Shortcuts and Important Results to Remember

1 '0' is neither positive nor negative even integer, '2' is the only even prime number and all other prime numbers are odd, '1' (i.e. unity) is neither a composite nor a prime number and 1, −1 are two units in the set of integers.

2 (i) If $a > 0, b > 0$ and $a < b \Rightarrow a^2 < b^2$

(ii) If $a < 0, b < 0$ and $a < b \Rightarrow a^2 > b^2$

(iii) If $a_1, a_2, a_3, \dots, a_n \in R$ and $a_1^2 + a_2^2 + a_3^2 + \dots + a_n^2 = 0$

$\Rightarrow a_1 = a_2 = a_3 = \dots = a_n = 0$

3 (i) $\text{Max}(a, b) = \frac{1}{2}(|a+b| + |a-b|)$

(ii) $\text{Min}(a, b) = \frac{1}{2}(|a+b| - |a-b|)$

4 If the equation $f(x) = 0$ has two real roots α and β, then $f'(x) = 0$ will have a real root lying between α and β.

5 If two quadratic equations $P(x) = 0$ and $Q(x) = 0$ have an irrational common root, both roots will be common.

6 In the equation $ax^2 + bx + c = 0$ $[a, b, c \in R]$, if $a + b + c = 0$, the roots are $1, \frac{c}{a}$ and if $a - b + c = 0$, the roots are -1 and $\frac{c}{a}$.

7 The condition that the roots of $ax^2 + bx + c = 0$ may be in the ratio $p : q$, is

$pq\,b^2 = ac\,(p+q)^2$ (here, $\alpha : \beta = p : q$)

i.e., $\sqrt{\frac{p}{q}} + \sqrt{\frac{q}{p}} = \pm\sqrt{\frac{b^2}{ac}}$

(i) If one root of $ax^2 + bx + c = 0$ is n times that of the other, then $nb^2 = ac\,(n+1)^2$, here $\alpha : \beta = n : 1$.

(ii) If one root of $ax^2 + bx + c = 0$ is double of the other here $n = 2$, then $2b^2 = 9ac$.

8 If one root of $ax^2 + bx + c = 0$ is nth power of the other, then $(a^n c)^{\frac{1}{n+1}} + (ac^n)^{\frac{1}{n+1}} = -b$.

9 If one root of $ax^2 + bx + c = 0$ is square of the other, then $a^2c + ac^2 + b^3 = 3abc$.

10 If the ratio of the roots of the equation $ax^2 + bx + c = 0$ is equal to the ratio of the roots of $Ax^2 + Bx + C = 0$ and $a \neq 0, A \neq 0$, then $\frac{b^2}{ac} = \frac{B^2}{AC}$.

11 If sum of the roots is equal to sum of their squares then $2ac = ab + b^2$.

12 If sum of roots of $ax^2 + bx + c = 0$ is equal to the sum of their reciprocals, then

$2a^2c = ab^2 + bc^2$, i.e. ab^2, bc^2, ca^2 are in AP

or $\frac{2a}{b} = \frac{b}{c} + \frac{c}{a}$ i.e. $\frac{c}{a}, \frac{a}{b}, \frac{b}{c}$ are in AP.

13 Given, $y = ax^2 + bx + c$

(i) If $a > 0$, $y_{\min} = \frac{4ac - b^2}{4a}$

(ii) If $a < 0$, $y_{\max} = \frac{4ac - b^2}{4a}$

14 If α, β are the roots of $ax^2 + bx + c = 0$ and $S_n = \alpha^n + \beta^n$, then $aS_{n+1} + bS_n + cS_{n-1} = 0$.

15 If D_1 and D_2 are discriminants of two quadratics $P(x) = 0$ and $Q(x) = 0$, then

(i) If $D_1D_2 < 0$, then the equation $P(x) \cdot Q(x) = 0$ will have two real roots.

(ii) If $D_1D_2 > 0$, then the equation $P(x) \cdot Q(x) = 0$ has either four real roots or no real root.

(iii) If $D_1D_2 = 0$, then the equation $P(x) \cdot Q(x) = 0$ will have

(a) two equal roots and two distinct roots such that $D_1 > 0$ and $D_2 = 0$ or $D_1 = 0$ and $D_2 > 0$.

(b) only one real solution such that $D_1 < 0$ and $D_2 = 0$ or $D_1 = 0$ and $D_2 < 0$.

16 If $a > 0$ and $x = \sqrt{a + \sqrt{a + \sqrt{a + \dots + \infty}}}$, then $x = \frac{1 + \sqrt{(4a+1)}}{2}$.

17 If $a_1, a_2, a_3, \dots, a_n$ are positive real numbers, then least value of $(a_1 + a_2 + a_3 + \dots + a_n)\left(\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \dots + \frac{1}{a_n}\right)$ is n^2.

(i) Least value of $(a + b + c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) = 3^2 = 9$

(ii) Least value of

$(a + b + c + d)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}\right) = 4^2 = 16$

18 **Law of Proportions** If $\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = \dots$, then each of these ratios is also equal to

(i) $\frac{a + c + e + \dots}{b + d + f + \dots}$

(ii) $\left(\frac{pa^n + qc^n + re^n + \dots}{pb^n + qd^n + rf^n + \dots}\right)^{1/n}$ (where, $p, q, r, \dots, n \in R$)

(iii) $\frac{\sqrt{ac}}{\sqrt{bd}} = \frac{\sqrt[n]{(ace \dots)}}{\sqrt[n]{(bdf \dots)}}$

19 **Lagrange's Mean Value Theorem** Let $f(x)$ be a function defined on $[a, b]$ such that

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(i) $f(x)$ is continuous on $[a, b]$ and

(ii) $f(x)$ is derivable on (a, b), then $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

20 Lagrange's Identity If $a_1, a_2, a_3, b_1, b_2, b_3 \in R$, then

$(a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) - (a_1b_1 + a_2b_2 + a_3b_3)^2$
$= (a_1b_2 - a_2b_1)^2 + (a_2b_3 - a_3b_2)^2 + (a_3b_1 - a_1b_3)^2$

or $(a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) - (a_1b_1 + a_2b_2 + a_3b_3)^2$

$$= \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}^2 + \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}^2 + \begin{vmatrix} a_3 & a_1 \\ b_3 & b_1 \end{vmatrix}^2$$

Remark

If $(a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) \le (a_1b_1 + a_2b_2 + a_3b_3)^2$,

then $$\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3}.$$

21 Horner's Method of Synthetic, Division When, we divide a polynomial of degree ≥ 1 by a linear monic polynomial, the quotient and remainder can be found by this method. Consider

$f(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \dots + a_n$

where $a_0 \neq 0$ and $a_0, a_1, a_2, \dots, a_n \in R$.

Let $g(x) = (x - \alpha)$ be a linear monic polynomial $\alpha \in R$.

When $g(x) \mid f(x)$; we can find quotient and remainder as follows :

α	a_0	a_1	a_2	$\dots$	a_n
	0	αa_0	$b_1\alpha$		αb_{n-1}
	$a_0 = b_0$	$a_1 + \alpha a_0 = b_1$	$a_2 + b_1\alpha = b_2$		$a_n + \alpha b_{n-1} = 0$

$\therefore \quad f(x) = (x - \alpha)(b_0 x^{n-1} + b_1 x^{n-2} + b_2 x^{n-3} + \dots + b_{n-1})$

e.g. Find all roots of $x^3 - 6x^2 + 11x - 6 = 0$.

$\because (x - 1)$ is a factor of $x^3 - 6x^2 + 11x - 6$, then

$x = 1$	1	–6	11	–6
	0	1	–5	6
	1	–5	6	0

$\therefore \quad x^3 - 6x^2 + 11x - 6 = (x - 1)(x^2 - 5x + 6)$

$= (x - 1)(x - 2)(x - 3)$

Hence, roots of $x^3 - 6x^2 + 11x - 6 = 0$ are 1, 2 and 3.