Session 5

Irrational Equations, Irrational Inequations, Exponential Equations, Exponential Inequations, Logarithmic Equations, Logarithmic Inequations

Irrational Equations

Here, we consider equations of the type which contain the unknown under the radical sign and the value under the radical sign is known as radicand.

- If roots are all even (i.e. $\sqrt{x}$, $\sqrt[4]{x}$, $\sqrt[6]{x}$,..., etc) of an equation are arithmetic. In other words, if the radicand is negative (i.e. $x<0$), then the root is imaginary, if the radicand is zero, then the root is also zero and if the radicand is positive, then the value of the root is also positive.
- If roots are all odd (i.e. $\sqrt[3]{x}, \sqrt[5]{x}, \sqrt[7]{x}$,... etc) of an equation, then it is defined for all real values of the radicand. If the radicand is negative, then the root is negative, if the radicand is zero, then the root is zero and if the radicand is positive, then the root is positive.

Some Standard Formulae to Solve Irrational Equations

If f and g be functions of $x, k \in N$. Then,

1. $\sqrt[2k]{f}\,\sqrt[2k]{g} = \sqrt[2k]{fg}, f \geq 0, g \geq 0$

2. $\sqrt[2k]{f} \,/\, \sqrt[2k]{g} = \sqrt[2k]{(f/g)}, f \geq 0, g > 0$

3. $|f|\sqrt[2k]{g} = \sqrt[2k]{(f^{2k} g)}, g \geq 0$

4. $\sqrt[2k]{(f/g)} = \sqrt[2k]{|f|} \,/\, \sqrt[2k]{|g|}, fg \geq 0, g \neq 0$

5. $\sqrt[2k]{fg} = \sqrt[2k]{|f|}\,\sqrt[2k]{g}, fg \geq 0$

Example 78. Prove that the following equations has no solutions.

(i) $\sqrt{(2x+7)} + \sqrt{(x+4)} = 0$ (ii) $\sqrt{(x-4)} = -5$

(iii) $\sqrt{(6-x)} - \sqrt{(x-8)} = 2$ (iv) $\sqrt{-2-x} = \sqrt[5]{(x-7)}$

(v) $\sqrt{x} + \sqrt{(x+16)} = 3$ (vi) $7\sqrt{x} + 8\sqrt{-x} + \dfrac{15}{x^3} = 98$

(vii) $\sqrt{(x-3)} - \sqrt{x+9} = \sqrt{(x-1)}$

Sol. (i) We have, $\sqrt{(2x+7)} + \sqrt{(x+4)} = 0$

This equation is defined for $2x + 7 \geq 0$

and $x + 4 \geq 0 \Rightarrow \begin{cases} x \geq -\dfrac{7}{2} \\ x \geq -4 \end{cases}$

$\therefore \quad x \geq -\dfrac{7}{2}$

For $x \geq -\dfrac{7}{2}$, the left hand side of the original equation is positive, but right hand side is zero. Therefore, the equation has no roots.

(ii) We have, $\sqrt{(x-4)} = -5$

The equation is defined for $x - 4 \geq 0$

$\therefore \quad x \geq 4$

For $x \geq 4$, the left hand side of the original equation is positive, but right hand side is negative.

Therefore, the equation has no roots.

(iii) We have, $\sqrt{(6-x)} - \sqrt{x-8} = 2$

The equation is defined for

$6 - x \geq 0$ and $x - 8 \geq 0$

$\therefore \quad \begin{cases} x \leq 6 \\ x \geq 8 \end{cases}$

Consequently, there is no x for which both expressions would have sense. Therefore, the equation has no roots.

(iv) We have, $\sqrt{(-2-x)} = \sqrt[5]{(x-7)}$

This equation is defined for

$-2 - x \geq 0 \quad \Rightarrow \quad x \leq -2$

For $x \leq -2$ the left hand side is positive, but right hand side is negative.

Therefore, the equation has no roots.

(v) We have, $\sqrt{x} + \sqrt{(x+16)} = 3$

The equation is defined for

$x \geq 0$ and $x + 16 \geq 0 \Rightarrow \begin{cases} x \geq 0 \\ x \geq -16 \end{cases}$

Hence, $\quad x \geq 0$

For $x \geq 0$ the left hand side ≥ 4, but right hand side is 3. Therefore, the equation has no roots.

(vi) We have, $7\sqrt{x} + 8\sqrt{-x} + \dfrac{15}{x^3} = 98$

For $x < 0$, the expression $7\sqrt{x}$ is meaningless,

For $x > 0$, the expression $8\sqrt{-x}$ is meaningless

and for $x = 0$, the expression $\frac{15}{x^3}$ is meaningless.

Consequently, the left hand side of the original equation is meaningless for any $x \in R$. Therefore, the equation has no roots.

(vii) We have, $\sqrt{(x-3)} - \sqrt{(x+9)} = \sqrt{x-1}$

This equation is defined for

$$\begin{cases} x - 3 \ge 0 \\ x + 9 \ge 0 \\ x - 1 \ge 0 \end{cases} \Rightarrow \begin{cases} x \ge 3 \\ x \ge -9 \\ x \ge 1 \end{cases}$$

Hence, $\quad x \ge 3$

For $x \ge 3$, $\sqrt{x-3} < \sqrt{x+9}$ i.e. $\sqrt{(x-3)} - \sqrt{(x+9)} < 0$

Hence, for $x \ge 3$, the left hand side of the original equation is negative and right hand side is positive. Therefore, the equation has no roots.

Some Standard Forms to Solve Irrational Equations

***Form* 1 An equation of the form**

$f^{2n}(x) = g^{2n}(x), n \in N$ is equivalent to $f(x) = g(x)$.

Then, find the roots of this equation. If root of this equation satisfies the original equation, then its root of the original equation, otherwise, we say that this root is its **extraneous root**.

Remark

Squaring an Equation May Give Extraneous Roots

Squaring should be avoided as for as possible. If squaring is necessary, then the roots found after squaring must be checked whether they satisfy the original equation or not. If some values of x which do not satisfy the original equation. These values of x are called extraneous roots and are rejected.

Example 79. Solve the equation $\sqrt{x} = x - 2$.

Sol. We have, $\sqrt{x} = x - 2$

On squaring both sides, we obtain

$$x = (x-2)^2$$

$$\Rightarrow \quad x^2 - 5x + 4 = 0 \Rightarrow (x-1)(x-4) = 0$$

$$\therefore \quad x_1 = 1 \text{ and } x_2 = 4$$

Hence, $x_1 = 4$ satisfies the original equation, but $x_2 = 1$ does not satisfy the original equation.

$\therefore \quad x_2 = 1$ is the extraneous root.

Example 80. Solve the equation

$$3\sqrt{(x+3)} - \sqrt{(x-2)} = 7.$$

Sol. We have, $3\sqrt{(x+3)} - \sqrt{x-2} = 7$

$$\Rightarrow \quad 3\sqrt{(x+3)} = 7 + \sqrt{(x-2)}$$

On squaring both sides of the equation, we obtain

$$9x + 27 = 49 + x - 2 + 14\sqrt{x-2}$$

$$\Rightarrow \quad 8x - 20 = 14\sqrt{(x-2)}$$

$$(4x - 10) = 7\sqrt{x-2}$$

Again, squaring both sides, we obtain

$$16x^2 + 100 - 80x = 49x - 98$$

$$\Rightarrow \quad 16x^2 - 129x + 198 = 0$$

$$\Rightarrow \quad (x-6)\left(x - \frac{33}{16}\right) = 0$$

$$x_1 = 6 \text{ and } x_2 = \frac{33}{16}$$

Hence, $x_1 = 6$ satisfies the original equation, but $x_2 = \frac{33}{16}$ does not satisfy the original equation.

$\therefore\ x_2 = \frac{33}{16}$ is the extraneous root.

***Form* 2 An equation in the form**

$\sqrt[2n]{f(x)} = g(x), n \in N$

is equivalent to the system $\begin{cases} g(x) \ge 0 \\ f(x) = g^{2n}(x) \end{cases}$

Example 81. ***Solve the equation***

$$\sqrt{(6 - 4x - x^2)} = x + 4.$$

Sol. We have, $\sqrt{(6 - 4x - x^2)} = x + 4$

This equation is equivalent to the system

$$\begin{cases} x + 4 \ge 0 \\ 6 - 4x - x^2 = (x+4)^2 \end{cases}$$

$$\Rightarrow \quad \begin{cases} x \ge -4 \\ x^2 + 6x + 5 = 0 \end{cases}$$

On solving the equation $x^2 + 6x + 5 = 0$

We find that, $x_1 = (-1)$ and $x_2 = (-5)$ only $x_1 = (-1)$ satisfies the condition $x \ge -4$.

Consequently, the number -1 is the only solution of the given equation.

***Form* 3 An equation in the form**

$$\sqrt[3]{f(x)} + \sqrt[3]{g(x)} = h(x) \quad \text{...(i)}$$

where $f(x)$, $g(x)$ are the functions of x, but $h(x)$ is a function of x or constant, can be solved as follows cubing both sides of the equation, we obtain

$$f(x) + g(x) + 3\sqrt[3]{f(x)\,g(x)}\,(\sqrt[3]{f(x)} + \sqrt[3]{g(x)}) = h^3(x)$$

$$\Rightarrow \quad f(x) + g(x) + 3\sqrt[3]{f(x)\,g(x)}\,(h(x)) = h^3(x)$$

[from Eq. (i)]

We find its roots and then substituting, then into the original equation, we choose those which are the roots of the original equation.

Example 82. ***Solve the equation***

$$\sqrt[3]{(2x-1)}+\sqrt[3]{(x-1)}=1.$$

Sol. We have, $\sqrt[3]{(2x-1)}+\sqrt[3]{(x-1)}=1$ ...(i)

Cubing both sides of Eq. (i), we obtain

$$2x-1+x-1+3\cdot\sqrt[3]{(2x-1)(x-1)}$$
$$(\sqrt[3]{(2x-1)}+\sqrt[3]{(x-1)})=1$$

$\Rightarrow$ $3x-2+3\cdot\sqrt[3]{(2x^2-3x+1)}\,(1)=1$ [from Eq. (i)]

$\Rightarrow$ $3\cdot\sqrt[3]{(2x^2-3x+1)}=3-3x$

$\Rightarrow$ $\sqrt[3]{(2x^2-3x+1)}=(1-x)$

Again cubing both sides, we obtain

$$2x^2-3x+1=(1-x)^3$$

$\Rightarrow$ $(2x-1)(x-1)=(1-x)^3$

$\Rightarrow$ $(2x-1)(x-1)=-(x-1)^3$

$\Rightarrow$ $(x-1)\{2x-1+(x-1)^2\}=0$

$\Rightarrow$ $(x-1)(x^2)=0$

$\therefore$ $x_1=0$ and $x_2=1$

$\because$ $x_1=0$ is not satisfies the Eq. (i), then $x_1=0$ is an extraneous root of the Eq. (i), thus $x_2=1$ is the only root of the original equation.

***Form* 4 An equation of the form**

$$\sqrt[n]{a-f(x)}+\sqrt[n]{b+f(x)}=g(x).$$

Let $u=\sqrt[n]{a-f(x)},\ v=\sqrt[n]{b+f(x)}$

Then, the given equation reduces to the solution of the system of algebraic equations.

$$\begin{cases} u+v=g(x) \\ u^n+v^n=a+b \end{cases}$$

Example 83. Solve the equation

$$\sqrt{(2x^2+5x-2)}-\sqrt{2x^2+5x-9}=1.$$

Sol. Let $u=\sqrt{(2x^2+5x-2)}$

and $v=\sqrt{(2x^2+5x-9)}$

$\therefore$ $u^2=2x^2+5x-2$

and $v^2=2x^2+5x-9$

Then, the given equation reduces to the solution of the system of algebraic equations.

$$u-v=1$$
$$u^2-v^2=7$$

$\Rightarrow$ $(u+v)(u-v)=7$

$\Rightarrow$ $u+v=7$ $[\because u-v=1]$

We get, $u=4,\ v=3$

$\therefore$ $\sqrt{2x^2+5x-2}=4$

$\therefore$ $2x^2+5x-18=0$

$\therefore$ $x_1=2$ and $x_2=-9/2$

Both roots satisfies the original equation.

Hence, $x_1=2$ and $x_2=-9/2$ are the roots of the original equation.

Irrational Inequations

We consider, here inequations which contain the unknown under the radical sign.

Some Standard Forms to Solve Irrational Inequations

***Form* 1** An inequation of the form

$$\sqrt[2n]{f(x)}<\sqrt[2n]{g(x)},\ n\in N$$

is equivalent to the system $\begin{cases} f(x)\geq 0 \\ g(x)>f(x) \end{cases}$

and inequation of the form $\sqrt[2n+1]{f(x)}<\sqrt[2n+1]{g(x)},\ n\in N$

is equivalent to the inequation $f(x)<g(x)$.

Example 84. Solve the inequation

$$\sqrt[5]{\left[\frac{3}{x+1}+\frac{7}{x+2}\right]}<\sqrt[5]{\frac{6}{x-1}}.$$

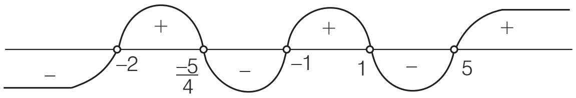
Sol. The given inequation is equivalent to

$$\frac{3}{x+1}+\frac{7}{x+2}<\frac{6}{x-1}$$

$\Rightarrow$ $\dfrac{4x^2-15x-25}{(x+1)(x+2)(x-1)}<0$

$\Rightarrow$ $\dfrac{(x+5/4)(x-5)}{(x+1)(x+2)(x-1)}<0$

From Wavy Curve Method :



$$x\in(-\infty,-2)\cup\left(-\frac{5}{4},1\right)\cup(1,5)$$

***Form* 2 An inequation of the form**

$$\sqrt[2n]{f(x)}<g(x),\ n\in N.$$

is equivalent to the system $\begin{cases} f(x)\geq 0 \\ g(x)>0 \\ f(x)<g^{2n}(x), \end{cases}$

and inequation of the form $\sqrt[2n+1]{f(x)} < g(x), n \in N$

is equivalent to the inequation $f(x) < g^{2n+1}(x)$.

Example 85. Solve the inequation $\sqrt{(x+14)} < (x+2)$.

Sol. We have, $\sqrt{(x+14)} < (x+2)$

This inequation is equivalent to the system

$$\begin{cases} x+14 \geq 0 \\ x+2 > 0 \\ x+14 < (x+2)^2 \end{cases} \Rightarrow \begin{cases} x \geq -14 \\ x > -2 \\ x^2 + 3x - 10 > 0 \end{cases}$$

$$\Rightarrow \begin{cases} x \geq -14 \\ x > -2 \\ (x+5)(x-2) > 0 \end{cases} \Rightarrow \begin{cases} x \geq -14 \\ x > -2 \\ x < -5 \text{ and } x > 2 \end{cases}$$

On combining all three inequation of the system, we get

$$x > 2, \text{ i.e. } x \in (2, \infty)$$

***Form* 3** An inequation of the form

$$\sqrt[2n]{f(x)} > g(x), n \in N$$

is equivalent to the collection of two systems of inequations

i.e. $\begin{cases} g(x) \geq 0 \\ f(x) > g^{2n}(x) \end{cases}$ and $\begin{cases} g(x) < 0 \\ f(x) \geq 0 \end{cases}$

and inequation of the form $\sqrt[2n+1]{f(x)} > g(x), n \in N$

is equivalent to the inequation $f(x) > g^{2n+1}(x)$.

Example 86. ***Solve the inequation***

$\sqrt{(-x^2+4x-3)} > 6-2x$.

Sol. We have, $\sqrt{(-x^2+4x-3)} > 6-2x$

This inequation is equivalent to the collection of two systems, of inequations

i.e. $\begin{cases} 6-2x \geq 0 \\ -x^2+4x-3 > (6-2x)^2 \end{cases}$ and $\begin{cases} 6-2x < 0 \\ -x^2+4x-3 \geq 0 \end{cases}$

$\Rightarrow \begin{cases} x \leq 3 \\ (x-3)(5x-13) < 0 \end{cases}$ and $\begin{cases} x > 3 \\ (x-1)(x-3) \leq 0 \end{cases}$

$\Rightarrow \begin{cases} x \leq 3 \\ \frac{13}{5} < x < 3 \end{cases}$ and $\begin{cases} x > 3 \\ 1 \leq x < 3 \end{cases}$

The second system has no solution and the first system has solution in the interval $\left(\frac{13}{5} < x < 3\right)$.

Hence, $x \in \left(\frac{13}{5}, 3\right)$ is the set of solution of the original inequation.

Exponential Equations

If we have an equation of the form $a^x = b\,(a > 0)$, then

(i) $x \in \phi$, if $b \leq 0$

(ii) $x = \log_a b$, if $b > 0, a \neq 1$

(iii) $x \in \phi$, if $a = 1, b \neq 1$

(iv) $x \in R$, if $a = 1, b = 1$ (since, $1^x = 1 \Rightarrow 1 = 1, x \in R$)

Example 87. ***Solve the equation***

$\sqrt{(6-x)}\,(3^{x^2-7.2x+3.9} - 9\sqrt{3}) = 0$.

Sol. We have,

$$\sqrt{(6-x)}\,(3^{x^2-7.2x+3.9} - 9\sqrt{3}) = 0$$

This equation is defined for

$$6-x \geq 0 \text{ i.e., } x \leq 6 \quad \text{...(i)}$$

This equation is equivalent to the collection of equations

$$\sqrt{6-x} = 0 \text{ and } 3^{x^2-7.2x+3.9} - 9\sqrt{3} = 0$$

$\therefore \quad x_1 = 6$ and $3^{x^2-7.2x+3.9} = 3^{2.5}$

then $\quad x^2 - 7.2x + 3.9 = 2.5$

$$x^2 - 7.2x + 1.4 = 0$$

We find that, $\quad x_2 = \frac{1}{5}$ and $x_3 = 7$

Hence, solution of the original equation are [which satisfies Eq. (i)]

$$x_1 = 6, x_2 = \frac{1}{5}.$$

Some Standard Forms to Solve Exponential Equations

***Form* 1 An equation in the form $a^{f(x)} = 1, a > 0, a \neq 1$**

is equivalent to the equation $f(x) = 0$

Example 88. ***Solve the equation*** $5^{x^2+3x+2} = 1$.

Sol. This equation is equivalent to

$$x^2 + 3x + 2 = 0$$

$$\Rightarrow \quad (x+1)(x+2) = 0$$

$\therefore x_1 = -1, x_2 = -2$ consequently, this equation has two roots $x_1 = -1$ and $x_2 = -2$.

***Form* 2** An equation in the form

$$f(a^x) = 0$$

is equivalent to the equation $f(t) = 0$, where $t = a^x$.

If $t_1, t_2, t_3, \ldots, t_k$ are the roots of $f(t) = 0$, then

$$a^x = t_1, a^x = t_2, a^x = t_3, \ldots, a^x = t_k$$

Example 89. Solve the equation $5^{2x} - 24\cdot 5^x - 25 = 0$.

Sol. Let $5^x = t$, then the given equation can reduce in the form

$$t^2 - 24t - 25 = 0$$

$\Rightarrow$ $(t-25)(t+1) = 0 \Rightarrow t \neq -1$,

$\therefore$ $t = 25$,

then $5^x = 25 = 5^2$, then $x = 2$

Hence, $x_1 = 2$ is only one root of the original equation.

***Form* 3** An equation of the form

$$\alpha a^{f(x)} + \beta b^{f(x)} + \gamma c^{f(x)} = 0,$$

where $\alpha, \beta, \gamma \in R$ and $\alpha, \beta, \gamma \neq 0$ and the bases satisfy the condition $b^2 = ac$ is equivalent to the equation

$$\alpha t^2 + \beta t + \gamma = 0, \text{ where } t = (a/b)^{f(x)}$$

If roots of this equation are t_1 and t_2, then

$$(a/b)^{f(x)} = t_1 \text{ and } (a/b)^{f(x)} = t_2$$

Example 90. Solve the equation $64\cdot 9^x - 84\cdot 12^x + 27\cdot 16^x = 0$.

Sol. Here, $9 \times 16 = (12)^2$.

Then, we divide its both sides by 12^x and obtain

$$\Rightarrow \quad 64\cdot\left(\frac{3}{4}\right)^x - 84 + 27\cdot\left(\frac{4}{3}\right)^x = 0 \quad \text{...(i)}$$

Let $\left(\frac{3}{4}\right)^x = t$, then Eq. (i) reduce in the form

$$64t^2 - 84t + 27 = 0$$

$\therefore$ $t_1 = \frac{3}{4}$ and $t_2 = \frac{9}{16}$

then, $\left(\frac{3}{4}\right)^x = \left(\frac{3}{4}\right)^1$ and $\left(\frac{3}{4}\right)^x = \left(\frac{3}{4}\right)^2$

$\therefore$ $x_1 = 1$ and $x_2 = 2$

Hence, roots of the original equation are $x_1 = 1$ and $x_2 = 2$.

***Form* 4 An equation in the form**

$$\alpha\cdot a^{f(x)} + \beta\cdot b^{f(x)} + c = 0,$$

where $\alpha, \beta, c \in R$ and $\alpha, \beta, c \neq 0$ and $ab = 1$ (a and b are inverse positive numbers) is equivalent to the equation

$$\alpha t^2 + ct + \beta = 0, \text{ where } t = a^{f(x)}.$$

If roots of this equation are t_1 and t_2, then $a^{f(x)} = t_1$ and $a^{f(x)} = t_2$.

Example 91. ***Solve the equation*** $15\cdot 2^{x+1} + 15\cdot 2^{2-x} = 135$.

Sol. This equation rewrite in the form

$$30.2^x + \frac{60}{2^x} = 135$$

Let $t = 2^x$,

Then, $30t^2 - 135t + 60 = 0$

$\Rightarrow$ $6t^2 - 27t + 12 = 0$

$\Rightarrow$ $6t^2 - 24t - 3t + 12 = 0$

$\Rightarrow$ $(t-4)(6t-3) = 0$

Then, $t_1 = 4$ and $t_2 = \frac{1}{2}$

Thus, given equation is equivalent to

$$2^x = 4 \text{ and } 2^x = \frac{1}{2}$$

Then, $x_1 = 2$ and $x_2 = -1$

Hence, roots of the original equation are $x_1 = 2$ and $x_2 = -1$.

***Form* 5 An equation of the form $a^{f(x)} + b^{f(x)} = c$,**

where $a, b, c \in R$ and a, b, c satisfies the condition $a^2 + b^2 = c$, then solution of this equation is $f(x) = 2$ and no other solution of this equation.

Example 92. Solve the equation $3^{x-4} + 5^{x-4} = 34$.

Sol. Here, $3^2 + 5^2 = 34$, then given equation has a solution $x - 4 = 2$.

$\therefore$ $x_1 = 6$ is a root of the original equation.

***Form* 6 An equation of the form $\{f(x)\}^{g(x)}$ is equivalent to the equation**

$$\{f(x)\}^{g(x)} = 10^{g(x)\log f(x)},$$

where $f(x) > 0$.

Example 93. ***Solve the equation*** $5^x \sqrt[x]{8^{x-1}} = 500$.

Sol. We have, $5^x \sqrt[x]{8^{x-1}} = 5^3\cdot 2^2$

$\Rightarrow$ $5^x\cdot 8^{\left(\frac{x-1}{x}\right)} = 5^3\cdot 2^2$

$\Rightarrow$ $5^x\cdot 2^{\frac{3x-3}{x}} = 5^3\cdot 2^2$

$\Rightarrow$ $5^{x-3}\cdot 2^{\left(\frac{x-3}{x}\right)} = 1$

$\Rightarrow$ $(5.2^{1/x})^{(x-3)} = 1$

is equivalent to the equation

$$10^{(x-3)\log(5\cdot 2^{1/x})} = 1$$

$\Rightarrow$ $(x-3)\log(5\cdot 2^{1/x}) = 0$

Thus, original equation is equivalent to the collection of equations

$$x - 3 = 0, \ \log(5\cdot 2^{1/x}) = 0$$

$\therefore$ $x_1 = 3, 5\cdot 2^{1/x} = 1 \Rightarrow 2^{1/x} = \left(\frac{1}{5}\right)$

$\therefore$ $x_2 = -\log_5 2$

Hence, roots of the original equation are $x_1 = 3$ and $x_2 = -\log_5 2$.

Exponential Inequations

When we solve exponential inequation

$a^{f(x)} > b\ (a > 0)$, we have

(i) $x \in D_f$, if $b \le 0$

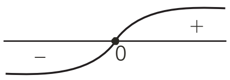
(ii) If $b > 0$, then we have $f(x) > \log_a b$, if $a > 1$ and $f(x) < \log_a b$, if $0 < a < 1$ for $a = 1$, then $b < 1$.

Remark

The inequation $a^{f(x)} \le b$ has no solution for $b \le 0$, $a > 0$, $a \ne 1$.

Example 94. ***Solve the inequation*** $3^{x+2} > \left(\dfrac{1}{9}\right)^{1/x}$.

Sol. We have, $\quad 3^{x+2} > (3^{-2})^{1/x} \Rightarrow 3^{x+2} > 3^{-2/x}$



Here, base $3 > 1$

$$\Rightarrow \quad x + 2 > -\frac{2}{x} \quad \Rightarrow \quad \frac{x^2 + 2x + 2}{x} > 0$$

$$\Rightarrow \quad \frac{(x+1)^2 + 1}{x} > 0 \quad \Rightarrow \quad x > 0$$

$$\therefore \quad x \in (0, \infty)$$

Some Standard Forms to Solve Exponential Inequations

Form 1 An inequation of the form

$$f(a^x) \ge 0 \quad \text{or } f(a^x) \le 0$$

is equivalent to the system of collection

$$\begin{cases} t > 0, & \text{where } t = a^x \\ f(t) \ge 0 \quad \text{or} & f(t) \le 0 \end{cases}$$

Example 95. ***Solve the inequation*** $4^{x+1} - 16^x < 2\log_4 8$.

Sol. Let $4^x = t$, then given inequation reduce in the form

$$4t - t^2 > 2 \cdot \frac{3}{2}$$

$$\Rightarrow \quad t^2 - 4t + 3 < 0 \Rightarrow (t-1)(t-3) < 0$$

$$\Rightarrow \quad 1 < t < 3 \qquad [\because t > 0]$$

$$\Rightarrow \quad 1 < 4^x < 3$$

$$\Rightarrow \quad 0 < x < \log_4 3$$

$$\therefore \quad x \in (0, \log_4 3)$$

Form 2 An inequation of the form

$$\alpha a^{f(x)} + \beta b^{f(x)} + \gamma c^{f(x)} \ge 0$$

or

$$\alpha a^{f(x)} + \beta b^{f(x)} + \gamma c^{f(x)} \le 0$$

where $\alpha, \beta, \gamma \in R$ and $\alpha, \beta, \gamma \ne 0$ and the bases satisfy the condition $b^2 = ac$ is equivalent to the inequation

$$\alpha t^2 + \beta t + \gamma \ge 0 \quad \text{or} \quad \alpha t^2 + \beta t + \gamma \le 0,$$

where $t = (a/b)^{f(x)}$.

Form 3 An inequation of the form

$$\alpha a^{f(x)} + \beta b^{f(x)} + \gamma \ge 0$$

or

$$\alpha a^{f(x)} + \beta b^{f(x)} + \gamma \le 0$$

where $\alpha, \beta, \gamma \in R$ and $\alpha, \beta, \gamma \ne 0$ and $ab = 1$(a and b are inverse (+ve) numbers) is equivalent to the inequation

$$\alpha t^2 + \beta t + \gamma \ge 0 \quad \text{or} \quad \alpha t^2 + \beta t + \gamma \le 0$$

where $\quad t = a^{f(x)}$

Form 4 If an inequation of the exponential form reduces to the solution of homogeneous algebraic inequation, i.e.

$$a_0 f^n(x) + a_1 f^{n-1}(x)\, g(x) + a_2 f^{n-2}(x) g^2(x) + \dots$$
$$+ a_{n-1} f(x)\, g^{n-1}(x) + a_n\, g^n(x) \ge 0,$$

where $a_0, a_1, a_2, \dots, a_n$ are constants $(a_0 \ne 0)$ and $f(x)$ and $g(x)$ are functions of x.

Example 96. Solve the inequation $2^{2x^2 - 10x + 3} + 6^{x^2 - 5x + 1} \ge 3^{2x^2 - 10x + 3}$.

Sol. The given inequation is equivalent to

$$8 \cdot 2^{2(x^2 - 5x)} + 6 \cdot 2^{x^2 - 5x} \cdot 3^{x^2 - 5x} - 27 \cdot 3^{2(x^2 - 5x)} \ge 0$$

Let $\quad 2^{x^2 - 5x} = f(x)$ and $3^{x^2 - 5x} = g(x)$,

then $\quad 8 \cdot f^2(x) + 6 f(x) \cdot g(x) - 27 g^2(x) \ge 0$

On dividing in each by $g^2(x)$ $\qquad [\because g(x) > 0]$

Then, $\quad 8\left(\dfrac{f(x)}{g(x)}\right)^2 + 6\left(\dfrac{f(x)}{g(x)}\right) - 27 \ge 0$

and let $\quad \dfrac{f(x)}{g(x)} = t \qquad [\because t > 0]$

then $\quad 8t^2 + 6t - 27 \ge 0$

$$\Rightarrow \quad \left(t - \frac{3}{2}\right)(t + 9/4) \ge 0$$

$$\Rightarrow \quad t \ge 3/2 \text{ and } t \le -9/4$$

The second inequation has no root. $\qquad [\because t > 0]$

From the first inequation, $t > 3/2$

$$\left(\frac{2}{3}\right)^{x^2 - 5x} \ge \left(\frac{2}{3}\right)^{-1} \qquad \left[\because \frac{2}{3} < 1\right]$$

$$\Rightarrow \quad x^2 - 5x \le -1 \Rightarrow x^2 - 5x + 1 \le 0$$

$$\therefore \quad \frac{5 - \sqrt{21}}{2} \le x \le \frac{5 + \sqrt{21}}{2}$$

Hence, $x \in \left[\dfrac{5 - \sqrt{21}}{2}, \dfrac{5 + \sqrt{21}}{2}\right]$.

Logarithmic Equations

If we have an equation of the form

$$\log_a f(x) = b, (a > 0), a \neq 1$$

is equivalent to the equation

$$f(x) = a^b \quad (f(x) > 0).$$

Example 97. Solve the equation $\log_3(5 + 4\log_3(x-1)) = 2$.

Sol. We have, $\log_3(5 + 4\log_3(x-1)) = 2$

is equivalent to the equation (here, base $\neq 1, > 0$).

$\therefore \quad 5 + 4\log_3(x-1) = 3^2$

$\Rightarrow \quad \log_3(x-1) = 1 \Rightarrow x - 1 = 3^1$

$\therefore \quad x = 4$

Hence, $x_1 = 4$ is the solution of the original equation.

Some Standard Formulae to Solve Logarithmic Equations

f and g are some functions and $a > 0, a \neq 1$, then, if $f > 0, g > 0$, we have

(i) $\log_a(fg) = \log_a f + \log_a g$

(ii) $\log_a(f/g) = \log_a f - \log_a g$

(iii) $\log_a f^{2\alpha} = 2\alpha \log_a |f|$ (iv) $\log_{a^\beta} f^\alpha = \dfrac{\alpha}{\beta}\log_a f$

(v) $f^{\log_a g} = g^{\log_a f}$ (vi) $a^{\log_a f} = f$

Example 98. Solve the equation $2x^{\log_4 3} + 3^{\log_4 x} = 27$.

Sol. The domain of the admissible values of the equation is $x > 0$. The given equation is equivalent to

$2.3^{\log_4 x} + 3^{\log_4 x} = 27$ [from above result (v)]

$\Rightarrow \quad 3.3^{\log_4 x} = 27$

$\Rightarrow \quad 3^{\log_4 x} = 9$

$\Rightarrow \quad 3^{\log_4 x} = 3^2$

$\Rightarrow \quad \log_4 x = 2$

$\Rightarrow \quad x_1 = 4^2 = 16$ is its only root.

Some Standard Forms to Solve Logarithmic Equations

Form 1 An equation of the form $\log_x a = b, a > 0$ has

(i) Only root $x = a^{1/b}$, if $a \neq 1$ and $b = 0$.

(ii) Any positive root different from unity, if $a = 1$ and $b = 0$.

(iii) No roots, if $a = 1, b \neq 0$.

(iv) No roots, if $a \neq 1, b = 0$.

Example 99. Solve the equation $\log_{(\log_5 x)} 5 = 2$.

Sol. We have, $\log_{(\log_5 x)} 5 = 2$

Base of logarithm > 0 and $\neq 1$.

$\therefore \quad \log_5 x > 0$ and $\log_5 x \neq 1$

$\Rightarrow \quad x > 1$ and $x \neq 5$

$\therefore$ The original equation is equivalent to

$$\log_5 x = 5^{1/2} = \sqrt{5}$$

$\therefore \quad x_1 = 5^{\sqrt{5}}$

Hence, $5^{\sqrt{5}}$ is the only root of the original equation.

Form 2 Equations of the form

(i) $f(\log_a x) = 0, a > 0, a \neq 1$ and

(ii) $g(\log_x A) = 0, A > 0$

Then, Eq. (i) is equivalent to

$f(t) = 0$, where $t = \log_a x$

If $t_1, t_2, t_3, \ldots, t_k$ are the roots of $f(t) = 0$, then

$\log_a x = t_1, \log_a x = t_2, \ldots, \log_a x = t_k$

and Eq. (ii) is equivalent to $f(y) = 0$, where $y = \log_x A$.

If $y_1, y_2, y_3, \ldots, y_k$ are the roots of $f(y) = 0$, then

$\log_x A = y_1, \log_x A = y_2, \ldots, \log_x A = y_k$

Example 100. Solve the equation $\dfrac{1 - 2(\log x^2)^2}{\log x - 2(\log x)^2} = 1.$

Sol. The given equation can rewrite in the form

$$\frac{1 - 2(2\log x)^2}{\log x - 2(\log x)^2} = 1$$

$$\Rightarrow \quad \frac{1 - 8(\log x)^2}{\log x - 2(\log x)^2} - 1 = 0$$

Let $\quad \log x = t,$

then $\quad \dfrac{1 - 8t^2}{t - 2t^2} - 1 = 0 \Rightarrow \dfrac{1 - 8t^2 - t + 2t^2}{t - 2t^2} = 0$

$$\Rightarrow \quad \frac{1 - t - 6t^2}{(t - 2t^2)} = 0 \Rightarrow \frac{(1 + 2t)(1 - 3t)}{t(1 - 2t)} = 0$$

$$\Rightarrow \quad \begin{cases} t = -\dfrac{1}{2} \\ t = \dfrac{1}{3} \end{cases} \Rightarrow \begin{cases} \log x = -\dfrac{1}{2} \\ \log x = \dfrac{1}{3} \end{cases} \Rightarrow \begin{cases} x_1 = 10^{-1/2} \\ x_2 = 10^{1/3} \end{cases}$$

Hence, $x_1 = \dfrac{1}{\sqrt{10}}$ and $x_2 = \sqrt[3]{10}$ are the roots of the original equation.

Example 101. *Solve the equation*

$$\log_x^3 10 - 6\log_x^2 10 + 11\log_x 10 - 6 = 0.$$

Sol. Put $\log_x 10 = t$ in the given equation, we get

$$t^3 - 6t^2 + 11t - 6 = 0 \Rightarrow (t-1)(t-2)(t-3) = 0,$$

then $\begin{cases} t = 1 \\ t = 2 \\ t = 3 \end{cases}$

It follows that

$$\begin{cases} \log_x 10 = 1 \\ \log_x 10 = 2 \\ \log_x 10 = 3 \end{cases} \Rightarrow \begin{cases} x = 10 \\ x^2 = 10 \\ x^3 = 10 \end{cases} \Rightarrow \begin{cases} x = 10 \\ x = \sqrt{10} \\ x = \sqrt[3]{10} \end{cases} \quad [\because x > 0 \text{ and} \neq 1]$$

$[\because x > 0 \text{ and } \neq 1]$

$\therefore$ $x_1 = 10$, $x_2 = \sqrt{10}$ and $x_3 = \sqrt[3]{10}$ are the roots of the original equation.

***Form* 3** Equations of the form

(i) $\boldsymbol{\log_a f(x) = \log_a g(x), a > 0, a \neq 1}$ is equivalent to two ways.

Method I $\begin{cases} g(x) > 0 \\ f(x) = g(x) \end{cases}$

Method II $\begin{cases} f(x) > 0 \\ f(x) = g(x) \end{cases}$

(ii) $\boldsymbol{\log_{f(x)} A = \log_{g(x)} A, A > 0}$ is equivalent to two ways.

Method I $\begin{cases} g(x) > 0 \\ g(x) \neq 1 \\ f(x) = g(x) \end{cases}$

Method II $\begin{cases} f(x) > 0 \\ f(x) \neq 1 \\ f(x) = g(x) \end{cases}$

Example 102. *Solve the equation*

$$\log_{1/3}\left[2\left(\frac{1}{2}\right)^x - 1\right] = \log_{1/3}\left[\left(\frac{1}{4}\right)^x - 4\right].$$

Sol. The given equation is equivalent to

$$\begin{cases} 2\left(\frac{1}{2}\right)^x - 1 > 0 \\ 2\left(\frac{1}{2}\right)^x - 1 = \left(\frac{1}{4}\right)^x - 4 \end{cases}$$

$$\Rightarrow \begin{cases} \left(\frac{1}{2}\right)^x > \frac{1}{2} \\ \left(\frac{1}{2}\right)^{2x} - 2\left(\frac{1}{2}\right)^x - 3 = 0 \end{cases}$$

$$\Rightarrow \begin{cases} x < 1 \\ \left[\left(\frac{1}{2}\right)^x - 3\right]\left[\left(\frac{1}{2}\right)^x + 1\right] = 0 \end{cases}$$

$$\Rightarrow \begin{cases} x < 1 \\ \left(\frac{1}{2}\right)^x = 3, \left(\frac{1}{2}\right)^x + 1 \neq 0 \end{cases} \Rightarrow \begin{cases} x < 1 \\ x = (-\log_2 3) \end{cases}$$

Hence, $x_1 = -\log_2 3$ is the root of the original equation.

Example 103. Solve the equation $\log_{\left(\frac{2+x}{10}\right)} 7 = \log_{\left(\frac{2}{x+1}\right)} 7$.

Sol. The given equation is equivalent to

$$\begin{cases} \frac{2}{x+1} > 0 \\ \frac{2}{x+1} \neq 1 \\ \frac{2+x}{10} = \frac{2}{x+1} \end{cases} \Rightarrow \begin{cases} x + 1 > 0 \\ x \neq 1 \\ x = -6, 3 \end{cases}$$

$\therefore$ $x_1 = 3$ is root of the original equation.

***Form* 4** Equations of the form

(i) $\boldsymbol{\log_{f(x)} g(x) = \log_{f(x)} h(x)}$ is equivalent to two ways.

Method I $\begin{cases} g(x) > 0 \\ f(x) > 0 \\ f(x) \neq 1 \\ g(x) = h(x) \end{cases}$ Method II $\begin{cases} h(x) > 0 \\ f(x) > 0 \\ f(x) \neq 1 \\ g(x) = h(x) \end{cases}$

(ii) $\boldsymbol{\log_{g(x)} f(x) = \log_{h(x)} f(x)}$ is equivalent to two ways.

Method I $\begin{cases} f(x) > 0 \\ g(x) > 0 \\ g(x) \neq 1 \\ g(x) = h(x) \end{cases}$

Method II $\begin{cases} f(x) > 0 \\ h(x) > 0 \\ h(x) \neq 1 \\ g(x) = h(x) \end{cases}$

Example 104. Solve the equation

$$\log_{(x^2-1)}(x^3 + 6) = \log_{(x^2-1)}(2x^2 + 5x).$$

Sol. This equation is equivalent to the system

$$\begin{cases} 2x^2 + 5x > 0 \\ x^2 - 1 > 0 \\ x^2 - 1 \neq 1 \\ x^3 + 6 = 2x^2 + 5x \end{cases} \Rightarrow \begin{cases} x < -\frac{5}{2} \text{ and } x > 0 \\ x < -1 \text{ and } x > 1 \\ x \neq \pm\sqrt{2} \\ x = -2, 1, 3 \end{cases}$$

Hence, $x_1 = 3$ is only root of the original equation.

Example 105. Solve the equation $\log_{(x^3+6)}(x^2-1)=\log_{(2x^2+5x)}(x^2-1)$.

Sol. This equation is equivalent to

$$\begin{cases} x^2-1>0 \\ 2x^2+5x>0 \\ 2x^2+5x\neq 1 \\ x^3+6=2x^2+5x \end{cases}$$

$$\Rightarrow \begin{cases} x<-1 \text{ and } x>1 \\ x<-\frac{5}{2} \text{ and } x>0 \\ x\neq \frac{-5\pm\sqrt{33}}{4} \\ x=-2,1,3 \end{cases}$$

Hence, $x_1=3$ is only root of the original equation.

***Form* 5 An equation of the form**
$\log_{h(x)}(\log_{g(x)} f(x))=0$ is equivalent to the system

$$\begin{cases} h(x)>0 \\ h(x)\neq 1 \\ g(x)>0 \\ g(x)\neq 1 \\ f(x)=g(x) \end{cases}$$

Example 106. ***Solve the equation*** $\log_{x^2-6x+8}[\log_{2x^2-2x+8}(x^2+5x)]=0$.

Sol. This equation is equivalent to the system

$$\begin{cases} x^2-6x+8>0 \\ x^2-6x+8\neq 1 \\ 2x^2-2x-8>0 \\ 2x^2-2x-8\neq 1 \\ x^2+5x=2x^2-2x-8 \end{cases}$$

Solve the equations of this system

$$\Rightarrow \begin{cases} x<2 \text{ and } x>4 \\ x\neq 3\pm\sqrt{2} \\ x<\frac{1-\sqrt{17}}{2} \text{ and } x>\frac{1+\sqrt{17}}{2} \\ x\neq\frac{1\pm\sqrt{19}}{2} \\ x=-1,8 \end{cases}$$

$x=-1$, does not satisfy the third relation of this system.
Hence, $x_1=8$ is only root of the original equation.

***Form* 6 An equation of the form**
$2m\log_a f(x)=\log_a g(x), a>0, a\neq 1, m\in N$ is equivalent to the system

$$\begin{cases} f(x)>0 \\ f^{2m}(x)=g(x) \end{cases}$$

Example 107. ***Solve the equation***

$$2\log 2x=\log(7x-2-2x^2).$$

Sol. This equation is equivalent to the system

$$\begin{cases} 2x>0 \\ (2x)^2=7x-2-2x^2 \end{cases}$$

$$\Rightarrow \begin{cases} x>0 \\ 6x^2-7x+2=0 \end{cases}$$

$$\Rightarrow \begin{cases} x>0 \\ (x-1/2)(x-2/3)=0 \end{cases}$$

$$\Rightarrow \begin{cases} x=1/2 \\ x=2/3 \end{cases}$$

Hence, $x_1=1/2$ and $x_2=2/3$ are the roots of the original equation.

***Form* 7 An equation of the form**

$$(2m+1)\log_a f(x)=\log_a g(x), a>0, a\neq 1, m\in N$$

is equivalent to the system $\begin{cases} g(x)>0 \\ f^{2m+1}(x)=g(x) \end{cases}$.

Example 108. ***Solve the equation*** $\log(3x^2+x-2)=3\log(3x-2)$.

Sol. This equation is equivalent to the system

$$\begin{cases} 3x^2+x-2>0 \\ 3x^2+x-2=(3x-2)^3 \end{cases}$$

$$\Rightarrow \begin{cases} (x-2/3)(x-2)>0 \\ (x-2/3)(9x^2-13x+3)=0 \end{cases}$$

$$\Rightarrow \begin{cases} x<2/3 \text{ and } x>2 \\ x=\frac{2}{3},\ x=\frac{13\pm\sqrt{61}}{18} \end{cases}$$

Original equation has the only root $x_1=\frac{13-\sqrt{61}}{18}$.

***Form* 8 An equation of the form**

$$\log_a f(x)+\log_a g(x)=\log_a m(x), a>0, a\neq 1$$

is equivalent to the system

$$\begin{cases} f(x)>0 \\ g(x)>0 \\ f(x)\,g(x)=m(x) \end{cases}$$

Example 109. ***Solve the equation*** $2\log_3 x+\log_3(x^2-3)=\log_3 0.5+5^{\log_5(\log_3 8)}$

Sol. This equation can be written as

$$\log_3 x^2+\log_3(x^2-3)=\log_3 0.5+\log_3 8$$

$$\log_3 x^2+\log_3(x^2-3)=\log_3(4)$$

This is equivalent to the system

$$\begin{cases} x^2 > 0 \\ x^2 - 3 > 0 \\ x^2(x^2 - 3) = 4 \end{cases} \Rightarrow \begin{cases} x < 0 \text{ and } x > 0 \\ x < -\sqrt{3} \text{ and } x > \sqrt{3} \\ (x^2 - 4)(x^2 + 1) = 0 \end{cases}$$

$$\Rightarrow \quad x^2 - 4 = 0 \quad \therefore \quad x = \pm 2, \text{but } x > 0$$

Consequently, $x_1 = 2$ is only root of the original equation.

***Form* 9 An equation of the form**

$\log_a f(x) - \log_a g(x) = \log_a h(x) - \log_a t(x), a > 0, a \neq 1$

is equivalent to the equation

$\log_a f(x) + \log_a t(x) = \log_a g(x) + \log_a h(x)$,

which is equivalent to the system

$$\begin{cases} f(x) > 0 \\ t(x) > 0 \\ g(x) > 0 \\ h(x) > 0 \\ f(x) \cdot t(x) = g(x) \cdot h(x) \end{cases}$$

Example 110. Solve the equation

$$\log_2(3 - x) - \log_2\left(\frac{\sin\frac{3\pi}{4}}{5 - x}\right) = \frac{1}{2} + \log_2(x + 7).$$

Sol. This equation is equivalent to

$$\log_2(3 - x) = \log_2\left(\frac{\sin\frac{3\pi}{4}}{5 - x}\right) + \frac{1}{2}\log_2 2 + \log_2(x + 7)$$

$$\Rightarrow \log_2(3 - x) = \log_2\left(\frac{1}{\sqrt{2}(5 - x)}\right) + \log_2\sqrt{2} + \log_2(x + 7)$$

which is equivalent to the system

$$\begin{cases} 3 - x > 0 \\ \dfrac{1}{\sqrt{2}(5 - x)} > 0 \\ x + 7 > 0 \\ (3 - x) = \dfrac{\sqrt{2}(x + 7)}{\sqrt{2}(5 - x)} \end{cases}$$

$$\Rightarrow \begin{cases} x < 3 \\ x < 5 \\ x > -7 \\ (x - 1)(x - 8) = 0 \end{cases}$$

Hence, $x_1 = 1$ is only root of the original equation.

Logarithmic Inequations

When we solve logarithmic inequations

(i) $\begin{cases} \log_a f(x) > \log_a g(x) \\ a > 1 \end{cases}$

$$\Rightarrow \begin{cases} g(x) > 0 \\ a > 1 \\ f(x) > g(x) \end{cases}$$

(ii) $\begin{cases} \log_a f(x) > \log_a g(x) \\ 0 < a < 1 \end{cases}$

$$\Rightarrow \begin{cases} f(x) > 0 \\ 0 < a < 1 \\ f(x) < g(x) \end{cases}$$

Example 111. Solve the inequation

$$\log_{2x+3} x^2 < \log_{2x+3}(2x + 3).$$

Sol. This inequation is equivalent to the collection of the systems

$$\left[\begin{array}{l} \begin{cases} 2x + 3 > 1 \\ x^2 < 2x + 3 \end{cases} \\ \begin{cases} 0 < 2x + 3 < 1 \\ x^2 > 2x + 3 \end{cases} \end{array}\right. \Rightarrow \left[\begin{array}{l} \begin{cases} x > -1 \\ (x - 3)(x + 1) < 0 \end{cases} \\ \begin{cases} -\frac{3}{2} < x < -1 \\ (x - 3)(x + 1) > 0 \end{cases} \end{array}\right.$$

$$\Rightarrow \left[\begin{array}{ll} \begin{cases} x > -1 \\ -1 < x < 3 \end{cases} & \Rightarrow -1 < x < 3 \\ \begin{cases} -\frac{3}{2} < x < -1 \\ x < -1 \text{ and } x > 3 \end{cases} & \Rightarrow -\frac{3}{2} < x < -1 \end{array}\right.$$

Hence, the solution of the original inequation is

$$x \in \left(-\frac{3}{2}, -1\right) \cup (-1, 3).$$

Canonical Logarithmic Inequalities

1. $\begin{cases} \log_a x > 0 \\ a > 1 \end{cases} \Rightarrow \begin{cases} x > 1 \\ a > 1 \end{cases}$

2. $\begin{cases} \log_a x > 0 \\ 0 < a < 1 \end{cases} \Rightarrow \begin{cases} 0 < x < 1 \\ 0 < a < 1 \end{cases}$

3. $\begin{cases} \log_a x < 0 \\ a > 1 \end{cases} \Rightarrow \begin{cases} 0 < x < 1 \\ a > 1 \end{cases}$

4. $\begin{cases} \log_a x < 0 \\ 0 < a < 1 \end{cases} \Rightarrow \begin{cases} x > 1 \\ 0 < a < 1 \end{cases}$

Some Standard Forms to Solve Logarithmic Inequations

***Form* 1 Inequations of the form**

Forms		Collection of systems
(a) $\log_{g(x)} f(x) > 0$	$\Leftrightarrow$	$\begin{cases} f(x)>1, \\ g(x)>1, \end{cases} \begin{cases} 0<f(x)<1 \\ 0<g(x)<1 \end{cases}$
(b) $\log_{g(x)} f(x) \ge 0$	$\Leftrightarrow$	$\begin{cases} f(x)\ge 1, \\ g(x)>1, \end{cases} \begin{cases} 0<f(x)\le 1 \\ 0<g(x)<1 \end{cases}$
(c) $\log_{g(x)} f(x) < 0$	$\Leftrightarrow$	$\begin{cases} f(x)>1, \\ 0<g(x)<1, \end{cases} \begin{cases} 0<f(x)<1 \\ g(x)>1 \end{cases}$
(d) $\log_{g(x)} f(x) \le 0$	$\Leftrightarrow$	$\begin{cases} f(x)\ge 1, \\ 0<g(x)<1, \end{cases} \begin{cases} 0<f(x)\le 1 \\ g(x)>1 \end{cases}$

Example 112. *Solve the inequation*

$$\log_{\left(\frac{x^2-12x+30}{10}\right)}\left(\log_2 \frac{2x}{5}\right) > 0.$$

Sol. This inequation is equivalent to the collection of two systems

$$\begin{cases} \dfrac{x^2-12x+30}{10} > 1, \\ \log_2\left(\dfrac{2x}{5}\right) > 1, \end{cases}$$

$$\begin{cases} 0 < \dfrac{x^2-12x+30}{10} < 1 \\ 0 < \log_2\left(\dfrac{2x}{5}\right) < 1 \end{cases}$$

On solving the first system, we have

$$\Rightarrow \begin{cases} x^2-12x+20>0 \\ \dfrac{2x}{5} > 2 \end{cases}$$

$$\Leftrightarrow \begin{cases} (x-10)(x-2)>0 \\ x>5 \end{cases}$$

$$\Leftrightarrow \begin{cases} x<2 \text{ and } x>10 \\ x>5 \end{cases}$$

Therefore, the system has solution $x > 10$.

On solving the second system, we have

$$\Rightarrow \begin{cases} 0<x^2-12x+30<10 \\ 1<\dfrac{2x}{5}<2 \end{cases}$$

$$\Leftrightarrow \begin{cases} x^2-12x+30>0 \text{ and } x^2-12x+20<0 \\ 5/2<x<5 \end{cases}$$

$$\Leftrightarrow \begin{cases} x<6-\sqrt{6} \text{ and } x>6+\sqrt{6} \text{ and } 2<x<10 \\ 0<x<5 \end{cases}$$

Therefore, the system has solution $2 < x < 6-\sqrt{6}$ combining both systems, then solution of the original inequations is

$x \in (2, 6-\sqrt{6}) \cup (10, \infty)$.

***Form* 2** Inequations of the form

Forms		Collection of systems
(a) $\log_{\phi(x)} f(x) > \log_{\phi(x)} g(x)$	$\Leftrightarrow$	$\begin{cases} f(x)>g(x), \\ g(x)>0, \\ \phi(x)>1, \end{cases}$ $\begin{cases} f(x)<g(x) \\ f(x)>0 \\ 0<\phi(x)<1 \end{cases}$
(b) $\log_{\phi(x)} f(x) \ge \log_{\phi(x)} g(x)$	$\Leftrightarrow$	$\begin{cases} f(x)\ge g(x), \\ g(x)>0, \\ \phi(x)>1, \end{cases}$ $\begin{cases} f(x)\le g(x) \\ f(x)>0 \\ 0<\phi(x)<1 \end{cases}$
(c) $\log_{\phi(x)} f(x) < \log_{\phi(x)} g(x)$	$\Leftrightarrow$	$\begin{cases} f(x)<g(x), \\ f(x)>0, \\ \phi(x)>1, \end{cases}$ $\begin{cases} f(x)>g(x) \\ g(x)>0 \\ 0<\phi(x)<1 \end{cases}$
(d) $\log_{\phi(x)} f(x) \le \log_{\phi(x)} g(x)$	$\Leftrightarrow$	$\begin{cases} f(x)\le g(x), \\ f(x)>0, \\ \phi(x)>1, \end{cases}$ $\begin{cases} f(x)\ge g(x) \\ g(x)>0 \\ 0<\phi(x)<1 \end{cases}$

Example 113. Solve the inequation

$$\log_{(x-3)}(2(x^2-10x+24)) \ge \log_{(x-3)}(x^2-9).$$

Sol. This inequation is equivalent to the collection of systems

$$\begin{cases} 2(x^2-10x+24) \ge x^2-9, \\ x^2-9>0, \\ x-3>1, \end{cases}$$

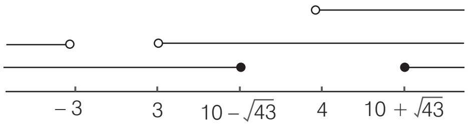
$$\begin{cases} 2(x^2 - 10x + 24) \le x^2 - 9 \\ 2(x^2 - 10x + 24) > 0 \\ 0 < x - 3 < 1 \end{cases}$$

On solving the first system, we have

$$\begin{cases} x^2 - 20x + 57 \ge 0, \\ (x+3)(x-3) > 0, \\ x > 4, \end{cases}$$

$$\Leftrightarrow \begin{cases} x \in (-\infty, 10 - \sqrt{43}] \cup [10 + \sqrt{43}, \infty) \\ x \in (-\infty, -3) \cup (3, \infty) \\ x \in (4, \infty) \end{cases}$$

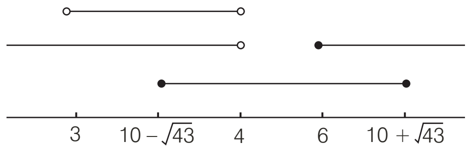
Therefore, the system has solution



$$x \ge 10 + \sqrt{43}$$

i.e. $x \in [10 + \sqrt{43}, \infty)$

On solving the second system, we have



$$\begin{cases} x^2 - 20x + 57 \le 0, \\ (x-6)(x-4) > 0, \\ 3 < x < 4, \end{cases}$$

$$\Leftrightarrow \begin{cases} x \in [10 - \sqrt{43}, 10 + \sqrt{43}] \\ x \in (-\infty, 4) \cup (6, \infty) \\ x \in (3, 4) \end{cases}$$

Therefore, the system has solution

$$10 - \sqrt{43} \le x < 4,$$

i.e., $x \in [10 - \sqrt{43}, 4)$

On combining the both systems, the solution of the original inequation is

$$x \in [10 - \sqrt{43}, 4) \cup [10 + \sqrt{43}, \infty).$$

Exercise for Session 5

1. The equation $\sqrt{(x+1)} - \sqrt{(x-1)} = \sqrt{(4x-1)}$ has

(a) no solution (b) one solution (c) two solutions (d) more than two solutions

2. The number of real solutions of $\sqrt{(x^2 - 4x + 3)} + \sqrt{(x^2 - 9)} = \sqrt{(4x^2 - 14x + 6)}$ is

(a) one (b) two (c) three (d) None of these

3. The number of real solutions of $\sqrt{(3x^2 - 7x - 30)} - \sqrt{(2x^2 - 7x - 5)} = x - 5$ is

(a) one (b) two (c) three (d) None of these

4. The number of integral values of x satisfying $\sqrt{(-x^2 + 10x - 16)} < x - 2$ is

(a) 0 (b) 1 (c) 2 (d) 3

5. The number of real solutions of the equation $\left(\frac{9}{10}\right)^x = -3 + x - x^2$ is

(a) 2 (b) 1 (c) 0 (d) None of these

6. The set of all x satisfying $3^{2x} - 3^x - 6 > 0$ is given by

(a) $0 < x < 1$ (b) $x > 1$ (c) $x > 3^{-2}$ (d) None of these

7. The number of real solutions of the equation $2^{x/2} + (\sqrt{2} + 1)^x = (3 + 2\sqrt{2})^{x/2}$ is

(a) one (b) two (c) four (d) infinite

8. The sum of the values of x satisfying the equation $(31 + 8\sqrt{15})^{x^2 - 3} + 1 = (32 + 8\sqrt{15})^{x^2 - 3}$ is

(a) 3 (b) 0 (c) 2 (d) None of these

9. The number of real solutions of the equation $\log_{0.5} x = |x|$ is

(a) 0 (b) 1 (c) 2 (d) None of these

10. The inequality $(x - 1)\ln(2 - x) < 0$ holds, if x satisfies

(a) $1 < x < 2$ (b) $x > 0$ (c) $0 < x < 1$ (d) None of these

Answers

Exercise for Session 5

1.(a) **2.** (a) **3.** (b) **4.** (c) **5.**(c) **6.** (b)
7.(a) **8.** (b) **9.** (b) **10.** (d)