = 1-\frac{x^4}{4}$$

which is the particular solution of the given differential equation.

46. Show that the differential equation

$x\cos\left(\frac{y}{x}\right)\frac{dy}{dx} = y\cos\left(\frac{y}{x}\right)+x$ is homogeneous.

SHORT ANSWER Type-II Questions (SA-II)

[3 marks]

47. Find the general solution of differential equation given below:

$$\frac{dy}{dx} = \frac{1}{x(1+x^2)}$$

Ans. Given differential equation is

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{x(1+x^2)}$$

$$\Rightarrow \quad dy = \frac{1}{x(1+x^2)}dx$$

Integrating both sides, we get

$$\int dy = \int \frac{1}{x(1+x^2)}dx \quad ...(i)$$

Let $\frac{1}{x(1+x^2)} = \frac{A}{x} + \frac{Bx+C}{1+x^2}$ ...(ii)

$$\Rightarrow \quad 1 = A(1+x^2) + (Bx+C)x$$

$$\Rightarrow \quad 1 = (A+B)x^2 + Cx + A$$

On comparing coefficients of x^2, x and constant terms both sides, we get

$$A + B = 0, C = 0, A = 1$$

$$\Rightarrow \quad A = 1, B = -1, C = 0$$

$$\therefore \quad \frac{1}{x(x+x^2)} = \frac{1}{x} - \frac{x}{1+x^2} \quad ...(iii)$$

From (i), (ii) and (iii), we get

$$\int dy = \int\left[\frac{1}{x} - \frac{x}{1+x^2}\right]dx$$

$$\Rightarrow \quad y = \log x - \frac{1}{2}\log(1+x^2) + C$$

$$\text{or,} \quad y = \frac{1}{2}\log\left(\frac{x^2}{1+x^2}\right) + C.$$

48. Find the particular solution of the differential equation $x\frac{dy}{dx} = y - x\tan\left(\frac{y}{x}\right)$, given that $y = \frac{\pi}{4}$ at $x = 1$. **[CBSE 2020]**

Ans. Given, $x\frac{dy}{dx} = y - x\tan\left(\frac{y}{x}\right)$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y}{x} - \tan\left(\frac{y}{x}\right)$$

Which is a homogeneous differential equation.
Now, put $y = vx$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$\therefore \quad v + x\frac{dv}{dx} = v - \tan v$$

$$\Rightarrow \quad \frac{dv}{\tan v} = \frac{-dx}{x}$$

$$\Rightarrow \quad \cot v\, dv + \frac{dx}{x} = 0$$

Now, on integrating both sides, we get

$\log|\sin v| + \log x = \log c$

$$\Rightarrow \quad x\sin v = c$$

$$\Rightarrow \quad x\sin\frac{y}{x} = c$$

When $x = 1$, $y = \frac{\pi}{4}$.

$$\therefore \quad 1.\sin\left(\frac{\pi}{4}\right) = c$$

$$\Rightarrow \quad c = \frac{1}{\sqrt{2}}$$

So, $x\sin\left(\frac{y}{x}\right) = \frac{1}{\sqrt{2}}$ is the required particular solution.

49. Solve the following differential equation:

$$x\frac{dy}{dx} + y - x + xy\cot x = 0$$ **[CBSE 2012, 11]**

Ans. Given, $x\frac{dy}{dx} + y - x + xy\cot x = 0$

$$\Rightarrow \quad x\frac{dy}{dx} + (1 + x\cot x).y = x$$

$$\Rightarrow \quad \frac{dy}{dx} + \frac{(1+x\cot x)}{x}y = 1 \quad ...(i)$$

This is linear D.E. of the form $\frac{dy}{dx} + Py = Q$

where, $P = \frac{1+x\cot x}{x}, Q = 1$

Now,
$$\text{I.F.} = e^{\int P.dx}$$
$$= e^{\int\left(\frac{1}{x}+\cot x\right)dx}$$
$$= e^{\log x + \log\sin x}$$
$$= e^{\log(x\sin x)}$$
$$= x\sin x$$

∴ The solution of equation (i) is

$$y.x\sin x = \int 1.x\sin x\,dx + c$$

$$\Rightarrow \quad y.x\sin x = x(-\cos x) + \int 1.\cos x\,dx + c$$

$$\Rightarrow \quad xy\sin x = -x\cos x + \sin x + c$$

50. If $y(x)$ is a solution of $\left(\frac{2+\sin x}{1+y}\right)\frac{dy}{dx} = -\cos x$ and $y(0) = 1$, then find the value of $y\left(\frac{\pi}{2}\right)$.

[NCERT Exemplar]

Ans. Given: $\left(\frac{2+\sin x}{1+y}\right)\frac{dy}{dx} = -\cos x$

$$\Rightarrow \quad \frac{dy}{1+y} = -\left(\frac{\cos x}{2+\sin x}\right)dx$$

On integrating both sides, we get

$$\int\frac{1}{1+y}\,dy = -\int\frac{\cos x}{2+\sin x}\,dx$$

$$\Rightarrow \quad \log(1+y) = -\log(2+\sin x) + \log c$$

$$\Rightarrow \quad \log(1+y)(2+\sin x) = \log c$$

$$\Rightarrow \quad (1+y)(2+\sin x) = c$$

$$\Rightarrow \quad y = \frac{c}{2+\sin x} - 1 \qquad \text{...(i)}$$

It is given that $y(0) = 1$.

$$\therefore \quad 1 = \frac{c}{2+0} - 1$$

$$\Rightarrow \quad c = 4$$

Putting $c = 4$ in equation (i), we get

$$y = \frac{4}{2+\sin x} - 1$$

$$\therefore \quad y\left(\frac{\pi}{2}\right) = \frac{4}{2+\sin\frac{\pi}{2}} - 1$$

$$= \frac{4}{3} - 1 = \frac{1}{3}$$

⚠ Caution

- *Separate the variables x and y by separating the variable technique.*

51. Shown below is a differential equation where the value of y is 0 when $x = 3$.

$$\frac{dy}{dx} + \frac{y}{x} = \frac{1}{x^2\log x}$$

Find the value of y when $x = 5$.

52. Find the general solution of the following differential equation:

$$x\frac{dy}{dx} = y - x\sin\left(\frac{y}{x}\right)$$

[CBSE Term-2 SQP 2022]

Ans. We have the differential equation:

$$x\frac{dy}{dx} = y - x\sin\left(\frac{y}{x}\right)$$

Or, $$\frac{dy}{dx} = \frac{y}{x} - \sin\left(\frac{y}{x}\right)$$

The equation is a homogeneous differential equation.

Putting $\quad y = vx$

$$\Rightarrow \quad \frac{dy}{dx} = v + x\frac{dv}{dx}$$

The given differential equation becomes

$$v + x\frac{dv}{dx} = v - \sin v$$

$$\Rightarrow \quad \frac{dv}{\sin v} = -\frac{dx}{x}$$

$$\Rightarrow \quad \text{cosec } v\,dv = -\frac{dx}{x}$$

Integrating both sides, we get

$\log|\text{cosec } v - \cot v| = -\log|x| + \log K,\ K > 0$

(Here, log K is an arbitrary constant)

$$\Rightarrow \quad \log|(\text{cosec } v - \cot v)x| = \log K$$

$$\Rightarrow \quad |(\text{cosec } v - \cot v)x| = K$$

$$\Rightarrow \quad (\text{cosec } v - \cot v)x = \pm K$$

$\Rightarrow \left(\text{cosec}\frac{y}{x} - \cot\frac{y}{x}\right)x = C$, which is the required general solution of the given differential equation.

[CBSE Marking Scheme Term-2 SQP 2022]

Explanation: Given, differential equation is:

$$x\frac{dy}{dx} = y - x\sin\left(\frac{y}{x}\right)$$

The given equation is a homogeneous differential equation.

$$y = vx$$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$\therefore \quad x\left(v + x\frac{dv}{dx}\right) = vx - x\sin v$$

$$\Rightarrow \quad v + x\frac{dv}{dx} = v - \sin v$$

$$\Rightarrow \quad x\frac{dv}{dx} = -\sin v$$

$$\Rightarrow \quad \frac{dv}{\sin v} = -\frac{dx}{x}$$

$$\Rightarrow \quad \text{cosec } v\, dv = -\frac{dx}{x}$$

on integrating both sides

$$\int \text{cosec } v\, dv = \int -\frac{dx}{x}$$

$$\Rightarrow \quad \log |\text{cosec } v - \cot v| = -\log x + \log c$$

$$\Rightarrow \log(\text{cosec } v - \cot v) + \log x = \log c$$

$$\Rightarrow \quad x(\text{cosec } v - \cot v) = c$$

$$\Rightarrow \quad x\left(\text{cosec}\frac{y}{x} - \cot\frac{y}{x}\right) = c$$

is the required general solution.

53. Find the general solution of the differential equation $(1 + y^2) + (x - e^{\tan^{-1}y})\frac{dy}{dx} = 0.$

[NCERT Exemplar]

Ans. Given, differential equation is

$$(1 + y^2) + (x - e^{\tan^{-1}y})\frac{dy}{dx} = 0$$

$$\Rightarrow \quad (1 + y^2) = -(x - e^{\tan^{-1}y})\frac{dy}{dx}$$

$$\Rightarrow \quad (1 + y^2)\frac{dx}{dy} = e^{\tan^{-1}y} - x$$

$$\Rightarrow \quad \frac{dx}{dy} = \frac{e^{\tan^{-1}y} - x}{1 + y^2}$$

$$\Rightarrow \quad \frac{dx}{dy} + \frac{x}{1+y^2} = \frac{e^{\tan^{-1}y}}{1+y^2}$$

It is a linear differential equation of the type

$\frac{dx}{dy} + Px = Q.$

Where, $\quad P = \frac{1}{1+y^2}$ and $Q = \frac{e^{\tan^{-1}y}}{1+y^2}$

$$\therefore \quad IF = e^{\int P.dx} = e^{\int \frac{1}{1+y^2}dy} = e^{\tan^{-1}y}$$

∴ The, general solution is:

$$x.IF = \int (Q.IF)dy + c$$

$$\Rightarrow \quad x.e^{\tan^{-1}y} = \int \frac{e^{\tan^{-1}y}}{1+y^2} \times e^{\tan^{-1}y} dy + c$$

Put $\quad \tan^{-1}y = t$

$$\Rightarrow \quad \frac{1}{1+y^2}dy = dt$$

$$\Rightarrow \quad x.e^{\tan^{-1}y} = \int e^{2t}\, dt + c$$

$$\Rightarrow \quad x.e^{\tan^{-1}y} = \frac{e^{2t}}{2} + c$$

$$\Rightarrow \quad 2xe^{\tan^{-1}y} = e^{2\tan^{-1}y} + 2c$$

$$\Rightarrow \quad 2xe^{\tan^{-1}y} = e^{2\tan^{-1}y} + k$$

where, $k = 2c$.

54. Find the particular solution of the following differential equation, given that $y = 0$ when $x = \frac{\pi}{4}$:

$$\frac{dy}{dx} + y\cot x = \frac{2}{1 + \sin x}$$

[CBSE Term-2 SQP 2022]

Ans. The differential equation is a linear differential equation.

$$IF = e^{\int \cot x dx} = e^{\log \sin x} = \sin x$$

The general solution is given by

$$y \sin x = \int 2\frac{\sin x}{1 + \sin x}dx$$

$$\Rightarrow \quad y \sin x = 2\int \frac{\sin x + 1 - 1}{1 + \sin x}dx$$

$$= 2\int \left(1 - \frac{1}{1 + \sin x}\right)dx$$

$$\Rightarrow \quad y \sin x = 2\int \left[1 - \frac{1}{1 + \cos\left(\frac{\pi}{2} - x\right)}\right]dx$$

$$\Rightarrow \quad y \sin x = 2\int \left[1 - \frac{1}{2\cos^2\left(\frac{\pi}{4} - \frac{x}{2}\right)}\right]dx$$

$$\Rightarrow \quad y \sin x = 2\int \left[1 - \frac{1}{2}\sec^2\left(\frac{\pi}{4} - \frac{x}{2}\right)\right]dx$$

$$\Rightarrow \quad y \sin x = 2\left[x + \tan\left(\frac{\pi}{4} - \frac{x}{2}\right)\right] + c$$

Given that $y = 0$, when $x = \frac{\pi}{4}$.

Hence, $\quad 0 = 2\left[\frac{\pi}{4} + \tan\frac{\pi}{8}\right] + c$

$$\Rightarrow \quad c = -\frac{\pi}{2} - 2\tan\frac{\pi}{8}$$

Hence, the particular solution is

$$y = \text{cosec } x\left[2\left\{x + \tan\left(\frac{\pi}{4} - \frac{x}{2}\right)\right\} - \left(\frac{\pi}{2} + 2\tan\frac{\pi}{8}\right)\right]$$

[CBSE Marking Scheme Term-2 SQP 2022]

Explanation: Given, differential equation

$$\frac{dy}{dx}+y\cot x = \frac{2}{1+\sin x}$$

The given differential equation is of the form

$$\frac{dy}{dx}+P(x) = Q(x)$$

Here, $P(x) = \cot x, Q(x) = \frac{2}{1+\sin x}$

Then, $\text{I.F.} = e^{\int P.dx} = e^{\int \cot x.dx}$

$$= e^{\log \sin x} = \sin x$$

Then, general solution is given by:

$$y.\text{ I.F.} = \int Q(x).\text{I.F. } dx$$

$$\Rightarrow \quad y \times \sin x = \int 2.\frac{\sin x}{1+\sin x}dx$$

$$\Rightarrow \quad y\sin x = 2\int \frac{\sin x+1-1}{1+\sin x}dx$$

$$\Rightarrow \quad y\sin x = 2\int\left[1-\frac{1}{1+\sin x}\right]dx$$

$$\Rightarrow \quad y\sin x = 2\int\left[1-\frac{1}{1+\cos\left(\frac{\pi}{2}-x\right)}\right]dx$$

$$\Rightarrow \quad y\sin x = 2\int\left[1-\frac{1}{2\cos^2\left(\frac{\pi}{4}-\frac{\pi}{2}\right)}\right]dx$$

$$\Rightarrow \quad y\sin x = 2\int\left[1-\frac{1}{2}\sec^2\left(\frac{\pi}{4}-\frac{x}{2}\right)\right]dx$$

$$\Rightarrow \quad y\sin x = 2\left[x+\tan\left(\frac{\pi}{4}-\frac{x}{2}\right)\right]+c$$

Here, $y = 0$, when $x = \frac{\pi}{4}$

$$\therefore \quad 0 = 2\left[\frac{\pi}{4}+\tan\left(\frac{\pi}{4}-\frac{\pi}{8}\right)\right]+c$$

$$\Rightarrow \quad 0 = 2\left[\frac{\pi}{4}+\tan\frac{\pi}{8}\right]+c$$

$$\Rightarrow \quad C = -\frac{\pi}{2}-2\tan\frac{\pi}{8}$$

$$\therefore \quad y\sin x = \left\{x+\tan\left(\frac{\pi}{4}-\frac{x}{2}\right)\right\} -\left\{\frac{\pi}{2}+2\tan\frac{\pi}{8}\right\}$$

$$y = \text{cosec}\,x\left[2\left\{x+\tan\left(\frac{\pi}{4}-\frac{x}{2}\right)\right\}\right] -\left\{\left(\frac{\pi}{2}+2\tan\frac{\pi}{3}\right)\right\}$$

Concept Applied:

- *The function $f(x, y)$ in a homogenious differential equation is a homogeneous function such that $f(\lambda x, \lambda y) = \lambda^n f(x, y)$ for any non-zero constant λ.*

55. **Find the particular solution of the differential equation**

$$\frac{dx}{dy}+x\cot y = 2y + y^2\cot y,\ y \neq 0, \text{ given that}$$

$x = 0$, when $y = \frac{\pi}{2}$. **[DIKSHA]**

56. **At any point (x, y) of a curve, the slope of the tangent is twice the slope of the line segment joining the point of contact to the point (–4, –3). Find the equation of the curve given that it passes through (–2, 1).**

Ans. It is given that (x, y) is the point of contact of tangent to the curve.

The slope of the line segment joining the points $(x_2, y_2) \equiv (x, y)$ and $(x_1, y_1) \equiv (-4, -3)$ is given as

$$\text{Slope of tangent} = \frac{y-(-3)}{x-(-4)} = \frac{y+3}{x+4}$$

Now, given that

$$\frac{dy}{dx} = 2\left(\frac{y+3}{x+4}\right)$$

$$\Rightarrow \quad \frac{dy}{y+3} = \left(\frac{2}{x+4}\right)dx$$

On integrating both sides we get,

$$\int\frac{dy}{y+3} = \int\frac{2}{x+4}dx$$

$$\Rightarrow \quad \log(y+3) = 2\log(x+4)+\log c$$

$$\Rightarrow \quad \log\frac{(y+3)}{(x+4)^2} = \log c$$

$$\Rightarrow \quad \frac{y+3}{(x+4)^2} = c \qquad ...(i)$$

Since, the curve passes through the point (– 2, 1) therefore, we have

$$\frac{(1+3)}{(-2+4)^2} = c \Rightarrow c = 1$$

On putting C = 1 in equation (i), we get

$$\Rightarrow \quad \frac{y+3}{(x+4)^2} = 1$$

$$\Rightarrow \quad y + 3 = (x+4)^2$$

which is the required equation of curve.

57. Solve the differential equation **$(1 + x)(1 + y^2)dx + (1 + y)(1 + x^2)dy = 0$**

58. Find the particular solution of the differential equation $e^x\sqrt{1-y^2}dx + \frac{y}{x}dy = 0$, given that $y = 1$, when $x = 0$. [CBSE 2014]

Ans. Given, differential equation is

$$e^x\sqrt{1-y^2}dx + \frac{y}{x}dy = 0$$

$$\Rightarrow e^x\sqrt{1-y^2}dx = \frac{-y}{x}dy$$

Now, separating the variables, we get

$$\Rightarrow \quad xe^x\,dx = \frac{-y}{\sqrt{1-y^2}}dy$$

On integrating both sides, we get

$$\int x e^x\,dx = \int \frac{-y}{\sqrt{1-y^2}}dy$$

Put $1 - y^2 = t$ in RHS

Then, $-2y\,dy = dt$ or $-y\,dy = \frac{dt}{2}$

$$\therefore \quad x\int e^x\,dx - \int\left[\frac{d}{dx}(x)\int e^x dx\right]dx = \int\frac{1}{2\sqrt{t}}dt$$

$$\Rightarrow \quad xe^x - \int e^x dx = \frac{1}{2}[2\sqrt{t}]$$

$$\Rightarrow \quad x\,e^x - e^x = \sqrt{t} + c$$

$$\Rightarrow \quad x\,e^x - e^x = \sqrt{1-y^2} + c$$

It is given that $y = 1$ when $x = 0$.

$$\therefore \quad 0 - e^0 = \sqrt{1-1} + c$$

$$\Rightarrow \quad c = -1$$

$$\therefore \quad x\,e^x - e^x = \sqrt{1-y^2} - 1$$

$$\Rightarrow \quad \sqrt{1-y^2} = e^x(x-1) + 1.$$

which is the required particular solution of the given differential equation.

LONG ANSWER Type Questions (LA)

[4 & 5 marks]

59. Solve: $y + \frac{d}{dx}(xy) = x(\sin x + \log x)$

[NCERT Exemplar]

Ans. Given:

$$y + \frac{d}{dx}(xy) = x(\sin x + \log x)$$

$$y + x\frac{dy}{dx} + y = x(\sin x + \log x)$$

$$x\frac{dy}{dx} + 2y = x(\sin x + \log x)$$

$$\frac{dy}{dx} + \frac{2y}{x} = \sin x + \log x$$

This is a linear differential equation of the type

$$\frac{dy}{dx} + Py = Q.$$

where, $P = \frac{2}{x}$ and $Q = \sin x + \log x$

$$\therefore \quad IF = e^{\int P.dx} = e^{\int\frac{2}{x}dx} = e^{2\log x}$$

$$= e^{\log x^2} = x^2$$

So, the general solution is:

$$y.IF = \int(Q.IF)dx + c$$

$$\Rightarrow \quad yx^2 = \int(\sin x + \log x)x^2\,dx + c$$

$$= \int x^2 \sin x\,dx + \int x^2 \log x\,dx + c \quad ...(i)$$

Now,

$$\int x^2 \sin x\,dx = x^2(-\cos x) + \int 2x\cos x\,dx$$

[Using integration by parts]

$$= -x^2\cos x + 2x\sin x - \int 2\sin x\,dx$$

$$= -x^2\cos x + 2x\sin x + 2\cos x \quad ...(ii)$$

and $\int x^2 \log x\,dx = \log x.\frac{x^3}{3} - \int\frac{1}{x}.\frac{x^3}{3}dx$

$$= \log x.\frac{x^3}{3} - \frac{1}{3}\int x^2\,dx$$

$$= \log x.\frac{x^3}{3} - \frac{1}{3}\frac{x^3}{3} \quad ...(iii)$$

Substitute the values of (ii) and (iii) in (i), we get

$$yx^2 = -x^2\cos x + 2x\sin x + 2\cos x + \frac{x^3}{3}\log x - \frac{x^3}{9} + c$$

$$\Rightarrow \quad y = -\cos x + \frac{2\sin x}{x} + \frac{2\cos x}{x^2} + \frac{x}{3}\log x - \frac{x}{9} + cx^2$$

⚠ Caution

➥ *Separately solve the integrals involved, as they have lengthy calculation.*

60. Solve the following differential equation

$$\left[y - x\cos\left(\frac{y}{x}\right)\right]dy + \left[y\cos\left(\frac{y}{x}\right) - 2x\sin\left(\frac{y}{x}\right)\right]dx$$ **[CBSE 2015]**

Ans. Given,

$$\left[y - x\cos\left(\frac{y}{x}\right)\right]dy + \left[y\cos\left(\frac{y}{x}\right) - 2x\sin\left(\frac{y}{x}\right)\right]dx = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{2x\sin\left(\frac{x}{y}\right) - y\cos\left(\frac{y}{x}\right)}{y - x\cos\left(\frac{y}{x}\right)}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{2\sin\left(\frac{x}{y}\right) - \frac{y}{x}\cos\left(\frac{y}{x}\right)}{\frac{y}{x} - \cos\left(\frac{y}{x}\right)} \quad ...(i)$$

Given equation is a homogeneous equation.

Put $\quad y = vx$

$$\Rightarrow \quad \frac{dy}{dx} = v + x\frac{dv}{dx}$$

∴ Equation (i) becomes

$$v + x\frac{dv}{dx} = \frac{2\sin v - v\cos v}{v - \cos v}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{2\sin v - v\cos v}{v - \cos v} - v$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{2\sin v - v^2}{v - \cos v}$$

$$\Rightarrow \quad \frac{v - \cos v}{2\sin v - v^2}dv = \frac{dx}{x}$$

On integrating both sides, we get

$$\Rightarrow \int -\frac{1}{2}\frac{(2\cos v - 2v)}{2\sin v - v^2}dv = \int \frac{dx}{x}$$

$$\Rightarrow \quad -\frac{1}{2}\log(2\sin v - v^2) = \log x + \log c$$

$$\Rightarrow \log x + \frac{1}{2}\log(2\sin v - v^2) = -\log c$$

$$\Rightarrow \log x^2 + \log(2\sin v - v^2) = -2\log c$$

$$\Rightarrow \log\left[x^2\left(2\sin\frac{y}{x} - \left(\frac{y}{x}\right)^2\right)\right] = -\log\left(\frac{1}{c^2}\right)$$

$$\Rightarrow 2x^2\sin\frac{y}{x} - y^2 = \left(\frac{1}{c^2}\right) = c' \text{ (say)}$$

which is the required solution.

61. Solve the following differential equation: $(1 + x^2)\, dy + 2xy\, dx = \cot x\, dx, x \neq 0$ **[CBSE 2012, 11]**

Ans. Given, differential equation is

$$(1 + x^2)dy + 2xy\, dx = \cot x\, dx$$

Dividing both sides by $(1 + x^2)\, dx$, we get

$$\frac{dy}{dx} + \frac{2x}{1+x^2}y = \frac{\cot x}{1+x^2} \quad ...(i)$$

which is a linear equation of the form

$$\frac{dy}{dx} + Py = Q$$

Here, $P = \frac{2x}{1+x^2}$, $Q = \frac{\cot x}{1+x^2}$

$$\text{Now, IF} = e^{\int P.dx} = e^{\int\frac{2x}{1+x^2}dx} = e^{\log(1+x^2)} = 1 + x^2$$

Then, the solution of the linear differential equation is given by

$$y \times \text{IF} = \int(Q \times \text{IF})dx + c$$

$$\Rightarrow \quad y(1 + x^2) = \int(1 + x^2)\frac{\cot x}{1+x^2}dx + c$$

$$\Rightarrow \quad y(1 + x^2) = \int\cot x\, dx + c$$

$$\Rightarrow \quad y(1 + x^2) = \log|\sin x| + c$$

$$\Rightarrow \quad y = \frac{\log|\sin x|}{1+x^2} + \frac{c}{1+x^2}$$

which is the required solution.

62. Find the particular solution of the differential equation $x^2\, dy + y(x + y)dx = 0$, given that $y = 1$ when $x = 1$.

63. Find the general solution of

$$\frac{dy}{dx} - 3y = \sin 2x.$$ **[NCERT Exemplar]**

TOPPER'S CORNER

VERY SHORT ANSWER Type Questions (VSA)

[**1** mark]

1. Find the order and the degree of the differential equation $x^2 \frac{d^2y}{dx^2} = \left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^4$

Ans.

$$x^2 \frac{d^2y}{dx^2} = \left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^4$$

ORDER = 2.

DEGREE = 1

[CBSE Topper 2019]