Session 3

Adjoint of a Matrix, Inverse of a Matrix (Reciprocal Matrix), **Elementary Row Operations** (Transformations), **Equivalent Matrices, Matrix Polynomial, Use of Mathematical Induction,**

Adjoint of a Matrix

Let $A=[a_{ij}]$ be a square matrix of order n and let C_{ij} be cofactor of a_{ij} in A. Then, the transpose of the matrix of cofactors of elements of A is called the adjoint of A and is denoted by adj (A).

Thus, $\text{adj}(A)=[C_{ij}]'$

$\Rightarrow \quad (\text{adj } A)_{ij}=C_{ji}=\text{Cofactor of } a_{ji} \text{ in } A$

i.e. if $$A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \text{then}$$

$$\text{adj } A=\begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}'=\begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$$

where C_{ij} denotes the cofactor of a_{ij} in A.

Here, $$C_{11}=\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}=a_{22}a_{33}-a_{23}a_{32},$$

$$C_{12}=-\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}=a_{31}a_{23}-a_{33}a_{21},$$

$$C_{13}=\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}=a_{21}a_{32}-a_{31}a_{22},$$

$$C_{21}=-\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}=a_{13}a_{32}-a_{12}a_{33},$$

$$C_{22}=\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}=a_{11}a_{33}-a_{31}a_{13},$$

$$C_{23}=-\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}=a_{12}a_{31}-a_{11}a_{32},$$

$$C_{31}=\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}=a_{12}a_{23}-a_{22}a_{13},$$

$$C_{32}=-\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}=a_{13}a_{21}-a_{11}a_{23}$$

and $$C_{33}=\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}=a_{11}a_{22}-a_{21}a_{12}$$

Rule to Write Cofactors of an Element a_{ij}

Cross the row and column intersection at the element a_{ij} and the determinant which is left be denoted by D, then

$$\text{Cofactors of } a_{ij}=\begin{cases} D, & \text{if } i+j=\text{even integer} \\ -D, & \text{if } i+j=\text{odd integer} \end{cases}$$

Example 30. Find the cofactor of a_{23} in $\begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & -1 \\ 1 & -3 & 5 \end{bmatrix}$.

Sol. Let $$A=\begin{bmatrix} 3 & 1 & 4 \\ & & \vdots \\ 0 & \cdots 2 & \cdots -1 \\ & & \vdots \\ 1 & -3 & 5 \end{bmatrix}$$

$\therefore$ Cofactor of $a_{23}=-D$ $\quad [\because 2+3=\text{odd}]$

where $D=\begin{vmatrix} 3 & 1 \\ 1 & -3 \end{vmatrix}$

[after crossing the 2nd row and 3rd column]

$=-9-1=-10$

Hence, cofactor of $a_{23}=-(-10)=10$

Note

The adjoint of a square matrix of order 2 is obtained by interchanging the diagonal elements and changing signs of off-diagonal elements.

If $$A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}, \text{then}$$

$$(\text{adj } A)=\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Example 31. ***Find the adjoint of the matrix***

$$A=\begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 0 \\ 2 & 4 & 3 \end{bmatrix}.$$

Sol. If C be the matrix of cofactors of the element in $|A|$, then

$$C=\begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$$

$$= \begin{bmatrix} \begin{vmatrix} 5 & 0 \\ 4 & 3 \end{vmatrix} & -\begin{vmatrix} 0 & 0 \\ 2 & 3 \end{vmatrix} & \begin{vmatrix} 0 & 5 \\ 2 & 4 \end{vmatrix} \\ -\begin{vmatrix} 2 & 3 \\ 4 & 3 \end{vmatrix} & \begin{vmatrix} 1 & 3 \\ 2 & 3 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} \\ \begin{vmatrix} 2 & 3 \\ 5 & 0 \end{vmatrix} & -\begin{vmatrix} 1 & 3 \\ 0 & 0 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 0 & 5 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} 12 & 0 & -10 \\ 6 & -3 & 0 \\ -15 & 0 & 5 \end{bmatrix}$$

$$\Rightarrow \quad \text{adj}A = C' = \begin{bmatrix} 12 & 6 & -15 \\ 0 & -3 & 0 \\ -10 & 0 & 5 \end{bmatrix}$$

Maha Shortcut for Adjoint

(Goyal's Method)

This method applied only for third order square matrix.

Method : Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 0 \\ 2 & 4 & 3 \end{bmatrix}$

Step I Write down the three rows of A and rewrite first two rows.

i.e.

$$\begin{array}{ccc} 1 & 2 & 3 \\ 0 & 5 & 0 \\ 2 & 4 & 3 \\ 1 & 2 & 3 \\ 0 & 5 & 0 \end{array}$$

Step II After Step I, rewrite first two columns.

i.e.

$$\begin{array}{ccccc} 1 & 2 & 3 & 1 & 2 \\ 0 & 5 & 0 & 0 & 5 \\ 2 & 4 & 3 & 2 & 4 \\ 1 & 2 & 3 & 1 & 2 \\ 0 & 5 & 0 & 0 & 5 \end{array}$$

Step III After Step II, deleting first row and first column, then we get all elements of adj A i.e.,

$$\begin{array}{ccccccccc} 1 & \cdots & 2 & \cdots & 3 & \cdots & 1 & \cdots & 2 \\ \vdots \\ 0 & & 5 & & 0 & & 0 & & 5 \\ \vdots & & & & & & & & \\ 2 & & 4 & & 3 & & 2 & & 4 \\ \vdots & & & & & & & & \\ 1 & & 2 & & 3 & & 1 & & 2 \\ \vdots & & & & & & & & \\ 0 & & 5 & & 0 & & 0 & & 5 \end{array}$$

$\Rightarrow 15 \quad 0 \quad -10$ first column of adj A

$\Rightarrow 6 \quad -3 \quad 0$ second column of adj A

$\Rightarrow -15 \quad 0 \quad 5$ third column of adj A

$$\text{or} \quad \text{adj } A = \begin{bmatrix} 15 & 6 & -15 \\ 0 & -3 & 0 \\ -10 & 0 & 5 \end{bmatrix}$$

Properties of Adjoint Matrix

Property 1 If A be a square matrix order n, then

$$A(\text{adj } A) = (\text{adj } A)A = |A| I_n$$

i.e., the product of a matrix and its adjoint is commutative.

Deductions of Property 1

Deduction 1 If A be a square singular matrix of order n, then $\quad A(\text{adj } A) = (\text{adj } A)\, A = O \quad$ [null matrix]

Since, for singular matrix, $|A| = 0$.

Deduction 2 If A be a square non-singular matrix of order n, then $\quad |\text{adj } A| = |A|^{n-1}$

Since, for non-singular matrix, $|A| \neq 0$.

Proof $\because \quad A(\text{adj } A) = |A| I_n$

Taking determinant on both sides, then

$$|A(\text{adj } A)| = ||A| I_n|$$

$$\Rightarrow \quad |A||\text{adj } A| = |A|^n |I_n| = |A|^n \quad [\because |I_n| = 1]$$

$$\Rightarrow \quad |\text{adj } A| = |A|^{n-1} \quad [\because |A| \neq 0]$$

Note

In general $\underbrace{|\text{adj } (\text{adj } (\text{adj} \ldots (\text{adj } A)))| = |A|}_{\text{adj repeat } m \text{ times}}{}^{(n-1)^m}$

Property 2 If A and B are square matrices of order n, then

$$\text{adj } (AB) = (\text{adj } B)(\text{adj } A)$$

Property 3 If A is a square matrix of order n, then

$$(\text{adj } A)' = \text{adj } A'$$

Property 4 If A be a square non-singular matrix of order n, then $\quad \text{adj } (\text{adj} A) = |A|^{n-2} A$

Proof $\because \quad A(\text{adj } A) = |A| I_n \quad \ldots\text{(i)}$

Replace A by adj A, then

$$(\text{adj } A)(\text{adj } (\text{adj } A)) = |\text{adj } A| I_n$$
$$= |A|^{n-1} I_n \quad [\because |\text{adj } A| = |A|^{n-1}]$$
$$= I_n |A|^{n-1}$$

Pre-multiplying both sides by matrix A, then

$$A(\text{adj } A)(\text{adj } (\text{adj } A)) = AI_n |A|^{n-1} = A|A|^{n-1}$$

$$\Rightarrow \quad |A| I_n (\text{adj}(\text{ adj} A)) = A|A|^{n-1}$$

$$\Rightarrow \quad (\text{adj } (\text{adj } A)) = A|A|^{n-2}$$

$$\text{or} \quad \text{adj } (\text{adj } A) = |A|^{n-2} A$$

Property 5 If A be a square non-singular matrix of order n, then

$$|\text{adj } (\text{adj } A)| = |A|^{(n-1)^2}$$

Proof $\because \quad \text{adj } (\text{adj } A) = |A|^{n-2} A$

Taking determinant on both sides, then

$$|\text{adj } (\text{adj } A)| = ||A|^{n-2} A|$$

$$= |A|^{n(n-2)}\,|A| \qquad [\because |kA| = k^n|A|]$$
$$= |A|^{n^2-2n+1} = |A|^{(n-1)^2}$$

Note

In general, $\underbrace{|\text{adj}\,(\text{adj}\,(\text{adj}\ \ldots\,(\text{adj}\,A)))|}_{\text{adj } m \text{ times}} = |A|^{(n-1)^m}$

Property 6 If A be a square matrix of order n and k is a scalar, then

$$\text{adj}\,(kA) = k^{n-1}\cdot(\text{adj}\,A)$$

Proof $\because$ $\quad A(\text{adj}\,A) = |A|I_n \qquad$ ...(i)

Replace A by kA, then

$$kA(\text{adj}\,(kA)) = |kA|I_n = k^n|A|I_n$$
$$\Rightarrow \quad A(\text{adj}\,(kA)) = k^{n-1}|A|I_n$$
$$= k^{n-1}A(\text{adj}\,A) \qquad [\text{from Eq. (i)}]$$

Hence, $\quad \text{adj}\,(kA) = k^{n-1}(\text{adj}\,A)$

Property 7 If A be a square matrix of order n and $m \in N$, then $\quad (\text{adj}\,A^m) = (\text{adj}\,A)^m$

Property 8 If A and B be two square matrices of order n such that B is the adjoint of A and k is a scalar, then

$$|AB + kI_n| = (|A| + k)^n$$

Proof $\because$ $\quad B = \text{adj}\ A$

$$\therefore \quad AB = A(\text{adj}\ A) = |A|I_n$$
$$\text{LHS} = |AB + kI_n| = ||A|I_n + kI_n| = |(|A| + k)I_n|$$
$$= (|A| + k)^n|I_n| = (|A| + k)^n = \text{RHS}$$

Property 9 Adjoint of a diagonal matrix is a diagonal matrix.

i.e. If $A = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$, then $\text{adj}\,A = \begin{bmatrix} bc & 0 & 0 \\ 0 & ca & 0 \\ 0 & 0 & ab \end{bmatrix}$

Note

$\text{adj}\,(I_n) = I_n$.

Example 32. If $A = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$, find the values of

(i) $|A||\text{adj}\ A|$ (ii) $|\text{adj}(\text{adj}\ (\text{adj}\ A))|$
(iii) $|\text{adj}\ (3A)|$ (iv) $\text{adj}\ (\text{adj}\ A)$

Sol. $\because$ $\quad A = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$

$$\therefore |A| = (-1)(1-1) - (1)(-1-1) + (1)(1+1) = 4 \neq 0$$

$\Rightarrow$ A is non-singular.

(i) $|A||\text{adj}\ A| = |A||A|^2 \qquad [\because n = 3]$

$$= |A|^3 = 4^3 = 64$$

(ii) $|\text{adj}\,(\text{adj}\,(\text{adj}\ A))| = |A|^{(3-1)^3} = |A|^8 = 4^8 = 2^{16}$

(iii) $|\text{adj}\,(3A)| = |3^2\ \text{adj}\ A| = (3^2)^3|\text{adj}\ A|$

$$= 3^6|A|^2 = 729 \times 4^2 = 11664$$

(iv) $\text{adj}(\ \text{adj}\,A) = |A|^2\,A = 16A$

Example 33. If $A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$ and B is the adjoint of A, find the value of $|AB + 2I|$, where I is the identity matrix of order 3.

Sol. $\because$ $\quad A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$

$$\therefore \quad |A| = \begin{vmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{vmatrix}$$
$$= 3(-3+4) + 3(2-0) + 4(-2-0) = 1 \neq 0$$
$$\therefore\ |AB + 2I| = (|A| + 2)^3 \qquad [\text{by property 8}]$$
$$= (1+2)^3 = 3^3 = 27$$

Inverse of a Matrix
(Reciprocal Matrix)

A square matrix A (non-singular) of order n is said to be invertible, if there exists a square matrix B of the same order such that $\quad AB = I_n = BA$,

then B is called the inverse (reciprocal) of A and is denoted by A^{-1}. Thus, $\quad A^{-1} = B \Leftrightarrow AB = I_n = BA$

We have, $\quad A(\text{adj}\ A) = |A|I_n$

$$\Rightarrow \quad A^{-1}A(\text{adj}\ A) = A^{-1}I_n|A|$$
$$\Rightarrow \quad I_n(\text{adj}\ A) = A^{-1}|A|I_n$$
$$\therefore \quad A^{-1} = \frac{\text{adj}\,A}{|A|};\ \text{provided}\ |A| \neq 0$$

Note The necessary and sufficient condition for a square matrix A to be invertible is that $|A| \neq 0$.

Properties of Inverse of a Matrix

Property 1 (Uniqueness of inverse) Every invertible matrix possesses a unique inverse.

Proof Let A be an invertible matrix of order $n \times n$. Let B and C be two inverses of A. Then,

$$AB = BA = I_n \qquad \text{...(i)}$$

and $\quad AC = CA = I_n \quad$ ...(ii)

Now, $\quad AB = I_n$

$\Rightarrow \quad C(AB) = CI_n \quad$ [pre-multiplying by C]

$\Rightarrow \quad (CA)B = CI_n \quad$ [by associativity]

$\Rightarrow \quad I_n B = CI_n \quad$ [$\because CA = I_n$ by Eq. (ii)]

$\Rightarrow \quad B = C$

Hence, an invertible matrix possesses a unique inverse.

Property 2 (Reversal law) If A and B are invertible matrices of order $n \times n$, then AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$.

Proof It is given that A and B are invertible matrices.

$$|A| \neq 0 \text{ and } |B| \neq 0 \Rightarrow |A||B| \neq 0$$

$\Rightarrow \quad |AB| \neq 0$

Hence, AB is an invertible matrix.

Now, $(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1}$ [by associativity]

$= (AI_n)A^{-1}$ [$\because BB^{-1} = I_n$]

$= AA^{-1}$ [$\because AI_n = A$]

$= I_n$ [$\because AA^{-1} = I_n$]

Also, $(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B$ [by associativity]

$= B^{-1}(I_n B)$ [$\because A^{-1}A = I_n$]

$= B^{-1}B$ [$\because I_n B = B$]

$= I_n$ [$\because B^{-1}B = I_n$]

Thus, $(AB)(B^{-1}A^{-1}) = I_n = (B^{-1}A^{-1})(AB)$

Hence, $(AB)^{-1} = B^{-1}A^{-1}$

Note

If $A, B, C, \ldots, Y, Z$ are invertible matrices, then

$(ABC \ldots YZ)^{-1} = Z^{-1}Y^{-1} \ldots C^{-1}B^{-1}A^{-1}$ [reversal law]

Property 3 Let A be an invertible matrix of order n, then A' is also invertible and $(A')^{-1} = (A^{-1})'$.

Proof $\because A$ is invertible matrix

$\therefore \quad |A| \neq 0 \Rightarrow |A'| \neq 0 \quad$ [$\because |A| = |A'|$]

Hence, A^{-1} is also invertible.

Now, $\quad AA^{-1} = I_n = A^{-1}A$

$\Rightarrow \quad (AA^{-1})' = (I_n)' = (A^{-1}A)'$

$\Rightarrow \quad (A^{-1})'A' = I_n = A'(A^{-1})'$

[by reversal law for transpose]

$\Rightarrow \quad (A')^{-1} = (A^{-1})' \quad$ [by definition of inverse]

Property 4 Let A be an invertible matrix of order n and $k \in N$, then

$$(A^k)^{-1} = (A^{-1})^k = A^{-k}$$

Proof We have,

$$(A^k)^{-1} = \underbrace{(A \times A \times A \times \ldots \times A)^{-1}}_{\text{repeat } k \text{ times}}$$

$$= \underbrace{A^{-1} \times A^{-1} \times A^{-1} \times \ldots \times A^{-1}}_{\text{repeat } k \text{ times}}$$

[by reversal law for inverse]

$$= (A^{-1})^k = A^{-k}$$

Property 5 Let A be an invertible matrix of order n, then $(A^{-1})^{-1} = A$.

Proof We have, $\quad A^{-1}A = I_n$

$\therefore$ Inverse of $\quad A^{-1} = A \Rightarrow (A^{-1})^{-1} = A$

Note

$I_n^{-1} = I_n$ as $I_n^{-1}\, I_n = I_n$

Property 6 Let A be an invertible matrix of order n, then $|A^{-1}| = \dfrac{1}{|A|}$.

Proof $\because A$ is invertible, then $|A| \neq 0$.

Now, $\quad AA^{-1} = I_n = A^{-1}A$

$\Rightarrow \quad |AA^{-1}| = |I_n|$

$\Rightarrow \quad |A||A^{-1}| = 1$

[$\because |AB| = |A||B|$ and $|I_n| = 1$]

$\Rightarrow \quad |A^{-1}| = \dfrac{1}{|A|} \quad$ [$\because |A| \neq 0$]

Property 7 Inverse of a non-singular diagonal matrix is a diagonal matrix.

i.e. If $A = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$ and $|A| \neq 0$, then

$$A^{-1} = \begin{bmatrix} \frac{1}{a} & 0 & 0 \\ 0 & \frac{1}{b} & 0 \\ 0 & 0 & \frac{1}{c} \end{bmatrix}$$

Note

The inverse of a non-singular square matrix A of order 2 is obtained by interchanging the diagonal elements and changing signs of off-diagonal elements and dividing by $|A|$.

For example,

If $\quad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $|A| = (ad - bc) \neq 0$, then

$$A^{-1} = \frac{1}{(ad - bc)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Example 34. Compute the inverse of the matrix $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$.

Sol. We have, $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$

Then, $|A| = \begin{vmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{vmatrix} = 0 \cdot (2-3) - 1(1-9) + 2(1-6)$

$= -2 \neq 0$

$\therefore$ A^{-1} exists.

Now, cofactors along $R_1 = -1, 8, -5$

cofactors along $R_2 = 1, -6, 3$

cofactors along $R_3 = -1, 2, -1$

Let C is a matrix of cofactors of the elements in $|A|$

$$\therefore \quad C = \begin{bmatrix} -1 & 8 & -5 \\ 1 & -6 & 3 \\ -1 & 2 & -1 \end{bmatrix}$$

$$\therefore \quad \text{adj } A = C' = \begin{bmatrix} -1 & 1 & -1 \\ 8 & -6 & 2 \\ -5 & 3 & -1 \end{bmatrix}$$

Hence, $$A^{-1} = \frac{\text{adj } A}{|A|} = -\frac{1}{2}\begin{bmatrix} -1 & 1 & -1 \\ 8 & -6 & 2 \\ -5 & 3 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -4 & 3 & -1 \\ \frac{5}{2} & -\frac{3}{2} & \frac{1}{2} \end{bmatrix}$$

Example 35. If A and B are symmetric non-singular matrices of same order, $AB = BA$ and $A^{-1}B^{-1}$ exist, prove that $A^{-1}B^{-1}$ is symmetric.

Sol. $\because$ $A' = A$, $B' = B$ and $|A| \neq 0$, $|B|' \neq 0$

$\therefore$ $(A^{-1}B^{-1})' = (B^{-1})'(A^{-1})'$ [by reversal law of transpose]

$= (B')^{-1}(A')^{-1}$ [by property 3]

$= B^{-1}A^{-1}$ [$\because A' = A$ and $B' = B$]

$= (AB)^{-1}$ [by reversal law of inverse]

$= (BA)^{-1}$ [$\because AB = BA$]

$= A^{-1}B^{-1}$ [by reversal law of inverse]

Hence, $A^{-1}B^{-1}$ is symmetric.

Example 36. Matrices A and B satisfy $AB = B^{-1}$, where $B = \begin{bmatrix} 2 & -2 \\ -1 & 0 \end{bmatrix}$, find the value of λ for which $\lambda A - 2B^{-1} + I = O$, without finding B^{-1}.

Sol. $\because$ $AB = B^{-1}$ or $AB^2 = I$

Now, $\lambda A - 2B^{-1} + I = O$

$\Rightarrow$ $\lambda AB - 2B^{-1}B + IB = O$ [post-multiplying by B]

$\Rightarrow$ $\lambda AB - 2I + B = O$

$\Rightarrow$ $\lambda AB^2 - 2IB + B^2 = O$ [again post-multiplying by B]

$\Rightarrow$ $\lambda AB^2 - 2B + B^2 = O$

$\Rightarrow$ $\lambda I - 2B + B^2 = O$ [$\because AB^2 = I$]

$$\Rightarrow \lambda\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - 2\begin{bmatrix} 2 & -2 \\ -1 & 0 \end{bmatrix} + \begin{bmatrix} 2 & -2 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 2 & -2 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 4 & -4 \\ -2 & 0 \end{bmatrix} + \begin{bmatrix} 6 & -4 \\ -2 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \lambda + 2 & 0 \\ 0 & \lambda + 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$\Rightarrow$ $\lambda + 2 = 0$

$\therefore$ $\lambda = -2$

Example 37. If A, B and C are three non-singular square matrices of order 3 satisfying the equation $A^2 = A^{-1}$ and let $B = A^8$ and $C = A^2$, find the value of det $(B - C)$.

Sol. $\because$ $B = A^8 = (A^2)^4 = (A^{-1})^4$ [$\because A^{-1} = A^2$]

$= (A^4)^{-1} = (A^{2 \cdot 2})^{-1}$

$= ((A^2)^2)^{-1} = ((A^2)^{-1})^2$

$= ((A^{-1})^{-1})^2 = A^2 = C$

So, $B = C \Rightarrow B - C = 0$

$\therefore$ $\det(B - C) = 0$

Elementary Row Operations

(Transformations)

The following three types of operations (transformations) on the rows of a given matrix are known as elementary row operation (transformations).

(i) The interchange of ith and jth rows is denoted by $R_i \leftrightarrow R_j$ or R_{ij}.

(ii) The multiplication of the ith row by a constant $k (k \neq 0)$ is denoted by $R_i \rightarrow kR_i$ or $R_i(k)$

(iii) The addition of the ith row to the elements of the jth row multiplied by constant $k(k \neq 0)$ is denoted by $R_i \to R_i + kR_j$ or $R_{ij}(k)$.

Note

Similarly, we can define the three column operations, $C_{ij}(C_i \leftrightarrow C_j)$, $C_i(k)(C_i \to kC_i)$ and $C_{ij}(k)(C_i \to C_i + kC_j)$.

Equivalent Matrices

Two matrices are said to be equivalent if one is obtained from the other by elementary operations (transformations). The symbol ~ is used for equivalence.

Properties of Equivalent Matrices

(i) If A and B are equivalent matrices, there exist non-singular matrices P and Q such that $B = PAQ$

(ii) If A and B are equivalent matrices such that $B = PAQ$, then $P^{-1}BQ^{-1} = A$

(iii) Every non-singular square matrix can be expressed as the product of elementary matrices.

Example 38. Transform $\begin{bmatrix} 1 & 3 & 3 \\ 2 & 4 & 10 \\ 3 & 8 & 4 \end{bmatrix}$ into a unit matrix.

Sol. Let $$A = \begin{bmatrix} 1 & 3 & 3 \\ 2 & 4 & 10 \\ 3 & 8 & 4 \end{bmatrix}$$

Applying $R_2 \to R_2 - 2R_1$ and $R_3 \to R_3 - 3R_1$, we get

$$A \sim \begin{bmatrix} 1 & 3 & 3 \\ 0 & -2 & 4 \\ 0 & -1 & -5 \end{bmatrix}$$

Applying $R_2 \to \left(-\dfrac{1}{2}\right)R_2$ and $R_2 \to (-1)R_2$, we get

$$A \sim \begin{bmatrix} 1 & 3 & 3 \\ 0 & 1 & -2 \\ 0 & 1 & 5 \end{bmatrix}$$

Applying $R_1 \to R_1 - 3R_2$ and $R_3 \to R_3 - R_2$, we get

$$A \sim \begin{bmatrix} 1 & 0 & 9 \\ 0 & 1 & -2 \\ 0 & 0 & 7 \end{bmatrix}$$

Applying $R_3 \to \left(\dfrac{1}{7}\right)R_3$, we get

$$A \sim \begin{bmatrix} 1 & 0 & 9 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

Applying $R_1 \to R_1 - 9R_3$ and $R_2 \to R_2 + 2R_3$, we get

$$A \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Hence, $A \sim I$

Example 39. Given $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 4 & 1 \\ 2 & 3 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}$. Find P such that $BPA = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$.

Sol. Given, $$BPA = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\therefore \quad P = B^{-1}\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}A^{-1} \quad \text{...(i) [by property]}$$

$$\because \quad B = \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix} \Rightarrow B^{-1} = \frac{1}{(-1)}\begin{bmatrix} 4 & -3 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} -4 & 3 \\ 3 & -2 \end{bmatrix}$$

$$\therefore \quad B^{-1} = \begin{bmatrix} -4 & 3 \\ 3 & -2 \end{bmatrix} \quad \text{...(ii)}$$

and $$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 4 & 1 \\ 2 & 3 & 1 \end{bmatrix}$$

$\therefore \quad |A| = 1(4-3) - 1(2-2) + 1(6-8) = -1 \neq 0$

$\Rightarrow \quad A^{-1}$ exists.

Now, $$\text{adj } A = \begin{bmatrix} 1 & 2 & -3 \\ 0 & -1 & 1 \\ -2 & -1 & 2 \end{bmatrix} \quad \text{[by shortcut method]}$$

$$\therefore \quad A^{-1} = \frac{\text{adj } A}{|A|} = \begin{bmatrix} -1 & -2 & 3 \\ 0 & 1 & -1 \\ 2 & 1 & -2 \end{bmatrix} \quad \text{...(iii)}$$

Substituting the values of A^{-1} and B^{-1} from Eqs. (ii) and (iii) in Eq. (i), then

$$P = \begin{bmatrix} -4 & 3 \\ 3 & -2 \end{bmatrix}\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} -1 & -2 & 3 \\ 0 & 1 & -1 \\ 2 & 1 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} -4 & 3 \\ 3 & -2 \end{bmatrix}\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix} = \begin{bmatrix} -4 & 7 & -7 \\ 3 & -5 & 5 \end{bmatrix}$$

To Compute the Inverse of a Non-Singular Matrix by Elementary Operations

(Gauss-Jordan Method)

If A be a non-singular matrix of order n, then write $A = I_n A$.

If A is reduced to I_n by elementary operations (LHS), then suppose I_n is reduced to P(RHS) and not change A in RHS, then after elementary operations, we get

$$I_n = PA,$$

then P is the inverse of A

$$\therefore \quad P = A^{-1}$$

Example 40. ***Find the inverse of the matrix*** $\begin{bmatrix} 1 & 2 & 5 \\ 2 & 3 & 1 \\ -1 & 1 & 1 \end{bmatrix}$***, using elementary row operations.***

Sol. Let $A = \begin{bmatrix} 1 & 2 & 5 \\ 2 & 3 & 1 \\ -1 & 1 & 1 \end{bmatrix}$

$$\therefore \quad |A| = \begin{vmatrix} 1 & 2 & 5 \\ 2 & 3 & 1 \\ -1 & 1 & 1 \end{vmatrix} = 1(3-1) - 2(2+1) + 5(2+3) = 21 \neq 0$$

$\therefore \quad A^{-1}$ exists.

We write $A = IA$

$$\Rightarrow \quad \begin{bmatrix} 1 & 2 & 5 \\ 2 & 3 & 1 \\ -1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

Applying $R_2 \to R_2 - 2R_1$ and $R_3 \to R_3 + R_1$, we get

$$\begin{bmatrix} 1 & 2 & 5 \\ 0 & -1 & -9 \\ 0 & 3 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} A$$

Applying $R_2 \to (-1)R_2$ and $R_3 \to \left(\frac{1}{3}\right)R_3$, we get

$$\begin{bmatrix} 1 & 2 & 5 \\ 0 & 1 & 9 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & -1 & 0 \\ \frac{1}{3} & 0 & \frac{1}{3} \end{bmatrix} A$$

Applying $R_1 \to R_1 - 2R_2$ and $R_3 \to R_3 - R_2$, we get

$$\begin{bmatrix} 1 & 0 & -13 \\ 0 & 1 & 9 \\ 0 & 0 & -7 \end{bmatrix} = \begin{bmatrix} -3 & 2 & 0 \\ 2 & -1 & 0 \\ -\frac{5}{3} & 1 & \frac{1}{3} \end{bmatrix} A$$

Applying $R_3 \to \left(-\frac{1}{7}\right)R_3$, we get

$$\begin{bmatrix} 1 & 0 & -13 \\ 0 & 1 & 9 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -3 & 2 & 0 \\ 2 & -1 & 0 \\ \frac{5}{21} & -\frac{1}{7} & -\frac{1}{21} \end{bmatrix} A$$

Applying $R_2 \to R_2 - 9R_3$ and $R_1 \to R_1 + 13R_3$, we get

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{2}{21} & \frac{1}{7} & -\frac{13}{21} \\ -\frac{1}{7} & \frac{2}{7} & \frac{3}{7} \\ \frac{5}{21} & -\frac{1}{7} & -\frac{1}{21} \end{bmatrix} A$$

Hence,
$$A^{-1} = \begin{bmatrix} \frac{2}{21} & \frac{1}{7} & -\frac{13}{21} \\ -\frac{1}{7} & \frac{2}{7} & \frac{3}{7} \\ \frac{5}{21} & -\frac{1}{7} & -\frac{1}{21} \end{bmatrix}$$

Matrix Polynomial

Let $f(x) = a_0 x^m + a_1 x^{m-1} + a_2 x^{m-2} + \ldots + a_{m-1}x + a_m$ be a polynomial in x and let $A = [a_{ij}]_{n \times n}$, then expression of the form

$f(A) = a_0 A^m + a_1 A^{m-1} + a_2 A^{m-2} + \ldots + a_{m-1} A + a_m I_n$ is called a matrix polynomial.

Thus, to obtain $f(A)$ replace x by A in $f(x)$ and the constant term is multiplied by the identity matrix of order equal to that of A.

For example, If $f(x) = x^2 - 7x + 32$ is a polynomial in x and A is a square matrix of order 3, then $f(A) = A^2 - 7A + 32 I_3$ is a matrix polynomial.

Note

1. The polynomial equation $f(x) = 0$ is satisfied by the matrix $A = [a_{ij}]_{n \times n}$, then $f(A) = 0$.
2. Let $A = [a_{ij}]_{n \times n}$ satisfies the equation $a_0 + a_1 x + a_2 x^2 + \ldots + a_r x^r = 0$, then A is invertible of $a_0 \neq 0, |A| = 0$ and its inverse is given by
$$A^{-1} = \frac{1}{a_0}(a_1 I_n + a_2 A + \ldots + a_r A^{r-1}).$$

Example 41. If $A = \begin{bmatrix} k & l \\ m & n \end{bmatrix}$ and $kn \neq lm$, show that $A^2 - (k+n)A + (kn - lm)I = O$. Hence, find A^{-1}.

Sol. We have, $A = \begin{bmatrix} k & l \\ m & n \end{bmatrix}$, then $|A| = \begin{vmatrix} k & l \\ m & n \end{vmatrix}$

$= kn - ml \neq 0$ [given]

$\therefore$ A^{-1} exists.

Now, $A^2 = A \cdot A = \begin{bmatrix} k & l \\ m & n \end{bmatrix}\begin{bmatrix} k & l \\ m & n \end{bmatrix} = \begin{bmatrix} k^2 + lm & kl + ln \\ mk + nm & ml + n^2 \end{bmatrix}$

$\therefore$ $A^2 - (k+n)A + (kn - lm)I$

$$= \begin{bmatrix} k^2 + lm & kl + ln \\ mk + nm & ml + n^2 \end{bmatrix} - (k+n)\begin{bmatrix} k & l \\ m & n \end{bmatrix} + (kn - lm)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} k^2 + lm & kl + ln \\ mk + nm & ml + n^2 \end{bmatrix} - \begin{bmatrix} k^2 + nk & kl + nl \\ km + nm & kn + n^2 \end{bmatrix} + \begin{bmatrix} kn - lm & 0 \\ 0 & kn - lm \end{bmatrix}$$

$$= \begin{bmatrix} k^2 + lm - k^2 - nk + kn - lm & kl + ln - kl - ln \\ mk + nm - km - nm & ml + n^2 - kn - n^2 + kn - lm \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O$$

As $A^2 - (k+n)\,A + (kn - lm)\,I = O$

$\Rightarrow$ $(kn - lm)I = (k+n)\,A - A^2$

$\Rightarrow$ $(kn - lm)IA^{-1} = ((k+n)\,A - A^2)A^{-1}$

$\Rightarrow$ $(kn - lm)A^{-1} = (k+n)\,AA^{-1} - A(AA^{-1})$

$= (k+n)I - AI$ $[\because AA^{-1} = I]$

$= (k+n)\,I - A$

$= (k+n)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} k & l \\ m & n \end{bmatrix}$

$= \begin{bmatrix} k+n & 0 \\ 0 & k+n \end{bmatrix} - \begin{bmatrix} k & l \\ m & n \end{bmatrix}$

$\Rightarrow$ $(kn - lm)A^{-1} = \begin{bmatrix} n & -l \\ -m & k \end{bmatrix}$

Hence, $A^{-1} = \dfrac{1}{(kn - lm)}\begin{bmatrix} n & -l \\ -m & k \end{bmatrix}$

Example 42. If $A = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$, find the value of $|a| + |b|$ such that $A^2 + aA + bI = O$. Hence, find A^{-1}.

Sol. We have, $A = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$, then $|A| = \begin{vmatrix} 3 & 1 \\ 2 & 1 \end{vmatrix} = 3 - 2 = 1 \neq 0$

$\therefore$ A^{-1} exists.

Now, $A^2 = A \cdot A = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 11 & 4 \\ 8 & 3 \end{bmatrix}$

Since, $A^2 + aA + bI = O$

$\Rightarrow$ $\begin{bmatrix} 11 & 4 \\ 8 & 3 \end{bmatrix} + a\begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} + b\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\Rightarrow$ $\begin{bmatrix} 11 + 3a + b & 4 + a \\ 8 + 2a & 3 + a + b \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Equating the corresponding elements, we get

$11 + 3a + b = 0$ ...(i)

$4 + a = 0$ ...(ii)

$8 + 2a = 0$ ...(iii)

$3 + a + b = 0$ ...(iv)

From Eqs. (ii) and (iv), we get $a = -4$ and $b = 1$

$\therefore$ $|a| + |b| = |-4| + |1| = 4 + 1 = 5$

As $A^2 + aA + bI = O$

$\Rightarrow$ $A^2 - 4A + I = O \Rightarrow I = 4A - A^2$

$\Rightarrow$ $IA^{-1} = (4A - A^2)\,A^{-1}$

$\Rightarrow$ $A^{-1} = 4(AA^{-1}) - A(AA^{-1})$

$= 4I - AI = 4I - A$

$= 4\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} - \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$

$\therefore$ $A^{-1} = \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}$

Use of Mathematical Induction

Example 43. Let $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, show that $(aI + bA)^n = a^n I + na^{n-1}\,bA,\ \forall\ n \in N.$

Sol. Let $P(n) : (aI + bA)^n = a^n I + na^{n-1}bA$

Step I For $n = 1$,

LHS $= (aI + bA)^1 = aI + bA$

and RHS $= a^1\ I + 1 \cdot a^0\ bA = aI + bA$

LHS $=$ RHS

Therefore, $P(1)$ is true.

Step II Assume that $P(k)$ is true, then

$P(k) : (aI + bA)^k = a^k\ I + ka^{k-1}\ bA$

Step III For $n = k + 1$, we have to prove that

$P(k+1) : (aI + bA)^{k+1} = a^{k+1}\ I + (k+1)\ a^k bA$

LHS $= (aI + bA)^{k+1} = (aI + bA)^k\ (aI + bA)$

$= (a^k I + ka^{k-1}bA)\,(aI + bA)$ [from step II]

$= a^{k+1}\ I^2 + a^k b\,(IA) + ka^k b\,(AI) + k\ a^{k-1}b^2\ A^2$

$= a^{k+1}\ I + (k+1)a^k b\ A + 0$

$[\because AI = A,\ A^2 = 0 \text{ and } I^2 = I\ ]$

$= a^{k+1}I + (k+1)a^k bA =$ RHS

Therefore, $P(k+1)$ is true.

Hence, by the principal of mathematical2 induction $P(n)$ is true for all $n \in N$.

Example 44. *If* $A = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$, *use mathematical induction to show that* $A^n = \begin{bmatrix} 1+2n & -4n \\ n & 1-2n \end{bmatrix}$, $\forall$ $n \in N$.

Sol. Let $P(n): A^n = \begin{bmatrix} 1+2n & -4n \\ n & 1-2n \end{bmatrix}$

Step I For $n = 1$,

LHS $= A^1 = A$

and RHS $= \begin{bmatrix} 1+2 & -4 \\ 1 & 1-2 \end{bmatrix} = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix} = A$

$\Rightarrow$ LHS = RHS

Therefore, $P(1)$ is true.

Step II Assume that $P(k)$ is true, then

$$P(k): A^k = \begin{bmatrix} 1+2k & -4k \\ k & 1-2k \end{bmatrix}$$

Step III For $n = k + 1$, we have to prove that

$$P(k+1): A^{k+1} = \begin{bmatrix} 3+2k & -4(k+1) \\ k+1 & -1-2k \end{bmatrix}$$

$$\text{LHS} = A^{k+1} = A^k \cdot A$$

$$= \begin{bmatrix} 1+2k & -4k \\ k & 1-2k \end{bmatrix} \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix} \quad \text{[from step II]}$$

$$= \begin{bmatrix} 3(1+2k) - 4k & -4(1+2k) + 4k \\ 3k + 1(1-2k) & -4k - 1(1-2k) \end{bmatrix}$$

$$= \begin{bmatrix} 3+2k & -4(k+1) \\ k+1 & -1-2k \end{bmatrix} = \text{RHS}$$

Therefore, $P(k+1)$ is true.

Hence, by the principal of mathematical induction $P(n)$ is true for all $n \in N$.

Exercise for Session 3

1 If $A = \begin{bmatrix} -1 & -2 & -2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$, then adj A equals to

(a) A (b) A^T (c) $3A$ (d) $3A^T$

2 If A is a 3×3 matrix and B is its adjoint such that $|B| = 64$, then $|A|$ is equal to

(a) 64 (b) $\pm$ 64 (c) $\pm$ 8 (d) 18

3 For any 2×2 matrix A, if $A(\text{adj } A) = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}$, then $|A|$ is equal to

(a) 0 (b) 10 (c) 20 (d) 100

4 If A is a singular matrix, then adj A is

(a) singular (b) non-singular (c) symmetric (d) not defined

5 If $A = \begin{bmatrix} 1 & 2 & -1 \\ -1 & 1 & 2 \\ 2 & -1 & 1 \end{bmatrix}$, then det (adj (adj A)) is

(a) 14^4 (b) 14^3 (c) 14^2 (d) 14

6 If $k \in R_0$, then det $(\text{adj}(k I_n))$ is equal to

(a) k^{n-1} (b) $k^{n(n-1)}$ (c) k^n (d) k

7 With $1, \omega, \omega^2$ as cube roots of unity, inverse of which of the following matrices exists?

(a) $\begin{bmatrix} 1 & \omega \\ \omega & \omega^2 \end{bmatrix}$ (b) $\begin{bmatrix} \omega^2 & 1 \\ 1 & \omega \end{bmatrix}$

(c) $\begin{bmatrix} \omega & \omega^2 \\ \omega^2 & 1 \end{bmatrix}$ (d) None of these

8 If the matrix A is such that $A\begin{bmatrix} -1 & 2 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} -4 & 1 \\ 7 & 7 \end{bmatrix}$, then A is equal to

(a) $\begin{bmatrix} 1 & 1 \\ 2 & -3 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 1 \\ -2 & 3 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$ (d) $\begin{bmatrix} -1 & 1 \\ 2 & 3 \end{bmatrix}$

9 If $A = \begin{bmatrix} \cos x & \sin x & 0 \\ -\sin x & \cos x & 0 \\ 0 & 0 & 1 \end{bmatrix} = f(x)$, then A^{-1} is equal to

(a) $f(-x)$ (b) $f(x)$
(c) $-f(x)$ (d) $-f(-x)$

10 The element in the first row and third column of the inverse of the matrix $\begin{bmatrix} 1 & 2 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$, is

(a) -2 (b) 0 (c) 1 (d) None of these

11 If $A = \begin{bmatrix} 0 & 1 & -1 \\ 2 & 1 & 3 \\ 3 & 2 & 1 \end{bmatrix}$, then $(A(\text{adj}\, A)\, A^{-1})\, A$ is equal to

(a) $\begin{bmatrix} -6 & 0 & 0 \\ 0 & -6 & 0 \\ 0 & 0 & -6 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & \frac{1}{6} & -\frac{1}{6} \\ \frac{1}{3} & \frac{1}{6} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{6} \end{bmatrix}$ (c) $2\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ (d) $2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

12 A is an involutory matrix given by $A = \begin{bmatrix} 0 & 1 & -1 \\ 4 & -3 & 4 \\ 3 & -3 & 4 \end{bmatrix}$, then the inverse of $\frac{A}{2}$ will be

(a) $2A$ (b) $\frac{A^{-1}}{2}$

(c) $\frac{A}{2}$ (d) A^2

13 If A satisfies the equation $x^3 - 5x^2 + 4x + \lambda = 0$, then A^{-1} exists, if

(a) $\lambda \neq 1$ (b) $\lambda \neq 2$
(c) $\lambda \neq -1$ (d) $\lambda \neq 0$

14 A square non-singular matrix A satisfies the equation $x^2 - x + 2 = 0$, then A^{-1} is equal to

(a) $I - A$ (b) $(I - A)/2$
(c) $I + A$ (d) $(I + A)/2$

15 Matrix A is such that $A^2 = 2A - I$, where I is the identity matrix, then for $n \geq 2$, A^n is equal to

(a) $nA - (n-1)I$ (b) $nA - I$
(c) $2^{n-1}A - (n-1)I$ (d) $2^{n-1}A - I$

16 If $X = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$, the value of X^n is

(a) $\begin{bmatrix} 3n & -4n \\ n & -n \end{bmatrix}$ (b) $\begin{bmatrix} 2n+n & 5-n \\ n & -n \end{bmatrix}$

(c) $\begin{bmatrix} 3^n & (-4)^n \\ 1^n & (-1)^n \end{bmatrix}$ (d) None of these

Answers

Exercise for Session 3

1. (d)	**2.** (c)	**3.** (b)	**4.** (d)	**5.** (a)	**6.** (b)
7. (d)	**8.** (c)	**9.** (a)	**10.** (d)	**11.** (c)	**12.** (a)
13. (d)	**14.** (b)	**15.** (a)	**16.** (d)		