JEE Type Solved Examples :
Single Option Correct Type Questions

- This section contains **10 multiple choice examples.** Each example has four choices (a), (b), (c) and (d) out of which **ONLY ONE** is correct.

● **Ex. 1** *If* $\binom{2n+1}{0}+\binom{2n+1}{3}+\binom{2n+1}{6}+\ldots=170$, *then n equals*

(a) 2 (b) 4 (c) 6 (d) 8

Sol. (b) $\because$ $(1+x)^{2n+1}={}^{2n+1}C_0+{}^{2n+1}C_1x+{}^{2n+1}C_2x^2+{}^{2n+1}C_3x^3$
$+{}^{2n+1}C_4x^4+{}^{2n+1}C_5x^5+{}^{2n+1}C_6x^6+\ldots$

Putting $x=1,\omega,\omega^2$ (where ω is cube root of unity) and adding, we get

$$2^{2n+1}+(1+\omega)^{2n+1}+(1+\omega^2)^{2n+1}=3({}^{2n+1}C_0+{}^{2n+1}C_3+{}^{2n+1}C_6+\ldots$$

$$\Rightarrow\quad 2^{2n+1}-\omega^{2(2n+1)}-\omega^{2n+1}=3({}^{2n+1}C_0+{}^{2n+1}C_3+{}^{2n+1}C_6+\ldots)\ [\because 1+\omega+\omega^2=0]$$

$$\Rightarrow\quad {}^{2n+1}C_0+{}^{1+2n}C_3+{}^{2n+1}C_6+\ldots=\frac{1}{3}(2^{2n+1}-\omega^{2(2n+1)}-\omega^{2n+1})$$

$$\Rightarrow\quad \binom{2n+1}{0}+\binom{2n+1}{3}+\binom{2n+1}{6}+\ldots=\frac{1}{3}(2^{2n+1}-\omega^{2(2n+1)}-\omega^{2n+1})$$

$$\Rightarrow\quad 170=\frac{1}{3}(2^{2n+1}-\omega^{2(2n+1)}-\omega^{2n+1})$$

For $n=4$, $170=\frac{1}{3}(512-1-1)=170$ $[\because \omega^3=1]$

Hence, $n=4$

● **Ex. 2** $({}^mC_0+{}^mC_1-{}^mC_2-{}^mC_3)+({}^mC_4+{}^mC_5-{}^mC_6-{}^mC_7)+\ldots=0$ *if and only if for some positive integer k, m is equal to*

(a) $4k$ (b) $4k+1$ (c) $4k-1$ (d) $4k+2$

Sol. (c) If $\theta\in R$ and $i=\sqrt{-1}$, then $(\cos\theta+i\sin\theta)^n$

$$={}^mC_0(\cos\theta)^m+{}^mC_1(\cos\theta)^{m-1}(i\sin\theta)+{}^mC_2(\cos\theta)^{m-2}(i\sin\theta)^2+\ldots+{}^mC_m(i\sin\theta)^m$$

$$(\cos m\theta+i\sin m\theta)=[{}^mC_0(\cos\theta)^m-{}^mC_2(\cos\theta)^{m-2}\cdot\sin^2\theta+{}^mC_4(\cos\theta)^{m-4}\sin^4\theta-\ldots]+i[{}^mC_1(\cos\theta)^{m-1}\cdot\sin\theta-{}^mC_3(\cos\theta)^{m-3}\sin^3\theta+\ldots]$$

[using Demoivre's theorem]

Comparing real and imaginary parts, we get

$$\cos m\theta={}^mC_0(\cos\theta)^m-{}^mC_2(\cos\theta)^{m-2}\sin^2\theta+{}^mC_4(\cos\theta)^{m-4}\sin^4\theta-\ldots\quad\ldots\text{(i)}$$

$$\sin m\theta={}^mC_1(\cos\theta)^{m-1}\cdot\sin\theta-{}^mC_3(\cos\theta)^{m-3}\cdot\sin^3\theta+\ldots\quad\ldots\text{(ii)}$$

On adding Eqs. (i) and (ii), we get

$$\cos m\theta+\sin m\theta={}^mC_0(\cos\theta)^m+{}^mC_1(\cos\theta)^{m-1}\cdot\sin\theta-{}^mC_2(\cos\theta)^{m-2}\sin^2\theta-{}^mC_3(\cos\theta)^{m-3}\sin^3\theta+{}^mC_4(\cos\theta)^{m-4}\sin^4\theta+\ldots\sin\left(m\theta+\frac{\pi}{4}\right)$$

$$=(\cos\theta)^m\left\{\begin{matrix}{}^mC_0+{}^mC_1\tan\theta-{}^mC_2\tan^2\theta-{}^mC_3\tan^3\theta\\+{}^mC_4\tan^4\theta+{}^mC_5\tan^5\theta-\ldots\end{matrix}\right\}$$

Putting $\theta=\frac{\pi}{4}$, $\sqrt{2}\sin\left(\frac{(m+1)\pi}{4}\right)=\frac{1}{2^{m/2}}$

$$\left\{\begin{matrix}({}^mC_0+{}^mC_1-{}^mC_2-{}^mC_3)+({}^mC_4+{}^mC_5-{}^mC_6-{}^mC_7)\\+\ldots+({}^mC_{m-3}+{}^mC_{m-2}-{}^mC_{m-1}-{}^mC_m)\end{matrix}\right\}$$

$\because({}^mC_0+{}^mC_1-{}^mC_2-{}^mC_3)+({}^mC_4+{}^mC_5-{}^mC_6-{}^mC_7)+\ldots=0$ [given]

$$\therefore\ \sin\left(\frac{(m+1)\pi}{4}\right)=0\Rightarrow\frac{(m+1)\pi}{4}=k\pi$$

or $\quad m=4k-1,\forall k\in I$

● **Ex. 3** *If coefficient of* x^n *in the expansion of* $(1+x)^{101}(1-x+x^2)^{100}$ *is non-zero, then n cannot be of the form*

(a) $3\lambda+1$ (b) 3λ (c) $3\lambda+2$ (d) $4\lambda+1$

Sol. (c) $\because(1+x)^{101}(1-x+x^2)^{100}=(1+x)((1+x)(1-x+x^2))^{100}$

$=(1+x)(1+x^3)^{100}$

$=(1+x)(1+{}^{100}C_1x^3+{}^{100}C_2x^6+{}^{100}C_3x^9+\ldots+\ldots+{}^{100}C_{10}x^{300})$

Clearly, in this expression x^3 will present if $n=3\lambda$ or $n=3\lambda+1$. So, n cannot be of the form $3\lambda+2$.

● **Ex. 4** *The sum* $\sum_{i=0}^{m}\binom{10}{i}\binom{20}{m-i}$, *(where* $\frac{p}{q}=0$, *if* $p<q$*) is maximum when m is*

(a) 5 (b) 10 (c) 15 (d) 20

Sol. (c) $\sum_{i=0}^{m}\binom{10}{i}\binom{20}{m-i}=\sum_{i=0}^{m}{}^{10}C_i\,{}^{20}C_{m-i}$

$={}^{10}C_0\cdot{}^{20}C_m+{}^{10}C_1\cdot{}^{20}C_{m-1}+{}^{10}C_2\cdot{}^{20}C_{m-2}+\ldots+{}^{10}C_m\cdot{}^{20}C_0$

= Coefficient of x^m in the expansion of product $(1+x)^{10}(1+x)^{20}$

= Coefficient of x^m in the expansion of $(1+x)^{30}={}^{30}C_m$

To get maximum value of the given sum, ${}^{30}C_m$ should be maximum. Which is so, when $m=\frac{30}{2}=15$

Ex. 5 *If* $^{n-1}C_r = (k^2-3)\cdot {}^nC_{r+1}$ *then k belongs to*

(a) $(-\infty, -2]$ (b) $[2, \infty)$
(c) $[-\sqrt{3}, \sqrt{3}]$ (d) $(\sqrt{3}, 2]$

Sol. (d) $\because {}^{n-1}C_r = (k^2-3)\cdot {}^nC_{r+1}$

$$\Rightarrow \quad k^2-3 = \frac{^{n-1}C_r}{^nC_{r+1}} = \frac{r+1}{n} \quad \text{...(i)}$$

$$\Rightarrow \quad 0 \le r \le n-1$$

$$\Rightarrow \quad 1 \le r+1 \le n$$

$$\Rightarrow \quad \frac{1}{n} \le \frac{r+1}{n} \le 1$$

$$\Rightarrow \frac{1}{n} \le (k^2-3) \le 1$$

$$\Rightarrow \quad 3+\frac{1}{n} \le k^2 \le 4 \quad \text{or} \quad 3 < k^2 \le 4 \qquad [\text{here, } n \ge 2]$$

$$\therefore \quad k \in [-2, \sqrt{3}) \cup (\sqrt{3}, 2]$$

Ex. 6 *If* $\left(x+\frac{1}{x}+1\right)^6 = a_0 + \left(a_1x+\frac{b_1}{x}\right) + \left(a_2x^2+\frac{b_2}{x^2}\right) + \ldots + \left(a_6x^6+\frac{b_6}{x^6}\right)$, *the value of* a_0 *is*

(a) 121 (b) 131
(c) 141 (d) 151

Sol. (c) $\because \left(x+\frac{1}{x}+1\right)^6 = \sum_{r=0}^{6} {}^6C_r\left(x+\frac{1}{x}\right)^r$ for constant term r must be even integer.

$\therefore a_0 = {}^6C_0 + {}^6C_2 \times {}^2C_1 + {}^6C_4 \times {}^4C_2 + {}^6C_6 \times {}^6C_3$

$= 1+30+90+20 = 141$

Ex. 7 *The coefficient of* x^{50} *in the series*

$$\sum_{r=1}^{101} rx^{r-1}(1+x)^{101-r} \text{ is}$$

(a) $^{100}C_{50}$ (b) $^{101}C_{50}$
(c) $^{102}C_{50}$ (d) $^{103}C_{50}$

Sol. (c) Let $S = \sum_{r=1}^{101} rx^{r-1}(1+x)^{101-r}$

$= (1+x)^{100} + 2x(1+x)^{99} + 3x^2(1+x)^{98} + \ldots + 101x^{100}$

$$S = (1+x)^{100}\left\{1+2\left(\frac{x}{1+x}\right)+3\left(\frac{x}{1+x}\right)^2+\ldots+101\left(\frac{x}{1+x}\right)^{100}\right\} \quad \text{...(i)}$$

$$\therefore \frac{Sx}{(1+x)} = (1+x)^{100}\left\{\left(\frac{x}{1+x}\right)+2\left(\frac{x}{1+x}\right)^2 +3\left(\frac{x}{1+x}\right)^3+\ldots+101\left(\frac{x}{1+x}\right)^{101}\right\} \quad \text{...(ii)}$$

On subtracting Eq. (ii) from Eq. (i), then we get

$$\frac{S}{(1+x)} = (1+x)^{100}\left\{1+\left(\frac{x}{1+x}\right)+\left(\frac{x}{1+x}\right)^2 +\ldots+\left(\frac{x}{1+x}\right)^{100} - 101\left(\frac{x}{1+x}\right)^{101}\right\}$$

$$= (1+x)^{100}\left\{\frac{1\cdot\left(1-\left(\frac{x}{1+x}\right)^{101}\right)}{1-\left(\frac{x}{1+x}\right)} - 101\left(\frac{x}{1+x}\right)^{101}\right\}$$

$$\therefore \quad S = (1+x)^{102} - x^{101}(1+x) - 101x^{101}$$

and coefficient of x^{50} in $S = {}^{102}C_{50}$.

Ex. 8 *The largest integer* λ *such that* 2^λ *divides* $3^{2^n}-1$, $n \in N$ *is*

(a) $n-1$ (b) n (c) $n+1$ (d) $n+2$

Sol. (d) $\because 3^{2^n}-1 = (4-1)^{2^n}-1$

$= (4^{2^n} - {}^{2^n}C_1\cdot 4^{2^n-1} + {}^{2^n}C_2\cdot 4^{2^n-2} - \ldots - {}^{2^n}C_{2^n-1}\cdot 4+1)-1$

$= 4^{2^n} - 2^n\cdot 4^{2^n-1} + \frac{2^n(2^n-1)}{2}\cdot 4^{2^n-2} - \ldots - 2^n\cdot 4$

$= 2^{n+2}(2^{2^{n+1}-n-2} - 2^{2^{n+1}-4} + \ldots - 1) = 2^{n+2}(\text{Integer})$

Hence, $3^{2^n}-1$ is divisible by $2^{n+2}\cdot \lambda = n+2$

Ex. 9 The last term in the binomial expansion of $\left(\sqrt[3]{2}-\frac{1}{\sqrt{2}}\right)^n$ is $\left(\frac{1}{3\sqrt[3]{9}}\right)^{\log_3 8}$, the 5th term from beginning is

(a) $^{10}C_6$ (b) $2\,{}^{10}C_4$
(c) $\frac{1}{2}\cdot{}^{10}C_4$ (d) None of the above

Sol. (a) Since, last term in the expansion of $\left(\sqrt[3]{2}-\frac{1}{\sqrt{2}}\right)^n$

$$= \left(\frac{1}{3\cdot\sqrt[3]{9}}\right)^{\log_3 8} \Rightarrow {}^nC_n\cdot\left(-\frac{1}{\sqrt{2}}\right)^n = \left(\frac{1}{3\cdot\sqrt[3]{9}}\right)^{\log_3 8}$$

$$\Rightarrow \quad (-1)^n\cdot\left(\frac{1}{2}\right)^{n/2} = \left(\frac{1}{3^{5/3}}\right)^{\log_3 8} = (3^{-5/3})^{\log_3 2^3}$$

$$= 3^{-\frac{5}{3}\times 3\times\log_3 2} = 3^{-5\log_3 2} = 3^{\log_3 2^{-5}} = 2^{-5} = \left(\frac{1}{2}\right)^5$$

$$\Rightarrow \quad (-1)^n\cdot\left(\frac{1}{2}\right)^{n/2} = \left(\frac{1}{2}\right)^5 \therefore \quad n = 10$$

Now, 5th term from beginning $= {}^{10}C_4(\sqrt[3]{2})^6\left(-\frac{1}{\sqrt{2}}\right)^4$

$$= {}^{10}C_4\cdot 2^2\cdot\frac{1}{2^2} = {}^{10}C_4 = {}^{10}C_6$$

● **Ex. 10** If $f(x)=\sum_{r=1}^{n}\{r^2({}^nC_r-{}^nC_{r-1})+(2r+1){}^nC_r\}$ *and* $f(30)=30(2)^{\lambda}$*, then the value of* λ *is*

(a) 3 (b) 4 (c) 5 (d) 6

Sol. (c) Here, $f(x)=\sum_{r=1}^{n}\{r^2({}^nC_r-{}^nC_{r-1})+(2r+1)^n C_r\}$

$$=\sum_{r=1}^{n}(r^2+2r+1)^nC_r-r^2\cdot{}^nC_{r-1}$$

$$=\sum_{r=1}^{n}((r+1)^2\cdot{}^nC_r-r^2\cdot{}^nC_{r-1})$$

$$=(n+1)^2\cdot{}^nC_n-1^2\cdot{}^nC_0$$

$$=(n+1)^2-1=(n^2+2n)$$

$\therefore$ $f(30)=(30)^2+2(30)=960$

$=30\times 32=30(2)^5=30(2)^{\lambda}$ [given]

Hence, $\lambda=5$

JEE Type Solved Examples :
More than One Correct Option Type Questions

- This section contains **5 multiple choice examples.** Each example has four choices (a), (b), (c) and (d) out of **which more than one** may be correct.

● **Ex. 11** *Let* $a_n=\left(1+\frac{1}{n}\right)^n$*. Then for each* $n\in N$

(a) $a_n\geq 2$ (b) $a_n<3$ (c) $a_n<4$ (d) $a_n<2$

Sol. (a, b, c)

$$\because\ a_n=\left(1+\frac{1}{n}\right)^n={}^nC_0+{}^nC_1\cdot\left(\frac{1}{n}\right)+\sum_{r=2}^{n}{}^nC_r\left(\frac{1}{n}\right)^2$$

$$=2+\sum_{r=2}^{n}{}^nC_r\left(\frac{1}{n}\right)^2$$

$\therefore$ $a_n\geq 2$ for all $n\in N$

Also, $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n=e=2.7182\ldots$

$\therefore$ $a_n<e$

Finally, $2\leq a_n<e$

● **Ex. 12** *Let* $S_n(x)=\sum_{k=0}^{n}{}^nC_k\sin(kx)\cos((n-k)x)$ *then*

(a) $S_5\left(\frac{\pi}{2}\right)=16$ (b) $S_7\left(\frac{-\pi}{2}\right)=64$

(c) $S_{50}(\pi)=0$ (d) $S_{51}(-\pi)=-2^{50}$

Sol. (a, b, c)

$$\because\ S_n(x)=\sum_{k=0}^{n}{}^nC_k\sin(kx)\cos((n-k)x) \quad \ldots\text{(i)}$$

Replace k by $n-k$ in Eq. (i), then

$$S_n(x)=\sum_{k=0}^{n}{}^nC_{n-k}\sin((n-k)x)\cos(kx)$$

or $$S_n(x)=\sum_{k=0}^{n}{}^nC_k\sin((n-k)x)\cos(kx) \quad \ldots\text{(ii)}$$

On adding Eqs. (i) and (ii), we get

$$2S_n(x)=\sum_{k=0}^{n}{}^nC_k\cdot\sin(nx)=2^n\cdot\sin(nx)$$

$$\Rightarrow\ S_n(x)=2^{n-1}\cdot\sin(nx)$$

$\therefore$ $S_5\left(\frac{\pi}{2}\right)=2^4\cdot\sin\left(\frac{5\pi}{2}\right)=16,$

$S_7\left(-\frac{\pi}{2}\right)=2^6\cdot\sin\left(-\frac{7\pi}{2}\right)=2^6\times -1\times -1=64$

$S_{50}(\pi)=2^{49}\cdot\sin(50\pi)=0$

and $S_{51}(-\pi)=2^{50}\cdot\sin(-51\pi)=0$

● **Ex. 13** *If* $a+b=k$*, when* $a,b>0$ *and* $S(k,n)=\sum_{r=0}^{n}r^2({}^nC_r)a^r\cdot b^{n-r}$*, then*

(a) $S(1,3)=3(3a^2+ab)$ (b) $S(2,4)=16(4a^2+ab)$

(c) $S(3,5)=25(5a^2+ab)$ (d) $S(4,6)=36(6a^2+ab)$

Sol. (a, b)

$$\because\ S(k,n)=\sum_{r=0}^{n}r^2\cdot({}^nC_r)a^r\cdot b^{n-r}$$

$$=b^n\sum_{r=0}^{n}r^2\cdot\left(\frac{n}{r}\cdot{}^{n-1}C_{r-1}\right)\cdot\left(\frac{a}{b}\right)^r$$

$$=nb^n\sum_{r=0}^{n}((r-1)+1)^{n-1}C_{r-1}\cdot\left(\frac{a}{b}\right)^r$$

$$=nb^n\sum_{r=0}^{n}((n-1)\cdot{}^{n-2}C_{r-2}+{}^{n-1}C_{r-1})\left(\frac{a}{b}\right)^r$$

$$=nb^n\cdot(n-1)\cdot\left(\frac{a}{b}\right)^2\sum_{r=0}^{n}{}^{n-2}C_{r-2}\left(\frac{a}{b}\right)^{r-2}$$

$$+nb^n\cdot\left(\frac{a}{b}\right)\sum_{r=0}^{n}{}^{n-1}C_{r-}\left(\frac{a}{b}\right)^{r-1}$$

$$=nb^n\cdot(n-1)\left(\frac{a}{b}\right)^2\left(1+\frac{a}{b}\right)^{n-2}+nb^n\cdot\left(\frac{a}{b}\right)\cdot\left(1+\frac{a}{b}\right)^{n-1}$$

$$=n(n-1)a^2k^{n-2}+nak^{n-1}$$

$$=n^2a^2k^{n-2}+nak^{n-2}(k-a)=n^2a^2k^{n-2}+nabk^{n-2}$$

$\therefore S(1,3)=9a^2+3ab=3(3a^2+ab)$ [$\because\ a+b=k$]

$S(2,4)=16(4a^2+ab)$

$S(3,5)=135(5a^2+ab)$

$S(4,6)=1536(6a^2+ab)$

Ex. 14 *The value of x, for which the ninth term in the expansion of* $\left\{\dfrac{\sqrt{10}}{(\sqrt{x})^{5\log_{10} x}} + x \cdot x^{\frac{1}{2\log_{10} x}}\right\}^{10}$ *is 450 is equal to*

(a) 10 (b) 10^2 (c) $\sqrt{10}$ (d) $10^{-2/5}$

Sol. (b, d) Let $\log_{10} x = \lambda \Rightarrow x = 10^{\lambda}$ ...(i)

Given, $T_9 = 450$

$$\Rightarrow {}^{10}C_8 \cdot \left(\frac{\sqrt{10}}{10^{\frac{5\lambda^2}{2}}}\right)^2 \cdot (10^{\lambda} \cdot 10^{1/2})^8 = 450$$

$$\Rightarrow {}^{10}C_2 \cdot \frac{10}{10^{5\lambda^2}} \cdot 10^{8\lambda} \cdot 10^4 = 450$$

$$\Rightarrow 10^{8\lambda + 4 - 5\lambda^2} = 1 = 10^0$$

$$\Rightarrow 8\lambda + 4 - 5\lambda^2 = 0$$

$$\Rightarrow 5\lambda^2 - 8\lambda - 4 = 0$$

$$\Rightarrow \lambda = 2, -2/5$$

$$\Rightarrow x = 10^2, 10^{-2/5} \quad \text{[from Eq. (i)]}$$

Ex. 15 *For a positive integer n, if the expansion of* $\left(\dfrac{5}{x^2} + x^4\right)$ *has a term independent of x, then n can be*

(a) 18 (b) 27 (c) 36 (d) 45

Sol. (a, b, c, d) Let $(r+1)$th term of $\left(\dfrac{5}{x^2} + x^4\right)^n$ be independent of x. We have, $T_{r+1} = {}^nC_r\left(\dfrac{5}{x^2}\right)^{n-r}(x^4)^r = {}^nC_r \cdot 5^{n-r} \cdot x^{6r-2n}$

For this term to be independent of x,

$$6r - 2n = 0 \text{ or } n = 3r$$

For $r = 6, 9, 12, 15,$

$$n = 18, 27, 36, 45.$$

JEE Type Solved Examples :
Passage Based Questions

- This section contains **2 solved passages** based upon each of the passage **3 multiple choice** examples have to be answered. Each of these examples has four choices (a), (b), (c) and (d) out of which **ONLY ONE** is correct.

Passage I (Ex. Nos. 16 to 18)

Consider $(1 + x + x^2)^n = \sum_{r=0}^{2n} a_r x^r$, *where* $a_0, a_1, a_2, \ldots, a_{2n}$ *are real numbers and n is a positive integer.*

16. The value of $\sum_{r=0}^{n-1} a_{2r}$ is

(a) $\dfrac{9^n - 2a_{2n} - 1}{4}$ (b) $\dfrac{9^n - 2a_{2n} + 1}{4}$

(c) $\dfrac{9^n + 2a_{2n} - 1}{4}$ (d) $\dfrac{9^n + 2a_{2n} + 1}{4}$

17. The value of $\sum_{r=1}^{n} a_{2r-1}$ is

(a) $\dfrac{9^n - 1}{2}$ (b) $\dfrac{9^n - 1}{4}$ (c) $\dfrac{9^n + 1}{2}$ (d) $\dfrac{9^n + 1}{4}$

18. The value of a_2 is

(a) ${}^{4n+1}C_2$ (b) ${}^{3n+1}C_2$

(c) ${}^{2n+1}C_2$ (d) ${}^{n+1}C_2$

Sol.

We have, $(1 + x + x^2)^{2n} = \sum_{r=0}^{4n} a_r x^r$ ...(i)

Replacing x by $\dfrac{1}{x}$ in Eq. (i), we get

$$\left(1 + \frac{1}{x} + \frac{1}{x^2}\right)^{2n} = \sum_{r=0}^{4n} a_r \left(\frac{1}{x}\right)^r$$

$$\Rightarrow (1 + x + x^2)^{2n} = \sum_{r=0}^{4n} a_r x^{4n-r} \quad \text{...(ii)}$$

From Eqs. (i) and (ii), we get $\sum_{r=0}^{4n} a_r x^r = \sum_{r=0}^{4n} a_r x^{4n-r}$

Equating the coefficient of x^{4n-r} on both sides, we get

$$a_{4n-r} = a_r \text{ for } 0 \le r \le 4n$$

Hence, $a_r = a_{4n-r}$

Putting $x = 1$ in Eq. (i), then

$$\sum_{r=0}^{4n} a_r = 3^{2n} = 9^n \quad \text{...(iii)}$$

Putting $x = -1$ in Eq. (i), then $\sum_{r=0}^{4n} (-1)^r a_r = 1$ ...(iv)

16. (b) On adding Eqs. (iii) and (iv), we get

$2(a_0 + a_2 + a_4 + \ldots + a_{2n-2} + a_{2n} + \ldots + a_{4n}) = 9^n + 1$

$\Rightarrow 2[2(a_0 + a_2 + a_4 + \ldots + a_{2n-2}) + a_{2n}) = 9^n + 1$

$[\because a_r = a_{4n-r}]$

$$\therefore \quad a_0 + a_2 + a_4 + \ldots + a_{2n-2} = \frac{9^n - 2a_{2n} + 1}{4}$$

$$\Rightarrow \sum_{r=0}^{n-1} a_{2r} = \frac{9^n - 2a_{2n} + 1}{4}$$

17. (b) On subtracting Eq. (iv) from Eq. (iii), we get

$2(a_1 + a_3 + a_5 + \ldots + a_{2n-1} + a_{2n+1} + \ldots + a_{4n-1}) = 9^n - 1$

$\Rightarrow 2[2(a_1 + a_3 + a_5 + \ldots + a_{2n-1}] = 9^n - 1 \quad [\because a_r = a_{4n-r}]$

$$\therefore\ a_1 + a_3 + a_5 + \dots + a_{2n-1} = \frac{9^n - 1}{4}$$

$$\Rightarrow \sum_{r=1}^{n} a_{2r-1} = \frac{9^n - 1}{4}$$

18. (c) $\because a_2 =$ Coefficient of x^2 in $(1 + x + x^2)^{2n}$

$$\therefore (1 + x + x^2)^{2n} = \sum_{\alpha+\beta+\gamma=2n} \frac{2n!}{\alpha!\beta!\gamma!}(1)^\alpha (x)^\beta (x^2)^\gamma$$

$$= \sum_{\alpha+\beta+\gamma=2n} \frac{2n!}{\alpha!\beta!\gamma!} x^{\beta+2\gamma}$$

For a_2, $\beta + 2\gamma = 2$

Possible values of α, β, γ are $(2n-2, 2, 0)$ and $(2n-1, 0, 1)$.

$$\therefore\ a_2 = \frac{2n!}{(2n-2)!\,2!\,0!} + \frac{2n!}{(2n-1)!\,0!\,1!}$$

$$= {}^{2n}C_2 + {}^{2n}C_1 = {}^{2n+1}C_2$$

Passage II

(Ex. Nos. 19 to 21)

Let $$S = \sum_{r=1}^{30} \frac{{}^{30+r}C_r\,(2r-1)}{{}^{30}C_r\,(30+r)}, K = \sum_{r=0}^{30} ({}^{30}C_r)^2$$

and $$G = \sum_{r=0}^{60} (-1)^r\,({}^{60}C_r)^2$$

19. The value of $(G - S)$ is

(a) 0 (b) 1 (c) 2^{30} (d) 2^{60}

20. The value of $(SK - SG)$ is

(a) 0 (b) 1

(c) 2^{30} (d) 2^{60}

21. The value of $K + G$ is

(a) $2S - 2$ (b) $2S - 1$

(c) $2S + 1$ (d) $2S + 2$

Sol.

$$\because S = \sum_{r=1}^{30} \frac{{}^{30+r}C_r\,(2r-1)}{{}^{30}C_r\,(30+r)} = \sum_{r=1}^{30} \frac{{}^{30+r}C_r}{{}^{30}C_r}\left(1 - \frac{30-r+1}{30+r}\right)$$

$$= \sum_{r=1}^{30}\left[\frac{{}^{30+r}C_r}{{}^{30}C_r} - \frac{{}^{30+r}C_r}{{}^{30}C_r}\cdot\frac{(30-r+1)}{(30+r)}\right]$$

$$= \sum_{r=1}^{30}\left[\frac{{}^{30+r}C_r}{{}^{30}C_r} - \frac{\frac{(30+r)}{r}\cdot {}^{29+r}C_{r-1}}{{}^{30}C_r}\cdot\frac{(31-r)}{30+r}\right]$$

$$= \sum_{r=1}^{30}\left[\frac{{}^{30+r}C_r}{{}^{30}C_r} - \frac{{}^{29+r}C_{r-1}}{{}^{30}C_{r-1}}\right] \quad \left[\because \frac{{}^nC_r}{{}^nC_{r-1}} = \frac{n-r+1}{r}\right]$$

For $n = 30\left(\frac{31-r}{r}\cdot {}^{30}C_r = {}^{30}C_{r-1}\right)$

$$= \frac{{}^{30+30}C_{30}}{{}^{30}C_{30}} - \frac{{}^{29+1}C_0}{{}^{30}C_0} = {}^{60}C_{30} - 1$$

$$K = \sum_{r=0}^{30} ({}^{30}C_r)^2 = {}^{60}C_{30} \text{ and } G = \sum_{r=0}^{60} (-1)^r ({}^{60}C_r)^2$$

$$= ({}^{60}C_0)^2 - ({}^{60}C_1)^2 + ({}^{60}C_2)^2 - \dots + ({}^{60}C_{60})^2 = {}^{60}C_{30}$$

[$\because n = 60$ is even]

19. (b) $G - S = {}^{60}C_{30} - ({}^{60}C_{30} - 1) = 1$

20. (a) $SK - SG = S(K - G) = S(G - G) = 0$ [$\because K = G$]

21. (d) $K + G = 2\cdot {}^{60}C_{30} = 2(S + 1) = 2S + 2$

JEE Type Solved Examples :
Single Integer Answer Type Questions

- This section contains **2 examples.** The answer to each example is a **single digit integer** ranging from **0** to **9** (both inclusive).

● ***Ex. 22*** *The digit at unit's place in* $2^{9^{100}}$ *is*

Sol. (2) $\because\ 9^{100} = (2\cdot 4 + 1)^{100} = 4n + 1$ [say]

[where n is positive integer]

$$\therefore\ 2^{9^{100}} = 2^{4n+1} = 2^{4n}\cdot 2 = (16)^n\cdot 2$$

The digit at unit's place in $(16)^n = 6$.

$\therefore$ The digit at unit's place in $(16)^n \cdot 2 = 2$

● ***Ex. 23*** *If* $(1+x)^n = \sum_{r=0}^{n} a_r x^r$, $b_r = 1 + \frac{a_r}{a_{r-1}}$ *and* $\prod_{r=1}^{n} b_r = \frac{(101)^{100}}{100!}$, *then the value of* $\frac{n}{20}$ *is*

Sol. (5) Here, $a_r = {}^nC_r$

$$\therefore\ b_r = 1 + \frac{a_r}{a_{r-1}} = 1 + \frac{{}^nC_r}{{}^nC_{r-1}}$$

$$= 1 + \frac{n-r+1}{r} = \frac{(n+1)}{r}$$

$$\Rightarrow \prod_{r=1}^{n} b_r = \prod_{r=1}^{n} \frac{(n+1)}{r}$$

$$= \frac{(n+1)}{1}\cdot\frac{(n+1)}{2}\cdot\frac{(n+1)}{3}\dots\frac{(n+1)}{n} = \frac{(n+1)^n}{n!}$$

$$= \frac{(101)^{100}}{100!} \quad \text{[given]}$$

$$\therefore\ n = 100 \Rightarrow \frac{n}{20} = 5$$

JEE Type Solved Examples : Matching Type Questions

- This section contains **2 examples.** Examples 24 and 25 have three statements (A, B and C) given in Column I and four statements (p, q, r and s) in **Column II.** Any given statement in **Column I** can have correct matching with one or more statement(s) given in **Column II.**

Ex. 24

	Column I		Column II
(A)	If m and n are the numbers of rational terms in the expansions of $(\sqrt{2}+3^{1/5})^{10}$ and $(\sqrt{3}+5^{1/8})^{256}$ respectively, then	(p)	$n-m=6$
(B)	If m and n are the numbers of irrational terms in the expansions of $(2^{1/2}+3^{1/5})^{40}$ and $(5^{1/10}+2^{1/6})^{100}$ respectively, then	(q)	$m+n=20$
(C)	If m and n are the numbers of rational terms in the expansions of $(1+\sqrt{2}+3^{1/3})^6$ and $(1+\sqrt[3]{2}+\sqrt[5]{3})^{15}$ respectively, then	(r)	$n-m=31$
		(s)	$m+n=35$
		(t)	$n-m=39$

Sol. (A) → (r, s); (B) → (t); (C) → (p, q)

(A) $\because$ $(\sqrt{2}+3^{1/5})^{10}=(2^{1/2}+3^{1/5})^{10}$

$$\therefore \quad T_{r+1}={}^{10}C_r\cdot 2^{\frac{10-r}{2}}\cdot 3^{\frac{r}{5}}$$

For rational terms, $r=0, 10$ $\quad[\because 0\le r\le 10]$

$\therefore$ Number of rational terms = 2

i.e., $m=2$ and $(\sqrt{3}+5^{1/8})^{256}=(3^{1/2}+5^{1/8})^{256}$

$$\therefore \quad T_{R+1}={}^{256}C_R\cdot 3^{\frac{256-R}{2}}\cdot 5^{R/8}$$

For rational terms, $r=0, 8, 16, 24, 32, \dots, 256$ $[\because 0\le r\le 256]$

$\therefore$ Number of rational terms $=1+32=33$

i.e., $n=33 \Rightarrow m+n=35$ (s) and $n-m=31$

(B) T_{r+1} in $(2^{1/3}+3^{1/5})^{40}={}^{40}C_r\cdot 2^{\frac{40-r}{3}}\cdot 3^{r/5}$

For rational terms, $r=10, 25, 40$ $[\because 0\le r\le 40]$

$\because$ Number of rational terms = 3

$\therefore$ Number of irrational terms

= Total terms − Number of rational terms

$=41-3=38$ i.e. $m=38$

and T_{R+1} in $(5^{1/10}+2^{1/6})^{100}={}^{100}C_R\cdot 5^{\frac{100-R}{10}}\cdot 2^{R/6}$

rational terms, $R=0, 30, 60, 90$ $\quad[\because 0\le R\le 100]$

$\because$ Number of rational terms = 4

$\therefore$ Number of irrational terms $=101-4=97$

i.e. $n=97 \Rightarrow m+n=100, n-m=97-38=39$

(C) $\because$ $(1+\sqrt{2}+3^{1/3})^6=(1+2^{1/2}+3^{1/3})^6$

$$=\sum_{\alpha+\beta+\gamma=6}\frac{6!}{\alpha!\,\beta!\,\gamma!}(1)^\alpha(2^{1/2})^\beta(3^{1/3})^\gamma$$

$$=\sum_{\alpha+\beta+\gamma=6}\frac{6!}{\alpha!\,\beta!\,\gamma!}2^{\beta/2}\cdot 3^{\gamma/3}$$

Values of (α,β,γ) for rational terms are $(0,0,6)$, $(1,2,3), (3,0,3), (0,6,0), (2,4,0), (4,2,0), (6,0,0)$.

$\therefore$ Number of rational terms = 7 i.e., $m=7$

and $(1+\sqrt[3]{2}+\sqrt[5]{3})^{15}=(1+2^{1/3}+3^{1/5})^{15}$

$$=\sum_{\alpha+\beta+\gamma=15}\frac{15!}{\alpha!\,\beta!\,\gamma!}(1)^\alpha(2^{1/3})^\beta(3^{1/5})^\gamma$$

$$=\sum_{\alpha+\beta+\gamma=15}\frac{15!}{\alpha!\,\beta!\,\gamma!}2^{\beta/3}\cdot 3^{\gamma/5}$$

of (α,β,γ) for rational terms are

$(5,0,10), (2,3,10), (10,0,5), (7,3,5), (4,6,5), (1,9,5), (15,0,0), (12,3,0), (9,6,0), (6,9,0), (3,12,0), (15,0,0)$.

$\therefore$ Number of rational terms = 13 i.e. $n=13$

Hence, $\quad m+n=20$ and $n-m=6$

Ex. 25 *If* $(1+x)^n=\sum_{r=0}^{n}C_r\,x^r$, *match the following.*

	Column I		Column II
(A)	If $S=\sum_{r=0}^{n}\lambda\, C_r$ and values of S are a, b, c for $\lambda=1, r, r^2$ respectively, then	(p)	$a=b+c$
(B)	If $S=\sum_{r=0}^{n}(-1)^r\ \lambda\, C_r$ and values of S are a, b, c for $\lambda=1, r, r^2$ respectively, then	(q)	$a+b=c+2$
(C)	If $S=\sum_{r=0}^{n}\frac{\lambda\, C_r}{(r+1)}$ and values of S are a, b, c for $\lambda=1, r, r^2$ respectively, then	(r)	$a^3+b^3+c^3=3abc$
		(s)	$b^{c-a}+(c-a)^b=1$
		(t)	$a+c=4b$

Sol. (A) → (p, q); (B) → (p, r, t); (C) → (s, t)

(A) For $\lambda=1$, $a=\sum_{r=0}^{n}C_r=2^n$

For $\lambda=r$, $b=\sum_{r=0}^{n}r\,C_r=\sum_{r=0}^{n}r\cdot\frac{n}{r}\cdot{}^{n-1}C_{r-1}$

$$=n\sum_{r=0}^{n}{}^{n-1}C_{r-1}=n\cdot 2^{n-1}$$

and for $\lambda=r^2$, $c=\sum_{r=0}^{n}r^2\,C_r=\sum_{r=0}^{n}r^2\cdot\frac{n}{r}\cdot{}^{n-1}C_{r-1}$

$$=n\sum_{r=0}^{n}r\cdot{}^{n-1}C_{r-1}=n\sum_{r=1}^{n}r\cdot{}^{n-1}C_{r-1}$$

$$= n\left[\sum_{r=1}^{n}\{(r-1)+1\}^{n-1}C_{r-1}\right]$$
$$= n\left[\sum_{r=1}^{n}(r-1)\cdot{}^{n-1}C_{r-1}+\sum_{r=1}^{n}{}^{n-1}C_{r-1}\right]$$
$$= n\left[\sum_{r=1}^{n}(r-1)\frac{(n-1)}{(r-1)}\cdot{}^{n-2}C_{r-2}+2^{n-1}\right]$$
$$= n\left[(n-1)\sum_{r=1}^{n}{}^{n-2}C_{r-2}+2^{n-1}\right]$$
$$= n\,[\,(n-1)\cdot 2^{n-2}+2^{n-1}] = n\,(n+1)\,2^{n-2}$$

For $n=1, a=2, b=1, c=1$ and for $n=2, a=4, b=4, c=6$ $\left.\right\}$ $a=b+c$, $a+b=c+2$

(B) For $\lambda = 1$, $a=\sum_{r=0}^{n}(-1)^r\cdot C_r = 0$

For $\lambda = r$,

$$b=\sum_{r=0}^{n}(-1)^r\cdot r\cdot C_r=\sum_{r=0}^{n}(-1)^r\cdot r\cdot\frac{n}{r}\,{}^{n-1}C_{r-1}$$
$$= n\sum_{r=1}^{n}(-1)^r\cdot{}^{n-1}C_{r-1} = n\,(1-1)^{n-1}=0$$

and for $\lambda = r^2$, $c=\sum_{r=0}^{n}(-1)^r\cdot r^2\cdot C_r$

$$=\sum_{r=0}^{n}(-1)^r\cdot r^2\cdot\frac{n}{r}\cdot{}^{n-1}C_{r-1}$$
$$=n\sum_{r=0}^{n}(-1)^r\cdot r\cdot{}^{n-1}C_{r-1}$$
$$=n\sum_{r=0}^{n}(-1)^r\,\{(r-1)+1\}\,{}^{n-1}C_{r-1}$$
$$= n\sum_{r=0}^{n}(-1)^r\,(r-1)\,{}^{n-1}C_{r-1}+n\sum_{r=0}^{n}(-1)^r\cdot{}^{n-1}C_{r-1}$$

$= 0+0 = 0$

$\therefore a=b=c=0 \Rightarrow a=b+c$

$\Rightarrow a^3+b^3+c^3=3abc \Rightarrow a+c=4b$

(C) For $\lambda = 1$, $a=\sum_{r=0}^{n}\frac{C_r}{(r+1)} = \frac{1}{(n+1)}\sum_{r=0}^{n}\left(\frac{n+1}{r+1}\right)\cdot{}^nC_r$

$$=\frac{1}{(n+1)}\sum_{r=0}^{n}{}^{n+1}C_{r+1} = \frac{1}{n+1}(2^{n+1}-1)$$
$$=\frac{2^{n+1}-1}{n+1}$$

For $\lambda = r$, $b=\sum_{r=0}^{n}\frac{r\cdot C_r}{(r+1)} = \sum_{r=0}^{n}\left(1-\frac{1}{r+1}\right)C_r$

$$=2^n-\left(\frac{2^{n+1}-1}{n+1}\right)=\frac{(n-1)\,2^n+1}{n+1}$$

For $\lambda = r^2$, $c=\sum_{r=0}^{n}\frac{r^2\cdot C_r}{(r+1)}=\sum_{r=0}^{n}\left((r-1)+\frac{1}{r+1}\right)C_r$

$$=\sum_{r=0}^{n}r\cdot C_r-\sum_{r=0}^{n}C_r+\sum_{r=0}^{n}\frac{C_r}{r+1}$$
$$=n\cdot 2^{n-1}-2^n+\frac{2^{n+1}-1}{n+1}$$
$$=\frac{(n^2-n+2)\,2^{n-1}-1}{(n+1)}$$

For $n=1, a=\frac{3}{2}, b=\frac{1}{2}, c=\frac{1}{2}$ and for $n=2, a=\frac{7}{3}, b=\frac{5}{3}, c=\frac{7}{3}$; $\left.\right\}$ $a+c=4b$, $b^{c-a}+(c-a)^b=1$

JEE Type Solved Examples :
Statement I and II Type Questions

- **Directions** Example numbers 26 and 27 are Assertion-Reason type examples. Each of these examples contains two statements:
Statement-1 (Assertion) and **Statement-2** (Reason)
Each of these examples also has four alternative choices, only one of which is the correct answer. You have to select the correct choice as given below.
(a) Statement-1 is true, Statement-2 is true; Statement-2 is a correct explanation for Statement-1
(b) Statement-1 is true, Statement-2 is true; Statement-2 is not a correct explanation for Statement-1
(c) Statement-1 is true, Statement-2 is false
(d) Statement-1 is false, Statement-2 is true

Ex. 26 **Statement-1** (7^9+9^7) *is divisible by 16*

Statement-2 (x^y+y^x) *is divisible by* $(x+y), \forall\ x, y$.

Sol. (c) $7^9+9^7=(8-1)^9+(8+1)^7$

$$=(8^9-{}^9C_1\cdot 8^8+{}^9C_2\cdot 8^7-{}^9C_3\cdot 8^6+\ldots+{}^9C_8\cdot 8-1)$$
$$+(8^7+{}^7C_1\cdot 8^6+{}^7C_2\cdot 8^5+\ldots+{}^7C_6\cdot 8+1)$$
$$=8^9-9\cdot 8^8+8^7\cdot({}^9C_2+1)+8^6(-{}^9C_3+7)$$
$$+8^5({}^9C_4+{}^7C_2)+\ldots+8({}^9C_8+{}^7C_6)$$

$= 64\,\lambda$ [λ is an integer]

$\therefore 7^9+9^7$ is divisible by 16.

$\therefore$ Statement-1 is true. Statement-2 is false.

Ex. 27. **Statement-1** *Number of distinct terms in the sum of expansion* $(1+ax)^{10}+(1-ax)^{10}$ *is 22.*

Statement-2 *Number of terms in the expansion of* $(1+x)^n$ *is* $n+1, \forall\ n\in N$.

Sol. (d) $\because$ $(1+ax)^{10}+(1-ax)^{10}=2\,\{1+{}^{10}C_2\,(ax)^2$
$+{}^{10}C_4\,(ax)^4+{}^{10}C_6\,(ax)^6+{}^{10}C_8\,(ax)^8+{}^{10}C_{10}\,(ax)^{10}\}$

$\therefore$ Number of distinct terms $= 6$

$\Rightarrow$ Statement-1 is false but Statement-2 is obviously true.

Subjective Type Examples

- **Ex. 28** *Find the coefficient independent of x in the expansion of* $(1+x+2x^3)\left(\frac{3}{2}x^2-\frac{1}{3x}\right)^9$.

Sol. $(r+1)$ th term in the expansion of $\left(\frac{3}{2}x^2-\frac{1}{3x}\right)^9$

i.e., $$T_{r+1}={}^9C_r\left(\frac{3}{2}x^2\right)^{9-r}\left(-\frac{1}{3x}\right)^r$$

$$={}^9C_r\left(\frac{3}{2}\right)^{9-r}\cdot x^{18-2r}\cdot\left(-\frac{1}{3}\right)^r\cdot x^{-r}$$

$$={}^9C_r\left(\frac{3}{2}\right)^{9-r}\cdot\left(-\frac{1}{3}\right)^r\cdot x^{18-3r}$$

Hence, general term in the expansion of $(1+x+2x^3)$

$$\left(\frac{3}{2}x^2-\frac{1}{3x}\right)^9={}^9C_r\left(\frac{3}{2}\right)^{9-r}\left(-\frac{1}{3}\right)^r\cdot x^{18-3r}$$
$$+{}^9C_r\left(\frac{3}{2}\right)^{9-r}\left(-\frac{1}{3}\right)^r\cdot x^{19-3r}$$
$$+2\,{}^9C_r\left(\frac{3}{2}\right)^{9-r}\left(-\frac{1}{3}\right)^r\cdot x^{21-3r}$$

For independent term, putting $18-3r=0, 19-3r=0$, $21-3r=0$ respectively, we get

$r=6, r=19/3$ [impossible] $r=7$, second term do not given the independent term.

Hence, coefficient independent of x

$$={}^9C_6\cdot\left(\frac{3}{2}\right)^3\cdot\left(-\frac{1}{3}\right)^6+0+2\cdot{}^9C_7\cdot\left(\frac{3}{2}\right)^2\left(-\frac{1}{3}\right)^7$$

$$={}^9C_3\cdot\frac{27}{8.729}-2\cdot{}^9C_2\cdot\frac{9}{4}\cdot\frac{1}{2187}=\frac{7}{18}-\frac{2}{27}=\frac{17}{54}$$

- **Ex. 29** *If* $(1+x)^n=C_0+C_1x+C_2x^2+\ldots+C_nx^n$, *show that* $\sum_{r=0}^{n}C_r^3$ *is equal to the coefficient of* x^ny^n *in the expansion of* $\{(1+x)(1+y)(x+y)\}^n$.

Sol. $(1+x)^n(y+1)^n(x+y)^n=\sum_{r=0}^{n}C_rx^r$

$$\sum_{s=0}^{n}C_sy^{n-s}\sum_{t=0}^{n}C_tx^{n-t}y^t \qquad \text{...(i)}$$

Since, C_0^3 is the coefficient of $x^0y^{n-0}x^{n-0}y^0$

i.e., $\qquad x^ny^n\ (r=s=t=0)$

Now, C_1^3 is the coefficient of $x^1y^{n-1}x^{n-1}y$

i.e., $\qquad x^ny^n\ (r=s=t=1)$

And C_k^3 is the coefficient of $x^ky^{n-k}x^{n-k}y^k$

i.e., $\qquad x^ny^n\ (r=s=t=k)$

Hence, the coefficient of x^ny^n in $(1+x)^n(y+1)^n(x+y)^n$

$$=C_0^3+C_1^3+C_2^3+\ldots+C_n^3=\sum_{r=0}^{n}C_r^3$$

- **Ex. 30** *Let* $(1+x^2)^2(1+x)^n=\sum_{k=0}^{n+4}a_kx^k$. *If a_1, a_2 and a_3 are in AP, find n.*

Sol. We have,

$$(1+x^2)^2(1+x)^n=(1+2x^2+x^4)\times({}^nC_0+{}^nC_1x+{}^nC_2x^2+{}^nC_3x^3+\ldots)$$
$$=a_0+a_1x+a_2x^2+a_3x^3+\ldots \qquad \text{[say]}$$

Now, comparing the coefficients of x, x^2 and x^3, we get

$a_1={}^nC_1, a_2=2\cdot{}^nC_0+{}^nC_2, a_3=2\cdot{}^nC_1+{}^nC_3$ ...(i)

In a_1, $n\ge1$, in a_2, $n\ge2$ and in a_3, $n\ge3$

$\therefore \qquad n\ge3$ ...(ii)

From Eq. (i),

$$a_1=n, a_2=2+\frac{n(n-1)}{1\cdot2}=\frac{n^2-n+4}{2}$$

and $$a_3=2n+\frac{n(n-1)(n-2)}{1\cdot2\cdot3}=\frac{n^3-3n^2+14n}{6}$$

Since, a_1, a_2, a_3 are in AP.

Therefore, $\qquad 2a_2=a_1+a_3$

$$\Rightarrow \quad n^2-n+4=n+\frac{n^3-3n^2+14n}{6}$$

$$\Rightarrow \quad n^3-9n^2+26n-24=0$$

or $\qquad (n-2)(n-3)(n-4)=0$

$\therefore \qquad n=2,3,4$

Hence, $\qquad n=3,4$ [from Eq. (ii)]

- **Ex. 31** *If* $(1-x^3)^n=\sum_{r=0}^{n}a_rx^r(1-x)^{3n-2r}$, *find a_r, where $n\in N$.*

Sol. We have, $\qquad (1-x^3)^n=\sum_{r=0}^{n}a_rx^r(1-x)^{3n-2r}$

$$\Rightarrow (1-x)^n(1+x+x^2)^n=\sum_{r=0}^{n}\frac{a_r\cdot x^r(1-x)^{3n}}{(1-x)^{2r}}$$

$$\Rightarrow \frac{(1-x)^n(1+x+x^2)^n}{(1-x)^{3n}}=\sum_{r=0}^{n}\frac{a_r\cdot x^r}{(1-x)^{2r}}$$

$$\Rightarrow \left[\frac{1+x+x^2}{(1-x)^2}\right]^n=\sum_{r=0}^{n}a_r\cdot\frac{x^r}{(1-x)^{2r}}$$

$$\Rightarrow \quad \left[\frac{(1-x)^2+3x}{(1-x)^2}\right]^n = \sum_{r=0}^{n} a_r \left[\frac{x}{(1-x)^2}\right]^r$$

$$\Rightarrow \quad \left[1+3\left(\frac{x}{(1-x)^2}\right)\right]^n = \sum_{r=0}^{n} a_r \left[\frac{x}{(1-x)^2}\right]^r \quad \text{...(i)}$$

Let $\quad A = \dfrac{x}{(1-x)^2}$

Then, Eq. (i) becomes $(1+3A)^n = \sum_{r=0}^{n} a_r A^r$

On comparing the coefficient of A^r, we get

$${}^nC_r \cdot 3^r = a_r$$

Hence, $\quad a_r = {}^nC_r \cdot 3^r$

Ex. 32 *If $a_0, a_1, a_2, \ldots, a_{2n}$ are the coefficients in the expansion of $(1+x+x^2)^n$ in ascending powers of x, show that $a_0^2 - a_1^2 - a_2^2 - \ldots + a_{2n}^2 = a_n$.*

Sol. We have, $(1+x+x^2)^n = a_0 + a_1 x + a_2 x^2 + a_{2n} x^{2n}$ ...(i)

Replacing x by $\left(-\dfrac{1}{x}\right)$ in Eq. (i), we get

$$\left(1-\frac{1}{x}+\frac{1}{x^2}\right)^n = a_0 - \frac{a_1}{x} + \frac{a_2}{x^2} - \ldots + \frac{a_{2n}}{x^{2n}} \quad \text{...(ii)}$$

On multiplying Eqs. (i) and (ii), we get

$$(1+x+x^2)^n \times \left(1-\frac{1}{x}+\frac{1}{x^2}\right)^n = (a_0 + a_1 x + a_2 x^2 + \ldots + a_{2n} x^{2n}) \times \left(a_0 - \frac{a_1}{x} + \frac{a_2}{x^2} - \ldots + \frac{a_{2n}}{x^{2n}}\right)$$

$$\Rightarrow \frac{(1+x^2+x^4)^n}{x^{2n}} = (a_0 + a_1 x + a_2 x^2 + \ldots + a_{2n} x^{2n}) \times \left(a_0 - \frac{a_1}{x} + \frac{a_2}{x^2} - \ldots + \frac{a_{2n}}{x^{2n}}\right) \text{...(iii)}$$

Constant term in RHS $= a_0^2 - a_1^2 + a_2^2 - \ldots + a_{2n}^2$

Now, constant term in $\dfrac{(1+x^2+x^4)^n}{x^{2n}} =$ Coefficient of x^{2n}

in $(1+x^2+x^4)^n = a_n$ [replacing x by x^2 in Eq. (i)]

But Eq. (iii) is an identity, therefore, the constant term in RHS = constant term in LHS.

$$a_0^2 - a_1^2 + a_2^2 - \ldots + a_{2n}^2 = a_n$$

Ex. 33 *Show that no three consecutive binomial coefficients can be in (i) GP and (ii) HP.*

Sol. (i) Suppose that the r th, $(r+1)$th and $(r+2)$th coefficients of $(1+x)^n$ are in GP.

i.e., $\quad {}^nC_{r-1}, \ {}^nC_r, \ {}^nC_{r+1}$ are in GP.

Then, $\quad \dfrac{{}^nC_r}{{}^nC_{r-1}} = \dfrac{{}^nC_{r+1}}{{}^nC_r}$

$$\Rightarrow \quad \frac{n-r+1}{r} = \frac{n-r}{r+1} \quad \left[\because \frac{{}^nC_r}{{}^nC_{r-1}} = \frac{n-r+1}{r}\right]$$

$$\Rightarrow \quad (n-r+1)(r+1) = r(n-r)$$

$$\Rightarrow \quad nr + n - r^2 - r + r + 1 = nr - r^2$$

$$\Rightarrow \quad n+1 = 0$$

$$\Rightarrow \quad n = -1$$

which is not possible, since n is a positive integer.

(ii) Suppose that rth, $(r+1)$th and $(r+2)$th coefficients of $(1+x)^n$ are in HP,

i.e. ${}^nC_{r-1}, {}^nC_r, {}^nC_{r+1}$ are in HP.

Then, $\quad \dfrac{2}{{}^nC_r} = \dfrac{1}{{}^nC_{r-1}} + \dfrac{1}{{}^nC_{r+1}}$

$$\Rightarrow \quad 2 = \frac{{}^nC_r}{{}^nC_{r-1}} + \frac{{}^nC_r}{{}^nC_{r+1}}$$

$$\left[\because \frac{{}^nC_r}{{}^nC_{r-1}} = \frac{n-r+1}{r}\right]$$

$$\Rightarrow \quad 2 = \frac{n-r+1}{r} + \frac{r+1}{n-r}$$

$$\Rightarrow \quad 2r(n-r) = (n-r+1)(n-r) + r(r+1)$$

$$\Rightarrow \quad 2nr - 2r^2 = n^2 - nr - nr + r^2 + n - r + r^2 + r$$

$$\Rightarrow \quad n^2 - 4nr + 4r^2 + n = 0 \quad \Rightarrow \quad (n-2r)^2 + n = 0$$

which is not possible, as $(n-2r)^2 \ge 0$ and n is a positive integer.

Ex. 34 *Evaluate $\sum_{i=0}^{n} \sum_{j=1}^{n} {}^nC_j \cdot {}^jC_i, i \le j$.*

Sol. We have, $\sum_{i=0}^{n} \sum_{j=1}^{n} {}^nC_j \cdot {}^jC_i$

$$= {}^nC_1({}^1C_0 + {}^1C_1) + {}^nC_2({}^2C_0 + {}^2C_1 + {}^2C_2) + {}^nC_3({}^3C_0 + {}^3C_1 + {}^3C_2 + {}^3C_3) + {}^nC_4({}^4C_0 + {}^4C_1 + {}^4C_2 + {}^4C_3 + {}^4C_4) + \ldots + {}^nC_n({}^nC_0 + {}^nC_1 + {}^nC_2 + \ldots + {}^nC_n)$$

$$= {}^nC_1(2) + {}^nC_2(2)^2 + {}^nC_3(2)^3 + \ldots + {}^nC_n(2)^n = (1+2)^n - 1$$

$$= 3^n - 1$$

Ex. 35 *Find the remainder, when 27^{40} is divided by 12.*

Sol. We have, $27^{40} = (3^3)^{40} = 3^{120} = 3 \cdot (3)^{119} = 3 \cdot (4-1)^{119}$

$= 3(4n - 1)$, where n is some integer

$= 12n - 3 = 12n - 12 + 9 = 12(n-1) + 9$

$= 12m + 9$, where m is some integer.

$$\therefore \quad \frac{27^{40}}{12} = m + \frac{9}{12}$$

Hence, the remainder is 9.

Ex. 36 *Show that* $[(\sqrt{3}+1)^{2n}]+1$ *is divisible by* 2^{n+1}, $\forall n \in N$, *where* $[\cdot]$ *denotes the greatest integer function.*

Sol. Let $x=(\sqrt{3}+1)^{2n}=[x]+f$ ...(i)

where, $0 \le f < 1$

and $(\sqrt{3}-1)^{2n}=f'$ ...(ii)

where, $0 < f' < 1$

On adding Eqs. (i) and (ii), we get

$$[x]+f+f'=(\sqrt{3}+1)^{2n}+(\sqrt{3}-1)^{2n}$$
$$=(4+2\sqrt{3})^n+(4-2\sqrt{3})^n$$
$$=2^n\{(2+\sqrt{3})^n+(2-\sqrt{3})^n\}$$
$$=2^n\cdot 2\{{}^nC_0(2)^n+{}^nC_2(2)^{n-2}(\sqrt{3})^2+{}^nC_4(2)^{n-4}(\sqrt{3})^4+\ldots\}$$

$\therefore$ $[x]+f+f'=2^{n+1}k$, where k is an integer. ...(iii)

Hence, $(f+f')$ is an integer.

i.e., $f+f'=1$ $[\because 0<(f+f')<2]$

From Eq. (iii), we get

$$[x]+1=2^{n+1}k$$

$\Rightarrow$ $[(\sqrt{3}+1)^{2n}]+1=2^{n+1}k$ [from Eq. (i)]

which shows that $[(\sqrt{3}+1)^{2n}]+1$ divisible by $2^{n+1}, \forall n \in N$.

Ex. 37 *Find the number of rational terms and also find the sum of rational terms in* $(\sqrt{2}+\sqrt[3]{3}+\sqrt[6]{5})^{10}$.

Sol. We have, $(\sqrt{2}+\sqrt[3]{3}+\sqrt[6]{5})^{10}=(2^{1/2}+3^{1/3}+5^{1/6})^{10}$

$$=\sum_{\alpha+\beta+\gamma=10}\frac{10!}{\alpha!\,\beta!\,\gamma!}2^{\alpha/2}\cdot 3^{\beta/3}\cdot 5^{\gamma/6}$$

For rational terms,

$\alpha=0, 2, 4, 6, 8, 10$, $\beta=0, 3, 6, 9$, $\gamma=0, 6$

Since, $0 \le \alpha, \beta, \gamma \le 10$.

$\therefore$ Possible triplets are $(4, 0, 6), (4, 6, 0), (10, 0, 0)$.

There exists three rational terms.

$\therefore$ Required sum

$$=\frac{10!}{4!\,0!\,6!}2^2\cdot 5+\frac{10!}{4!\,6!\,0!}2^2\cdot 3^2+\frac{10!}{10!\,0!\,0!}2^5$$

$=4200+7560+32=11792$

Ex. 38 *Find the remainder, when* $(1690^{2608}+2608^{1690})$ *is divided by* 7.

Sol. We have, $1690^{2608}+2608^{1690}=(1690^{2608}-3^{2608})+(2608^{1690}-4^{1690})+(3^{2608}+4^{1690})$

The number $(1690^{2608}-3^{2608})$ is divisible by $1690-3=1687=7\times 241$ which is divisible by 7, the difference $(2608^{1690}-4^{1690})$ is also divisible by 7, since it is divisible by $2608-4=2604=7\times 372$.

As to sum $3^{2608}+4^{1690}$, it can be rewritten as

$$3\cdot(3^3)^{869}+4\cdot(4^3)^{563}$$
$$=3(28-1)^{869}+4(63+1)^{563}$$
$$=3(7m-1)+4(7n+1)$$

[where, m and n are some positive integers]

where p is some positive integer.

Hence, the remainder is 1.

Ex. 39 *If* $C_0, C_1, C_2, \ldots, C_n$ *are the binomial coefficients in the expansion of* $(1+x)^n$, *prove that*

$$(C_0+2C_1+C_2)(C_1+2C_2+C_3)\ldots(C_{n-1}+2C_n+C_{n+1})=\frac{(n+2)^n}{(n+1)!}\prod_{r=1}^{n}(C_{r-1}+C_r).$$

Sol. LHS $=(C_0+2C_1+C_2)(C_1+2C_2+C_3)\ldots(C_{n-1}+2C_n+C_{n+1})$

$$=\prod_{r=1}^{n}({}^nC_{r-1}+2\,{}^nC_r+{}^nC_{r+1})$$
$$=\prod_{r=1}^{n}\{({}^nC_{r-1}+{}^nC_r)+({}^nC_r+{}^nC_{r+1})\}$$
$$=\prod_{r=1}^{n}({}^{n+1}C_r+{}^{n+1}C_{r+1}) \quad \text{[by Pascal's rule]}$$
$$=\prod_{r=1}^{n}({}^{n+2}C_{r+1})=\prod_{r=1}^{n}\left(\frac{n+2}{r+1}\right){}^{n+1}C_r \quad \left[\because {}^nC_r=\frac{n}{r}\cdot{}^{n-1}C_{r-1}\right]$$
$$=\prod_{r=1}^{n}\left(\frac{n+2}{r+1}\right)({}^nC_{r-1}+{}^nC_r)=\prod_{r=1}^{n}\left(\frac{n+2}{r+1}\right)\prod_{r=1}^{n}(C_{r-1}+C_r)$$
$$=\frac{(n+2)}{2}\cdot\frac{(n+2)}{3}\cdot\frac{(n+2)}{4}\cdots\frac{(n+2)}{(n+1)}\prod_{r=1}^{n}(C_{r-1}+C_r)$$
$$=\frac{(n+2)^n}{(n+1)!}\prod_{r=1}^{n}(C_{r-1}+C_r)=\text{RHS}$$

Ex. 40 *If* $\sum_{r=0}^{2n}a_r(x-2)^r=\sum_{r=0}^{2n}b_r(x-3)^r$ *and* $a_k=1$, $\forall k \ge n$, *show that* $b_n={}^{2n+1}C_{n+1}$.

Sol. $\because$ $\sum_{r=0}^{2n}a_r(x-2)^r=\sum_{r=0}^{2n}b_r(x-3)^r$

Let $y=x-3 \Rightarrow y+1=x-2$

So, the given expression reduces to

$$\sum_{r=0}^{2n}a_r(1+y)^r=\sum_{r=0}^{2n}b_ry^r$$

$$\Rightarrow \quad a_0 + a_1(1+y) + a_2(1+y)^2 + \dots + a_{2n}(1+y)^{2n}$$
$$= b_0 + b_1 y + \dots + b_{2n} y^{2n}$$

Using $a_k = 1, \forall\, k \geq n$, we get

$$a_0 + a_1(1+y) + a_2(1+y)^2 + \dots + a_{n-1}(1+y)^{n-1}$$
$$+ (1+y)^n + (1+y)^{n+1} + \dots + (1+y)^{2n}$$
$$= b_0 + b_1 y + \dots + b_n y^n + \dots + b_{2n} y^{2n}$$

On comparing the coefficient of y^n on both sides, we get

$${}^nC_n + {}^{n+1}C_n + {}^{n+2}C_n + \dots + {}^{2n}C_n = b_n$$
$$\Rightarrow \quad {}^{n+1}C_{n+1} + {}^{n+1}C_n + {}^{n+2}C_n + \dots + {}^{2n}C_n = b_n$$
$$[\because\ {}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r]$$
$$\Rightarrow {}^{n+2}C_{n+1} + {}^{n+2}C_n + \dots + {}^{2n}C_n = b_n$$

[adding first two terms]

If we combine terms on LHS finally, we get

$${}^{2n+1}C_{n+1} = b_n$$

Ex. 41 *(i) If n is an odd natural number, prove that*

$$\sum_{r=0}^{n} \frac{(-1)^r}{{}^nC_r} = 0.$$

(ii) If n is an even natural number, find the value of

$$\sum_{r=0}^{n} \frac{(-1)^r}{{}^nC_r}.$$

Sol. (i) We have, $\displaystyle\sum_{r=0}^{n} \frac{(-1)^r}{{}^nC_r} = \sum_{r=0}^{\frac{n+1}{2}} \left[\frac{(-1)^r}{{}^nC_r} + \frac{(-1)^{n-r}}{{}^nC_{n-r}}\right]$

$$= \sum_{r=0}^{\frac{n+1}{2}} (-1)^r \left[\frac{1}{{}^nC_r} + \frac{(-1)^n}{{}^nC_{n-r}}\right] = \sum_{r=0}^{\frac{n+1}{2}} (-1)^r \left[\frac{1}{{}^nC_r} - \frac{1}{{}^nC_r}\right]$$

$= 0$ $\qquad [\because n$ is odd and ${}^nC_r = {}^nC_{n-r}]$

(ii) We have,

$$\sum_{r=0}^{n} \frac{(-1)^r}{{}^nC_r} = \sum_{r=0}^{\frac{n}{2}-1} \left[\frac{(-1)^r}{{}^nC_r} + \frac{(-1)^{n-r}}{{}^nC_{n-r}}\right] + \frac{(-1)^{n/2}}{{}^nC_{n/2}}$$
$$= \sum_{r=0}^{\frac{n}{2}-1} (-1)^r \left[\frac{1}{{}^nC_r} + \frac{(-1)^n}{{}^nC_r}\right] + \frac{(-1)^{n/2}}{{}^nC_{n/2}}$$
$$= \sum_{r=0}^{\frac{n}{2}-1} (-1)^r \left[\frac{1}{{}^nC_r} + \frac{1}{{}^nC_r}\right] + \frac{(-1)^{n/2}}{{}^nC_{n/2}}$$
$$= \left[\sum_{r=0}^{\frac{n}{2}-1} (-1)^r \cdot \frac{2}{{}^nC_r}\right] + \frac{(-1)^{n/2}}{{}^nC_{n/2}}$$

Ex. 42 *If* $(1+x)^n = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \dots + C_n x^n$, *show that*

$$C_1 - \frac{C_2}{2} + \frac{C_3}{3} - \dots (-1)^{n-1} \frac{C_n}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}.$$

Sol. We know that,

$$(1-x)^n = C_0 - C_1 x + C_2 x^2 - \dots + (-1)^n C_n x^n$$

or $C_0 - (1-x)^n = C_1 x - C_2 x^2 + C_3 x^3 - \dots + (-1)^{n-1} C_n x^n$

$$\Rightarrow 1 - (1-x)^n = C_1 x - C_2 x^2 + C_3 x^3 - \dots + (-1)^{n-1} C_n x^n$$

Dividing in each side by x, then

$$\frac{1-(1-x)^n}{x} = C_1 - C_2 x + C_3 x^2 - \dots + (-1)^{n-1} C_n x^{n-1}$$

On integrating within limits 0 to 1, we have

$$\int_0^1 \frac{1-(1-x)^n}{x}\,dx = \int_0^1 (C_1 - C_2 x + C_3 x^2 - \dots + (-1)^{n-1} C_n x^{n-1})\,dx$$
$$= \left[C_1 x - \frac{C_2 x^2}{2} + \frac{C_3 x^3}{3} - \dots + (-1)^{n-1} C_n \frac{x^n}{n}\right]_0^1$$
$$\int_0^1 \frac{1-(1-x)^n}{x}\,dx = C_1 - \frac{C_2}{2} + \frac{C_3}{3} - \dots + \frac{(-1)^{n-1}}{n} C_n$$

Putting $1 - x = t$ in integral,

$\Rightarrow \quad dx = -\,dt$

when $x \to 1, t \to 0$ and when $x \to 0, t \to 1$

$$\therefore \int_0^1 \frac{(1-t^n)}{(1-t)}(-\,dt) = C_1 - \frac{C_2}{2} + \frac{C_3}{3} - \dots + (-1)^{n-1} \frac{C_n}{n}$$
$$\Rightarrow \int_0^1 \frac{(1-t^n)}{(1-t)}\,dt = C_1 - \frac{C_2}{2} + \frac{C_3}{3} - \dots + (-1)^{n-1} \frac{C_n}{n}$$
$$\Rightarrow \int_0^1 (1 + t + t^2 + \dots + t^{n-1})\,dt = C_1 - \frac{C_2}{2} + \frac{C_3}{3} - \dots + (-1)^{n-1} \frac{C_n}{n}$$
$$\Rightarrow \left[t + \frac{t^2}{2} + \frac{t^3}{3} + \dots + \frac{t^n}{n}\right]_0^1 = C_1 - \frac{C_2}{2} + \frac{C_3}{3} - \dots + (-1)^{n-1} \frac{C_n}{n}$$
$$\Rightarrow 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} = C_1 - \frac{C_2}{2} + \frac{C_3}{3} - \dots + (-1)^{n-1} \frac{C_n}{n}$$

Hence, $\displaystyle C_1 - \frac{C_2}{2} + \frac{C_3}{3} - \dots + (-1)^{n-1} \frac{C_n}{n}$

$$= 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}$$

Ex. 43 *If* $(1+x)^n = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \dots + C_n x^n$, *find the sum of the seriesd*

$$\frac{C_0}{2} - \frac{C_1}{6} + \frac{C_2}{10} - \frac{C_3}{14} + \dots + (-1)^n \frac{C_n}{4n+2}.$$

Sol. Let $S = \frac{C_0}{2} - \frac{C_1}{6} + \frac{C_2}{10} - \frac{C_3}{14} + \ldots + (-1)^n \frac{C_n}{4n+2}$

$$= \frac{1}{2}\left(\frac{C_0}{1} - \frac{C_1}{3} + \frac{C_2}{5} - \frac{C_3}{7} + \ldots + (-1)^n \frac{C_n}{2n+1}\right) \quad \ldots\text{(i)}$$

Consider, $(1-x^2)^n = C_0 - C_1 x^2 + C_2 x^4 - C_3 x^6 + \ldots + (-1)^n C_n x^{2n}$

$$\Rightarrow \int_0^1 (1-x^2)^n \, dx = \int_0^1 (C_0 - C_1 x^2 + C_2 x^4 - C_3 x^6 + \ldots + (-1)^n C_n x^{2n})\, dx$$

$$\Rightarrow \int_0^1 (1-x^2)^n \, dx = \left[C_0 x - \frac{C_1 x^3}{3} + \frac{C_2 x^5}{5} - \frac{C_3 x^7}{7} + \ldots + (-1)^n \frac{C_n x^{2n+1}}{2n+1}\right]$$

$$\Rightarrow \int_0^1 (1-x^2)^n \, dx = C_0 - \frac{C_1}{3} + \frac{C_2}{5} - \frac{C_3}{7} + \ldots + (-1)^n \frac{C_n}{2n+1}$$

From Eq. (i),

$$\int_0^1 (1-x^2)^n \, dx = 2S \quad \text{or} \quad S = \frac{1}{2}\int_0^1 (1-x^2)^n \, dx$$

Put $x = \sin\theta$ i.e., $dx = \cos\theta \, d\theta$

$$S = \frac{1}{2}\int_0^{\pi/2} \cos^{2n+1}\theta \, d\theta$$

By using Walli's formula,

$$S = \frac{1}{2} \cdot \frac{2n(2n-2)(2n-4)\ldots 4\cdot 2}{(2n+1)(2n-1)(2n-3)\ldots 3\cdot 1} \cdot 1$$

$$= \frac{1}{2} \cdot \frac{\{2n(2n-2)(2n-4)\ldots 4\cdot 2\}^2}{(2n+1)!}$$

$$= \frac{1}{2} \cdot \frac{(2^n n!)^2}{(2n+1)!} = 2^{2n-1} \frac{(n!)^2}{(2n+1)!}$$

Ex. 44 *If* $(1+x)^n = \sum_{r=0}^{n} C_r x^r$, *then prove that*

$$\sum\sum_{0 \le i < j \le n} \left(\frac{i}{C_i} + \frac{j}{C_j}\right) = \frac{n^2}{2} \sum_{r=0}^{n} \frac{1}{C_r}.$$

Sol. Let $S = \sum\sum_{0 \le i < j \le n} \left(\frac{i}{C_i} + \frac{j}{C_j}\right)$ ...(i)

Replacing i by $n-i$ and j by $n-j$, we get

$$S = \sum\sum_{0 \le i < j \le n} \left(\frac{n-i}{C_{n-i}} + \frac{n-j}{C_{n-j}}\right) = \sum\sum_{0 \le i < j \le n} \left(\frac{n-i}{C_i} + \frac{n-j}{C_j}\right)$$

$[\because C_r = C_{n-r}]$ ...(ii)

On adding Eqs. (i) and (ii), we get

$$2S = n \sum\sum_{0 \le i < j \le n} \left(\frac{1}{C_i} + \frac{1}{C_j}\right)$$

$$\therefore \quad S = \frac{n}{2} \sum\sum_{0 \le i < j \le n} \left(\frac{1}{C_i} + \frac{1}{C_j}\right) = \frac{n}{2}\left(\sum_{r=0}^{n-1} \frac{n-r}{C_r} + \sum_{r=1}^{n} \frac{r}{C_r}\right)$$

$$= \frac{n}{2}\left(\sum_{r=0}^{n} \frac{n-r}{C_r} + \sum_{r=0}^{n} \frac{r}{C_r}\right) = \frac{n}{2}\left(\sum_{r=0}^{n} \frac{n}{C_r}\right) = \frac{n^2}{2} \sum_{r=0}^{n} \frac{1}{C_r}$$

Ex. 45 *If* $(1+x)^n = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \ldots + C_n x^n$, *show that*

$$\sum_{r=0}^{n} \frac{C_r 3^{r+4}}{(r+1)(r+2)(r+3)(r+4)} = \frac{1}{(n+1)(n+2)(n+3)(n+4)}\left(4^{n+4} - \sum_{t=0}^{3} {}^{n+4}C_t 3^t\right).$$

Sol. LHS $= \sum_{r=0}^{n} \frac{C_r \cdot 3^{r+4}}{(r+1)(r+2)(r+3)(r+4)}$

$$= \sum_{r=0}^{n} \frac{C_r \cdot 3^{r+4}}{\frac{(r+1)(r+2)(r+3)(r+4)}{4!} 4!}$$

$$= \sum_{r=0}^{n} \frac{C_r \cdot 3^{r+4}}{{}^{r+4}C_4 \cdot 4!} = \sum_{r=0}^{n} \frac{n!}{r!(n-r)!} \cdot \frac{3^{r+4}}{\frac{(r+4)!}{4!r!} \cdot 4!}$$

$$= \sum_{r=0}^{n} \frac{n! \cdot 3^{r+4}}{(n-r)! \cdot (n+4)!}$$

$$= \sum_{r=0}^{n} \frac{n! \cdot 3^{r+4}}{(n-r)! \cdot (r+4)!} \cdot \frac{(n+1)(n+2)(n+3)(n+4)}{(n+1)(n+2)(n+3)(n+4)}$$

$$= \sum_{r=0}^{n} \frac{(n+4)!\, 3^{r+4}}{(n-r)!\,(r+4)!\,(n+1)(n+2)(n+3)(n+4)}$$

$$= \frac{1}{(n+1)(n+2)(n+3)(n+4)}\left[\sum_{r=0}^{n} \frac{(n+4)! \cdot 3^{r+4}}{(n-r)! \cdot (r+4)!}\right]$$

$$= \frac{1}{(n+1)(n+2)(n+3)(n+4)}\left[\sum_{r=0}^{n} {}^{n+4}C_{r+4}\, 3^{r+4}\right]$$

$$= \frac{1}{(n+1)(n+2)(n+3)(n+4)}\left\{\sum_{t=4}^{n+4} {}^{n+4}C_t\, 3^t\right\}$$

[put $r + 4 = t$]

$$= \frac{1}{(n+1)(n+2)(n+3)(n+4)} \left\{\sum_{t=0}^{n+4} {}^{n+4}C_t\, 3^t - \sum_{t=0}^{3} {}^{n+4}C_t\, 3^t\right\}$$

$$= \frac{1}{(n+1)(n+2)(n+3)(n+4)}\left\{(1+3)^{n+4} - \sum_{t=0}^{3} {}^{n+4}C_t\, 3^t\right\}$$

$$= \frac{1}{(n+1)(n+2)(n+3)(n+4)}\left\{4^{n+4} - \sum_{t=0}^{3} {}^{n+4}C_t\, 3^t\right\}$$

= RHS

- **Ex. 46** *Prove that* $\sum_{k=0}^{9} x^k$ *divides* $\sum_{k=0}^{9} x^{kkkk}$.

Sol. Let $S_1 = \sum_{k=0}^{9} x^{kkkk} = x^0 + x^{1111} + x^{2222} + \dots + x^{9999}$

and $\quad S_2 = \sum_{k=0}^{9} x^k = x^0 + x^1 + x^2 + \dots + x^9$

Now, $S_1 - S_2 = \sum_{k=0}^{9} (x^{kkkk} - x^k) = \sum_{k=0}^{9} x^k\,((x^{10})^{kkk} - 1)$

$= [(x^{10})^{kkk} - 1] \sum_{k=0}^{9} x^k = \lambda \sum_{k=0}^{9} x^k$

$\Rightarrow \quad S_1 - S_2 = \lambda S_2 \;\Rightarrow\; S_1 = (1+\lambda) S_2$

Hence, $\sum_{k=0}^{9} x^{kkkk}$ is divisible by $\sum_{k=0}^{9} x^k$.

- **Ex. 47** *Prove that* $\sum_{r=1}^{k} (-3)^{r-1} \cdot {}^{3n}C_{2r-1} = 0$, *where* $k = \frac{3n}{2}$ *and* n *is an even positive integer.*

Sol. Given, n is an even positive integer.

Let $n = 2m$; $\therefore\; k = 3m,\; m \in N$

$$\text{LHS} = \sum_{r=1}^{k} (-3)^{r-1}\,{}^{3n}C_{2r-1} = \sum_{r=1}^{3m} (-3)^{r-1}\,{}^{6m}C_{2r-1}$$

$$= {}^{6m}C_1 - 3\cdot {}^{6m}C_3 + 3^2 \cdot {}^{6m}C_5 - \dots + (-3)^{3m-1}\,{}^{6m}C_{6m-1} \quad \text{...(i)}$$

Consider
$$(1+i\sqrt{3})^{6m} = {}^{6m}C_0 + {}^{6m}C_1 (i\sqrt{3}) + {}^{6m}C_2 (i\sqrt{3})^2 + {}^{6m}C_3 (i\sqrt{3})^3 + {}^{6m}C_4 (i\sqrt{3})^4 + {}^{6m}C_5 (i\sqrt{3})^5 + \dots + {}^{6m}C_{6m-1} (i\sqrt{3})^{6m-1} + {}^{6m}C_{6m} (i\sqrt{3})^{6m} \quad \text{...(ii)}$$

Now, $(1+i\sqrt{3})^{6m} = \left\{(-2)\left(\frac{-1-i\sqrt{3}}{2}\right)\right\}^{6m} = (-2\omega^2)^{6m}$

$= 2^{6m}$, where ω^2 is cube root of unity.

Then, Eq. (ii) can be written as

$$2^{6m} = \{{}^{6m}C_0 - {}^{6m}C_2 \cdot 3 + {}^{6m}C_4 \cdot 3^2 - \dots + (-3)^{3m} \cdot {}^{6m}C_{6m}\} + i\sqrt{3}\,\{{}^{6m}C_1 - {}^{6m}C_3 \cdot 3 + {}^{6m}C_5 \cdot 3^2 - \dots + (-3)^{3m-1} \cdot {}^{6m}C_{6m-1}\}$$

On comparing the imaginary part on both sides, we get

$$\sqrt{3}\,({}^{6m}C_1 - 3\cdot {}^{6m}C_3 + 3^2 \cdot {}^{6m}C_5 - \dots + (-3)^{3m-1} \cdot {}^{6m}C_{6m-1}) = 0$$

or $\quad {}^{6m}C_1 - 3\cdot {}^{6m}C_3 + 3^2 \cdot {}^{6m}C_5 - \dots + (-3)^{3m-1} \cdot {}^{6m}C_{6m-1} = 0$

$$\Rightarrow \quad \sum_{r=1}^{3m} (-3)^{r-1} \cdot {}^{6m}C_{2r-1} = 0$$

or $\quad \sum_{r=1}^{k} (-3)^{r-1} \cdot {}^{3n}C_{2r-1} = 0$, where $n = 2m$ and $k = 3m$

- **Ex. 48** *Prove that*

$${}^nC_3 + {}^nC_7 + {}^nC_{11} + \dots = \frac{1}{2}\left\{2^{n-1} - 2^{n/2} \sin \frac{n\pi}{4}\right\}$$

Sol. In given series difference in lower suffices is 4.

i.e., $\quad 7 - 3 = 11 - 7 = \dots = 4$

Now, $\quad (1)^{1/4} = (\cos 0 + i \sin 0)^{1/4}$

$= (\cos 2r\pi + i \sin 2r\pi)^{1/4}$

$= \cos \frac{r\pi}{2} + i \sin \frac{r\pi}{2}$, where $r = 0, 1, 2, 3$

Four roots of unity $= 1, i, -1, -i = 1, \alpha, \alpha^2, \alpha^3$ [say]

and $\quad (1+x)^n = \sum_{r=0}^{n} {}^nC_r\, x^r$

Putting $x = 1, \alpha, \alpha^2, \alpha^3$, we get $2^n = \sum_{r=0}^{n} {}^nC_r \quad$ ...(i)

$$(1+\alpha)^n = \sum_{r=0}^{n} {}^nC_r\, \alpha^r \quad \text{...(ii)}$$

$$(1+\alpha^2)^n = \sum_{r=0}^{n} {}^nC_r\, \alpha^{2r} \quad \text{...(iii)}$$

and $\quad (1+\alpha^3)^n = \sum_{r=0}^{n} {}^nC_r\, \alpha^{3r} \quad$ ...(iv)

On multiplying Eq. (i) by 1, Eq. (ii) by α, Eq. (iii) by α^2 and Eq. (iv) by α^3 and adding, we get

$\Rightarrow 2^n + \alpha(1+\alpha)^n + \alpha^2 (1+\alpha^2)^n + \alpha^3 (1+\alpha^3)^n$

$$= \sum_{r=0}^{n} {}^nC_r\, (1 + \alpha^{r+1} + \alpha^{2r+2} + \alpha^{3r+3}) \quad \text{...(v)}$$

For $r = 3, 7, 11, \dots$ RHS of Eq. (v)

$$= {}^nC_3 (1 + \alpha^4 + \alpha^8 + \alpha^{12}) + {}^nC_7 (1 + \alpha^8 + \alpha^{16} + \alpha^{24}) + {}^nC_{11} (1 + \alpha^{12} + \alpha^{24} + \alpha^{36}) + \dots$$

$= 4\,({}^nC_3 + {}^nC_7 + {}^nC_{11} + \dots) \quad [\because \alpha^4 = 1]$

and LHS of Eq. (v)

$= 2^n + i(1+i)^n + i^2 (1+i^2)^n + i^3 (1+i^3)^n$

$= 2^n + i(1+i)^n + 0 - i(1-i)^n$

$= 2^n + i\{(1+i)^n - (1-i)^n\}$

Since, $\left[(1+i)^n = \left[\sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)\right]^n\right]$

$= 2^n + i2^{n/2} \cdot 2i \sin \frac{n\pi}{4} = 2^{n/2}\left\{\cos \frac{\pi}{4} + i \sin \frac{\pi}{4}\right\}^n$

$= 2^n - 2^{n/2} \cdot 2 \sin \frac{n\pi}{4} = 2^{n/2}\left\{\cos \frac{n\pi}{4} + i \sin \frac{n\pi}{4}\right\}$

Hence, $4\,({}^nC_3 + {}^nC_7 + {}^nC_{11} + \dots) = 2\left(2^{n-1} - 2^{n/2} \sin \frac{n\pi}{4}\right)$

$$\Rightarrow \quad {}^nC_3 + {}^nC_7 + {}^nC_{11} + \dots = \frac{1}{2}\left(2^{n-1} - 2^{n/2} \sin \frac{n\pi}{4}\right)$$

- **Ex. 49** *Evaluate* $\sum_{i=0}^{n-1} \sum_{j=1+i}^{n+1} {}^nC_i \; {}^{n+1}C_j$.

Sol. Let $P = \sum_{i=0}^{n-1} \sum_{j=1+i}^{n+1} {}^nC_i \; {}^{n+1}C_j$

$$= \sum_{j=1}^{n+1} {}^nC_0 \; {}^{n+1}C_j + \sum_{j=2}^{n+1} {}^nC_1 \; {}^{n+1}C_j + \sum_{j=3}^{n+1} {}^nC_2 \; {}^{n+1}C_j + \dots + \sum_{j=n}^{n+1} {}^nC_{n-1} \; {}^{n+1}C_j$$

$$= {}^nC_0 \sum_{j=1}^{n+1} {}^{n+1}C_j + {}^nC_1 \sum_{j=2}^{n+1} {}^{n+1}C_j + {}^nC_2 \sum_{j=3}^{n+1} {}^{n+1}C_j + \dots + {}^nC_{n-1} \sum_{j=n}^{n+1} {}^{n+1}C_j$$

$$= {}^nC_0 ({}^{n+1}C_1 + {}^{n+1}C_2 + {}^{n+1}C_3 + \dots + {}^{n+1}C_{n+1}) + {}^nC_1({}^{n+1}C_2 + {}^{n+1}C_3 + {}^{n+1}C_4 + \dots + {}^{n+1}C_{n+1}) + {}^nC_2 ({}^{n+1}C_3 + {}^{n+1}C_4 + {}^{n+1}C_5 + \dots + {}^{n+1}C_{n+1}) + \dots + {}^nC_{n-1} ({}^{n+1}C_n + {}^{n+1}C_{n+1})$$

$$= {}^{n+1}C_1 \cdot {}^nC_0 + {}^{n+1}C_2 ({}^nC_0 + {}^nC_1) + {}^{n+1}C_3 ({}^nC_0 + {}^nC_1 + {}^nC_2) + \dots + {}^{n+1}C_{n+1} ({}^nC_0 + {}^nC_1 + {}^nC_2 + \dots + {}^nC_{n-1})$$

$$= ({}^nC_0 + {}^nC_1) \cdot {}^nC_0 + ({}^nC_1 + {}^nC_2)({}^nC_0 + {}^nC_1) + ({}^nC_2 + {}^nC_3)({}^nC_0 + {}^nC_1 + {}^nC_2) + \dots + ({}^nC_n + {}^nC_{n-1})({}^nC_0 + {}^nC_1 + {}^nC_2 + \dots + {}^nC_{n-1}) + n$$

$$= ({}^nC_0)^2 + ({}^nC_1)^2 + ({}^nC_2)^2 + \dots + ({}^nC_{n-1})^2 + 2\{{}^nC_0 \cdot {}^nC_1 + {}^nC_0 \cdot {}^nC_2 + {}^nC_0 \cdot {}^nC_3 + \dots + {}^nC_0 \cdot {}^nC_{n-1} + \dots + {}^nC_{n-2} \; {}^nC_{n-1}\} + 2^n - 1 + n$$

$$= ({}^nC_0 + {}^nC_1 + {}^nC_2 + \dots + {}^nC_{n-1})^2 + 2^n - 1 + n$$

$$= (2^n - 1)^2 + 2^n - 1 + n = 2^{2n} - 2^n + n$$

- **Ex. 50** *If* $(9 + 4\sqrt{5})^n = I + f$, *n and I being positive integers and f is a proper fraction, show that* $(I - 1) f + f^2$ *is an even integer.*

Sol. $(9 + 4\sqrt{5})^n = I + f$ ...(i)

$0 \le f < 1$ ...(ii)

Let $f' = (9 - 4\sqrt{5})^n$ ...(iii)

and $0 < f' < 1$ ...(iv)

From Eqs. (i) and (iii), we get

$$I + f + f' = (9 + 4\sqrt{5})^n + (9 - 4\sqrt{5})^n$$
$$= 2\{9^n + {}^nC_2 \, 9^{n-2} (4\sqrt{5})^2 + \dots\}$$
$$= 2N, \text{ where } N \text{ is a positive integer.}$$

and from Eqs. (ii) and (iii), we get $0 < f + f' < 2$

Since, $f + f'$ is an integer.

$\therefore \quad f + f' = 1$

Now, $I + 1 = 2N \Rightarrow 1 = 2N - 1$ ...(v)

$\because \quad (I + f)(1 - f) = (9 + 4\sqrt{5})^n f'$

$$= (9 + 4\sqrt{5})^n (9 - 4\sqrt{5})^n = 1^n = 1$$

$\therefore \quad (I - 1) f + f^2 = I - 1 = 2N - 1 - 1 = 2N - 2$ [from Eq. (v)]

$= $ An even integer

- **Ex. 51** *If* P_r *is the coefficient of* x^r *in the expansion of* $(1+x)^2 \left(1 + \frac{x}{2}\right)^2 \left(1 + \frac{x}{2^2}\right)^2 \left(1 + \frac{x}{2^3}\right)^2 \dots$, *prove that* $P_r = \frac{2^2}{(2^r - 1)}(P_{r-1} + P_{r-2})$ *and* $P_4 = \frac{1072}{315}$.

Sol. Let $(1 + x)^2 \left(1 + \frac{x}{2}\right)^2 \left(1 + \frac{x}{2^2}\right)^2 \left(1 + \frac{x}{2^3}\right)^2 \dots$,

$$= 1 + P_1 x + P_2 x^2 + P_3 x^3 + P_4 x^4 + \dots + P_{r-1} x^{r-1} + P_r x^r + \dots \quad \dots(i)$$

Replacing x by $\frac{x}{2}$, we get

$$\left(1 + \frac{x}{2}\right)^2 \left(1 + \frac{x}{2^2}\right)^2 \left(1 + \frac{x}{2^3}\right)^2 \left(1 + \frac{x}{2^4}\right)^2 \dots = \left[1 + P_1 \left(\frac{x}{2}\right) + P_2 \left(\frac{x}{2}\right)^2 + P_3 \left(\frac{x}{2}\right)^3 + \dots\right]$$

On multiplying both sides by $(1 + x)^2$, we get

$$(1 + x)^2 \left(1 + \frac{x}{2}\right)^2 \left(1 + \frac{x}{2^2}\right)^2 \left(1 + \frac{x}{2^3}\right)^2 \dots = (1 + x)^2 \left[1 + P_1 \left(\frac{x}{2}\right) + P_2 \left(\frac{x}{2}\right)^2 + P_3 \left(\frac{x}{2}\right)^3 + \dots\right] \quad \dots(ii)$$

From Eqs. (i) and (ii), we get

$$1 + P_1 x + P_2 x^2 + P_3 x^3 + P_4 x^4 + \dots + P_{r-1} x^{r-1} + P_r x^r + \dots = (1 + x)^2 \left[1 + P_1 \left(\frac{x}{2}\right) + P_2 \left(\frac{x}{2}\right)^2 + P_3 \left(\frac{x}{2}\right)^3 + \dots\right]$$

On equating coefficient of x^r, we get

$$P_r = P_r \left(\frac{1}{2^r}\right) + 2 P_{r-1} \left(\frac{1}{2^{r-1}}\right) + P_{r-2} \left(\frac{1}{2^{r-2}}\right)$$

$$\Rightarrow \quad P_r \left(1 - \frac{1}{2^r}\right) = \frac{1}{2^{r-2}} (P_{r-1} + P_{r-2})$$

$$\Rightarrow \quad P_r = \frac{2^2}{(2^r - 1)} (P_{r-1} + P_{r-2})$$

Now, $P_0 = 1, \; P_1 = 2 + 1 + \frac{1}{2} + \frac{1}{2^2} + \dots = 4$

$$P_2 = \frac{2^2 (P_1 + P_0)}{2^2 - 1} = \frac{20}{3},$$

$$P_3 = \frac{2^2 (P_2 + P_1)}{2^3 - 1} = \frac{128}{21}$$

and $$P_4 = \frac{2^2 (P_3 + P_2)}{2^4 - 1} = \frac{4\left(\frac{128}{21} + \frac{20}{3}\right)}{15} = \frac{1072}{315}$$