4 DETERMINANTS

The moving power of mathematical invention is not reasoning but imagination.

—A. Demorgan

Objectives

After studying the material of this chapter, you should be able to :

- Understand the determinants and their expansions.
- Understand the properties and their applications.
- Understand to find the area of a triangle by using determinants.
- Understand Minors and Co-factors.
- Understand Adjoint and Inverse of a Matrix.
- Understand to solve linear equations by Matrix method.

INTRODUCTION

Gratified Wilhelm Von Leibnitz originated Determinants and gave a written notation for them. **Arthur Cayley** gave **"Vertical line rotation"** in the year 1841 A.D. **Gabriel Cramer** invented **'Determinants'** independently. **Alexandre Theophile, Vandermonde, Pierre Simon Laplace, Joseph Maria Wronski** and **Augustri Louis Cauchy** were prominent who made contributions to Determinant theory.

In the previous chapter, we have studied about Matrices and Matrix Algebra.

Now we shall deal with Determinants, which are very much applicable in Engineering, Science, Economics, Social Science ; etc.

4.1. DETERMINANT

To each square matrix $A = [a_{ij}]$ of order n, there is associated a number (real or complex), which is called the determinant of square matrix A, where $a_{ij} = (i, j)$ th element in A. This can be thought as a function, which associates each square matrix with a unique number (real or complex).

Symbolically : $f : M \to K$

given by $f(A) = k$, where M is the set of matrices and K that of numbers (real or complex)

and $A \in M$ and $k \in K$.

Symbolically : $f(A) = |A| = \det A = \Delta = k$.

$|A|$ or det A is read as determinant of A.

When $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $\det A = |A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$.

KEY POINT

1. $|A|$ is not modulus of A but is read as determinant of A.
2. Only square matrices have determinants.

(*a*) **Determinant of a matrix of order one.**

Let $A = [a]$ be a matrix of order 1×1.

Then $\det A = |A| = |a| = a$.

For Examples : (I) If $A = [5]$, then $\det A = |5| = 5$.

(II) If $A = [-5]$, then $\det A = |-5| = -5$.

Chapter at Glance

(*b*) **Determinant of a matrix of order two.**

Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ be a matrix of order 2×2.

Then $\det A = |A| = \Delta = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$

$\Rightarrow \qquad \Delta = a_{11}a_{22} - a_{12}a_{21}.$

For Example : $\begin{vmatrix} 2 & 4 \\ -1 & 2 \end{vmatrix} = (2)(2) - (4)(-1) = 4 + 4 = 8.$

(*c*) **Determinant of a matrix of order three.**

Let $A = [a_{ij}]_{3 \times 3}$ be a matrix of order 3×3.

Then $|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}.$

There are six ways of expanding a determinant of order 3×3, which corresponds to each of three rows (R_1, R_2 and R_3) and three columns (C_1, C_2 and C_3).

(*i*) **Expansion along first Row (R_1) :**

Step I. Multiply first element a_{11} of R_1 by $(-1)^{1+1}$ with the second order determinant, obtained by deleting elements of first row (R_1) and first column (C_1) of $|A|$ [$\because$ *a_{11} lies in R_1 and C_1*]

i.e. $\quad (-1)^{1+1} a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}.$

Step II. Multiply second element a_{12} of R_1 by $(-1)^{1+2}$ with the second order determinant, obtained by deleting elements of first row (R_1) and second column (C_2) of $|A|$ [$\because$ *a_{12} lies in R_1 and C_2*]

i.e. $\quad (-1)^{1+2} a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}.$

Step III. Multiply third element a_{13} of R_1 by $(-1)^{1+3}$ with the second order determinant, obtained by deleting elements of first row (R_1) and third column (C_3) of $|A|$ [$\because$ *a_{13} lies in R_1 and C_3*]

i.e. $\quad (-1)^{1+3} a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}.$

Step IV. Expansion of $\det A = |A|$ is sum of all three terms, obtained in steps I – III.

Thus $\quad \det A = |A| = (-1)^{1+1} a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + (-1)^{1+2} a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + (-1)^{1+3} a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$

$$= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

$$= a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}.$$

(*ii*) – (*iii*) Similarly, we can find expansions along second row (R_2) and along third row (R_3). [*Do yourself*]

(*iv*) **Expansion along first column (C_1) :**

$$\det A = |A| = (-1)^{1+1} a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + (-1)^{2+1} a_{21} \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + (-1)^{3+1} a_{31} \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$$

$$= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{21}(a_{12}a_{33} - a_{13}a_{32}) + a_{31}(a_{12}a_{23} - a_{13}a_{22})$$

$$= a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{13}a_{21}a_{32} + a_{12}a_{23}a_{31} - a_{13}a_{22}a_{31}.$$

(*v*) – (*vi*) Similarly, we can find expansions along second column (C_2) and along third column (C_3). [*Do yourself*]

It is observed that expansion of determinant is same in each case.

KEY POINT

1. *For easier calculation, expand the determinant along that row (or column), which contains maximum number of zeros.*

2. *Signs for expansion are shown in the following pattern :*

$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}.$$

For Example : *Evaluate :* $\Delta = \begin{vmatrix} 1 & -3 & 2 \\ 4 & -1 & 2 \\ 3 & 5 & 2 \end{vmatrix}$.

By *definition*, $\Delta = 1.\begin{vmatrix} -1 & 2 \\ 5 & 2 \end{vmatrix} + 3\begin{vmatrix} 4 & 2 \\ 3 & 2 \end{vmatrix} + 2\begin{vmatrix} 4 & -1 \\ 3 & 5 \end{vmatrix}$ [*Expanding by* R_1]

$= (-2 - 10) + 3(8 - 6) + 2(20 + 3) = -12 + 6 + 46 = 40.$

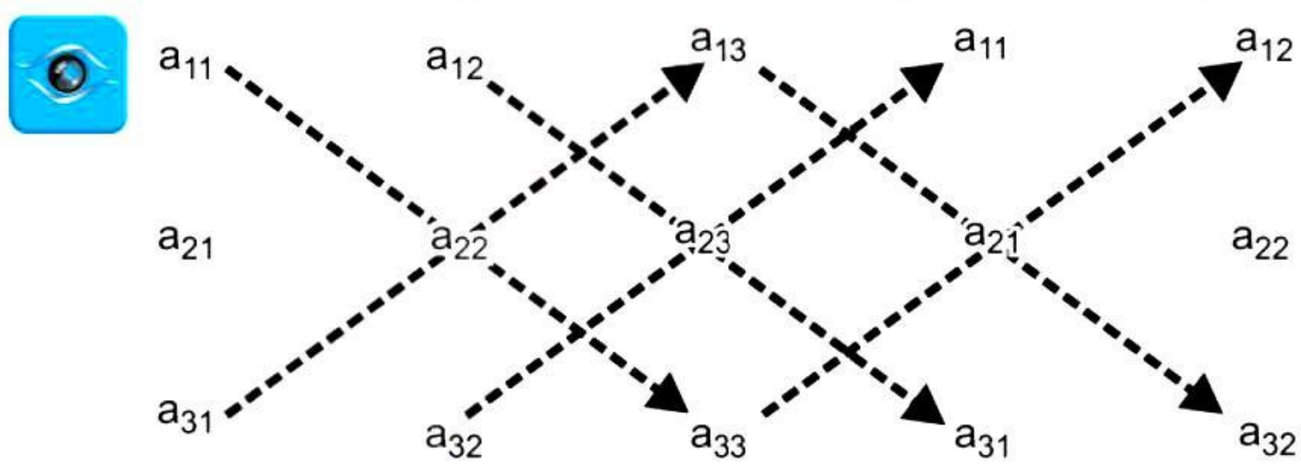
4.2. SARUS RULE

Consider $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$.

Write the three columns and repeat first and second columns.

Draw arrows as shown in the figure below.

Add the products on the downward moving arrows and subtract the products on the upward moving arrows.



Caution. Above method does not apply to determinants of order greater than 3.

ILLUSTRATIVE EXAMPLES

Example 1. Evaluate : $\begin{vmatrix} \cos 15^\circ & \sin 15^\circ \\ \sin 75^\circ & \cos 75^\circ \end{vmatrix}$.

(A.I.C.B.S.E. 2011)

Solution. $\begin{vmatrix} \cos 15^\circ & \sin 15^\circ \\ \sin 75^\circ & \cos 75^\circ \end{vmatrix}$

$= \cos 75^\circ \cos 15^\circ - \sin 75^\circ \sin 15^\circ$

$= \cos(75^\circ + 15^\circ) = \cos 90^\circ = 0.$

Example 2. If $\begin{vmatrix} 3x & 7 \\ -2 & 4 \end{vmatrix} = \begin{vmatrix} 8 & 7 \\ 6 & 4 \end{vmatrix}$**, find the value of 'x'.**

(A.I.C.B.S.E. 2014)

Solution. We have : $\begin{vmatrix} 3x & 7 \\ -2 & 4 \end{vmatrix} = \begin{vmatrix} 8 & 7 \\ 6 & 4 \end{vmatrix}$

$\Rightarrow \quad 12x + 14 = 32 - 42 \quad \Rightarrow \quad 12x = -14 - 10$

$\Rightarrow \quad 12x = -24.$ Hence, $x = -2.$

Example 3. If $\begin{vmatrix} 2x & 5 \\ 8 & x \end{vmatrix} = \begin{vmatrix} 6 & -2 \\ 7 & 3 \end{vmatrix}$**, write the value of 'x'.** *(C.B.S.E. 2014)*

Solution. We have : $\begin{vmatrix} 2x & 5 \\ 8 & x \end{vmatrix} = \begin{vmatrix} 6 & -2 \\ 7 & 3 \end{vmatrix}$

$\Rightarrow \quad 2x^2 - 40 = 18 + 14 \quad \Rightarrow \quad 2x^2 - 40 = 32$

$\Rightarrow \quad 2x^2 = 72 \quad \Rightarrow \quad x^2 = 36.$

Hence, $\quad x = \pm 6.$

Example 4. Write the value of $\begin{vmatrix} x & x+1 \\ x-1 & x \end{vmatrix}$**.**

(N.C.E.R.T. ; Uttarakhand B. 2013)

Solution. $\begin{vmatrix} x & x+1 \\ x-1 & x \end{vmatrix}$

$= (x)(x) - (x-1)(x+1) = x^2 - (x^2 - 1) = 1.$

Example 5. Expand $\begin{vmatrix} 3 & -5 & 4 \\ 7 & 6 & 1 \\ 1 & 2 & 3 \end{vmatrix}$ **by two methods.**

Solution. Method I. $\begin{vmatrix} 3 & -5 & 4 \\ 7 & 6 & 1 \\ 1 & 2 & 3 \end{vmatrix}$

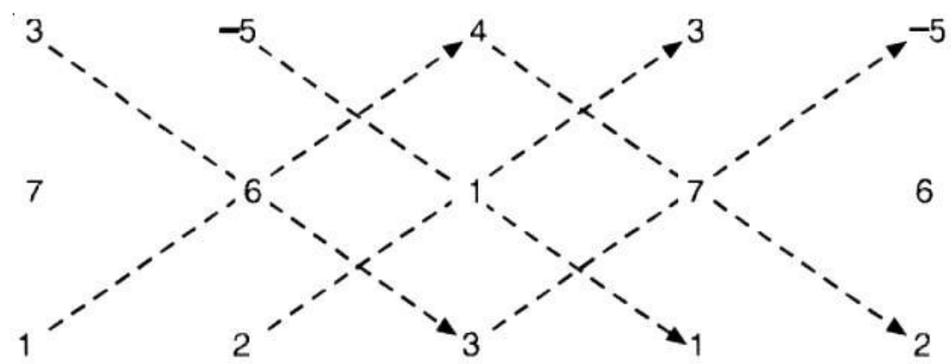
$= 3\begin{vmatrix} 6 & 1 \\ 2 & 3 \end{vmatrix} - (-5)\begin{vmatrix} 7 & 1 \\ 1 & 3 \end{vmatrix} + 4\begin{vmatrix} 7 & 6 \\ 1 & 2 \end{vmatrix}$

[*Expanding by* R_1]

$= 3(18 - 2) + 5(21 - 1) + 4(14 - 6)$

$= 48 + 100 + 32 = 180.$

Method II. By Sarus Rule :



The value of the given determinant

$= (3.6.3 + (-5).1.1 + 4.7.2)$

$- (1.6.4 + 2.1.3 + 3.7.(-5))$

$= (54 - 5 + 56) - (24 + 6 - 105)$

$= 105 + 75 = 180.$

Example 6. Evaluate :

$$\Delta = \begin{vmatrix} 0 & \sin\alpha & -\cos\alpha \\ -\sin\alpha & 0 & \sin\beta \\ \cos\alpha & -\sin\beta & 0 \end{vmatrix}.$$

(N.C.E.R.T.)

Solution. $\Delta = \begin{vmatrix} 0 & \sin\alpha & -\cos\alpha \\ -\sin\alpha & 0 & \sin\beta \\ \cos\alpha & -\sin\beta & 0 \end{vmatrix}$

$= (0)\begin{vmatrix} 0 & \sin\beta \\ -\sin\beta & 0 \end{vmatrix} - \sin\alpha\begin{vmatrix} -\sin\alpha & \sin\beta \\ \cos\alpha & 0 \end{vmatrix}$

$- \cos\alpha\begin{vmatrix} -\sin\alpha & 0 \\ \cos\alpha & -\sin\beta \end{vmatrix}$

[*Expanding by* R_1]

$= 0 - \sin\alpha(0 - \sin\beta\cos\alpha) - \cos\alpha(\sin\alpha\sin\beta - 0)$

$= \sin\alpha\sin\beta\cos\alpha - \sin\alpha\sin\beta\cos\alpha = 0.$

EXERCISE 4 (a)

Fast Track Answer Type Questions

FTATQ

1. If $\begin{vmatrix} x & 2 \\ 8 & x \end{vmatrix} = \begin{vmatrix} 3 & 2 \\ 9 & 6 \end{vmatrix}$, then the value of '$x$' is.....

(***Fill in the Blanks***) *(Jammu B. 2016)*

2. If $\begin{vmatrix} x & 2 \\ 18 & x \end{vmatrix} = \begin{vmatrix} 6 & 2 \\ 18 & 6 \end{vmatrix}$, then '$x$' is equal to $\pm$ 6.

(***True/False***) *(Jammu B. 2012)*

3. (*i*) If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, find $|2A|$. *(Karnataka B. 2014)*

(*ii*) Let A be a 3 × 3 determinant and $|A| = 7$. Find the value of $|2A|$. *(Assam B. 2017)*

4. (*i*) If $A = \begin{bmatrix} 1 & 2 \\ 4 & 2 \end{bmatrix}$, then find the value of 'k' if $|2A| = k|A|$. *(Jammu B. 2012)*

(*ii*) If $A = \begin{bmatrix} 1 & 2 \\ 4 & 2 \end{bmatrix}$, then show that $|2A| = 4|A|$.

(N.C.E.R.T.)

5. Evaluate the following determinants :

(*i*) $\begin{vmatrix} 2 & 4 \\ -1 & 2 \end{vmatrix}$ *(N.C.E.R.T.)* (*ii*) $\begin{vmatrix} -2 & 3 \\ 4 & -9 \end{vmatrix}$

(*iii*) $\begin{vmatrix} \sqrt{6} & \sqrt{5} \\ \sqrt{20} & \sqrt{24} \end{vmatrix}$

(*iv*) $\begin{vmatrix} 2\cos\theta & -2\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix}$

(*v*) $\begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix}$ **(N.C.E.R.T.)**

(*vi*) $\begin{vmatrix} x^2 - x + 1 & x - 1 \\ x + 1 & x + 1 \end{vmatrix}$. **(N.C.E.R.T.)**

6. (*a*) Find the value of 'x' if :

(*i*) $\begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} = \begin{vmatrix} x & 3 \\ 2x & 5 \end{vmatrix}$ **(N.C.E.R.T.)**

(*ii*) $\begin{vmatrix} 2 & 4 \\ 5 & 1 \end{vmatrix} = \begin{vmatrix} 2x & 4 \\ 6 & x \end{vmatrix}$

(N.C.E.R.T.; Jammu B. 2017; Kashmir B. 2016)

(*iii*) $\begin{vmatrix} 3 & x \\ x & 1 \end{vmatrix} = \begin{vmatrix} 3 & 2 \\ 4 & 1 \end{vmatrix}$

(N.C.E.R.T.; Jaharkhand B. 2016; Kerala 2013)

(*iv*) $\begin{vmatrix} 3 & x \\ x & 1 \end{vmatrix} = \begin{vmatrix} 5 & 2 \\ 4 & -1 \end{vmatrix}$ **(Kashmir B. 2016)**

(*v*) $\begin{vmatrix} 2x & 3 \\ 5 & 2 \end{vmatrix} = \begin{vmatrix} 16 & 3 \\ 5 & 2 \end{vmatrix}, x > 0$ **(A.I.C.B.S.E. 2010)**

(*vi*) $\begin{vmatrix} x+1 & x-1 \\ x-3 & x+2 \end{vmatrix} = \begin{vmatrix} 4 & -1 \\ 1 & 3 \end{vmatrix}$ **(C.B.S.E. 2013)**

(*vii*) $\begin{vmatrix} -1 & 2 \\ 4 & 8 \end{vmatrix} = \begin{vmatrix} 2 & x \\ x & -4 \end{vmatrix}$. **(Jammu B. 2012)**

(*b*) If $\begin{vmatrix} x & x \\ 1 & x \end{vmatrix} = \begin{vmatrix} 3 & 4 \\ 1 & 2 \end{vmatrix}$, write the positive value of 'x'. **(C.B.S.E. (F) 2011)**

7. If $\begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} = \begin{vmatrix} x & 3 \\ 2x & 5 \end{vmatrix}$, then the value of '$x$' is .....

(Jammu B. 2014; Kashmir B. 2012 ; H.B. 2010)

8. If $A = \begin{bmatrix} 1 & 3 \\ 4 & 1 \end{bmatrix}$, then find $|3A'|$. **(Kerala B. 2017)**

Very Short Answer Type Questions

VSATQ

9. Evaluate the following determinants :

(*i*) $\begin{vmatrix} 3 & -1 & -2 \\ 0 & 0 & -1 \\ 3 & -5 & 0 \end{vmatrix}$ **(N.C.E.R.T.)**

(*ii*) $\begin{vmatrix} 4 & 9 & 7 \\ 3 & 5 & 7 \\ 5 & 4 & 5 \end{vmatrix}$ **(Jharkhand B. 2013)**

(*iii*) $\begin{vmatrix} 2 & -1 & -2 \\ 0 & 2 & -1 \\ 3 & -5 & 0 \end{vmatrix}$ **(N.C.E.R.T.)**

(*iv*) $\begin{vmatrix} 0 & 1 & 2 \\ -1 & 0 & -3 \\ -2 & 3 & 0 \end{vmatrix}$ **(N.C.E.R.T.)**

(*v*) $\begin{vmatrix} 3 & -4 & 5 \\ 1 & 1 & -2 \\ 2 & 3 & 1 \end{vmatrix}$. **(N.C.E.R.T.)**

10. Evaluate the following determinants by two methods :

(*i*) $\begin{vmatrix} 1 & 2 & 4 \\ -1 & 3 & 0 \\ 4 & 1 & 0 \end{vmatrix}$ **(N.C.E.R.T.)**

(*ii*) $\begin{vmatrix} 0 & 2 & 0 \\ 2 & 3 & 4 \\ 4 & 5 & 6 \end{vmatrix}$. **(Jammu B. 2012 ; C.B.S.E. 2010)**

11. If $A = \begin{bmatrix} 1 & 1 & -2 \\ 2 & 1 & -3 \\ 5 & 4 & -9 \end{bmatrix}$, find $|A|$. **(N.C.E.R.T.)**

12. If $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$, then show that :

$|4A| = 64\ |A|$. **(P.B. 2011)**

13. Prove that :

$\begin{vmatrix} 1! & 2! & 3! \\ 2! & 3! & 4! \\ 3! & 4! & 5! \end{vmatrix} = 4!$. **(Kerala B. 2015)**

14. If $\begin{vmatrix} x+1 & 1 & 1 \\ 1 & x+1 & 1 \\ -1 & 1 & x+1 \end{vmatrix} = 0$, find the value of x.

(W. Bengal B. 2018)

Answers

1. ± 4. **2.** True. **3.** (*i*) -8 (*ii*) 56.

4. (*i*) $k = 4$.

5. (*i*) 8 (*ii*) 6 (*iii*) – (*iv*) 2 (*v*) 1 (*vi*) $x^3 - x^2 + 2$.

6. (*a*) (*i*) $x = 2$ (*ii*) $x = \pm\sqrt{3}$ (*iii*) $x = \pm 2\sqrt{2}$ (*iv*) ± 4 (*v*) $x = 4$ (*vi*) $x = 2$ (*vii*) $x = \pm 2\sqrt{2}$
(*b*) $x = 2$.

7. $x = 2$.

8. -99.

9. (*i*) -12 (*ii*) 77 (*iii*) 5 (*iv*) 0 (*v*) 46.

10. (*i*) -52 (*ii*) 8.

11. 0. **14.** $x = -2, -1, 0$.

Hints to Selected Questions

11. $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \Rightarrow 4A = \begin{bmatrix} 8 & 4 & 4 \\ 4 & 8 & 4 \\ 4 & 4 & 8 \end{bmatrix}.$

$$\therefore \quad |4A| = \begin{vmatrix} 8 & 4 & 4 \\ 4 & 8 & 4 \\ 4 & 4 & 8 \end{vmatrix} = (4)^3 \begin{vmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{vmatrix}$$

$$= 64\ |A|.$$

4.3. PROPERTIES

The following properties of determinants are true for determinants of any order. However, we shall prove them for determinants of order 3. These properties are used in order to simplify the determinant before expanding it.

Property I. (Reflection Property) The value of the determinant remains unaltered by interchanging its rows and columns. *(Jammu B. 2012)*

Verification. Let $\Delta = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$

Expanding along first row, we get : $\Delta = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$

$$= a_1 (b_2c_3 - b_3c_2) - a_2 (b_1c_3 - b_3c_1) + a_3 (b_1c_2 - b_2c_1) \qquad ...(1)$$

Let us interchange the rows and columns of Δ. Then :

$$\Delta' = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

Expanding along first column, we get :

$$\Delta' = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$$

$$= a_1 (b_2c_3 - b_3c_2) - a_2 (b_1c_3 - b_3c_1) + a_3 (b_1c_2 - b_2c_1) \qquad ...(2)$$

From (1) and (2), $\Delta = \Delta'$.

For Example : *Consider* $\Delta = \begin{vmatrix} 2 & -3 & 5 \\ 6 & 0 & 4 \\ 1 & 5 & -7 \end{vmatrix}.$ *(N.C.E.R.T.)*

Expanding by first row, $\Delta = 2\begin{vmatrix} 0 & 4 \\ 5 & -7 \end{vmatrix} - (-3)\begin{vmatrix} 6 & 4 \\ 1 & -7 \end{vmatrix} + 5\begin{vmatrix} 6 & 0 \\ 1 & 5 \end{vmatrix}$

$$= 2(-0-20) + 3(-42-4) + 5(30-0)$$

$$= -40 - 138 + 150 = -28.$$

Interchanging rows and columns ($R \leftrightarrow C$), we get :

$$\Delta' = \begin{vmatrix} 2 & 6 & 1 \\ -3 & 0 & 5 \\ 5 & 4 & -7 \end{vmatrix}$$

$$= 2\begin{vmatrix} 0 & 5 \\ 4 & -7 \end{vmatrix} - (-3)\begin{vmatrix} 6 & 1 \\ 4 & -7 \end{vmatrix} + 5\begin{vmatrix} 6 & 1 \\ 0 & 5 \end{vmatrix} \qquad [\textit{Expanding by } C_1]$$

$$= 2(-0-20) + 3(-42-4) + 5(30-0) = -40 - 138 + 150 = -28.$$

Hence, $\Delta = \Delta'$.

KEY POINT

1. The interchange of the rows and columns does not change the value of the determinant. Whatever is true for rows, that is also true for columns.
2. If A is a square matrix, then det A = det A′, where A′ is the transpose of A.

Property II. (Switching Property) If two adjacent rows (or columns) of a determinant are interchanged, then the sign of the determinant is changed.

Verification. Let $\Delta = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$ ...(1)

Let us interchange first and second rows*. Then the new determinant is :

$$\Delta' = \begin{vmatrix} b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \quad ...(2)$$

Expanding (1) along first row, we get :

$$\Delta = a_1 (b_2c_3 - b_3c_2) - a_2 (b_1c_3 - b_3c_1) + a_3 (b_1c_2 - b_2c_1) \quad ...(3)$$

Expanding (2) along second row, we get :

$$\Delta' = -a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} + a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} - a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

$$= -a_1 (b_2c_3 - b_3c_2) + a_2 (b_1c_3 - b_3c_1) - a_3 (b_1c_2 - b_2c_1)$$

$$= -[a_1 (b_2c_3 - b_3c_2) - a_2 (b_1c_3 - b_3c_1) + a_3 (b_1c_2 - b_2c_1)]$$

$$= -\Delta. \quad [\textit{Using (3)}]$$

For Example : *Consider* $\Delta = \begin{vmatrix} 2 & -3 & 5 \\ 6 & 0 & 4 \\ 1 & 5 & -7 \end{vmatrix}$. **(N.C.E.R.T.)**

Here $\Delta = -28$. [*See Ex. of Property I*]

Interchanging R_2 and R_3 ($R_2 \leftrightarrow R_3$), we get :

$$\Delta' = \begin{vmatrix} 2 & -3 & 5 \\ 1 & 5 & -7 \\ 6 & 0 & 4 \end{vmatrix}$$

$$= 2\begin{vmatrix} 5 & -7 \\ 0 & 4 \end{vmatrix} - (-3)\begin{vmatrix} 1 & -7 \\ 6 & 4 \end{vmatrix} + 5\begin{vmatrix} 1 & 5 \\ 6 & 0 \end{vmatrix} \quad [\textit{Expanding by } R_1]$$

$$= 2(20 + 0) + 3(4 + 42) + 5(0 - 30)$$

$$= 40 + 138 - 150 = 28.$$

Hence, $\Delta' = -\Delta$.

Cor. *If any row of a determinant is carried over n rows, then Δ′, the resultant determinant* $= (-1)^n \Delta$.

If n is even, then $\Delta' = \Delta$ and if n is odd, then $\Delta' = -\Delta$.

Property III. (Repetition Property) If two rows (or columns) of a determinant are identical, then its value is zero.

Verification. If the identical rows (or columns) of Δ are interchanged, then the value of the determinant remains unchanged.

$\therefore \quad \Delta' = \Delta$ ...(1)

By *Property* II, the interchange changes Δ to – Δ

$\therefore \quad \Delta' = -\Delta$ ...(2)

From (1) and (2), $\Delta = -\Delta \Rightarrow 2\Delta = 0 \Rightarrow \Delta = 0$.

Hence, the result.

*The arguments for other interchanges are similar.

For Example : *Consider* $\Delta = \begin{vmatrix} 3 & 2 & 3 \\ 2 & 2 & 3 \\ 3 & 2 & 3 \end{vmatrix}$. **(N.C.E.R.T.)** [*Here* R_1 & R_3 *are identical*]

Expanding along first row, $\Delta = 3\begin{vmatrix} 2 & 3 \\ 2 & 3 \end{vmatrix} - 2\begin{vmatrix} 2 & 3 \\ 3 & 3 \end{vmatrix} + 3\begin{vmatrix} 2 & 2 \\ 3 & 2 \end{vmatrix}$

$= 3(6-6) - 2(6-9) + 3(4-6)$

$= 0 - 2(-3) + 3(-2) = 6 - 6 = 0.$

Hence, $\Delta = 0.$

Property IV. (Scalar Multiple Property) If each element of a row (or column) of a determinant is multiplied by a scalar '*k*', then its value gets multiplied by the scalar '*k*'.

Verification. Let $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$.

Let Δ' be the determinant obtained by multiplying the elements of the first row by the scalar 'k'. Then :

$$\Delta' = \begin{vmatrix} ka_1 & kb_1 & kc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$= ka_1(b_2c_3 - b_3c_2) - kb_1(a_2c_3 - a_3c_2) + kc_1(a_2b_3 - a_3b_2)$ [*Expanding by* R_1]

$= k[a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2)]$

$= k\Delta.$

Similar is the result when all elements of any other row or column are multiplied by 'k'.

KEY POINT

In case of matrices, kA is the matrix obtained when each entry of A has been multiplied by 'k'. Thus if A is square matrix of order n, then $|kA| = k^n |A|$ *i.e.* $\det(kA) = k^n \det A$.

An Important Results. (I) *Determinant of a skew-symmetric matrix of odd order is zero.*

If A is a skew-symmetric matrix, then $A' = -A$.

Now $|A'| = |-A| = (-1)^n |A|$

$= -|A|.$ [$\because$ *n is odd*]

But $|A'| = |A|.$ [*Property I*]

$\therefore$ $|A| = -|A| \Rightarrow 2|A| = 0 \Rightarrow |A| = 0.$

Hence, the result.

(II) *Determinant of a skew-symmetric matrix of even order is a perfect square.*

Cor. (Proportionality Property) *If two rows (or columns) of a determinant are proportional, then its value is zero.*

Verification. Let $\Delta = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ kb_1 & kb_2 & kb_3 \end{vmatrix} = k\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$ [*Property IV*]

$= k(0)$

$= 0.$ [*Property III*]

For Example : *Consider* $\begin{vmatrix} 102 & 18 & 36 \\ 1 & 3 & 4 \\ 17 & 3 & 6 \end{vmatrix}$. **(N.C.E.R.T.)**

Here elements of R_1 and R_3 are proportional. $\left[\because \frac{102}{17} = \frac{18}{3} = \frac{36}{6}\right]$

Hence, $\Delta = 0$.

Property V. (Sum Property) If each element of a row (or column) of a determinant is expressed as the sum of two or more terms, then the determinant can be expressed as the sum of two or more determinants.

Thus $\Delta = \begin{vmatrix} a_1 + \alpha_1 & a_2 + \alpha_2 & a_3 + \alpha_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + \begin{vmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$.

Verification. Expanding determinant on the left along the first row, we get :

$$\Delta = (a_1 + \alpha_1)(b_2c_3 - b_3c_2) - (a_2 + \alpha_2)(b_1c_3 - b_3c_1) + (a_3 + \alpha_3)(b_1c_2 - b_2c_1)$$

$$= \{a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1)\} + \{\alpha_1(b_2c_3 - b_3c_2) - \alpha_2(b_1c_3 - b_3c_1) + \alpha_3(b_1c_2 - b_2c_1)\}$$

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + \begin{vmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

Similar is the case with other rows and columns.

For Example : *Consider* $\Delta = \begin{vmatrix} a & b & c \\ a + 2x & b + 2y & c + 2z \\ x & y & z \end{vmatrix}$. **(N.C.E.R.T.)**

Here $\Delta = \begin{vmatrix} a & b & c \\ a & b & c \\ x & y & z \end{vmatrix} + \begin{vmatrix} a & b & c \\ 2x & 2y & 2z \\ x & y & z \end{vmatrix}$ [*Property V*]

$= 0 + 0$ [*Using Properties III & IV*]

$= 0.$

Property VI. (Invariance Property) If to any row (or column) of a determinant, a multiple of another row (or column) is added, the value of the determinant remains the same.

Verification. Let $\Delta = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$.

Suppose $\Delta' = \begin{vmatrix} a_1 + kc_1 & a_2 + kc_2 & a_3 + kc_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$.

[*Δ' is obtained from Δ when k–times the third row is added to its first row*]

Now $\Delta' = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + \begin{vmatrix} kc_1 & kc_2 & kc_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + k \begin{vmatrix} c_1 & c_2 & c_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

$= \Delta + 0.$ [$\because$ *In second Det on RHS first and third rows are identical*]

Hence, $\Delta' = \Delta$.

KEY POINT

While altering the parallel lines (rows or columns), one line must be left unchanged.

Property VII. (Factor Property) If a determinant Δ vanishes when for 'x' is put 'a' in those elements of Δ, which are polynomials in x, then (x – a) is a factor of Δ.

Verification. Since the elements of Δ are polynomials in x, [*Given*]

$\therefore$ its expansion is also a polynomial in x.

Let $\Delta = f(x)$.

Since 'a' is a zero of the polynomial $f(x)$,

$\therefore$ $(x - a)$ must be a factor of $f(x)$ [*Factor Theorem*]

i.e. $(x - a)$ must be a factor of Δ.

Hence, the result.

NOTATIONS

We denote :

(*i*) $R_1, R_2, \ldots\ldots$ as first, second,.....rows

(*ii*) $C_1, C_2, \ldots\ldots$as first, second, ...columns

(*iii*) "$R_1 \to R_1 - 2R_2$" means "*From the elements of first row, subtract twice those of second row*"

(*iv*) " $C_1 \to C_1 + 2C_2 + 3C_3$" means "*To the elements of first column, add twice those of second column and thrice those of the third column.*"

Frequently Asked Questions

FAQs

Example 1. If A is a 3 × 3 matrix and $|3A| = k|A|$, then write the value of 'k'. *(A.I.C.B.S.E. 2016)*

Solution. We have : $|3A| = k|A|$

$\Rightarrow$ $3^3|A| = k|A|$.

Hence, $k = 3^3 = 27$.

Example 2. Let A be a square matrix of order 3 × 3. Write the value of $|2A|$, where $|A| = 4$.

(A.I.C.B.S.E. 2012)

Solution. $|2A| = 2^3 |A|$ [$\because$ *A is 3 × 3 matrix*]

$= 8|A| = 8(4) = 32$.

Example 3. Find the maximum value of :

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1+\sin\theta & 1 \\ 1 & 1 & 1+\cos\theta \end{vmatrix}.$$

(C.B.S.E. 2016)

Solution. Operating $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, we get :

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 0 & \sin\theta & 0 \\ 0 & 0 & \cos\theta \end{vmatrix}$$

$= 1.(\sin\theta\cos\theta - 0)$

[*Expanding by* C_1]

$= \sin\theta\cos\theta = \frac{1}{2}\sin 2\theta$.

Hence, the maximum value of $\Delta = \frac{1}{2}(1) = \frac{1}{2}$.

[$\because$ *Max. value of* $\sin 2\theta = 1$]

Example 4. Using properties of determinants, prove that :

$$\begin{vmatrix} 1 & 1 & 1+3x \\ 1+3y & 1 & 1 \\ 1 & 1+3z & 1 \end{vmatrix} = 9(3xyz + xy + yz + zx).$$

(C.B.S.E. 2018)

Solution. LHS $= \begin{vmatrix} 1 & 1 & 1+3x \\ 1+3y & 1 & 1 \\ 1 & 1+3z & 1 \end{vmatrix}$

$= \begin{vmatrix} 1 & 0 & 3x \\ 1+3y & -3y & -3y \\ 1 & 3z & 0 \end{vmatrix}$

[*Operating* $C_2 \to C_2 - C_1$ *and* $C_3 \to C_3 - C_1$]

$= 1.\begin{vmatrix} -3y & -3y \\ 3z & 0 \end{vmatrix} + 3x\begin{vmatrix} 1+3y & -3y \\ 1 & 3z \end{vmatrix}$

[*Expanding by* R_1]

$= 1.(0 + 9yz) + 3x(3z + 9yz + 3y)$

$= 9(3xyz + xy + yz + zx) =$ **RHS.**

Example 5. If $\Delta = \begin{vmatrix} 1 & a & a^2 \\ a & a^2 & 1 \\ a^2 & 1 & a \end{vmatrix} = -4$, **then find the value of :**

$$\begin{vmatrix} a^3-1 & 0 & a-a^4 \\ 0 & a-a^4 & a^3-1 \\ a-a^4 & a^3-1 & 0 \end{vmatrix}.$$

(C.B.S.E. Sample Paper 2018)

Solution. We have $\Delta = \begin{vmatrix} 1 & a & a^2 \\ a & a^2 & 1 \\ a^2 & 1 & a \end{vmatrix} = -4$

$\Rightarrow \quad \Delta = 1.\,(a^3 - 1) - a(a^2 - a^2) + a^2(a - a^4) = -4$

$\Rightarrow \quad (a^3 - 1) - 0 - a^3(a^3 - 1) = -4$

$\Rightarrow \quad (a^3 - 1)\,(1 - a^3) = -4$

$\Rightarrow \quad (a^3 - 1)^2 = 4 \qquad ...(1)$

Now, $\begin{vmatrix} a^3-1 & 0 & a-a^4 \\ 0 & a-a^4 & a^3-1 \\ a-a^4 & a^3-1 & 0 \end{vmatrix}$

$$= (a^3-1)^3 \begin{vmatrix} 1 & 0 & -a \\ 0 & -a & 1 \\ -a & 1 & 0 \end{vmatrix}$$

$= (a^3 - 1)^3\,[(1.\,(-0 - 1) - 0 - a\,(0 - a^2)]$

$= (a^3 - 1)^3\,(a^3 - 1) = (a^3 - 1)^4$

$= (4)^2 = 16.$ [*Using (1)*]

Example 6. Prove that :

$$\begin{vmatrix} \mathbf{a+b} & \mathbf{b+c} & \mathbf{c+a} \\ \mathbf{b+c} & \mathbf{c+a} & \mathbf{a+b} \\ \mathbf{c+a} & \mathbf{a+b} & \mathbf{b+c} \end{vmatrix} = \mathbf{2}\begin{vmatrix} \mathbf{a} & \mathbf{b} & \mathbf{c} \\ \mathbf{b} & \mathbf{c} & \mathbf{a} \\ \mathbf{c} & \mathbf{a} & \mathbf{b} \end{vmatrix}.$$

Solution. **LHS** $= \begin{vmatrix} a+b & b+c & c+a \\ b+c & c+a & a+b \\ c+a & a+b & b+c \end{vmatrix}$

$$= \begin{vmatrix} 2\,(a+b+c) & b+c & c+a \\ 2\,(a+b+c) & c+a & a+b \\ 2\,(a+b+c) & a+b & b+c \end{vmatrix}$$

[*Operating* $C_1 \to C_1 + C_2 + C_3$]

$$= 2\begin{vmatrix} a+b+c & b+c & c+a \\ a+b+c & c+a & a+b \\ a+b+c & a+b & b+c \end{vmatrix}$$

[*Taking 2 common from* C_1]

$$= 2\begin{vmatrix} a+b+c & -a & -b \\ a+b+c & -b & -c \\ a+b+c & -c & -a \end{vmatrix}$$

[*Operating* $C_2 \to C_2 - C_1$ & $C_3 \to C_3 - C_1$]

$$= 2\begin{vmatrix} c & -a & -b \\ a & -b & -c \\ b & -c & -a \end{vmatrix}$$

[*Operating* $C_1 \to C_1 + C_2 + C_3$]

$$= 2(-1)^2\begin{vmatrix} c & a & b \\ a & b & c \\ b & c & a \end{vmatrix}$$

[*Taking (–1) common from* C_2 *and* C_3]

$$= 2\begin{vmatrix} c & a & b \\ a & b & c \\ b & c & a \end{vmatrix}$$

$$= -2\begin{vmatrix} a & c & b \\ b & a & c \\ c & b & a \end{vmatrix}$$ [*Operating* $C_1 \leftrightarrow C_2$]

$$= 2\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$$ [*Operating* $C_2 \leftrightarrow C_3$]

= RHS.

Example 7. Without expanding, prove that the following determinants vanish :

(i) $\begin{vmatrix} \mathbf{b^2c^2} & \mathbf{bc} & \mathbf{b+c} \\ \mathbf{c^2a^2} & \mathbf{ca} & \mathbf{c+a} \\ \mathbf{a^2b^2} & \mathbf{ab} & \mathbf{a+b} \end{vmatrix}$

(ii) $\begin{vmatrix} \mathbf{a} & \mathbf{b} & \mathbf{a+b} \\ \mathbf{b} & \mathbf{c} & \mathbf{b+c} \\ \mathbf{c} & \mathbf{a} & \mathbf{c+a} \end{vmatrix}.$ ***(H.B. 2011)***

Solution. (*i*) $\begin{vmatrix} b^2c^2 & bc & b+c \\ c^2a^2 & ca & c+a \\ a^2b^2 & ab & a+b \end{vmatrix}$

$$= \frac{1}{abc}\begin{vmatrix} ab^2c^2 & abc & ab+ac \\ bc^2a^2 & abc & bc+ba \\ ca^2b^2 & abc & ac+bc \end{vmatrix}$$

[*Multiplying* R_1 *by a,* R_2 *by b,* R_3 *by c and taking* $\frac{1}{abc}$ *outside*]

$$= \frac{(abc)^2}{abc}\begin{vmatrix} bc & 1 & ab+ac \\ ca & 1 & bc+ba \\ ab & 1 & ac+bc \end{vmatrix}$$

[*Taking (abc) common from* C_1 & C_2]

$$= abc\begin{vmatrix} bc & 1 & ab+bc+ca \\ ca & 1 & ab+bc+ca \\ ab & 1 & ab+bc+ca \end{vmatrix}$$

[*Operating* $C_3 \to C_3 + C_1$]

$$= abc\,(ab+bc+ca)\begin{vmatrix} bc & 1 & 1 \\ ca & 1 & 1 \\ ab & 1 & 1 \end{vmatrix}$$

[*Taking (ab + bc + ca) common from* C_3]

$= abc\,(ab + bc + ca)\,(0)$

[$\because$ C_2 & C_3 *are identical*]

$= 0$, which is true.

(*ii*) Given determinant

$$= \begin{vmatrix} a & b & a+b \\ b & c & b+c \\ c & a & c+a \end{vmatrix}$$

$$= \begin{vmatrix} a+b & b & a+b \\ b+c & c & b+c \\ c+a & a & c+a \end{vmatrix} \quad [\textit{Operating } C_1 \to C_1 + C_2]$$

$$= 0. \quad [\because C_1 \ \& \ C_3 \textit{ are identical}]$$

Example 8. If $f(x) = \begin{vmatrix} \mathbf{a} & \mathbf{-1} & \mathbf{0} \\ \mathbf{ax} & \mathbf{a} & \mathbf{-1} \\ \mathbf{ax^2} & \mathbf{ax} & \mathbf{a} \end{vmatrix}$, **using properties of determinants, find the value of f(2x) –f(x).**

(C.B.S.E. 2015)

Solution. We have : $f(x) = \begin{vmatrix} a & -1 & 0 \\ ax & a & -1 \\ ax^2 & ax & a \end{vmatrix}$

$$= a \begin{vmatrix} 1 & -1 & 0 \\ x & a & -1 \\ x^2 & ax & a \end{vmatrix} \quad [\textit{Taking } a \textit{ common from } C_1]$$

$$= a \begin{vmatrix} 1 & 0 & 0 \\ x & a+x & -1 \\ x^2 & ax+x^2 & a \end{vmatrix} \quad [\textit{Operating } C_2 \to C_2 + C_1]$$

$$= a[(a + x)\, a + (ax + x^2)]$$

$$= a[a^2 + ax + ax + x^2] = a(x^2 + 2ax + a^2)$$

$$= a(x + a)^2.$$

$$\therefore \ f(2x) = a(2x + a)^2.$$

$$\therefore \ f(2x) - f(x) = a[(2x + a)^2 - (x + a)^2]$$

$$= a[(4x^2 + 4ax + a^2) - (x^2 + 2ax + a^2)]$$

$$= a(3x^2 + 2ax) = ax(3x + 2a).$$

Example 9. Using the properties of determinants, prove that : $\begin{vmatrix} \mathbf{x+y} & \mathbf{x} & \mathbf{x} \\ \mathbf{5x+4y} & \mathbf{4x} & \mathbf{2x} \\ \mathbf{10x+8y} & \mathbf{8x} & \mathbf{3x} \end{vmatrix} = \mathbf{x^3}.$

(A.I.C.B.S.E. 2009)

Solution. LHS $= \begin{vmatrix} x+y & x & x \\ 5x+4y & 4x & 2x \\ 10x+8y & 8x & 3x \end{vmatrix}$

$$= \begin{vmatrix} x & x & x \\ 5x & 4x & 2x \\ 10x & 8x & 3x \end{vmatrix} + \begin{vmatrix} y & x & x \\ 4y & 4x & 2x \\ 8y & 8x & 3x \end{vmatrix} \quad (\textit{Property V})$$

$$= x^3 \begin{vmatrix} 1 & 1 & 1 \\ 5 & 4 & 2 \\ 10 & 8 & 3 \end{vmatrix} + x^2 y \begin{vmatrix} 1 & 1 & 1 \\ 4 & 4 & 2 \\ 8 & 8 & 3 \end{vmatrix}$$

$$= x^3 \begin{vmatrix} 1 & 0 & 0 \\ 5 & -1 & -2 \\ 10 & -2 & -5 \end{vmatrix} + x^2 y\,(0)$$

[*Operating* $C_2 \to C_2 - C_1$ & $C_3 \to C_3 - C_2$ *in 1st determinant* & C_1, C_2 *are idential in 2nd determinant*]

$$= x^3 \begin{vmatrix} 1 & 0 & 0 \\ 0 & -1 & -2 \\ 0 & 0 & -1 \end{vmatrix}$$

[*Operating* $R_2 \to R_2 - 5R_1$ & $R_3 \to R_3 - 2R_2$]

$$= (-1)^2\, x^3 \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{vmatrix}$$

[*Taking* (-1) *common from* C_2 & C_3]

$$= x^3(1) \begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} = x^3(1-0) = x^3 = \textbf{RHS.}$$

Example 10. Prove that :

$$\begin{vmatrix} \mathbf{a+b+2c} & \mathbf{a} & \mathbf{b} \\ \mathbf{c} & \mathbf{b+c+2a} & \mathbf{b} \\ \mathbf{c} & \mathbf{a} & \mathbf{c+a+2b} \end{vmatrix} = \mathbf{2\,(a+b+c)^3.}$$

(H.P.B. 2016; Kashmir B. 2015; C.B.S.E. 2014; H.B. 2013; J. & K.B. 2011)

Solution.

$$\textbf{LHS} = \begin{vmatrix} a+b+c & -(a+b+c) & 0 \\ 0 & a+b+c & -(a+b+c) \\ c & a & c+a+2b \end{vmatrix}$$

[*Operating* $R_1 \to R_1 - R_2$ *and* $R_2 \to R_2 - R_3$]

$$= (a+b+c)^2 \begin{vmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ c & a & c+a+2b \end{vmatrix}$$

[*Taking* $(a + b + c)$ *common from* R_1 *and* R_2]

$$= (a+b+c)^2 \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ c & a+c & c+a+2b \end{vmatrix}$$

[*Operating* $C_2 \to C_2 + C_1$]

$$= (a+b+c)^2\ (1)\begin{vmatrix} 1 & -1 \\ a+c & c+a+2b \end{vmatrix}$$

[*Expanding by* R_1]

$= (a+b+c)^2\ [(c+a+2b)+(a+c)]$

$= (a+b+c)^2\ (2a+2b+2c)$

$= 2\ (a+b+c)^2\ (a+b+c)$

$= 2\ (a+b+c)^3 =$ **RHS.**

Example 11. If x, y, z are different and

$$\Delta = \begin{vmatrix} x & x^2 & 1+x^3 \\ y & y^2 & 1+y^3 \\ z & z^2 & 1+z^3 \end{vmatrix} = 0, \text{ show that :}$$

(*i*) 1 + xyz = 0

(N.C.E.R.T.; H.P.B. 2013 ; Kashmir B. 2012)

(*ii*) xyz = – 1. *(P.B. 2018 ; Assam B. 2015)*

Solution. Here $\Delta = \begin{vmatrix} x & x^2 & 1+x^3 \\ y & y^2 & 1+y^3 \\ z & z^2 & 1+z^3 \end{vmatrix} = 0$

$$\Rightarrow \quad \Delta = \begin{vmatrix} x & x^2 & 1+x^3 \\ y-x & y^2-x^2 & y^3-x^3 \\ z-y & z^2-y^2 & z^3-y^3 \end{vmatrix} = 0$$

[*Operating* $R_2 \to R_2 - R_1$ & $R_3 \to R_3 - R_2$]

$$\Rightarrow \quad \Delta = \begin{vmatrix} x & x^2 & 1+x^3 \\ y-x & (y-x)(y+x) & (y-x)(y^2+xy+x^2) \\ z-y & (z-y)(z+y) & (z-y)(z^2+yz+y^2) \end{vmatrix} = 0$$

$$\Rightarrow \Delta = (y-x)(z-y)\begin{vmatrix} x & x^2 & 1+x^3 \\ 1 & y+x & y^2+xy+x^2 \\ 1 & z+y & z^2+yz+y^2 \end{vmatrix} = 0$$

[*Taking* $(y-x)$ *common from* R_2 *and* $(z-y)$ *from* R_3]

$\Rightarrow \quad \Delta = (y-x)(z-y)$

$$\begin{vmatrix} x & x^2 & 1+x^3 \\ 1 & y+x & y^2+xy+x^2 \\ 0 & z-x & z^2+yz-xy-x^2 \end{vmatrix} = 0$$

[*Operating* $R_3 \to R_3 - R_2$]

$$\Rightarrow \quad \Delta = (y-x)(z-y)\begin{vmatrix} x & x^2 & 1+x^3 \\ 1 & y+x & y^2+xy+x^2 \\ 0 & z-x & z+x+y \end{vmatrix} = 0$$

$\Rightarrow \quad \Delta = (y-x)(z-y)(z-x)$

$$\begin{vmatrix} x & x^2 & 1+x^3 \\ 1 & y+x & y^2+xy+x^2 \\ 0 & 1 & z+x+y \end{vmatrix} = 0$$

[*Taking* $(z-x)$ *common from* R_3]

$$\Rightarrow \begin{vmatrix} x & x^2 & 1+x^3 \\ 1 & x+y & x^2+xy+y^2 \\ 0 & 1 & x+y+z \end{vmatrix} = 0$$

[*Dividing by* $(x-y)(y-z)(z-x)$ *because* $x \neq y,\ y \neq z$ & $z \neq x$]

$$\Rightarrow x\begin{vmatrix} x+y & x^2+xy+y^2 \\ 1 & x+y+z \end{vmatrix} - (1)\begin{vmatrix} x^2 & 1+x^3 \\ 1 & x+y+z \end{vmatrix} = 0$$

[*Expanding by* C_1]

$\Rightarrow \quad x\ [(x+y)(x+y+z) - (1)(x^2+xy+y^2)]$

$\qquad - [x^2(x+y+z) - (1)(1+x^3)] = 0$

$\Rightarrow \quad x\ [x^2+xy+zx+xy+y^2+yz-x^2-xy-y^2]$

$\qquad - [x^3+x^2y+x^2z-1-x^3] = 0$

$\Rightarrow \quad x\ [xy+yz+zx] - [x^2y+x^2z-1] = 0$

$\Rightarrow \quad x^2y+xyz+zx^2-x^2y-x^2z+1 = 0$

$\Rightarrow \quad 1+xyz = 0$ *or* $xyz = -1$, which is true.

Example 12. Prove that :

$$\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} = abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$$

(N.C.E.R.T.; P.B. 2017; Assam B. 2016; Nagaland B. 2016)

= abc + ab + bc + ca.

(N.C.E.R.T. ; H.P.B. 2017, 13; A.I.C.B.S.E. 2014; C.B.S.E. 2012, 10 C)

Solution. $\Delta = \begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix}$

$$= abc \begin{vmatrix} \frac{1}{a}+1 & \frac{1}{a} & \frac{1}{a} \\ \frac{1}{b} & \frac{1}{b}+1 & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & \frac{1}{c}+1 \end{vmatrix}$$

[*Taking a, b, c common from 1st, 2nd, 3rd row respectively*]

$$= abc \begin{vmatrix} \frac{1}{a}+1 & -1 & 0 \\ \frac{1}{b} & 1 & -1 \\ \frac{1}{c} & 0 & 1 \end{vmatrix}$$

[*Operating* $C_2 \to C_2 - C_1$, $C_3 \to C_3 - C_2$]

$$= abc \left[\left(\frac{1}{a}+1\right) \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} + (1) \begin{vmatrix} \frac{1}{b} & -1 \\ \frac{1}{c} & 1 \end{vmatrix} \right]$$

[*Expanding by* R_1]

$$= abc \left[\left(\frac{1}{a}+1\right)(1+0) + \left(\frac{1}{b}+\frac{1}{c}\right) \right]$$

$$= \boldsymbol{abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)}$$ **...First Result**

$$= abc + bc + ca + ab = \boldsymbol{abc + ab + bc + ca.}$$

...Second Result

Example 13. Prove that :

$$\begin{vmatrix} 1 & 1 & 1 \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix} = (b-c)(c-a)(a-b)(bc+ca+ab).$$

Solution. LHS $= \begin{vmatrix} 1 & 1 & 1 \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix}$

$$= \begin{vmatrix} 1 & 0 & 0 \\ a^2 & b^2-a^2 & c^2-b^2 \\ a^3 & b^3-a^3 & c^3-b^3 \end{vmatrix}$$

[*Operating* $C_2 \to C_2 - C_1$ & $C_3 \to C_3 - C_2$]

$$= \begin{vmatrix} 1 & 0 & 0 \\ a^2 & (b-a)(b+a) & (c-b)(c+b) \\ a^3 & (b-a)(b^2+ba+a^2) & (c-b)(c^2+cb+b^2) \end{vmatrix}$$

$$= (b-a)(c-b) \begin{vmatrix} 1 & 0 & 0 \\ a^2 & a+b & b+c \\ a^3 & a^2+ab+b^2 & b^2+bc+c^2 \end{vmatrix}$$

[*Taking* $(b-a)$ *common from* C_2 & $(c-b)$ *from* C_3]

$$= (a-b)(b-c)(1) \begin{vmatrix} a+b & b+c \\ a^2+ab+b^2 & b^2+bc+c^2 \end{vmatrix}$$

[*Expanding by* R_1]

$$= (a-b)(b-c) \begin{vmatrix} a+b & c-a \\ a^2+ab+b^2 & (bc+c^2)-(ab+a^2) \end{vmatrix}$$

[*Operating* $C_2 \to C_2 - C_1$]

$$= (a-b)(b-c) \begin{vmatrix} a+b & c-a \\ a^2+ab+b^2 & (c-a)(a+b+c) \end{vmatrix}$$

$$= (a-b)(b-c)(c-a) \begin{vmatrix} a+b & 1 \\ a^2+ab+b^2 & a+b+c \end{vmatrix}$$

[*Taking* $(c-a)$ *common from* C_2]

$$= (a-b)(b-c)(c-a)\,[(a+b)(a+b+c) - (1)(a^2+ab+b^2)]$$

$$= (a-b)(b-c)(c-a)\,[a^2+ab+ca+ab+b^2 + bc - a^2 - ab - b^2]$$

$$= (a-b)(b-c)(c-a)(bc+ca+ab) = \textbf{RHS.}$$

Example 14. Prove that :

$$\begin{vmatrix} a^2+1 & ab & ac \\ ab & b^2+1 & bc \\ ac & bc & c^2+1 \end{vmatrix}$$

$$= \begin{vmatrix} a^2+1 & b^2 & c^2 \\ a^2 & b^2+1 & c^2 \\ a^2 & b^2 & c^2+1 \end{vmatrix} = \mathbf{1 + a^2 + b^2 + c^2.}$$

(N.C.E.R.T. ; W. Bengal B. 2016; C.B.S.E. 2014 ; Kashmir B. 2013 ; Assam B. 2013)

Solution. First Term $= \begin{vmatrix} a^2+1 & ab & ac \\ ab & b^2+1 & bc \\ ac & bc & c^2+1 \end{vmatrix}$

$$= abc \begin{vmatrix} a+\frac{1}{a} & b & c \\ a & b+\frac{1}{b} & c \\ a & b & c+\frac{1}{c} \end{vmatrix}$$

[*Taking a common from R_1, b from R_2 & c from R_3*]

$$= \begin{vmatrix} a^2+1 & b^2 & c^2 \\ a^2 & b^2+1 & c^2 \\ a^2 & b^2 & c^2+1 \end{vmatrix}$$

[*Multiplying C_1 by a, C_2 by b & C_3 by c*]

= Second Term

$$= \begin{vmatrix} 1+a^2+b^2+c^2 & b^2 & c^2 \\ 1+a^2+b^2+c^2 & b^2+1 & c^2 \\ 1+a^2+b^2+c^2 & b^2 & c^2+1 \end{vmatrix}$$

[*Operating $C_1 \to C_1 + C_2 + C_3$*]

$$= (1+a^2+b^2+c^2) \begin{vmatrix} 1 & b^2 & c^2 \\ 1 & b^2+1 & c^2 \\ 1 & b^2 & c^2+1 \end{vmatrix}$$

[*Taking $(1 + a^2 + b^2 + c^2)$ common from C_1*]

$$= (1+a^2+b^2+c^2) \begin{vmatrix} 1 & b^2 & c^2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

[*Operating $R_2 \to R_2 - R_1$ & $R_3 \to R_3 - R_1$*]

$= (1 + a^2 + b^2 + c^2)\,(1)\,(1 - 0)$ [*Expanding by C_1*]

$= 1 + a^2 + b^2 + c^2 =$ **Third Term.**

Hence, the result.

Example 15. Prove that :

$$\begin{vmatrix} yz-x^2 & zx-y^2 & xy-z^2 \\ zx-y^2 & xy-z^2 & yz-x^2 \\ xy-z^2 & yz-x^2 & zx-y^2 \end{vmatrix}$$ **is divisible by (x + y + z),**

and hence find the quotient. ***(C.B.S.E. 2016)***

Solution.

$$\Delta = \begin{vmatrix} yz-x^2 & zx-y^2 & xy-z^2 \\ zx-y^2 & xy-z^2 & yz-x^2 \\ xy-z^2 & yz-x^2 & zx-y^2 \end{vmatrix}$$

$$= \begin{vmatrix} -(x^2+y^2+z^2-xy-yz-zx) & zx-y^2 & xy-z^2 \\ -(x^2+y^2+z^2-xy-yz-zx) & xy-z^2 & yz-x^2 \\ -(x^2+y^2+z^2-xy-yz-zx) & yz-x^2 & zx-y^2 \end{vmatrix}$$

[*Operating $C_1 \to C_1 + C_2 + C_3$*]

$$= -(x^2+y^2+z^2-xy-yz-zx) \begin{vmatrix} 1 & zx-y^2 & xy-z^2 \\ 1 & xy-z^2 & yz-x^2 \\ 1 & yz-x^2 & zx-y^2 \end{vmatrix}$$

[*Taking $-(x^2 + y^2 + z^2 - xy - yz - zx)$ common from C_1*]

$= -(x^2 + y^2 + z^2 - xy - yz - zx)$

$$\begin{vmatrix} 0 & (x-y)(x+y+z) & (y-z)(x+y+z) \\ 0 & (x-z)(x+y+z) & (y-x)(x+y+z) \\ 1 & yz-x^2 & zx-y^2 \end{vmatrix}$$

[*Operating $R_1 \to R_1 - R_3$ and $R_2 \to R_2 - R_3$*]

$= -(x^2 + y^2 + z^2 - xy - yz - zx)\,(x + y + z)^2$

$$\begin{vmatrix} 0 & x-y & y-z \\ 0 & x-z & y-x \\ 1 & yz-x^2 & zx-y^2 \end{vmatrix}$$

[*Taking $(x + y + z)$ common from R_1 and R_2*]

$= -(x + y + z)(x^3 + y^3 + z^3 - 3xyz)$

$$\begin{vmatrix} 0 & x-y & y-z \\ 0 & x-z & y-x \\ 1 & yz-x^2 & zx-y^2 \end{vmatrix}$$

$= -(x + y + z)(x^3 + y^3 + z^3 - 3xyz)\,[(x - y)(y - x) - (x - z)(y - z)]$

$= -(x + y + z)(x^3 + y^3 + z^3 - 3xyz)(xy - x^2 - y^2 + xy - xy + xz + zy - z^2)$

$= (x + y + z)\,(x^3 + y^3 + z^3 - 3xyz)\,(x^2 + y^2 + z^2 - xy - yz - zx).$

Hence, Δ is divisible by $(x + y + z)$ and quotient is :

$(x^3 + y^3 + z^3 - 3xyz)(x^2 + y^2 + z^2 - xy - yz - zx).$

Example 16. Using properties of determinants, prove that :

$$\begin{vmatrix} -a^2 & ab & ac \\ ba & -b^2 & bc \\ ca & cb & -c^2 \end{vmatrix} = 4a^2b^2c^2.$$

(N.C.E.R.T; C.B.S.E. 2011; H.P.B. 2010)

Solution. LHS = $\begin{vmatrix} -a^2 & ab & ac \\ ba & -b^2 & bc \\ ca & cb & -c^2 \end{vmatrix}$

$$= abc \begin{vmatrix} -a & b & c \\ a & -b & c \\ a & b & -c \end{vmatrix}$$

[*Taking a, b and c common from R_1, R_2 and R_3 respectively*]

$$= a^2b^2c^2 \begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$

[*Taking a, b and c common from C_1, C_2 and C_3 respectively*]

$$= a^2b^2c^2 \begin{vmatrix} 0 & 0 & 1 \\ 2 & -2 & 1 \\ 0 & 2 & -1 \end{vmatrix}$$

[*Operating $C_1 \to C_1 + C_3$ & $C_2 \to C_2 - C_3$*]

$$= a^2b^2c^2\,(1) \begin{vmatrix} 2 & -2 \\ 0 & 2 \end{vmatrix}$$ [*Expanding by R_1*]

$= a^2b^2c^2\,(4 + 0) = 4a^2b^2c^2$ = **RHS.**

Example 17. Using the properties of determinants, solve the following for 'x' :

$$\begin{vmatrix} x+2 & x+6 & x-1 \\ x+6 & x-1 & x+2 \\ x-1 & x+2 & x+6 \end{vmatrix} = 0.$$ *(A.I.C.B.S.E. 2015)*

Solution. We have :

$$\begin{vmatrix} x+2 & x+6 & x-1 \\ x+6 & x-1 & x+2 \\ x-1 & x+2 & x+6 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 3x+7 & 3x+7 & 3x+7 \\ x+6 & x-1 & x+2 \\ x-1 & x+2 & x+6 \end{vmatrix} = 0$$

[*Operating $R_1 \to R_1 + R_2 + R_3$*]

$$\Rightarrow (3x+7) \begin{vmatrix} 1 & 1 & 1 \\ x+6 & x-1 & x+2 \\ x-1 & x+2 & x+6 \end{vmatrix} = 0$$

[*Taking (3x + 7) common from R_1*]

$$\Rightarrow (3x+7) \begin{vmatrix} 1 & 0 & 0 \\ x+6 & -7 & 3 \\ x-1 & 3 & 4 \end{vmatrix} = 0$$

[*Operating $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_2$*]

$\Rightarrow$ $(3x + 7)\,(1)\,[-28 - 9] = 0$ $\Rightarrow -37(3x + 7) = 0$

$\Rightarrow$ $3x + 7 = 0.$ Hence, $x = \dfrac{-7}{3}$.

Example 18. Using properties of determinants, prove that:

$$\begin{vmatrix} (x+y)^2 & zx & zy \\ zx & (z+y)^2 & xy \\ zy & xy & (z+x)^2 \end{vmatrix} = 2xyz\,(x + y + z)^3.$$

(A.I.C.B.S.E. 2016)

Solution. A = $\begin{vmatrix} (x+y)^2 & zx & zy \\ zx & (z+y)^2 & xy \\ zy & xy & (z+x)^2 \end{vmatrix}$

$$= \frac{1}{xyz} \begin{vmatrix} z(x+y)^2 & z^2x & z^2y \\ zx^2 & x(z+y)^2 & x^2y \\ zy^2 & xy^2 & y(z+x)^2 \end{vmatrix}$$

[*Operating $R_1 \to zR_1$, $R_2 \to xR_2$, $R_3 \to yR_3$ and dividing by xyz*]

$$= \frac{xyz}{xyz} \begin{vmatrix} (x+y)^2 & z^2 & z^2 \\ x^2 & (z+y)^2 & x^2 \\ y^2 & y^2 & (z+x)^2 \end{vmatrix}$$

[*Taking z, x, y common from C_1, C_2, C_3 respectively*]

$$= \begin{vmatrix} (x+y)^2 & z^2-(x+y)^2 & z^2-(x+y)^2 \\ x^2 & (z+y)^2-x^2 & 0 \\ y^2 & 0 & (z+x)^2-y^2 \end{vmatrix}$$

[*Operating* $C_2 \to C_2 - C_1$ *and* $C_3 \to C_3 - C_1$]

$$= (x+y+z)^2 \begin{vmatrix} (x+y)^2 & z-(x+y) & z-(x+y) \\ x^2 & (z+y)-x & 0 \\ y^2 & 0 & (z+x)-y \end{vmatrix}$$

[*Taking* $(x + y + z)$ *common from* C_2 *and* C_3]

$$= (x+y+z)^2 \begin{vmatrix} 2xy & -2y & -2x \\ x^2 & y+z-x & 0 \\ y^2 & 0 & z+x-y \end{vmatrix}$$

[*Operating* $R_1 \to R_1 - R_2 - R_3$]

$$= (x+y+z)^2 \begin{vmatrix} 2xy & 0 & 0 \\ x^2 & y+z & \frac{x^2}{y} \\ y^2 & \frac{y^2}{x} & z+x \end{vmatrix}$$

$\left[\text{Operating } C_2 \to C_2 + \frac{1}{x}C_1 \text{ and } C_3 \to C_3 + \frac{1}{y}C_1\right]$

$= (x + y + z)^2 (2xy)[(y + z)(z + x) - xy]$

[*Expanding by* R_1]

$= (x + y + z)^2 (2xy) (yz + xy + z^2 + zx - xy]$

$= (x + y + z)^2 (2xy) (yz + z^2 + zx)$

$= (x + y + z)^2 (2xyz) (y + z + x) = 2xyz(x + y + z)^3$,

which is true.

EXERCISE 4 (b)

Fast Track Answer Type Questions

FTATQ

1. (*i*) Let it be a square matrix of order 3 × 3, then $|k\,A| = k\,|A|$. ***(True/False)*** ***(Kashmir B. 2013)***

(*ii*) If any two rows (or columns) of a determinant are identical, the value of the determinant is one. ***(True/False)*** ***(Jammu B. 2013)***

2. If A is a skew symmetric matrix of order 3, then prove that det A = 0. ***(C.B.S.E. 2017; A.I.C.B.S.E. 2017)***

3. Answer in one word :

$$\begin{vmatrix} 3 & 1 & 6 \\ 5 & 2 & 10 \\ 7 & 4 & 14 \end{vmatrix}.$$ ***(H.B. 2014)***

4. (*i*) If A is a square matrix of order 3 and $|3A| = k\,|A|$, then write the value of '*k*'. ***(C.B.S.E. 2010)***

(*ii*) If A is a square matrix with $|A| = 8$, then find the value of $|AA'|$. ***(Karnataka B. 2017)***

5. Use properties of determinants to evaluate :

(*a*) (*i*) $\begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix}$

(N.C.E.R.T. ; Meghalaya B. 2018, 15; Jammu B. 2017;)

(*ii*) $\begin{vmatrix} 2 & a & abc \\ 2 & b & bca \\ 2 & c & cab \end{vmatrix}$ ***(H.B. 2014, 12, 10)***

(*iii*) $\begin{vmatrix} x+y & y+z & z+x \\ z & x & y \\ 1 & 1 & 1 \end{vmatrix}$ ***(H.B. 2012)***

(*b*) (*i*) $\begin{vmatrix} 102 & 18 & 36 \\ 1 & 3 & 4 \\ 17 & 3 & 6 \end{vmatrix}$

(Jammu B. 2017; C.B.S.E. (F) 2012)

(*ii*) $\begin{vmatrix} 2 & 3 & 1 \\ 4 & 6 & 2 \\ 1 & 3 & 2 \end{vmatrix}$ ***(Uttarakhand B. 2015)***

(*iii*) $\begin{vmatrix} 2 & 3 & 5 \\ 261 & 592 & 127 \\ 20 & 30 & 50 \end{vmatrix}$. ***(Meghalaya B. 2015)***

6. Show that $\begin{vmatrix} ma_1 & b_1 & nc_1 \\ ma_2 & b_2 & nc_2 \\ ma_3 & b_3 & nc_3 \end{vmatrix}$

$= -mn \begin{vmatrix} c_1 & b_1 & a_1 \\ c_2 & b_2 & a_2 \\ c_3 & b_3 & a_3 \end{vmatrix}$. ***(J. & K.B. 2010)***

7. (*a*) Show that $x = 1$ is a root of the equation :

$$\begin{vmatrix} x+1 & 2x & 11 \\ 2x & x+1 & -4 \\ -3 & 4x-7 & 6 \end{vmatrix} = 0.$$ *(Bihar B. 2014)*

(*b*) If p, q, r are in A.P., write the value of :

$$\begin{vmatrix} x+1 & x+2 & x+p \\ x+2 & x+3 & x+q \\ x+3 & x+4 & x+r \end{vmatrix}.$$ *(Tripura B. 2016)*

(*c*) Prove that $\begin{vmatrix} a & b & c \\ a+2x & b+2y & c+2z \\ x & y & z \end{vmatrix} = 0.$

(Kerala B. 2018)

Very Short Answer Type Questions

VSATQ

8. Without expanding the determinant, show that the determinant $\begin{vmatrix} a^2+10 & ab & ac \\ ab & b^2+10 & bc \\ ac & bc & c^2+10 \end{vmatrix}$ is divisible by 100. *(W. Bengal B. 2017)*

Without actual expansion, prove that the following (9 – 10) determinants vanish :

9. (*i*) $\begin{vmatrix} 2 & 7 & 65 \\ 3 & 8 & 75 \\ 5 & 9 & 86 \end{vmatrix}.$ *(N.C.E.R.T.)*

(*ii*) $\begin{vmatrix} a & b & c \\ a-b & b-c & c-a \\ b & c & a \end{vmatrix}$ *(H.B. 2011)*

(*iii*) $\begin{vmatrix} x & a & x+a \\ y & b & y+b \\ z & c & z+c \end{vmatrix}$ *(N.C.E.R.T. ; H.B. 2014)*

(*iv*) $\begin{vmatrix} 1 & a & a^2-bc \\ 1 & b & b^2-ca \\ 1 & c & c^2-ab \end{vmatrix}$ *(Kashmir B. 2011)*

(*v*) $\begin{vmatrix} 0 & a & -b \\ -a & 0 & -c \\ b & c & 0 \end{vmatrix}.$ *(N.C.E.R.T.)*

10. (*i*) $\begin{vmatrix} x+y & y+z & z+x \\ z & x & y \\ 1 & 1 & 1 \end{vmatrix}$

(N.C.E.R.T. ; H.B. 2014)

(*ii*) $\begin{vmatrix} 1 & bc & a(b+c) \\ 1 & ca & b(c+a) \\ 1 & ab & c(a+b) \end{vmatrix}.$

(N.C.E.R.T. ; H.B. 2012)

Without expanding the determinants (11-13), show that :

11. $\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+1\right)$ is a factor of :

$$\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix}.$$ *(P.B. 2010)*

12. $(x + y + z)$ is a factor of :

$$\begin{vmatrix} x-y-z & 2x & 2x \\ 2y & y-z-x & 2y \\ 2z & 2z & z-x-y \end{vmatrix}.$$

13. $\begin{vmatrix} a & a^2 & bc \\ b & b^2 & ca \\ c & c^2 & ab \end{vmatrix} = \begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix}.$

(N.C.E.R.T. ; H.B. 2015)

14. Evaluate : $\begin{vmatrix} 0 & \sin\alpha & -\cos\alpha \\ -\sin\alpha & 0 & \sin\beta \\ \cos\alpha & -\sin\beta & 0 \end{vmatrix}.$

(H.B. 2017)

15. Without expanding the determinant at any stage,

show that $\begin{vmatrix} x^2+x & x+1 & x-2 \\ 2x^2+3x-1 & 3x & 3x-3 \\ x^2+2x+3 & 2x-1 & 2x-1 \end{vmatrix} = Ax + B,$

where A and B are determinants of order 3 not involving x. *(Assam B. 2017)*

Short Answer Type Questions

SATQ

16. Evaluate the following :

(*i*) $\begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix}$

(Uttarakhand B. 2013 ; A.I.C.B.S.E. 2012)

(*ii*) $\begin{vmatrix} x+\lambda & x & x \\ x & x+\lambda & x \\ x & x & x+\lambda \end{vmatrix}$ *(N.C.E.R.T.)*

(*iii*) $\begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix}$ *(N.C.E.R.T.)*

(*iv*) $\begin{vmatrix} x & y & x+y \\ y & x+y & x \\ x+y & x & y \end{vmatrix}$ *(N.C.E.R.T.)*

(*v*) $\begin{vmatrix} 1 & x & y \\ 1 & x+y & y \\ 1 & x & x+y \end{vmatrix}$. *(N.C.E.R.T.; H.B. 2010)*

Prove the following (17–31) :

17. (*a*) (*i*) $\begin{vmatrix} b+c & c+a & a+b \\ q+r & r+p & p+q \\ y+z & z+x & x+y \end{vmatrix} = 2\begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix}$

(A.I.C.B.S.E. 2014)

(*ii*) $\begin{vmatrix} b+c & q+r & y+z \\ c+a & r+p & z+x \\ a+b & p+q & x+y \end{vmatrix} = 2\begin{vmatrix} a & p & x \\ b & q & y \\ c & r & z \end{vmatrix}$.

(N.C.E.R.T.; C.B.S.E. 2012)

(*b*) (*i*) $\begin{vmatrix} x+4 & 2x & 2x \\ 2x & x+4 & 2x \\ 2x & 2x & x+4 \end{vmatrix} = (5x+4)(4-x)^2$

(N.C.E.R.T. ; H.P.B. 2017, 15, 12 ; C.B.S.E. 2011)

(*ii*) $\begin{vmatrix} x+4 & x & x \\ x & x+4 & x \\ x & x & x+4 \end{vmatrix} = 16(3x+4)$ *(H.B. 2011)*

(*iii*) $\begin{vmatrix} y+k & y & y \\ y & y+k & y \\ y & y & y+k \end{vmatrix} = k^2(3y+k)$

(N.C.E.R.T.; H.P.B. 2017, 15, 12 ; C.B.S.E. (F) 2011)

(*iv*) $\begin{vmatrix} x & y & x+y \\ y & x+y & x \\ x+y & x & y \end{vmatrix} = -2(x^3+y^3)$

(H.P.B. 2016)

(*v*) $\begin{vmatrix} 1 & \log_x y & \log_x z \\ \log_y x & 1 & \log_y z \\ \log_z x & \log_z y & 1 \end{vmatrix} = 0.$

(W. Bengal B. 2017)

18. (*i*) $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$

(N.C.E.R.T. ; W. Bengal B. 2018; H.P.B. 2016, 14, 10; Negaland B. 2015; Uttarakhand B. 2015 ; H.B. 2013, 10; J. & K.B. 2011)

(*ii*) $\begin{vmatrix} a & b-c & c+b \\ a+c & b & c-a \\ a-b & b+a & c \end{vmatrix} = (a+b+c)(a^2+b^2+c^2)$

(C.B.S.E. Sample Paper 2019)

(*iii*) $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c).$

(Kashmir B. 2016)

19. (*i*) $\begin{vmatrix} 1 & a & a \\ a & 1 & a \\ a & a & 1 \end{vmatrix} = (2a+1)(1-a)^2$

(H.B. 2010)

(*ii*) $\begin{vmatrix} 1 & x & x^3 \\ 1 & y & y^3 \\ 1 & z & z^3 \end{vmatrix} = (x-y)(y-z)(z-x)(x+y+z).$

(H.B. 2016, 15 ; Kerala B. 2015)

20. (*i*) $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ bc & ca & ab \end{vmatrix} = (a-b)(b-c)(c-a)$

(Meghalya B. 2016, 13 ; H.B. 2012)

(*ii*) $\begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix} = (a-b)(b-c)(c-a)$

(N.C.E.R.T.; H.P.B. 2013 ; Kashmir B. 2012)

(*iii*) $\begin{vmatrix} bc & a & 1 \\ ca & b & 1 \\ ab & c & 1 \end{vmatrix} = (a-b)(b-c)(a-c)$. *(H.B. 2011)*

21. $\begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ bc & ca & ab \end{vmatrix} = \begin{vmatrix} a & a^2 & bc \\ b & b^2 & ca \\ c & c^2 & ab \end{vmatrix}$

$= (ab + bc + ca)(a - b)(b - c)(c - a).$

(N.C.E.R.T. ; H.B. 2014 ; P.B. 2012)

22. (*i*) $\begin{vmatrix} a^2 & a & b+c \\ b^2 & b & c+a \\ c^2 & c & a+b \end{vmatrix}$

$= -(a + b + c)(a - b)(b - c)(c - a)$ *(Jammu B. 2013)*

(*ii*) $\begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ b+c & c+a & a+b \end{vmatrix}$

$= (a + b + c)(a - b)(b - c)(c - a)$

(Tripura B. 2016; W. Bengal B. 2016; C.B.S.E. 2010 C)

(*iii*) $\begin{vmatrix} x & x^2 & y+z \\ y & y^2 & z+x \\ z & z^2 & x+y \end{vmatrix} = (y-z)(z-x)(x-y)(x+y+z)$

(Assam B. 2018)

(*iv*) $\begin{vmatrix} \alpha & \alpha^2 & \beta+\gamma \\ \beta & \beta^2 & \gamma+\alpha \\ \gamma & \gamma^2 & \alpha+\beta \end{vmatrix} = \begin{vmatrix} \alpha & \beta & r \\ \alpha^2 & \beta^2 & r^2 \\ \beta+\gamma & \gamma+\alpha & \alpha+\beta \end{vmatrix}$

$= (\beta - \gamma)(\gamma - \alpha)(\alpha - \beta)(\alpha + \beta + \gamma)$

(N.C.E.R.T. ; Assam B. 2016; H.B. 2014)

(*v*) $\begin{vmatrix} b+c & a-b & a \\ c+a & b-c & b \\ a+b & c-a & c \end{vmatrix} = 3abc - a^3 - b^3 - c^3.$

(J. & K.B. 2011)

23. (*i*) $\begin{vmatrix} a^2+2a & 2a+1 & 1 \\ 2a+1 & a+2 & 1 \\ 3 & 3 & 1 \end{vmatrix} = (a - 1)^3$

(A.I.C.B.S.E. 2017)

(*ii*) $\begin{vmatrix} b^2+c^2 & ab & ca \\ ab & c^2+a^2 & bc \\ ca & bc & a^2+b^2 \end{vmatrix} = 4a^2b^2c^2$

(P.B. 2010 S)

(*iii*) $\begin{vmatrix} 1+a^2-b^2 & 2ab & -2b \\ 2ab & 1-a^2+b^2 & 2a \\ 2b & -2a & 1-a^2-b^2 \end{vmatrix}.$

$= (1 + a^2 + b^2)^3.$

(N.C.E.R.T. ; Mizogram B. 2018; Jammu B. 2013 ; Rajasthan B. 2013 ; Kashmir B. 2012)

24. (*i*) $\begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ x^3 & y^3 & z^3 \end{vmatrix} = \begin{vmatrix} x & x^2 & x^3 \\ y & y^2 & y^3 \\ z & z^2 & z^3 \end{vmatrix}$

$= xyz(x - y)(y - z)(z - x)$

(P.B. 2015 ; C.B.S.E. 2011, 10 C)

(*ii*) $\begin{vmatrix} x & x^2 & yz \\ y & y^2 & zx \\ z & z^2 & xy \end{vmatrix} = (x - y)(y - z)(z - x).$

$(xy + yz + zx).$

(N.C.E.R.T.; Karnataka B. 2017; Jammu B. 2016, 15W, 14 ; Kashmir B. 2011 ; H.P.B. 2010 S)

25. $\begin{vmatrix} x+y+2z & x & y \\ z & 2x+y+z & y \\ z & x & x+2y+z \end{vmatrix}$

$= 2(x + y + z)^3.$

(N.C.E.R.T. ; Jammu B. 2013 ; J. & K.B. 2011; H.P.B. 2010)

26. $\begin{vmatrix} b+c & c+a & a+b \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix}$

$= 2(a + b + c)(ab + bc + ca - a^2 - b^2 - c^2).$

(C.B.S.E. 2009 C)

27. (*i*) $\begin{vmatrix} x & x+y & x+2y \\ x+2y & x & x+y \\ x+y & x+2y & x \end{vmatrix} = 9y^2(x + y)$

(C.B.S.E. 2017; A.I.C.B.S.E. 2013)

(*ii*) $\begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix} = 4abc$

(N.C.E.R.T.; H.P.B. 2018, 14; Karnataka B. 2014)

(*iii*) $\begin{vmatrix} 2ab & a^2 & b^2 \\ a^2 & b^2 & 2ab \\ b^2 & 2ab & a^2 \end{vmatrix} = -(a^3 + b^3)^2.$

(W. Bengal B. 2017)

28. $\begin{vmatrix} 1 & x & x^2-yz \\ 1 & y & y^2-zx \\ 1 & z & z^2-xy \end{vmatrix} = 0.$ *(P.B. 2013)*

29. $\begin{vmatrix} x & x^2 & 1+px^3 \\ y & y^2 & 1+py^3 \\ z & z^2 & 1+pz^3 \end{vmatrix}$

$= (1 + pxyz)(x - y)(y - z)(z - x)$, where '$p$' is any scalar.
(N.C.E.R.T.; Assam B. 2017; A.I.C.B.S.E. 2010)

30. $\begin{vmatrix} x+y+z & -z & -y \\ -z & x+y+z & -x \\ -y & -x & x+y+z \end{vmatrix}$
$= 2(x + y)(y + z)(z + x)$. *(P.B. 2011)*

31. $\begin{vmatrix} 2y & y-z-x & 2y \\ 2z & 2z & z-x-y \\ x-y-z & 2x & 2x \end{vmatrix}$
$= (x + y + z)^3$. *(C.B.S.E. 2014)*

32. Using properties of determinants, prove that :

(*i*) $\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} = (a+b+c)^3$

(N.C.E.R.T.; Nagaland B. 2018; H.P.B. 2015 ; P.B. 2014; Bihar B. 2014; H.B. 2013; Jammu B. 2013 ; Jharkhand B. 2013 ; Kashmir B. 2012 ; C.B.S.E. (F) 2011)

(*ii*) $\begin{vmatrix} 3a & -a+b & -a+c \\ -b+a & 3b & -b+c \\ -c+a & -c+b & 3c \end{vmatrix} = 3(a+b+c)$
$(ab + bc + ca)$.
(Mizoram B. 2016 ; A.I.C.B.S.E. 2011)

33. Solve for x, the equations :

(*i*) $\begin{vmatrix} a+x & a-x & a-x \\ a-x & a+x & a-x \\ a-x & a-x & a+x \end{vmatrix} = 0$ *(A.I.C.B.S.E. 2011)*

(*ii*) $\begin{vmatrix} x-2 & 2x-3 & 3x-4 \\ x-4 & 2x-9 & 3x-16 \\ x-8 & 2x-27 & 3x-64 \end{vmatrix} = 0$.
(A.I.C.B.S.E. 2011)

34. Prove that $\begin{vmatrix} x & \sin\theta & \cos\theta \\ -\sin\theta & -x & 1 \\ \cos\theta & 1 & x \end{vmatrix}$ is independent of 'θ'. *(N.C.E.R.T.)*

Long Answer Type Questions

LATQ

35. Using properties of determinants, show that :

$$\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} = abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right).$$

(Jammu B. 2018)

36. Prove that :

(*i*) $\begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix} = 2abc(a+b+c)^3$

(N.C.E.R.T. ;Mizoram B. 2017)

(*ii*) $\begin{vmatrix} (b+c)^2 & ab & ca \\ ab & (a+c)^2 & bc \\ ac & bc & (a+b)^2 \end{vmatrix} = 2abc\ (a+b+c)^3$.

(N.C.E.R.T., Jammu B. 2017; C.B.S.E. 2010)

(*iii*) $\begin{vmatrix} (y+z)^2 & xy & zx \\ xy & (x+z)^2 & yz \\ xz & yz & (x+y)^2 \end{vmatrix} = 2xyz(x+y+z)^3$.

(Kashmir B. 2017)

37. If a, b, c are positive and unequal, show that the value of the determinant :

$$\Delta = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \text{ is negative.}$$

(N.C.E.R.T.; A.I.C.B.S.E. 2010)

38. Prove that :

(*i*) $\begin{vmatrix} a & a+b & a+b+c \\ 2a & 3a+2b & 4a+3b+2c \\ 3a & 6a+3b & 10a+6b+3c \end{vmatrix} = a^3$

(N.C.E.R.T. ; Jharkhand B. 2016; P.B. 2014 S; Jammu B. 2012)

(*ii*) $\begin{vmatrix} 1 & 1+p & 1+p+q \\ 2 & 3+2p & 4+3p+2q \\ 3 & 6+3p & 10+6p+3q \end{vmatrix} = 1$ *(H.B. 2013)*

(*iii*) $\begin{vmatrix} x+y & x & x \\ 5x+4y & 4x & 2x \\ 10x+8y & 8x & 3x \end{vmatrix} = x^3$. *(A.I.C.B.S.E. 2014)*

39. Prove that :

$$\begin{vmatrix} -bc & b^2+bc & c^2+bc \\ a^2+ac & -ac & c^2+ac \\ a^2+ab & b^2+ab & -ab \end{vmatrix} = (ab+bc+ca)^3.$$

(W. Bengal B. 2018)

40. Prove that :

$$\begin{vmatrix} a^2 & a^2-(b-c)^2 & bc \\ b^2 & b^2-(c-a)^2 & ca \\ c^2 & c^2-(a-b)^2 & ab \end{vmatrix}$$

$= (b-c)(c-a)(a-b)(a+b+c)(a^2+b^2+c^2).$

(J & K.B. 2010)

41. If p, q, r are not in G.P. and

$$\begin{vmatrix} 1 & \frac{q}{p} & \alpha+\frac{q}{p} \\ 1 & \frac{r}{q} & \alpha+\frac{r}{q} \\ p\alpha+q & q\alpha+r & 0 \end{vmatrix} = 0,$$

then prove that $p\alpha^2 + 2q\alpha + r = 0$.

(W. Bengal B. 2016)

Answers

1. (*i*) False. (*ii*) False. **3.** 0. **4.** (*i*) $k = 27$; (*ii*) 64.
5. (All parts) 0. **7.** (*b*) 0.

14. 0 **16.** (*i*) $4abc$ (*ii*) $\lambda^2(3x+\lambda)$ (*iii*) 0 (*iv*) $-2(x^3+y^3)$ (*v*) xy.
33. (*i*) 0, 0, $3a$ (*ii*) 4.

Hints to Selected Questions

5. (*a*) (*i*) Operate $C_1 \to C_1 + C_2 + C_3$

(*ii*) $\begin{vmatrix} 2 & a & abc \\ 2 & b & bca \\ 2 & c & cab \end{vmatrix} = (2)(abc)\begin{vmatrix} 1 & a & 1 \\ 1 & b & 1 \\ 1 & c & 1 \end{vmatrix}$

$= 2abc\,(0) = 0.$

10. (*i*) Operate $R_1 \to R_1 + R_2$.

12. Operate $R_1 \to R_1 + R_2 + R_3$.

16. (*ii*) Operate $C_1 \to C_1 + C_2 + C_3$.

18. (*i*) Operate $R_1 \to R_1 - R_2$ & $R_2 \to R_2 - R_3$.

19. (*i*) Operate $R_1 \to R_1 - R_2$ & $R_2 \to R_2 - R_3$.

20. (*i*) Operate $C_2 \to C_2 - C_1$ & $C_3 \to C_3 - C_2$.

21. Operate $C_1 \to C_1 - C_2$ & $C_2 \to C_2 - C_3$.

22. (*iii*) Operate $C_3 \to C_3 + C_1$ and take $(\alpha+\beta+\gamma)$ common from C_3.

23. (*i*) Operate $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_2$

(*iii*) Operate $R_1 \to R_1 + b\,R_3$ and take $(1 + a^2 + b^2)$ common from R_1.

36. (*i*) Operate $C_1 \to C_1 - C_3$; $C_2 \to C_2 - C_3$ and then operate $R_3 \to R_3 - R_1 - R_2$.

37. As usual, $\Delta = -\frac{1}{2}(a+b+c)[(a-b)^2 + (b-c)^2 + (c-a)^2]$. This is negative. $[\because a+b+c>0]$

38. (*i*) Operate $R_2 \to R_2 - R_1$ & $R_3 \to R_3 - R_2$; etc.

4.4. PRODUCT OF DETERMINANTS

Two determinants ; say Δ_1 and Δ_2 are conformable for the product $\Delta_1\,\Delta_2$ when these are determinants of square matrices of the same order.

Moreover, $\Delta_1\,\Delta_2$ is the same as $\Delta_2\,\Delta_1$.

By Row-column Multiplication Rule :

Let $\Delta_1 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and $\Delta_2 = \begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix}$.

Then $\Delta_1\,\Delta_2 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}\begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix} = \begin{vmatrix} a_1\alpha_1+b_1\alpha_2+c_1\alpha_3 & a_1\beta_1+b_1\beta_2+c_1\beta_3 & a_1\gamma_1+b_1\gamma_2+c_1\gamma_3 \\ a_2\alpha_1+b_2\alpha_2+c_2\alpha_3 & a_2\beta_1+b_2\beta_2+c_2\beta_3 & a_2\gamma_1+b_2\gamma_2+c_2\gamma_3 \\ a_3\alpha_1+b_3\alpha_2+c_3\alpha_3 & a_3\beta_1+b_3\beta_2+c_3\beta_3 & a_3\gamma_1+b_3\gamma_2+c_3\gamma_3 \end{vmatrix}$.

This rule is same as that of two matrices.

But the value of the determinant does not change if we interchange rows and columns.

Thus we may also follow the rule of row-by row or column-by column.

$$\text{Thus } \Delta_1 \Delta_2 = \begin{vmatrix} a_1\alpha_1 + b_1\beta_1 + c_1\gamma_1 & a_1\alpha_2 + b_1\beta_2 + c_1\gamma_2 & a_1\alpha_3 + b_1\beta_3 + c_1\gamma_3 \\ a_2\alpha_1 + b_2\beta_1 + c_2\gamma_1 & a_2\alpha_2 + b_2\beta_2 + c_2\gamma_2 & a_2\alpha_3 + b_2\beta_3 + c_2\gamma_3 \\ a_3\alpha_1 + b_3\beta_1 + c_3\gamma_1 & a_3\alpha_2 + b_3\beta_2 + c_3\gamma_2 & a_3\alpha_3 + b_3\beta_3 + c_3\gamma_3 \end{vmatrix}.$$

ILLUSTRATIVE EXAMPLES

Example 1. Find the product of the determinants :

$$\begin{vmatrix} \mathbf{1} & \mathbf{2} & \mathbf{1} \\ \mathbf{-1} & \mathbf{0} & \mathbf{0} \\ \mathbf{1} & \mathbf{0} & \mathbf{1} \end{vmatrix} \times \begin{vmatrix} \mathbf{1} & \mathbf{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{2} \\ \mathbf{2} & \mathbf{-1} & \mathbf{1} \end{vmatrix}.$$

Solution. $$\begin{vmatrix} 1 & 2 & 1 \\ -1 & 0 & 0 \\ 1 & 0 & 1 \end{vmatrix} \times \begin{vmatrix} 1 & -1 & 0 \\ 0 & 1 & 2 \\ 2 & -1 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 1+0+2 & -1+2-1 & 0+4+1 \\ -1+0+0 & 1+0+0 & 0+0+0 \\ 1+0+2 & -1+0-1 & 0+0+1 \end{vmatrix} = \begin{vmatrix} 3 & 0 & 5 \\ -1 & 1 & 0 \\ 3 & -2 & 1 \end{vmatrix}.$$

Example 2. Express the determinant :

$$\begin{vmatrix} \mathbf{1} & \mathbf{\cos(\beta-\alpha)} & \mathbf{\cos(\gamma-\alpha)} \\ \mathbf{\cos(\alpha-\beta)} & \mathbf{1} & \mathbf{\cos(\gamma-\beta)} \\ \mathbf{\cos(\alpha-\gamma)} & \mathbf{\cos(\beta-\gamma)} & \mathbf{1} \end{vmatrix}$$

as the product of two determinants and hence prove that the determinant vanishes.

Solution. $$\begin{vmatrix} 1 & \cos(\beta-\alpha) & \cos(\gamma-\alpha) \\ \cos(\alpha-\beta) & 1 & \cos(\gamma-\beta) \\ \cos(\alpha-\gamma) & \cos(\beta-\gamma) & 1 \end{vmatrix}$$

$$= \begin{vmatrix} \cos^2\alpha + \sin^2\alpha & \cos\beta\cos\alpha + \sin\beta\sin\alpha & \cos\gamma\cos\alpha + \sin\gamma\sin\alpha \\ \cos\alpha\cos\beta + \sin\alpha\sin\beta & \cos^2\beta + \sin^2\beta & \cos\gamma\cos\beta + \sin\gamma\sin\beta \\ \cos\alpha\cos\gamma + \sin\alpha\sin\gamma & \cos\beta\cos\gamma + \sin\beta\sin\gamma & \cos^2\gamma + \sin^2\gamma \end{vmatrix}$$

$$= \begin{vmatrix} \cos^2\alpha + \sin^2\alpha & \cos\beta\cos\alpha + \sin\beta\sin\alpha & \cos\alpha\cos\gamma + \sin\alpha\sin\gamma \\ \cos\alpha\cos\beta + \sin\alpha\sin\beta & \cos^2\beta + \sin^2\beta & \cos\beta\cos\gamma + \sin\beta\sin\gamma \\ \cos\alpha\cos\gamma + \sin\alpha\sin\gamma & \cos\beta\cos\gamma + \sin\beta\sin\gamma & \cos^2\gamma + \sin^2\gamma \end{vmatrix}$$

$$= \begin{vmatrix} \cos\alpha & \sin\alpha & 0 \\ \cos\beta & \sin\beta & 0 \\ \cos\gamma & \sin\gamma & 0 \end{vmatrix} \begin{vmatrix} \cos\alpha & \sin\alpha & 0 \\ \cos\beta & \sin\beta & 0 \\ \cos\gamma & \sin\gamma & 0 \end{vmatrix}$$

$= (0)\,(0) = 0.$

EXERCISE 4 (c)

Fast Track Answer Type Questions

FTATQ

1. Prove that :

$$\begin{vmatrix} a_1x_1 + b_1y_1 & a_1x_2 + b_1y_2 & a_1x_3 + b_1y_3 \\ a_2x_1 + b_2y_1 & a_2x_2 + b_2y_2 & a_2x_3 + b_2y_3 \\ a_3x_1 + b_3y_1 & a_3x_2 + b_3y_2 & a_3x_3 + b_3y_3 \end{vmatrix} = 0.$$

2. Prove that :

$$\begin{vmatrix} 2y_1z_1 & y_1z_2 + y_2z_1 & y_1z_3 + y_3z_1 \\ y_1z_2 + y_2z_1 & 2y_2z_2 & y_2z_3 + y_3z_2 \\ y_1z_3 + y_3z_1 & y_2z_3 + y_3z_2 & 2y_3z_3 \end{vmatrix} = 0.$$

3. Express the determinant :

$$\Delta = \begin{vmatrix} \cos(\alpha_1-\beta_1) & \cos(\alpha_1-\beta_2) & \cos(\alpha_1-\beta_3) \\ \cos(\alpha_2-\beta_1) & \cos(\alpha_2-\beta_2) & \cos(\alpha_2-\beta_3) \\ \cos(\alpha_3-\beta_1) & \cos(\alpha_3-\beta_2) & \cos(\alpha_3-\beta_3) \end{vmatrix}$$

as the product of two determinants and hence prove that $\Delta = 0$.

4. If $< l_r, m_r, n_r >$; $r = 1, 2, 3$ are direction-cosines of three mutually perpendicular lines, then prove that :

$$\begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix} = \pm 1.$$

Hints to Selected Questions

1. Given det. $$= \begin{vmatrix} a_1 & b_1 & 0 \\ a_2 & b_2 & 0 \\ a_3 & b_3 & 0 \end{vmatrix} \begin{vmatrix} x_1 & y_1 & 0 \\ x_2 & y_2 & 0 \\ x_3 & y_3 & 0 \end{vmatrix}$$

$= (0)\,(0) = 0.$

4. Find Δ^2, where Δ is given determinant.

Also $l_1^2 + m_1^2 + n_1^2 = 1$; etc. and $l_1l_2 + m_1m_2 + n_1n_2 = 0$; etc.

4.5. AREA OF A TRIANGLE

In the previous class we learnt that the area of a triangle whose vertices are (x_1, y_1), (x_2, y_2), (x_3, y_3) is given by :

$$\frac{1}{2}[x_1 (y_2 - y_3) - y_1 (x_2 - x_3) + x_2 y_3 - x_3 y_2].$$

This is clearly the expansion of the determinant $\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$.

Thus the area of the triangle whose vertices are (x_1, y_1), (x_2, y_2), (x_3, y_3) is given by :

$$\Delta = \frac{1}{2}\begin{vmatrix} \mathbf{x_1} & \mathbf{y_1} & \mathbf{1} \\ \mathbf{x_2} & \mathbf{y_2} & \mathbf{1} \\ \mathbf{x_3} & \mathbf{y_3} & \mathbf{1} \end{vmatrix}.$$

Since the area is always a positive quantity, therefore, we take the absolute value of the above determinant for the area.

KEY POINT

(1) We always take absolute value of the determinant for area because area is a positive quantity.
(2) When area is given, then take positive and negative values for calculation.
(3) When three points are collinear, then the area of the triangle formed by them will be zero.

ILLUSTRATIVE EXAMPLES

Example 1. Find the area of the triangle with vertices : (3, 8), (– 4, 2), (5, 1). *(N.C.E.R.T.)*

Solution. Δ, the area of the triangle

$$= \frac{1}{2}\begin{vmatrix} 3 & 8 & 1 \\ -4 & 2 & 1 \\ 5 & 1 & 1 \end{vmatrix}$$

$$= \frac{1}{2}[(1)(-4-10) - (1)(3-40) + (1)(6+32)]$$

[*Expanding along* C_3]

$$= \frac{1}{2}[-14 + 37 + 38] = \frac{1}{2}(61) = 30 \cdot 5 \text{ sq. units.}$$

Example 2. By using determinants, find the value of 'y' for which the points (1, 3), (2, 5) and (3, y) are collinear. *(J & K.B. 2011)*

Solution. The given points are collinear

$$\Rightarrow \frac{1}{2}\begin{vmatrix} 1 & 3 & 1 \\ 2 & 5 & 1 \\ 3 & y & 1 \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} 1 & 3 & 1 \\ 2 & 5 & 1 \\ 3 & y & 1 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 1 & 3 & 1 \\ 1 & 2 & 0 \\ 1 & y-5 & 0 \end{vmatrix} = 0$$

[*Operating* $R_2 \to R_2 - R_1$ & $R_3 \to R_3 - R_2$]

$$\Rightarrow (1)\begin{vmatrix} 1 & 2 \\ 1 & y-5 \end{vmatrix} = 0 \qquad [\textit{Expanding by } C_3]$$

$\Rightarrow \quad (y - 5) - 2 = 0.$

Hence, $\quad y = 7.$

Example 3. A triangle has its three sides equal to a, b and c. If the co-ordinates of its vertices are A (x_1, y_1), B (x_2, y_2) and C (x_3, y_3), show that :

$$\begin{vmatrix} \mathbf{x_1} & \mathbf{y_1} & \mathbf{2} \\ \mathbf{x_2} & \mathbf{y_2} & \mathbf{2} \\ \mathbf{x_3} & \mathbf{y_3} & \mathbf{2} \end{vmatrix}^2$$

$$\mathbf{= (a + b + c)\,(b + c - a)\,(c + a - b)\,(a + b - c).}$$

Solution. If Δ be the area of triangle ABC, then :

$$\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \Rightarrow 2\Delta = \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

$$\Rightarrow 4\Delta = \begin{vmatrix} x_1 & y_1 & 2 \\ x_2 & y_2 & 2 \\ x_3 & y_3 & 2 \end{vmatrix} \Rightarrow 16\,\Delta^2 = \begin{vmatrix} x_1 & y_1 & 2 \\ x_2 & y_2 & 2 \\ x_3 & y_3 & 2 \end{vmatrix}^2 \qquad ...(1)$$

Also $\quad \Delta = \sqrt{s\,(s-a)\,(s-b)\,(s-c)},$

where $s = \frac{1}{2}(a + b + c).$

$\therefore \quad s - a = \frac{1}{2}(a + b + c) - a = \frac{1}{2}(b + c - a).$

Similarly $\quad s - b = \frac{1}{2}(c + a - b)$

and $\quad s - c = \frac{1}{2}(a + b - c).$

Now $\Delta^2 = s(s-a)(s-b)(s-c)$.

$$\therefore \quad \Delta^2 = \frac{1}{2}(a+b+c) \cdot \frac{1}{2}(b+c-a) \cdot \frac{1}{2}(c+a-b) \cdot \frac{1}{2}(a+b-c)$$

$$\Rightarrow \quad 16\,\Delta^2 = (a+b+c)(b+c-a)(c+a-b)(a+b-c) \quad ...(2)$$

From (1) and (2), $\begin{vmatrix} x_1 & y_1 & 2 \\ x_2 & y_2 & 2 \\ x_3 & y_3 & 2 \end{vmatrix}^2$

$= (a+b+c)(b+c-a)(c+a-b)(a+b-c)$,

which is true.

EXERCISE 4 (d)

Very Short Answer Type Questions

VSATQ

1. Using determinants, find the area of the triangle with vertices :

(*i*) (1, 0), (6, 0), (4, 3) ***(N.C.E.R.T.; H.P.B. 2013 S ; Jammu B. 2012)***

(*ii*) (1, – 1), (2, 4), (– 3, 5). ***(H.B. 2018, 12)***

2. Find the area of the triangle with vertices at the points given in each of the following. Are the following points collinear ?

(*i*) (0, 0), (6, 0), (4, 3) ***(N.C.E.R.T. ; H.P.B. 2013 S)***

(*ii*) (2, 7), (1, 1), (10, 8) ***(N.C.E.R.T.; Kashmir B. 2017)***

(*iii*) (– 2, – 3), (3, 2), (– 1, – 8). ***(N.C.E.R.T. ; H.P.B. 2013 S)***

3. Using determinants, show that the following points are collinear :

(*i*) (11, 7), (5, 5) and (– 1, 3)

(*ii*) (– 2, 5), (– 6, – 7) and (– 5, – 4).

4. If the points (3, – 2), (*x*, 2), (8, 8) are collinear, find '*x*', using determinants.

Short Answer Type Questions

SATQ

5. If the area of triangle is 35 sq. units with vertices (2, – 6), (5, 4) and (*k*, 4). Find the values of '*k*'. ***(Jammu B. 2013 ; Kashmir B. 2011)***

6. (*i*) Find the values of '*k*' if area of the triangle is 4 square units and vertices are (– 2, 0), (0, 4), (0, *k*). ***(N.C.E.R.T. ; Kashmir B. 2017, 13; H.B. 2013; Jammu B. 2013)***

(*ii*) Find the value of *k*, if area of triangle is 4 sq. units and vertices are (*k*, 0), (4, 0) and (0, 2), using determinant. ***(Karnataka B. 2017)***

7. Determine the value of '*x*' for which the area of the triangle formed by joining the points (*x*, 4), (0, 8) and (–1, 3) will be $4\frac{1}{2}$ square units. ***(J & K.B. 2011)***

8. Find the equation of the line joining the following points, using determinants :

(*i*) (1, 2) and (3, 6) ***(Karnataka B.2014)***

(*ii*) (3, 1) and (9, 3). ***(N.C.E.R.T.)***

9. Find the equation of the line joining A (1, 3) and B (0, 0), using determinants and find '*k*' if D (*k*, 0) is a point such that area of ΔABD is 3 square units. ***(N.C.E.R.T.)***

10. Prove that the area of the triangle whose vertices are :

$$\left(at_1^2, 2at_1\right); \left(at_2^2, 2at_2\right); \left(at_3^2, 2at_3\right)$$

is $a^2(t_1 - t_2)(t_2 - t_3)(t_3 - t_1)$. ***(J & K.B. 2010)***

11. Show that the points A(a, $b + c$), B(b, $c + a$), C(c, $a + b$) are collinear.

(N.C.E.R.T.; Jammu B. 2018; Kerala B. 2013; Kashimir B. 2011)

12. If the points (a_1, b_1), (a_2, b_2) and $(a_1 - a_2, b_1 - b_2)$ are collinear, then show that $a_1 b_2 = a_2 b_1$.

13. If (x, y) is any point on the line joining (a, 0) and (0, b), show that $\frac{x}{a} + \frac{y}{b} = 1$.

14. An equilateral triangle has each side equal to a. If the co-ordinates of its vertices are (x_1, y_1), (x_2, y_2) and (x_3, y_3), show that :

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}^2 = \frac{3a^4}{4}.$$

Answers

1. (*i*) 7·5 sq. units (*ii*) 13 sq. units.

2. (*i*) 9 sq. units ; No (*ii*) 23·5 sq. units ; No (*iii*) 15 sq. units ; No.

4. $x = 5$.

5. $k = -2, 12$. **6.** (*i*) – (*ii*) $k = 0, 8$.

7. $x = 1, -\frac{13}{5}$.

8. (*i*) $y = 2x$ (*ii*) $x = 3y$.

9. $y = 3x$; $k \pm 2$.

Hints to Selected Questions

8. (*i*) Let (x, y) be any point on the line joining $(1, 2)$ and $(3, 6)$.

Then the points (x, y), $(1, 2)$ and $(3, 6)$ are collinear

$$\Rightarrow \frac{1}{2}\begin{vmatrix} x & y & 1 \\ 1 & 2 & 1 \\ 3 & 6 & 1 \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} x & y & 1 \\ 1 & 2 & 1 \\ 3 & 6 & 1 \end{vmatrix} = 0; \text{ etc.}$$

13. The points (x, y), $(a, 0)$ and $(0, b)$ are collinear.

14. Here $A = \dfrac{\sqrt{3}\,a^2}{4} \Rightarrow A^2 = \dfrac{3a^4}{16}$.

4.6. MINORS AND CO-FACTORS

Now we shall learn to write the expansion of determinant in compact form by using minors and co-factors.

We define the **minor** and **co–factor** of the element a_{ij}, which appears in the ith row and jth column of the determinant.

The **minor** *of the element a_{ij} is a determinant, which is obtained by supressing the ith row and jth column.*

Thus the minor of an element a_{ij} of a determinant of order n is a determinant of order $(n - 1)$ and is denoted as M_{ij}.

The **co–factor** *of the element a_{ij} in a determinant is its* **minor** *with proper sign.*

The co–factor A_{ij} of an element a_{ij} is defined as $\mathbf{A_{ij} = (-1)^{i+j} M_{ij}}$.

KEY POINT

Minor of an element of a determinant of order n is a determinant of order $n - 1$.

Explanation : Let $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$.

$\therefore$ $M_{11} = \text{Minor of } a_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$, $A_{11} = \text{Co–factor of } a_{11} = (-1)^{1+1}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$.

Similarly, we have : $M_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$, $A_{12} = (-1)^{1+2}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$,

$M_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$, $A_{13} = (-1)^{1+3}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$.

And so on.

Now, we easily define the value of the determinant Δ of order n.

$$\Delta = \sum_{j=1}^{n} (-1)^{i+j}\, a_{ij}\, M_{ij} = \sum_{j=1}^{n} a_{ij}\, A_{ij}.$$

The above expression is obtained by expanding the determinant along the ith row.

If we expand the determinant along jth column, we get :

$$\Delta = \sum_{i=1}^{n} (-1)^{i+j}\, a_{ij}\, M_{ij} = \sum_{i=1}^{n} a_{ij}\, A_{ij}.$$

The value of the determinant does not depend on along which row or column it is expanded. We do not prove it here, but let us verify for the determinant $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$.

Expanding by first row :

$$\Delta = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}\,a_{22}\,a_{33} - a_{11}\,a_{32}\,a_{23} - a_{12}\,a_{21}\,a_{33} + a_{12}\,a_{31}\,a_{23} + a_{13}\,a_{21}\,a_{32} - a_{13}\,a_{31}\,a_{22}.$$

Expanding by first column :

$$\Delta = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{21}\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{31}\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$$

$$= a_{11}\,a_{22}\,a_{33} - a_{11}\,a_{32}\,a_{23} - a_{21}\,a_{12}\,a_{33} + a_{21}\,a_{32}\,a_{13} + a_{31}\,a_{12}\,a_{23} - a_{31}\,a_{22}\,a_{13}.$$

Clearly the two expressions of Δ are equal.

Thus ultimately we have : $\Delta = a_{11} A_{11} + a_{12} A_{12} + a_{13} A_{13}$

$= a_{21} A_{21} + a_{22} A_{22} + a_{23} A_{23}$ [*Expansion by second row*]

$= a_{31} A_{31} + a_{32} A_{32} + a_{33} A_{33}$ [*Expansion by third row*]

$= a_{11} A_{11} + a_{21} A_{21} + a_{31} A_{31}$ [*Expansion by first column*]

$= a_{12} A_{12} + a_{22} A_{22} + a_{32} A_{32}$ [*Expansion by second column*]

$= a_{13} A_{13} + a_{23} A_{23} + a_{33} A_{33}$. [*Expansion by third column*]

Cor. *The sum of the product of elements of any row (column) with co–factors of the corresponding elements of some other row (column) is zero.*

Since $$A_{31} = (-1)^{3+1}\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} = (a_{12}a_{23} - a_{22}a_{13}).$$

$$A_{32} = (-1)^{3+2}\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} = -(a_{11}a_{23} - a_{21}a_{13}) = (-a_{11}a_{23} + a_{21}a_{13})$$

$$A_{33} = (-1)^{3+3}\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = (a_{11}a_{22} - a_{21}a_{12})$$

so that $a_{11}A_{31} + a_{12}A_{32} + a_{13}A_{33} = a_{11}(a_{12}a_{23} - a_{22}a_{13}) + a_{12}(-a_{11}a_{23} + a_{21}a_{13}) + a_{13}(a_{11}a_{22} - a_{21}a_{12})$

$= a_{11}a_{12}a_{23} - a_{11}a_{22}a_{13} - a_{11}a_{12}a_{23} + a_{12}a_{13}a_{21} + a_{13}a_{11}a_{22} - a_{13}a_{12}a_{21}$.

Hence, $a_{11} A_{31} + a_{12} A_{32} + a_{13} A_{33} = 0$; etc.

ILLUSTRATIVE EXAMPLES

Example 1. If $\Delta = \begin{vmatrix} 5 & 3 & 8 \\ 2 & 0 & 1 \\ 1 & 2 & 3 \end{vmatrix}$, write :

(i) the minor of the element a_{23} *(C.B.S.E. 2012)*

(ii) the co-factor of the element a_{32}. *(C.B.S.E. 2012)*

Solution. (*i*) $M_{23} = \begin{vmatrix} 5 & 3 \\ 1 & 2 \end{vmatrix}$

$= (5)(2) - (1)(3) = 10 - 3 = 7.$

(*i*) $A_{32} = (-1)^{3+2}\begin{vmatrix} 5 & 8 \\ 2 & 1 \end{vmatrix}$

$= (-1)^5 [(5)(1) - (2)(8)]$

$= (-1)^5 (5 - 16)$

$= (-1)(-11) = 11.$

Example 2. Find the minor of the element $\boxed{6}$ in the determinant $\Delta = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$. *(N.C.E.R.T.)*

Solution. We have : $\Delta = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & \boxed{6} \\ 7 & 8 & 9 \end{vmatrix}$.

$\therefore$ Minor of 6 = Minor of a_{23}

$= \begin{vmatrix} 1 & 2 \\ 7 & 8 \end{vmatrix} = (1)(8) - (7)(2)$

$= 8 - 14 = -6.$

Example 3. Find the minors and co-factors of all elements of the determinant $\begin{vmatrix} 1 & -2 \\ 4 & 3 \end{vmatrix}$. *(N.C.E.R.T.)*

Solution. By *definition*,

M_{11} = Minor of $a_{11} = 3$

A_{11} = Co-factor of $a_{11} = (-1)^{1+1} M_{11} = (-1)^2\ 3 = 3$

M_{12} = Minor of $a_{12} = 4$

A_{12} = Co-factor of $a_{12} = (-1)^{1+2} M_{12} = (-1)^3\ 4 = -4$

M_{21} = Minor of $a_{21} = -2$

A_{21} = Co-factor of $a_{21} = (-1)^{2+1} M_{21}$

$= (-1)^3 (-2) = (-1)(-2) = 2$

M_{22} = Minor of $a_{22} = 1$

A_{22} = Co-factor of $a_{22} = (-1)^{2+2} M_{22} = (-1)^4 (1) = 1.$

Example 4. If $A = [a_{ij}]$ is a matrix of order 2×2, such that $|A| = -15$ and c_{ij} represents co-factor of a_{ij}, then find $a_{21}c_{21} + a_{22}c_{22}$. *(C.B.S.E. Sample Paper 2018)*

Solution. We have : $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$.

$\therefore$ $|A| = -15$

$\Rightarrow$ $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = -15$

$\Rightarrow$ $a_{11}a_{22} - a_{21}a_{12} = -15$ ...(1)

Now $a_{21}c_{21} + a_{22}c_{22} = a_{21}(-a_{12}) + a_{22}a_{11}$

$= a_{11}a_{22} - a_{21}a_{12} = -15.$

[*Using (1)*]

Example 5. If A_{ij} is the co-factor of the element a_{ij} of the determinant $\begin{vmatrix} 2 & -3 & 5 \\ 6 & 0 & 4 \\ 1 & 5 & -7 \end{vmatrix}$, then write the value of $a_{32} \cdot A_{32}$. *(A.I.C.B.S.E. 2013)*

Solution. In the determinant $\begin{vmatrix} 2 & -3 & 5 \\ 6 & 0 & 4 \\ 1 & 5 & -7 \end{vmatrix}$, $a_{32} = 5$.

And A_{32}, co-factor of $a_{32} = (-1)^{3+2} \begin{vmatrix} 2 & 5 \\ 6 & 4 \end{vmatrix}$

$= (-1)^5 [(2)(4) - (6)(5)]$

$= (-1)(8 - 30) = (-1)(-22) = 22.$

Hence, $a_{32} . A_{32} = (5)(22) = 110.$

Example 6. Find the minors and co–factors of the elements a_{11}, a_{21} and a_{31} of the determinant :

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}.$$ *(N.C.E.R.T.)*

Solution. By *definition,*

$$M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = a_{22}\, a_{33} - a_{23}\, a_{32}$$

$$A_{11} = (-1)^{1+1} M_{11} = M_{11} = a_{22}\, a_{33} - a_{23}\, a_{32}\ ;$$

$$M_{21} = \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} = a_{12}\, a_{33} - a_{13}\, a_{32}$$

$$A_{21} = (-1)^{2+1} M_{21} = -M_{21}$$

$$= -(a_{12}\, a_{33} - a_{13}\, a_{32})$$

$$= -a_{12}a_{33} + a_{13}a_{32}$$

and $$M_{31} = \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} = a_{12}\, a_{23} - a_{13}\, a_{22}$$

$$A_{31} = (-1)^{3+1} M_{31} = M_{31} = a_{12}\, a_{23} - a_{13}\, a_{22}.$$

EXERCISE 4 (e)

Fast Track Answer Type Questions

FTATQ

1. (*i*) If $\Delta = \begin{vmatrix} 1 & a \\ 1 & b \end{vmatrix}$, then minor of '$b$' is ..... . *(Jammu B. 2012)*

(*ii*) Minor of 'd' in determinant $\begin{vmatrix} a & c \\ b & d \end{vmatrix}$ is ... *(Kashmir B. 2011)*

2. Co-factor of 'b' in $\begin{vmatrix} a & c \\ b & d \end{vmatrix}$ is ..... . *(Kashmir B. 2016)*

3 If $\Delta = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 0 & 1 \\ 5 & 3 & 8 \end{vmatrix}$, write the minor of the element a_{22}. *(C.B.S.E. 2012)*

4. Find all the co–factors of $\begin{vmatrix} 1 & 3 \\ -2 & 4 \end{vmatrix}$. *(N.C.E.R.T.)*

5. Write the minors and co-factors of elements of the following determinants :

(*i*) $\begin{vmatrix} 2 & -4 \\ 0 & 3 \end{vmatrix}$ (*ii*) $\begin{vmatrix} a & c \\ b & d \end{vmatrix}$. *(N.C.E.R.T.)*

6. (*i*) Write the co-factor of '7' in the determinant $\begin{vmatrix} 4 & 5 & 6 \\ 5 & 6 & 7 \\ 13 & 15 & 17 \end{vmatrix}$. *(Assam B. 2015)*

(*ii*) Write co-factor of 3 in $\begin{vmatrix} 1 & 2 & 5 \\ -1 & 1 & 6 \\ 4 & 3 & 2 \end{vmatrix}$. *(Kashmir B. 2017)*

(*iii*) In the determinant $\begin{vmatrix} 5 & -1 & 4 \\ 2 & 3 & -5 \\ -1 & -2 & 6 \end{vmatrix}$, find the co-factors of the elements –5, 3 and 6. *(N.C.E.R.T.)*

Very Short Answer Type Questions

VSATQ

7. Find the minor and co–factor of each element of the first column of the following determinants :

(*i*) $\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$ (*ii*) $\begin{vmatrix} 0 & 2 & 6 \\ 1 & 5 & 0 \\ 3 & 7 & 1 \end{vmatrix}$ (*iii*) $\begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix}$.

Hence or otherwise evaluate them. *(N.C.E.R.T.)*

8. Write the minor and co-factor of each element of the following determinants and also evaluate the determinant in each case :

(*i*) $\begin{vmatrix} 5 & -10 \\ 0 & 3 \end{vmatrix}$ (*ii*) $\begin{vmatrix} 1 & 3 & -2 \\ 4 & -5 & 6 \\ 3 & 5 & 2 \end{vmatrix}$ (*iii*) $\begin{vmatrix} 1 & 0 & 4 \\ 3 & 5 & -1 \\ 0 & 1 & 2 \end{vmatrix}$. *(N.C.E.R.T.)*

9. Using co-factors of elements of third column, evaluate $\Delta = \begin{vmatrix} 1 & x & yz \\ 1 & y & zx \\ 1 & z & xy \end{vmatrix}$. *(N.C.E.R.T.; Kashmir B. 2017)*

10. Write down the co-factors of the elements of the first row of the following determinant and hence evaluate the determinant :

$$\begin{vmatrix} 1 & 3 & -3 \\ 2 & -1 & 0 \\ 4 & -2 & 5 \end{vmatrix}.$$ *(J& K.B. 2010)*

11. Find minors and co-factors of the elements of the determinant :

$$\begin{vmatrix} 2 & -3 & 5 \\ 6 & 0 & 4 \\ 1 & 5 & -7 \end{vmatrix}$$

and verify that $a_{11}A_{31} + a_{12}A_{32} + a_{13}A_{33} = 0$.

(N.C.E.R.T.)

Answers

1. (*i*) 1 (*ii*) *a*. **2.** $-c$. **3.** -7.

4. 4, 2 ; –3, 1.

5. (*i*) 3, 0, – 4, 2 ; 3, 0, 4, 2

(*ii*) *d, b, c, a ; d, – b, – c, a.*

6. (*i*) 5 (*ii*) – 11 (*iii*) 11, 34, 17.

7. (*i*) 1, 0, 0 ; 1, 0, 0 ; 1

(*ii*) 5, – 40, –30 ; 5, 40, –30 ; –50

(*iii*) $a(b^2 - c^2), b(a^2 - c^2), c(a^2 - b^2)$;

$a(b^2 - c^2), -b(a^2 - c^2), c(a^2 - b^2)$;

$a^2(c - b) + b^2(a - c) + c^2(b - a)$.

8. (*i*) $M_{11} = 3, M_{12} = 0, \quad M_{21} = -10, M_{22} = 5$;

$A_{11} = 3, \ A_{12} = 0, \ A_{21} = 10, \quad A_{22} = 5$; 15

(*ii*) $M_{11} = -40, M_{12} = -10, M_{13} = 35$

$M_{21} = 16, M_{22} = 8, M_{23} = -4$

$M_{31} = 8, M_{32} = 14, M_{33} = -17$;

$A_{11} = -40, A_{12} = 10, A_{13} = 35$

$A_{21} = -16, A_{22} = 8, A_{23} = 4$

$A_{31} = 8, A_{32} = -14, A_{33} = -17$; – 80

(*iii*) $M_{11} = 11, M_{12} = 6, M_{13} = 3$

$M_{21} = -4, M_{22} = 2, M_{23} = 1$

$M_{31} = -20, M_{32} = -13, M_{33} = 5$;

$A_{11} = 11, A_{12} = -6, A_{13} = 3$

$A_{21} = 4, A_{22} = 2, A_{23} = -1$

$A_{31} = -20, A_{32} = 13, A_{33} = 5$; 23.

9. $A_{13} = z - y$

$A_{23} = x - z$

$A_{33} = y - x$; $\Delta = xy^2 - z^2x - x^2y + zx^2 + yz^2 - y^2z$.

10. $A_{11} = -5, A_{12} = -10, \ A_{13} = 0$; – 35.

4.7. SOLUTION OF SYSTEMS OF LINEAR EQUATIONS

(*a*) To solve, by determinants, the system of equations :

$$\mathbf{a_1x + b_1y = c_1, \ a_2x + b_2y = c_2.}$$

The given equations are : $a_1x + b_1y = c_1$ ....(1)

and $a_2x + b_2y = c_2$ ....(2)

b_2 (1) – b_1 (2) gives : $(a_1b_2 - a_2b_1)\,x = b_2c_1 - b_1c_2$

$$\Rightarrow \qquad x = \frac{b_2c_1 - b_1c_2}{a_1b_2 - a_2b_1}.$$

a_2 (1) – a_1 (2) gives : $(a_2b_1 - a_1b_2)\,y = a_2c_1 - a_1c_2$

$$\Rightarrow \qquad y = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}.$$

Writing in determinants, we can express the above as :

$$x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}, \ y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}.$$

KEY POINT

The denominator in the values of x and y is the determinant of coefficients.

The numerators of x and y are the determinants obtained by replacing the coefficients of x and y respectively by the constant terms.

(*b*) To solve, by determinants, the system of equations :

$a_1x + b_1y + c_1z = d_1$, $a_2x + b_2y + c_2z = d_2$, $a_3x + b_3y + c_3z = d_3$.

The given equations are :

$$a_1x + b_1y + c_1z = d_1 \quad ...(1)$$
$$a_2x + b_2y + c_2z = d_2 \quad ...(2)$$
and
$$a_3x + b_3y + c_3z = d_3 \quad ...(3)$$

Let
$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

Then
$$xD = \begin{vmatrix} a_1x & b_1 & c_1 \\ a_2x & b_2 & c_2 \\ a_3x & b_3 & c_3 \end{vmatrix}.$$

Multiplying C_2 by y and C_3 by z and adding to C_1 ($C_1 \to C_1 + y\,C_2 + z\,C_3$), we get :

$$xD = \begin{vmatrix} a_1x + b_1y + c_1z & b_1 & c_1 \\ a_2x + b_2y + c_2z & b_2 & c_2 \\ a_3x + b_3y + c_3z & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix} = D_1 \text{ (Say)[}\textit{Using (1), (2) \& (3)}\text{]}$$

$\therefore$
$$x = \frac{D_1}{D}, \text{ where } D \neq 0.$$

Similarly
$$y = \frac{D_2}{D} \text{ and } z = \frac{D_3}{D}, \text{ where } D \neq 0$$

and
$$D_2 = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}, \; D_3 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}.$$

Key Points are similar to those of part (*a*).

In the same way, we can solve a system of n linear equations in n unknowns. Below we state the theorem for general case and the theorem is known as **Cramer's Rule** after the name of the Swiss Mathematician **Gabriel Cramer** (1704–1752).

(*c*) Cramer's Rule :

To solve, by determinants, a system of n linear equations in n unknowns :

$$\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \ldots\ldots\ldots\ldots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots\ldots\ldots\ldots + a_{2n}x_n = b_2 \\ \ldots\ldots \quad \ldots\ldots \quad \ldots\ldots \quad \ldots\ldots\ldots \\ \ldots\ldots \quad \ldots\ldots \quad \ldots\ldots \quad \ldots\ldots\ldots \\ a_{n1}x_1 + a_{n2}x_2 + \ldots\ldots\ldots\ldots + a_{nn}x_n = b_n. \end{array}$$

Let
$$D = \begin{vmatrix} a_{11} & a_{12} & \ldots\ldots & a_{1n} \\ a_{21} & a_{22} & \ldots\ldots & a_{2n} \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ a_{n1} & a_{n2} & \ldots\ldots & a_{nn} \end{vmatrix}.$$

Let D_j be the determinant obtained from D after replacing the jth column by :

$$\begin{bmatrix} b_1 \\ b_2 \\ \ldots\ldots \\ \ldots\ldots \\ \ldots\ldots \\ b_n \end{bmatrix}.$$

Then $x_1 = \dfrac{D_1}{D}$, $x_2 = \dfrac{D_2}{D}$, ..., $x_n = \dfrac{D_n}{D}$, where $D \neq 0$.

KEY POINT

(1) In the above theorem, if

$$b_1 = b_2 = \ldots\ldots\ldots = b_n = 0,$$

then $D_i = 0$ and if $D \neq 0$, then the system has only **trivial solution**

$$x_1 = x_2 = \ldots\ldots\ldots = x_n = 0.$$

(2) Cramer's Rule does not apply if $D = 0$.

(3) In the case of system of three equations in three unknowns, we have :

$$x = \frac{D_1}{D},\ y = \frac{D_2}{D},\ z = \frac{D_3}{D}.$$

Here if $D = 0$ but $D_i \neq 0$ (for some i), then the system has **no solution.**
Same thing holds good in general case also.

(4) If $D = 0$, and $D_i = 0$ for all i, then the system has either **infinite number of solutions or no solution.**

(5) If $b_1 = b_2 = b_3 = 0$, then system of equations is called system of **Homogeneous Linear Equations.**
If $D \neq 0$, the system has only **trivial solution** *i.e.* $x = y = z = 0$.
If $D = 0$, the system has **infinite number of solutions.**

Definition

***(i)* Consistent Equations.** A system of equations is said to be consistent if its solution exists.

***(ii)* Inconsistent Equations.** A system of equations is said to be inconsistent if its solution does not exist.

ILLUSTRATIVE EXAMPLES

Example 1. For what value of 'k', the system of linear equations :

$x + y + z = 2$, $2x + y - z = 3$ and $3x + 2y + kz = 4$ has a unique solution. *(A.I.C.B.S.E. 2016)*

Solution. For unique solution,

$$\begin{vmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 3 & 2 & k \end{vmatrix} \neq 0$$

$$\Rightarrow \quad \begin{vmatrix} 1 & 0 & 0 \\ 2 & -1 & -2 \\ 3 & -1 & k-2 \end{vmatrix} \neq 0$$

[*Operating* $C_2 \to C_2 - C_1$ *and* $C_3 \to C_3 - C_2$]

$$\Rightarrow \quad (1)\begin{vmatrix} -1 & -2 \\ -1 & k-2 \end{vmatrix} \neq 0$$

$$\Rightarrow \quad -(k-2) - 2 = 0$$

$$\Rightarrow \quad -k \neq 0.$$

Hence, $k \neq 0$.

Example 2. Using Cramer's Rule, solve the following linear equations :

$3x - 2y = 5$, $x - 3y = -3$.

Solution. The given equations are :

$$3x - 2y = 5 \quad \ldots(1)$$

and $$x - 3y = -3 \quad \ldots(2)$$

Here $$D = \begin{vmatrix} 3 & -2 \\ 1 & -3 \end{vmatrix} = -9 + 2 = -7 \neq 0.$$

By *Cramer's Rule,*

$$x = \frac{D_1}{D} = \frac{\begin{vmatrix} 5 & -2 \\ -3 & -3 \end{vmatrix}}{\begin{vmatrix} 3 & -2 \\ 1 & -3 \end{vmatrix}}$$

$$= \frac{-15-6}{-9+2} = \frac{-21}{-7} = 3$$

and $$y = \frac{D_2}{D} = \frac{\begin{vmatrix} 3 & 5 \\ 1 & -3 \end{vmatrix}}{\begin{vmatrix} 3 & -2 \\ 1 & -3 \end{vmatrix}} = \frac{-9-5}{-9+2}$$

$$= \frac{-14}{-7} = 2.$$

Hence, $x = 3$ and $y = 2$.

Example 3. Verify whether the system of equations : $3x - y - 2z = 2$, $2y - z = -1$, $3x - 5y = 3$ is consistent or inconsistent.

Solution. The given equations are $3x - y - 2z = 2 \quad \ldots(1)$

$$2y - z = -1 \quad \ldots(2)$$

and $$3x - 5y = 3 \quad \ldots(3)$$

Here $$D = \begin{vmatrix} 3 & -1 & -2 \\ 0 & 2 & -1 \\ 3 & -5 & 0 \end{vmatrix}$$

$$= 3\,(0-5) + 1\,.\,(0+3) - 2\,(-0-6)$$

$$= -15 + 3 + 12 = 0$$

$$D_1 = \begin{vmatrix} 2 & -1 & -2 \\ -1 & 2 & -1 \\ 3 & -5 & 0 \end{vmatrix}$$

$$= 2(0-5) + 1(-0+3) - 2(5-6)$$

$$= -10 + 3 + 2 = -5 \neq 0.$$

Since $D = 0$ and $D_i \neq 0$ for $i = 1$,

$\therefore$ the system of equations has no solution.

Hence, the system of equations is inconsistent.

Example 4. Solve :

$x + y + z = 1$, $2x + 2y + 2z = 2$, $3x + 3y + 3z = 4$.

Solution. The given equations are $x + y + z = 1$ ....(1)

$2x + 2y + 2z = 2$ ...(2)

and $3x + 3y + 3z = 4$ ...(3)

Here $$D = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{vmatrix} = 0.$$

$\therefore$ *Cramer's rule* does not apply.

But $$D_1 = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 4 & 3 & 3 \end{vmatrix} = 0.$$

Similarly $D_2 = D_3 = 0$.

$\therefore$ The system has either infinite number of solutions or no solution.

Let $z = k$. From (1), $x + y = 1 - k$ ...(1)′

From (3), $3x + 3y = 4 - 3k$ ...(3)′

Now $$D = \begin{vmatrix} 1 & 1 \\ 3 & 3 \end{vmatrix} = 0$$

$$D_1 = \begin{vmatrix} 1-k & 1 \\ 4-3k & 3 \end{vmatrix}$$

$$= 3 - 3k - 4 + 3k = -1 \neq 0.$$

$\therefore$ The system of equations given by (1)′ and (3)′ has no solution.

Hence, the given system of equations has no solution.

Example 5. Using Cramer's Rule, solve the following system of equations :

$3x - 4y + 5z = -6$

$x + y - 2z = -1$

$2x + 3y + z = 5$.

Solution. The given system of equations is :

$3x - 4y + 5z = -6$ ...(1)

$x + y - 2z = -1$ ...(2)

and $2x + 3y + z = 5$ ...(3)

By *Cramer's Rule,*

$$x = \frac{D_1}{D} = \frac{\begin{vmatrix} -6 & -4 & 5 \\ -1 & 1 & -2 \\ 5 & 3 & 1 \end{vmatrix}}{\begin{vmatrix} 3 & -4 & 5 \\ 1 & 1 & -2 \\ 2 & 3 & 1 \end{vmatrix}},$$

$$y = \frac{D_2}{D} = \frac{\begin{vmatrix} 3 & -6 & 5 \\ 1 & -1 & -2 \\ 2 & 5 & 1 \end{vmatrix}}{\begin{vmatrix} 3 & -4 & 5 \\ 1 & 1 & -2 \\ 2 & 3 & 1 \end{vmatrix}}$$

and $$z = \frac{D_3}{D} = \frac{\begin{vmatrix} 3 & -4 & -6 \\ 1 & 1 & -1 \\ 2 & 3 & 5 \end{vmatrix}}{\begin{vmatrix} 3 & -4 & 5 \\ 1 & 1 & -2 \\ 2 & 3 & 1 \end{vmatrix}}.$$

Now $$\begin{vmatrix} 3 & -4 & 5 \\ 1 & 1 & -2 \\ 2 & 3 & 1 \end{vmatrix}$$

$$= 3(1+6) + 4(1+4) + 5(3-2)$$

$$= 3(7) + 4(5) + 5(1)$$

$$= 21 + 20 + 5 = 46 \neq 0$$

and $$\begin{vmatrix} -6 & -4 & 5 \\ -1 & 1 & -2 \\ 5 & 3 & 1 \end{vmatrix}$$

$$= -6(1+6) + 4(-1+10) + 5(-3-5)$$

$$= -6(7) + 4(9) + 5(-8)$$

$$= \quad 42 + 36 \quad 40 = \quad 46;$$

$$\begin{vmatrix} 3 & -6 & 5 \\ 1 & -1 & -2 \\ 2 & 5 & 1 \end{vmatrix} = 3(-1+10) + 6(1+4) + 5(5+2)$$

$$= 3(9) + 6(5) + 5(7)$$

$$= 27 + 30 + 35 = 92;$$

$$\begin{vmatrix} 3 & -4 & -6 \\ 1 & 1 & -1 \\ 2 & 3 & 5 \end{vmatrix} = 3(5+3) + 4(5+2) - 6(3-2)$$

$$= 3(8) + 4(7) - 6(1)$$

$$= 24 + 28 - 6 = 46.$$

$\therefore$ $$x = \frac{-46}{46} = -1,\ y = \frac{92}{46} = 2,\ z = \frac{46}{46} = 1.$$

Hence, $x = -1$, $y = 2$ and $z = 1$.

Example 6. Solve :

$2x - y + z = 4$, $x + 3y + 2z = 12$, $3x + 2y + 3z = 16$.

Solution. The given equations are :

$2x - y + z = 4$ ....(1)

$x + 3y + 2z = 12$ ....(2)

and $3x + 2y + 3z = 16$ ....(3)

Here $$D = \begin{vmatrix} 2 & -1 & 1 \\ 1 & 3 & 2 \\ 3 & 2 & 3 \end{vmatrix}$$

$= 2(9-4) + 1(3-6) + 1(2-9)$
$= 10 - 3 - 7 = 0,$

$\therefore$ *Cramer's rule* does not apply.

Now $\quad D_1 = \begin{vmatrix} 4 & -1 & 1 \\ 12 & 3 & 2 \\ 16 & 2 & 3 \end{vmatrix}$

$= 4(9-4) + 1(36-32) + 1(24-48)$
$= 20 + 4 - 24 = 0.$

Similarly $\quad D_2 = D_3 = 0.$

$\therefore$ The system has either infinite number of solutions or no solution.

Let $\quad z = k$

From (1), $\quad 2x - y = 4 - k \quad$ ....(1)′

From (2), $\quad x + 3y = 12 - 2k \quad$ ....(2)′

Here $\quad D = \begin{vmatrix} 2 & -1 \\ 1 & 3 \end{vmatrix} = 6 + 1 = 7 \neq 0,$

Also $\quad D_1 = \begin{vmatrix} 4-k & -1 \\ 12-2k & 3 \end{vmatrix}$

$= (12 - 3k) + (12 - 2k) = 24 - 5k$

and $\quad D_2 = \begin{vmatrix} 2 & 4-k \\ 1 & 12-2k \end{vmatrix}$

$= (24 - 4k) - (4 - k) = 20 - 3k.$

$\therefore$ By *Cramer's rule*, $x = \dfrac{D_1}{D} = \dfrac{24-5k}{7},$

$$y = \frac{D_2}{D} = \frac{20-3k}{7}.$$

Putting these in (3), we get :

$$\textbf{LHS} = 3x + 2y + 3z$$
$$= 3\left(\frac{24-5k}{7}\right) + 2\left(\frac{20-3k}{7}\right) + 3k$$
$$= \frac{1}{7}[72 - 15k + 40 - 6k + 21k]$$
$$= \frac{1}{7}[112] = 16 = \textbf{RHS.}$$

Since (3) is satisfied,

$\therefore \quad x = \dfrac{24-5k}{7},\ y = \dfrac{20-3k}{7},\ z = k$

give the solution for all k.

Hence, the given system has infinite number of solutions.

Example 7. Solve the following system of homogeneous equations :

$x + y - 2z = 0,\ 2x + y - 3z = 0,\ 5x + 4y - 9z = 0.$

Solution. The given system of equations is :

$x + y - 2z = 0 \quad$ ...(1)

$2x + y - 3z = 0 \quad$ ...(2)

and $\ 5x + 4y - 9z = 0 \quad$ ...(3)

Here $\quad D = \begin{vmatrix} 1 & 1 & -2 \\ 2 & 1 & -3 \\ 5 & 4 & -9 \end{vmatrix}$

$= 1(-9 + 12) - 1(-18 + 15) - 2(8 - 5)$

$= 3 + 3 - 6 = 0.$

$\therefore$ The given system has infinite number of solutions.

Let $\quad z = k$. From (1), $x + y = 2k \quad$ ...(1)′

From (2), $\quad 2x + y = 3k \quad$ ...(2)′

Now $\quad D = \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} = 1 - 2 = -1 \neq 0.$

Also $\quad D_1 = \begin{vmatrix} 2k & 1 \\ 3k & 1 \end{vmatrix} = 2k - 3k = -k$

and $\quad D_2 = \begin{vmatrix} 1 & 2k \\ 2 & 3k \end{vmatrix} = 3k - 4k = -k.$

$\therefore$ By *Cramer's rule*, $x = \dfrac{D_1}{D} = \dfrac{-k}{-1} = k$

and $\quad y = \dfrac{D_2}{D} = \dfrac{-k}{-1} = k.$

Putting these in (3), we get :

LHS $= 5k + 4k - 9k = 0 =$ **RHS.**

Thus (3) is satisfied by $x = k,\ y = k,\ z = k,$ which give the solution for all values of k.

EXERCISE 4 (f)

Very Short Answer Type Questions

VSATQ

With the help of determinants, solve the following (1 – 2) systems of equations :

1. $3x + ay = 4\ ;\ 2x + ay = 2,\ a \neq 0.$

2. (*i*) $2x + 3y = 9\ ;\ 3x - 2y = 7$

(*ii*) $2x - y = -2\ ;\ 3x + 4y + 8 = 0$

(*iii*) $3x + y = 19\ ;\ 3x - y = 23.$

3. Classify the following system of equations as consistent or inconsistent. If consistent, solve them :

(*i*) $x + y = -1,\ 2x - 3y = 8$

(*ii*) $3x - 2y = 4,\ 6x - 4y = 10$

(*iii*) $x + 2y = 5\ ;\ 3x + 6y = 15.$

4. Use Cramer's rule to show that the following systems of equations have infinite number of solutions :

(*i*) $x + y + 3z = 6$; $x - 3y - 3z = -4$; $5x - 3y + 3z = 8$

(*ii*) $2x - 3y - z = 0$, $x + 3y - 2z = 0$, $x - 3y = 0$.

5. Solve the following system of equations, using Cramer's rule :

$x + y + 3z = 6$, $x - 3y - 3z = -4$, $5x - 3y + 3z = 8$.

6. Show that the following systems of equations are inconsistent :

(*i*) $3x + y = 5$
$-6x - 2y = 9$

(*ii*) $2x - y = 5$
$4x - 2y = 7$.

7. Find a quadratic function defined by the equation :

$f(x) = ax^2 + bx + c$

if (*i*) $f(1) = 0$, $f(2) = -2$ and $f(3) = -6$

(*ii*) $f(0) = f(-1) = 0$ and $f(1) = 2$.

8. Find the value of 'λ', for which the given system of equations :

$\lambda x + y + z = 0$, $-x + \lambda y + z = 0$, $-x - y + \lambda z = 0$

will have a non-zero solution.

Short Answer Type Questions

SATQ

Solve the following (9 – 10) systems of equations by Cramer's Rule :

9. (*i*) $x + 3y = 4$; $y + 3z = 7$; $4x + z = 6$ *(W. Bengal B. 2017)*

(*ii*) $x + y + z = -1$; $x + 2y + 3z = -4$; $x + 3y + 4z = -6$. *(J & K.B. 2011)*

10. (*i*) $3x - 4y + 5z = 6$; $x + y - 2z + 1 = 0$; $2x + 3y + z = 0$

(*ii*) $(b + c)(y + z) - ax = b - c$
$(c + a)(z + x) - by = c - a$
$(a + b)(x + y) - cz = a - b$.

11. Classify the following system of equations as consistent or inconsistent. If consistent, then solve them :

(*i*) $x - y + 3z = 6$; $x + 3y - 3z = -4$; $5x + 3y + 3z = 14$

(*ii*) $x - y + 3z = 6$; $x + 3y - 3z = -4$; $5x + 3y + 3z = 10$

(*iii*) $2x + 5y - z = 9$; $3x - 3y + 2z = 7$; $2x - 4y + 3z = 1$.

12. Solve the following system of homogeneous linear equations :

(*i*) $2x + 3y + 4z = 0$; $x + y + z = 0$; $2x - y + 3z = 0$

(*ii*) $x + y - z = 0$; $x - 2y + z = 0$; $3x + 6y - 5z = 0$.

13. Find the value of λ for which the homogeneous system of equations :

$2x + 3y - 2z = 0$; $2x - y + 3z = 0$; $7x + \lambda y - z = 0$

has non-trivial solution. Find the solution.

14. Find the cost of sugar and wheat per kg., if the cost of 7 kg. of sugar and 3 kg. of wheat is ₹ 240 and cost of 7 kg. of wheat and 3 kg. of sugar is ₹ 160. (Use determinants).

15. Represent the following problem by a system of equations :

"₹ 240 is the cost of 5 kg. sugar, 5 kg. of wheat and 2 kg. of rice. The cost of 4 kg. rice. 2 kg. sugar and 5 kg. wheat is ₹ 190. The cost of 3 kg. wheat, 2 kg. rice and 4 kg. sugar is ₹ 190". Use determinants to find the cost of each per kg.

Answers

1. $x = 2$, $y = \dfrac{-2}{a}$, $a \neq 0$.

2. (*i*) $x = 3$, $y = 1$

(*ii*) $x = -\dfrac{16}{11}$, $y = -\dfrac{10}{11}$

(*iii*) $x = 7$, $y = -2$.

3. (*i*) Consistent ; $x = 1$, $y = -2$

(*ii*) Inconsistent

(*iii*) Consistent ; infinitely many solutions.

5. No solution.

7. (*i*) $f(x) = -x^2 + x$ (*ii*) $f(x) = x^2 + x$.

8. $\lambda = 0$.

9. (*i*) $x = 1$, $y = 1$, $z = 2$

(*ii*) $x = 1$, $y = -1$, $z = -1$.

10. (*i*) $x = \dfrac{1}{2}$, $y = -\dfrac{1}{2}$, $z = \dfrac{1}{2}$

(*ii*) $x = -\dfrac{b-c}{a+b+c}$, $y = -\dfrac{c-a}{a+b+c}$, $z = -\dfrac{a-b}{a+b+c}$.

11. (*i*) Inconsistent

(*ii*) Consistent, infinitely many solutions :

$x = \dfrac{7-3k}{2}$, $y = \dfrac{3k-5}{2}$, $z = k$

(*iii*) Consistent :

$x = \dfrac{79}{21}$, $y = -\dfrac{4}{21}$, $z = -\dfrac{17}{7}$.

12. (*i*) $x = y = z = 0$

(*ii*) $x = k$, $y = 2k$, $z = 3k$

13. $\lambda = \dfrac{57}{10}$; $x = -7k$, $y = 10k$, $z = 8k$.

14. Sugar : ₹ 30 per kg. ; Wheat : ₹ 10 per kg.

15. Sugar : ₹ 30 per kg. ; Wheat : ₹ 10 per kg. ; Rice ₹ 20 per kg.

Hints to Selected Questions

8. The given system has non-zero solution if D = 0.

15. If ₹ x, ₹ y and ₹ z be the cost of sugar, wheat and rice respectively, then :

$5x + 5y + 2z = 240$, $2x + 5y + 4z = 190$ and $4x + 3y + 2z = 190$.

4.8. ADJOINT AND INVERSE OF A MATRIX

In Chapter 3, we studied about the inverse of a matrix. Now we shall discuss the condition for the existence of inverse of a matrix, which is similar to the concept of the inverse of a number.

For Example : When $ab = 1$, then $b = \frac{1}{a} = a^{-1}$, where a, b are numbers and $a \neq 0$.

On similar lines, if we recall the definition of a matrix :
If B is the inverse of a matrix A, then AB = BA = I,
where A, B are square matrices of the same order.
For the existence of A^{-1}, $|A| \neq 0$ (will be shown later on)
First of all, we define the adjoint of a matrix before finding its inverse.

4.9. ADJOINT OF A MATRIX

Let A be a square matrix $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$. Then the matrix formed by the co–factors in $|A|$ is $\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$, which is called the **Co–factor Matrix of A.**

Its transpose *i.e.* $\begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$ is called the **Adjoint of A.**

Hence, *the adjoint of a matrix A is defined to be the transpose of the matrix* $[A_{ij}]$ *and is denoted by adj A (or Adj A).*

For Example : *Take* $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

Here $A_{11} = d$, $A_{12} = -c$, $A_{21} = -b$, $A_{22} = a$.

$$\therefore \text{ adj. } A = \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}' = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

OBSERVATION

The adjoint of a square matrix of order 2 can be obtained by interchanging the diagonal elements and changing the signs of non–diagonal elements.

Theorem. *For any given square marix A, A. (adj A) = |A|I = (adj A) . A, where I is the identity matrix of order n.*

Verification : Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$.

$$\text{Then adj. } A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}' = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}.$$

Since sum of products of elements of a row (or a column) with corresponding co-factors is equal to $|A|$ and zero, otherwise,

$$\therefore \quad A(\text{adj. } A) = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} & a_{11}A_{21} + a_{12}A_{22} + a_{13}A_{23} & a_{11}A_{31} + a_{12}A_{32} + a_{13}A_{33} \\ a_{21}A_{11} + a_{22}A_{12} + a_{23}A_{13} & a_{21}A_{21} + a_{22}A_{22} + a_{23}A_{23} & a_{21}A_{31} + a_{22}A_{32} + a_{23}A_{33} \\ a_{31}A_{11} + a_{32}A_{12} + a_{33}A_{13} & a_{31}A_{21} + a_{32}A_{22} + a_{33}A_{23} & a_{31}A_{31} + a_{32}A_{32} + a_{33}A_{33} \end{bmatrix}$$

$$= \begin{bmatrix} |A| & 0 & 0 \\ 0 & |A| & 0 \\ 0 & 0 & |A| \end{bmatrix} = |A| \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = |A|\,I.$$

Similarly, (adj. A) A = $|A|I$.

Hence, A (adj. A) = $|A|I$ = (adj. A) A. which is verified.

4.10. INVERTIBLE MATRIX AND INVERSE OF A MATRIX

Definition

Any n–rowed square matrix A is said to be invertible if there exists an n–rowed matrix B such that

$$AB = BA = I_n,$$

where I_n is the unit matrix of order n.

B is also called the **inverse** or **reciprocal** of A.

Inverse of A is generally denoted by A^{-1}.

Note. A non–square matrix does not possess an inverse.

KEY POINT

If A^{-1} is the inverse of A, then A is the inverse of A^{-1}.

Theorem I. *Inverse of a square matrix, if it exists, is unique.*

Proof. Let A be any n–rowed square invertible matrix.

Let B and C be two inverses of A.

Then $AB = BA = I_n$ ...(1) [$\because$ *B is the inverse of A*]

and $AC = CA = I_n$ ...(2) [$\because$ *C is the inverse of A*]

From (1), we have :

$$AB = I_n.$$

Pre-multiplying by C, we have :

$$C(AB) = CI_n = C \quad ...(3)$$

From (2), we have :

$$CA = I_n.$$

Post-multiplying by B, we have :

$$(CA)B = I_nB = B \quad ...(4)$$

Since $C(AB) = (CA)B$,

$\therefore$ from (3) and (4), we have : C = B.

Hence, the inverse of A is unique.

Theorem II. *The necessary and sufficient condition for a square matrix A to be invertible is that $|A| \neq 0$.*

Proof.

The condition is necessary.

Given : A is invertible.

To prove : $|A| \neq 0$.

Since A is invertible, [*Given*]

$\therefore$ there exists another matrix B such that $AB = BA = I_n$.

Taking determinants, we have :

$$|AB| = |BA| = |I_n| = 1$$

$\Rightarrow \quad |A|.\ |B| = 1 \neq 0.$

But $|A|$ and $|B|$ are scalars,

$\therefore$ if $|A|.|B| \neq 0$, then $|A| \neq 0, |B| \neq 0$.

Hence, $|A| \neq 0$.

The condition is sufficient.

Given : $|A| \neq 0$.

To prove : A is invertible.

Let $B = \dfrac{1}{|A|}$ (adj. A).

Then $\quad AB = A.\left(\dfrac{\text{adj. A}}{|A|}\right) = \dfrac{1}{|A|}.|A|.\ I_n = I_n \qquad$ ...(1)

Also $\quad BA = \left(\dfrac{1}{|A|}\text{adj. A}\right)A = \dfrac{1}{|A|}\ (\text{adj. A})\ A = \dfrac{1}{|A|}\ |A|.\ I_n = I_n \qquad$ ...(2)

(1) and (2) $\Rightarrow$ $AB = BA = I_n$.

Hence, A is invertible.

$$\boxed{A^{-1} = \frac{\text{adj. A}}{|A|} \text{ if } |A| \neq 0.}$$

4.11. SINGULAR AND NON–SINGULAR MATRICES

Definition

A square matrix A is said to be singular if $|A| = 0$ and non–singular if $|A| \neq 0$.

In view of this def., above Th. II can also be restated as :

The necessary and sufficient condition for a square matrix A to be invertible is that it is non–singular.

For Examples : (I) Matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$ is singular because $|A| = \begin{vmatrix} 1 & 2 \\ 3 & 6 \end{vmatrix} = 6 - 6 = 0.$

(II) Matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ is non-singular because $|A| = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 4 - 6 = -2 \neq 0.$

Theorem. *If A and B be two non–singular matrices of the same order n, prove that :*

(*i*) **Reversal law.** $(AB)^{-1} = B^{-1} A^{-1}$ ***(Kashmir B. 2015)*** (*ii*) $(A^k)^{-1} = (A^{-1})^k$, *where 'k' is any positive integer*

(*iii*) $(A')^{-1} = (A^{-1})'$.

Proof. (*i*) We know that $|AB| = |A|\,|B|$ ...(1)

But as A and B are non–singular,

$\therefore \quad |A| \neq 0$ and $|B| \neq 0$.

$\therefore$ From (1), $|AB| \neq 0$.

Thus the matrix AB is invertible.

Let us define a matrix C such that $C = B^{-1} A^{-1}$.

Then $\quad C\,(AB) = (B^{-1} A^{-1})\,(AB) = B^{-1}\,(A^{-1} A)\,B = B^{-1} I_n B = B^{-1} B = I_n$.

Also $\quad (AB)C = (AB)\,(B^{-1} A^{-1}) = A\,(BB^{-1})\,A^{-1} = AI_n A^{-1} = AA^{-1} = I_n$.

Thus $\quad C(AB) = (AB)\,C = I_n$

$\Rightarrow \quad C\ (= B^{-1} A^{-1})$ is the inverse of AB.

Hence, $(AB)^{-1} = B^{-1} A^{-1}$.

Cor. $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$, if A, B, C are non–singular.

(*ii*) The above result (*i*) can be generalised as :

$$(A_1A_2 \ldots A_k)^{-1} = A_k^{-1} \ldots A_2^{-1} A_1^{-1}.$$

Putting $A_1 = A_2 = \ldots = A_k = A$, we get :

$$(A.A \ldots A)^{-1} = A^{-1} A^{-1} \ldots k \text{ times.}$$

Hence, $(A^k)^{-1} = (A^{-1})^k$.

(*iii*) We have : $AA^{-1} = A^{-1} A = I$.

$\therefore$ $(AA^{-1})' = (A^{-1} A)' = I'$ [*Taking transposes*]

$\Rightarrow$ $(A^{-1})' A' = A' (A^{-1})' = I$

$\Rightarrow$ $(A^{-1})'$ is the inverse of A'.

Hence, $(A')^{-1} = (A^{-1})'$.

Cor. 1. *If A is invertible symmetric matrix, then A^{-1} is also symmetric.*

Since A is symmetric, therefore, $A' = A$ ...(1)

Since A is invertible, therefore, A' is also invertible.

Now $(A^{-1})' = (A')^{-1} = A^{-1}$. [*Using (1)*]

Hence, A^{-1} is symmetric.

Cor. 2. *Every skew–symmetric matrix of odd order is singular.*

Let A be a skew–symmetric matrix of order *n*, where *n* is odd.

$\therefore$ $A' = -A$.

Now $-A = A'$ $\Rightarrow$ $|-A| = |A'|$

$\Rightarrow$ $|-A| = |A|$ $\Rightarrow$ $(-1)^n |A| = |A|$

$\Rightarrow$ $-|A| = |A|$ [$\because$ *n is odd*]

$\Rightarrow$ $2|A| = 0$ $\Rightarrow$ $|A| = 0$.

Hence, A is singular.

KEY POINT

(1) $(A^{-1})^{-1} = A$ (2) $(A^{-1})^2 = (A^2)^{-1}$.

4.12. MORE THEOREMS

Theorem I. *If A and B are non-singular matrices of same order, then AB and BA are also non-singular matrices of the same order.*

Theorem II. $|AB| = |A|\,|B|$, *where A and B are square matrices of same order.*

Theorem III. *Let A, B, C be three square matrices of the same order n and A be non–singular.*

(*i*) *If AB = AC, then B = C.* (*Left Cancellation Law*)

(*ii*) *If BA = CA, then B = C.* (*Right Cancellation Law*)

Proof. Since A is non–singular, therefore, A^{-1} exists.

(*i*) $AB = AC \Rightarrow A^{-1}(AB) = A^{-1}(AC)$

$\Rightarrow (A^{-1} A) B = (A^{-1} A) C$ [*Associative Law*]

$\Rightarrow I_n B = I_n C \Rightarrow B = C$.

(*ii*) **Similar to part (i).**

Theorem IV. *If A is invertible square matrix, then (adj. A)′ = (adj. A′).*

Proof. Let A be an invertible square matrix of order n.

Then $|A| \neq 0$.

And $|A'| = |A| \neq 0$

$\Rightarrow$ A′ is non-singular $\Rightarrow$ A′ is invertible.

Since A. (adj. A) = $|A|I_n$, $\therefore$ (A. (adj. A))′ = $(|A|I_n)'$

$\Rightarrow$ (adj. A)′.A′ = $|A|(I_n)'$ $\Rightarrow$ (adj. A)′.A′ = $|A|I_n$ ...(1) [$\because I_n' = I_n$]

Also (adj. A′).A′ = $|A'|.I_n$ $\Rightarrow$ (adj. A′).A′ = $|A|I_n$ ...(2) [$\because |A'| = |A|$]

From (1) and (2),

(adj. A)′.A′ = (adj. A′).A′

$\Rightarrow$ (adj. A)′ = (adj. A′). [*By Right Cancellation Law because A′ is invertible*]

Cor. *If A is symmetric matrix, then (adj. A) is also symmetric.*

Since A is symmetric, therefore, A′ = A ...(1)

$\therefore$ (adj. A)′ = (adj. A′) $\Rightarrow$ (adj. A)′ = (adj. A) [*Using (1)*]

$\Rightarrow$ (adj. A) is symmetric.

Theorem V. *If A is an invertible square matrix of order n, then* $|adj.\ A| = |A|^{n-1}$.

Proof. Since A is invertible, therefore, $|A| \neq 0$.

Now A. (adj. A) = $|A|.I_n$

$\Rightarrow$ $|A.(adj.\ A)| = ||A|.I_n|$

$\Rightarrow$ $|A||adj.\ A| = |A|^n$. [$\because ||A|.I_n| = |A|^n |I_n| = |A|^n$]

Hence, $|adj.\ A| = |A|^{n-1}$.

KEY POINT

If A is an invertible square matrix of order 3, then $|adj.\ A| = |A|^2$.

Theorem VI. *If A is an invertible square matrix of order n, then adj. (adj. A) =* $|A|^{n-2}.A$.

Proof. We know that B (adj. B) = $|B|I_n$, where B is a square matrix of order n.

Taking B = adj. A, we get :

(adj. A) [adj. (adj. A)] = $|adj.\ A|I_n \Rightarrow$ (adj. A) [adj. (adj. A)] = $|A|^{n-1} I_n$ [*Theorem V*]

$\Rightarrow$ A (adj. A) [adj. (adj. A)] = $|A|^{n-1}A$ [$\because A I_n = A$]

$\Rightarrow$ $|A| I_n$ [adj. (adj. A)] = $|A|^{n-1} A$ [$\because A\ (adj.\ A) = |A| I_n$]

$\Rightarrow$ $|A|$[adj. (adj. A)] = $|A|^{n-1}A$.

Hence, adj. (adj. A) = $|A|^{n-2}A$.

Frequently Asked Questions

FAQs

Example 1. If A is a 3 × 3 invertible matrix, then what will be the value of k if

det (A^{-1}) = (det A)^k. *(C.B.S.E. 2017)*

Solution. Here det (A^{-1}) = (det A)^k

$\Rightarrow$ $|A^{-1}| = |A|^k$.

We know that $|A^{-1}| = \dfrac{1}{|A|} = |A|^{-1}$.

Hence, $k = -1$.

Example 2. If for any 2 × 2 square matrix A,

A (adj A) = $\begin{bmatrix} 8 & 0 \\ 0 & 8 \end{bmatrix}$, then write the value of | A |.

(A.I.C.B.S.E. 2017)

Solution. We know that A.(adj A) = | A | I

$$\Rightarrow \quad \begin{bmatrix} 8 & 0 \\ 0 & 8 \end{bmatrix} = |A|\ I$$

$$\Rightarrow \quad 8\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = |A|\ I.$$

Hence $\quad |A| = 8.$

Example 3. Write A^{-1} for $A = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$.

(C.B.S.E. 2011)

Solution. *Given :* $A = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$.

$$\therefore \quad |A| = \begin{vmatrix} 2 & 5 \\ 1 & 3 \end{vmatrix} = 6 - 5 = 1 \neq 0.$$

$$\therefore \quad \text{adj } A = \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}' = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}.$$

$$\text{Hence,} \quad A^{-1} = \frac{\text{adj } A}{|A|} = \frac{1}{1}\begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}.$$

Example 4. For what value of 'x', the matrix $\begin{bmatrix} 5-x & x+1 \\ 2 & 4 \end{bmatrix}$ is singular ? *(C.B.S.E. 2011)*

Solution. The matrix $\begin{bmatrix} 5-x & x+1 \\ 2 & 4 \end{bmatrix}$ is singular

$$\Rightarrow \begin{vmatrix} 5-x & x+1 \\ 2 & 4 \end{vmatrix} = 0$$

$\Rightarrow 4(5 - x) - 2(x + 1) = 0$

$\Rightarrow 20 - 4x - 2x - 2 = 0$

$\Rightarrow \quad 18 - 6x = 0$

$\Rightarrow \quad 6x = 18.$

Hence, $\quad x = 3.$

Example 5. If A and B are invertible matrices of order 3, | A | = 2 and $|(AB)^{-1}| = -\frac{1}{6}$, find | B |.

(C.B.S.E. Sample Paper 2019)

Solution. We have : $|A| = 2$ and $|(AB)^{-1}| = -\frac{1}{6}$.

$$\text{Now, } |(AB)^{-1}| = \frac{1}{6} \Rightarrow \frac{1}{|AB|} = \frac{1}{6}$$

$$\Rightarrow \quad \frac{1}{|A||B|} = \frac{1}{6} \Rightarrow \frac{1}{2|B|} = \frac{1}{6}.$$

Hence, $\quad |B| = 3.$

Example 6. If $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{bmatrix}$, find adj. A.

Solution. Here $|A| = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{vmatrix}$.

Now A_{11} = Co–factor of 1 = $\begin{vmatrix} 3 & 2 \\ 3 & 4 \end{vmatrix} = 12 - 6 = 6$;

A_{12} = Co–factor of 2 = $-\begin{vmatrix} 2 & 2 \\ 3 & 4 \end{vmatrix} = -(8 - 6) = -2$;

A_{13} = Co–factor of 3 = $\begin{vmatrix} 2 & 3 \\ 3 & 3 \end{vmatrix} = 6 - 9 = -3.$

A_{21} = Co–factor of 2 = $-\begin{vmatrix} 2 & 3 \\ 3 & 4 \end{vmatrix} = -(8 - 9) = 1$;

A_{22} = Co–factor of 3 = $\begin{vmatrix} 1 & 3 \\ 3 & 4 \end{vmatrix} = 4 - 9 = -5$;

A_{23} = Co–factor of 2 = $-\begin{vmatrix} 1 & 2 \\ 3 & 3 \end{vmatrix} = -(3 - 6) = 3.$

A_{31} = Co–factor of 3 = $\begin{vmatrix} 2 & 3 \\ 3 & 2 \end{vmatrix} = 4 - 9 = -5$;

A_{32} = Co–factor of 3 = $-\begin{vmatrix} 1 & 3 \\ 2 & 2 \end{vmatrix} = -(2 - 6) = 4$;

A_{33} = Co–factor of 4 = $\begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} = 3 - 4 = -1.$

$$\therefore \text{ Co–factor matrix} = \begin{bmatrix} 6 & -2 & -3 \\ 1 & -5 & 3 \\ -5 & 4 & -1 \end{bmatrix}.$$

$$\text{Hence, adj. A} = \begin{bmatrix} 6 & -2 & -3 \\ 1 & -5 & 3 \\ -5 & 4 & -1 \end{bmatrix}' = \begin{bmatrix} 6 & 1 & -5 \\ -2 & -5 & 4 \\ -3 & 3 & -1 \end{bmatrix}.$$

Example 7. Given $A = \begin{bmatrix} 2 & -3 \\ -4 & 7 \end{bmatrix}$, compute A^{-1} and show that $2A^{-1} = 9I - A$. *(C.B.S.E. 2018)*

Solution. (*i*) We have : $A = \begin{bmatrix} 2 & -3 \\ -4 & 7 \end{bmatrix}$.

$$\therefore \quad |A| = \begin{vmatrix} 2 & -3 \\ -4 & 7 \end{vmatrix}$$
$$= (2)(7) - (-4)(-3)$$
$$= 14 - 12 = 2 \neq 0.$$

$$\therefore \quad A^{-1} \text{ exists and } A^{-1} = \frac{1}{|A|}\text{adj } A$$
$$= \frac{1}{2}\begin{bmatrix} 7 & 3 \\ 4 & 2 \end{bmatrix}.$$

(*ii*) **LHS** $= 2A^{-1} = 2 \times \frac{1}{2}\begin{bmatrix} 7 & 3 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 3 \\ 4 & 2 \end{bmatrix}$.

RHS $= 9I - A$
$$= 9\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & -3 \\ -4 & 7 \end{bmatrix}$$
$$= \begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix} + \begin{bmatrix} -2 & 3 \\ 4 & -7 \end{bmatrix}$$
$$= \begin{bmatrix} 9-2 & 0+3 \\ 0+4 & 9-7 \end{bmatrix} = \begin{bmatrix} 7 & 3 \\ 4 & 2 \end{bmatrix}.$$

Hence, **LHS = RHS**.

Example 8. For the matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}$, show that :

$A^3 - 6A^2 + 5A + 11\,I = O$. Hence, find A^{-1}.

(H.B. 2016; Jammu B. 2015 W)

Solution. We have : $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}$.

$$\therefore \quad A^2 = AA = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1+1+2 & 1+2-1 & 1-3+3 \\ 1+2-6 & 1+4+3 & 1-6-9 \\ 2-1+6 & 2-2-3 & 2+3+9 \end{bmatrix} = \begin{bmatrix} 4 & 2 & 1 \\ -3 & 8 & -14 \\ 7 & -3 & 14 \end{bmatrix}.$$

And $A^3 = A^2 A = \begin{bmatrix} 4 & 2 & 1 \\ -3 & 8 & -14 \\ 7 & -3 & 14 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}$

$$= \begin{bmatrix} 4+2+2 & 4+4-1 & 4-6+3 \\ -3+8-28 & -3+16+14 & -3-24-42 \\ 7-3+28 & 7-6-14 & 7+9+42 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 7 & 1 \\ -23 & 27 & -69 \\ 32 & -13 & 58 \end{bmatrix}.$$

Now $A^3 - 6A^2 + 5A + 11\,I$

$$= \begin{bmatrix} 8 & 7 & 1 \\ -23 & 27 & -69 \\ 32 & -13 & 58 \end{bmatrix} - 6\begin{bmatrix} 4 & 2 & 1 \\ -3 & 8 & -14 \\ 7 & -3 & 14 \end{bmatrix}$$
$$+ 5\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix} + 11\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 7 & 1 \\ -23 & 27 & -69 \\ 32 & -13 & 58 \end{bmatrix} + \begin{bmatrix} -24 & -12 & -6 \\ 18 & -48 & 84 \\ -42 & 18 & -84 \end{bmatrix}$$
$$+ \begin{bmatrix} 5 & 5 & 5 \\ 5 & 10 & -15 \\ 10 & -5 & 15 \end{bmatrix} + \begin{bmatrix} 11 & 0 & 0 \\ 0 & 11 & 0 \\ 0 & 0 & 11 \end{bmatrix}$$

$$= \begin{bmatrix} 8-24+5+11 & 7-12+5+0 & 1-6+5+0 \\ -23+18+5+0 & 27-48+10+11 & -69+84-15+0 \\ 32-42+10+0 & -13+18-5+0 & 58-84+15+11 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O, \text{ which is true.}$$

Now $A^3 - 6A^2 + 5A + 11I = O$.

Multiplying by A^{-1}, we get :

$$(A^{-1}A)A^2 - 6(A^{-1}A)A + 5(A^{-1}A) + 11(A^{-1}I) = O$$
$$\Rightarrow \quad IA^2 - 6IA + 5I + 11A^{-1} = O$$
$$\Rightarrow \quad 11\,A^{-1} = -A^2 + 6A - 5\,I$$

$$= -\begin{bmatrix} 4 & 2 & 1 \\ -3 & 8 & -14 \\ 7 & -3 & 14 \end{bmatrix} + 6\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -4 & -2 & -1 \\ 3 & -8 & 14 \\ -7 & 3 & -14 \end{bmatrix} + \begin{bmatrix} 6 & 6 & 6 \\ 6 & 12 & -18 \\ 12 & -6 & 18 \end{bmatrix} + \begin{bmatrix} -5 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & -5 \end{bmatrix}$$

$$= \begin{bmatrix} -4+6-5 & -2+6+0 & -1+6+0 \\ 3+6+0 & -8+12-5 & 14-18+0 \\ -7+12+0 & 3-6+0 & -14+18-5 \end{bmatrix}$$

$$= \begin{bmatrix} -3 & 4 & 5 \\ 9 & -1 & -4 \\ 5 & -3 & -1 \end{bmatrix}.$$

$$\text{Hence, } A^{-1} = \begin{bmatrix} -3/11 & 4/11 & 5/11 \\ 9/11 & -1/11 & -4/11 \\ 5/11 & -3/11 & -1/11 \end{bmatrix}.$$

Example 9. Find the inverse of the matrix :

$$\mathbf{A} = \begin{bmatrix} \mathbf{0} & \mathbf{1} & \mathbf{2} \\ \mathbf{0} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} & \mathbf{2} \end{bmatrix}.$$ ***(Kerala B. 2015)***

Solution. Here

$$|A| = \begin{vmatrix} 0 & 1 & 2 \\ 0 & 1 & 1 \\ 1 & 0 & 2 \end{vmatrix}$$

$$= (1)(1-2) = (1)(-1) = -1 \neq 0$$

$\Rightarrow A^{-1}$ exists.

$$\text{Now} \quad A_{11} = \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} = 2-0 = 2\,;$$

$$A_{12} = -\begin{vmatrix} 0 & 1 \\ 1 & 2 \end{vmatrix} = -(0-1) = 1\,;$$

$$A_{13} = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = 0-1 = -1.$$

$$A_{21} = -\begin{vmatrix} 1 & 2 \\ 0 & 2 \end{vmatrix} = -(2-0) = -2\,;$$

$$A_{22} = \begin{vmatrix} 0 & 2 \\ 1 & 2 \end{vmatrix} = 0-2 = -2\,;$$

$$A_{23} = -\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -(0-1) = 1.$$

$$A_{31} = \begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} = 1-2 = -1\,;$$

$$A_{32} = -\begin{vmatrix} 0 & 2 \\ 0 & 1 \end{vmatrix} = -(0-0) = 0\,;$$

$$A_{33} = \begin{vmatrix} 0 & 1 \\ 0 & 1 \end{vmatrix} = 0-0 = 0.$$

$$\therefore \quad \text{adj. } A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}'$$

$$= \begin{bmatrix} 2 & 1 & -1 \\ -2 & -2 & 1 \\ -1 & 0 & 0 \end{bmatrix}' = \begin{bmatrix} 2 & -2 & -1 \\ 1 & -2 & 0 \\ -1 & 1 & 0 \end{bmatrix}.$$

$$\text{Hence, } A^{-1} = \frac{\text{adj. } A}{|A|} = \frac{1}{-1}\begin{bmatrix} 2 & -2 & -1 \\ 1 & -2 & 0 \\ -1 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 2 & 1 \\ -1 & 2 & 0 \\ 1 & -1 & 0 \end{bmatrix}.$$

Example 10. If $\mathbf{A} = \begin{bmatrix} \mathbf{1} & \mathbf{-2} & \mathbf{3} \\ \mathbf{0} & \mathbf{-1} & \mathbf{4} \\ \mathbf{-2} & \mathbf{2} & \mathbf{1} \end{bmatrix}$**, find** $\mathbf{(A')^{-1}}$**.**

(C.B.S.E. 2015)

Solution. We have :

$$A = \begin{bmatrix} 1 & -2 & 3 \\ 0 & -1 & 4 \\ -2 & 2 & 1 \end{bmatrix}.$$

$$\therefore \quad B = A' = \begin{bmatrix} 1 & 0 & -2 \\ -2 & -1 & 2 \\ 3 & 4 & 1 \end{bmatrix}.$$

$$\text{Now} \quad |B| = \begin{vmatrix} 1 & 0 & -2 \\ -2 & -1 & 2 \\ 3 & 4 & 1 \end{vmatrix}$$

$$= (1)(-1-8) - 0 - 2(-8+3)$$

$$= -9 + 10 = 1 \neq 0$$

$\Rightarrow B^{-1}$ exists.

Now $B_{11} = \begin{vmatrix} -1 & 2 \\ 4 & 1 \end{vmatrix} = -1 - 8 = -9$;

$B_{12} = -\begin{vmatrix} -2 & 2 \\ 3 & 1 \end{vmatrix}$
$= -(-2-6) = 8$;

$B_{13} = \begin{vmatrix} -2 & -1 \\ 3 & 4 \end{vmatrix} = -8+3 = -5.$

$B_{21} = -\begin{vmatrix} 0 & -2 \\ 4 & 1 \end{vmatrix} = -(0+8) = -8$;

$B_{22} = \begin{vmatrix} 1 & -2 \\ 3 & 1 \end{vmatrix} = 1+6 = 7$;

$B_{23} = -\begin{vmatrix} 1 & 0 \\ 3 & 4 \end{vmatrix} = -(4-0) = -4.$

$B_{31} = \begin{vmatrix} 0 & -2 \\ -1 & 2 \end{vmatrix} = 0-2 = -2$;

$B_{32} = -\begin{vmatrix} 1 & -2 \\ -2 & 2 \end{vmatrix} = -(2-4) = 2$;

$B_{33} = \begin{vmatrix} 1 & 0 \\ -2 & -1 \end{vmatrix} = -1+0 = -1.$

$$\therefore \quad \text{adj. B} = \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{bmatrix}' = \begin{bmatrix} -9 & 8 & -5 \\ -8 & 7 & -4 \\ -2 & 2 & -1 \end{bmatrix}'$$

$$= \begin{bmatrix} -9 & -8 & -2 \\ 8 & 7 & 2 \\ -5 & -4 & -1 \end{bmatrix}.$$

$$\therefore \quad B^{-1} = \frac{\text{adj. B}}{|B|} = \frac{1}{1}\begin{bmatrix} -9 & -8 & -2 \\ 8 & 7 & 2 \\ -5 & -4 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -9 & -8 & -2 \\ 8 & 7 & 2 \\ -5 & -4 & -1 \end{bmatrix}.$$

Hence, $(A')^{-1} = B^{-1} = \begin{bmatrix} -9 & -8 & -2 \\ 8 & 7 & 2 \\ -5 & -4 & -1 \end{bmatrix}.$

Example 11. If $A = \begin{bmatrix} 2 & 3 \\ 1 & -4 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$, verify that $(AB)^{-1} = B^{-1}A^{-1}$.

Solution. $AB = \begin{bmatrix} 2 & 3 \\ 1 & -4 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$

$$= \begin{bmatrix} 2-3 & -4+9 \\ 1+4 & -2-12 \end{bmatrix} = \begin{bmatrix} -1 & 5 \\ 5 & -14 \end{bmatrix}.$$

Since $|AB| = \begin{vmatrix} -1 & 5 \\ 5 & -14 \end{vmatrix} = 14 - 25 = -11 \neq 0,$

$\therefore$ $(AB)^{-1}$ exists and is given by :

$$(AB)^{-1} = \frac{1}{|AB|} \cdot \text{adj. (AB)} \qquad ...(1)$$

But adj. $(AB) = \begin{bmatrix} -14 & -5 \\ -5 & -1 \end{bmatrix}' = \begin{bmatrix} -14 & -5 \\ -5 & -1 \end{bmatrix}.$

$$\therefore \text{From (1)}, (AB)^{-1} = -\frac{1}{11}\begin{bmatrix} -14 & -5 \\ -5 & -1 \end{bmatrix} \qquad ...(2)$$

Further, $|A| = \begin{vmatrix} 2 & 3 \\ 1 & -4 \end{vmatrix} = -8-3 = -11 \neq 0$

and $|B| = \begin{vmatrix} 1 & -2 \\ -1 & 3 \end{vmatrix} = 3-2 = 1 \neq 0,$

$\therefore$ A^{-1} and B^{-1} both exist and are given by :

$$A^{-1} = \frac{1}{-11}\begin{bmatrix} -4 & -1 \\ -3 & 2 \end{bmatrix}' \text{ and } B^{-1} = \frac{1}{1}\begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}'$$

$$\Rightarrow A^{-1} = \frac{1}{-11}\begin{bmatrix} -4 & -3 \\ -1 & 2 \end{bmatrix} \text{ and } B^{-1} = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}.$$

$$\therefore B^{-1}A^{-1} = -\frac{1}{11}\begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -4 & -3 \\ -1 & 2 \end{bmatrix}$$

$$= -\frac{1}{11}\begin{bmatrix} -12-2 & -9+4 \\ -4-1 & -3+2 \end{bmatrix} = -\frac{1}{11}\begin{bmatrix} -14 & -5 \\ -5 & -1 \end{bmatrix} \qquad ...(3)$$

From (2) and (3), $(AB)^{-1} = B^{-1}A^{-1}$,

which verifies the result.

Example 12. If $A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$,

verify A. (adj. A) = | A | I and find A^{-1}. *(N.C.E.R.T.)*

Solution. We have : $|A| = \begin{vmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{vmatrix}$

$= 1.(16-9) - 3(4-3) + 3(3-4)$
$= 7 - 3 - 3 = 1 \neq 0$

$\Rightarrow$ A^{-1} exists.

Now $\quad A_{11} = \begin{vmatrix} 4 & 3 \\ 3 & 4 \end{vmatrix} = 16 - 9 = 7$;

$$A_{12} = -\begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} = -(4-3) = -1;$$

$$A_{13} = \begin{vmatrix} 1 & 4 \\ 1 & 3 \end{vmatrix} = 3 - 4 = -1.$$

Similarly $A_{21} = -3, A_{22} = 1, A_{23} = 0$
and $\quad A_{31} = -3, A_{32} = 0, A_{33} = 1$ **(Do it)**

$$\therefore \quad \text{adj. } A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}'$$

$$= \begin{bmatrix} 7 & -1 & -1 \\ -3 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}' = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}.$$

(*i*) $$A\,(\text{adj. } A) = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}\begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 7-3-3 & -3+3+0 & -3+0+3 \\ 7-4-3 & -3+4+0 & -3+0+3 \\ 7-3-4 & -3+3+0 & -3+0+4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = (1)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = |A|\,I.$$

(*ii*) $$A^{-1} = \frac{\text{adj. } A}{|A|} = \frac{1}{1}\begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}.$$

Example 13. Compute $(AB)^{-1}$:

$$\mathbf{A} = \begin{bmatrix} \mathbf{1} & \mathbf{1} & \mathbf{2} \\ \mathbf{0} & \mathbf{2} & \mathbf{-3} \\ \mathbf{3} & \mathbf{-2} & \mathbf{4} \end{bmatrix}, \mathbf{B}^{-1} = \begin{bmatrix} \mathbf{1} & \mathbf{2} & \mathbf{0} \\ \mathbf{0} & \mathbf{3} & \mathbf{-1} \\ \mathbf{1} & \mathbf{0} & \mathbf{2} \end{bmatrix}.$$

Solution. We know that :

$$(AB)^{-1} = B^{-1}A^{-1} \qquad ...(1)$$

Now $$|A| = \begin{vmatrix} 1 & 1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{vmatrix}$$

$$= 1\,(8-6) + 3\,(-3-4)$$
$$= 2 - 21 = -19 \neq 0$$

$\Rightarrow A^{-1}$ exists.

Now $\quad A_{11} = \begin{vmatrix} 2 & -3 \\ -2 & 4 \end{vmatrix} = 8 - 6 = 2$;

$$A_{12} = -\begin{vmatrix} 0 & -3 \\ 3 & 4 \end{vmatrix} = -(0+9) = -9;$$

$$A_{13} = \begin{vmatrix} 0 & 2 \\ 3 & -2 \end{vmatrix} = -0 - 6 = -6.$$

$$A_{21} = -\begin{vmatrix} 1 & 2 \\ -2 & 4 \end{vmatrix} = -(4+4) = -8;$$

$$A_{22} = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 4 - 6 = -2;$$

$$A_{23} = -\begin{vmatrix} 1 & 1 \\ 3 & -2 \end{vmatrix} = -(-2-3) = 5.$$

$$A_{31} = \begin{vmatrix} 1 & 2 \\ 2 & -3 \end{vmatrix} = -3 - 4 = -7;$$

$$A_{32} = -\begin{vmatrix} 1 & 2 \\ 0 & -3 \end{vmatrix} = -(-3-0) = 3;$$

$$A_{33} = \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} = 2 - 0 = 2.$$

$$\therefore \quad \text{adj. } A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}'$$

$$= \begin{bmatrix} 2 & -9 & -6 \\ -8 & -2 & 5 \\ -7 & 3 & 2 \end{bmatrix}' = \begin{bmatrix} 2 & -8 & -7 \\ -9 & -2 & 3 \\ -6 & 5 & 2 \end{bmatrix}.$$

$$\therefore \quad A^{-1} = \frac{\text{adj. } A}{|A|} = \frac{1}{-19}\begin{bmatrix} 2 & -8 & -7 \\ -9 & -2 & 3 \\ -6 & 5 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{-2}{19} & \frac{8}{19} & \frac{7}{19} \\ \frac{9}{19} & \frac{2}{19} & \frac{-3}{19} \\ \frac{6}{19} & \frac{-5}{19} & \frac{-2}{19} \end{bmatrix}.$$

$\therefore$ From (1), $(AB)^{-1}$

$$= \begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & -1 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} \frac{-2}{19} & \frac{8}{19} & \frac{7}{19} \\ \frac{9}{19} & \frac{2}{19} & \frac{-3}{19} \\ \frac{6}{19} & \frac{-5}{19} & \frac{-2}{19} \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{2}{19}+\frac{18}{19}+0 & \frac{8}{19}+\frac{4}{19}-0 & \frac{7}{19}-\frac{6}{19}-0 \\ 0+\frac{27}{19}-\frac{6}{19} & 0+\frac{6}{19}+\frac{5}{19} & 0-\frac{9}{19}+\frac{2}{19} \\ -\frac{2}{19}+0+\frac{12}{19} & \frac{8}{19}+0-\frac{10}{19} & \frac{7}{19}-0-\frac{4}{19} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{16}{19} & \frac{12}{19} & \frac{1}{19} \\ \frac{21}{19} & \frac{11}{19} & -\frac{7}{19} \\ \frac{10}{19} & -\frac{2}{19} & \frac{3}{19} \end{bmatrix}.$$

Example 14. Find A^{-1}, if $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$.

Also show that $A^{-1} = \dfrac{A^2 - 3I}{2}$.

Solution. (*i*) Here $|A| = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix}$

$= 0(0-1) - 1(0-1) + 1(1-0)$

$= 0 + 1 + 1 = 2 \neq 0$

$\Rightarrow A^{-1}$ exists.

Now $A_{11} = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = 0 - 1 = -1$;

$A_{12} = -\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = -(0-1) = 1$;

$A_{13} = \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 1 - 0 = 1.$

$A_{21} = -\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = -(0-1) = 1$;

$A_{22} = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = 0 - 1 = -1$;

$A_{23} = -\begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} = -(0-1) = 1.$

$A_{31} = \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1 - 0 = 1$;

$A_{32} = -\begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} = -(0-1) = 1$;

$A_{33} = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = 0 - 1 = -1.$

$$\therefore \text{ adj. } A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}'$$

$$= \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}'$$

$$= \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}.$$

$$\therefore \quad A^{-1} = \frac{\text{adj. } A}{|A|} = \frac{1}{2}\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & 1/2 \\ 1/2 & 1/2 & -1/2 \end{bmatrix} \qquad ...(1)$$

(*ii*) $A^2 = AA = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

$$= \begin{bmatrix} 0+1+1 & 0+0+1 & 0+1+0 \\ 0+0+1 & 1+0+1 & 1+0+0 \\ 0+1+0 & 1+0+0 & 1+1+0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}.$$

$$\therefore \quad \frac{A^2 - 3I}{2} = \frac{1}{2}A^2 - \frac{3}{2}I$$

$$= \frac{1}{2}\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} - \frac{3}{2}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1/2 & 1/2 \\ 1/2 & 1 & 1/2 \\ 1/2 & 1/2 & 1 \end{bmatrix} + \begin{bmatrix} -3/2 & 0 & 0 \\ 0 & -3/2 & 0 \\ 0 & 0 & -3/2 \end{bmatrix}$$

$$= \begin{bmatrix} -1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & 1/2 \\ 1/2 & 1/2 & -1/2 \end{bmatrix} \qquad ...(2)$$

From (1) and (2), $A^{-1} = \dfrac{A^2 - 3I}{2}$, which is true.

EXERCISE 4 (g)

Fast Track Answer Type Questions

FTATQ

1. (*a*) If A and B are invertible matrices of same order, then $(AB)^{-1} = B^{-1}A^{-1}$. **(True/False)** *(Kashmir B. 2017)*

(*b*) A square matrix A has inverse if and only if A is..... **(Fill in the blank)** *(Jammu B. 2015)*

2. If A is a square matrix of order 3 such that $|\text{adj } A| = 225$, find $|A'|$.

3. (*a*) For what value of '*x*', is the following matrix singular ?

(*i*) $\begin{bmatrix} 3-2x & x+1 \\ 2 & 4 \end{bmatrix}$ *(Meghalaya B. 2015)*

(*ii*) $\begin{bmatrix} 5-x & x+1 \\ 2 & 4 \end{bmatrix}$. *(Meghalaya B. 2016)*

(*b*) For what value of '*x*', the matrix $\begin{bmatrix} 2-x & 3 \\ -5 & 1 \end{bmatrix}$ is not invertible ? *(Assam B. 2016)*

4. Find the adjoint of the following matrices :

(*i*) $\begin{bmatrix} 2 & -1 \\ 4 & 3 \end{bmatrix}$ (*ii*) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

(A.I.C.B.S.E. 2010) (N.C.E.R.T. ; Jammu B. 2012)

(*iii*) $\begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$ *(N.C.E.R.T. ; Assam B. 2018)*

(*iv*) $\begin{bmatrix} 1 & -1 & 2 \\ 2 & 3 & 5 \\ -2 & 0 & 1 \end{bmatrix}$. *(N.C.E.R.T.; H.B. 2010)*

5. (*a*) If $A = \begin{pmatrix} 3 & 1 \\ 2 & -3 \end{pmatrix}$, then find $|\text{adj. } A|$.

(C.B.S.E. 2010 C)

(*b*) If A is a square matrix of order 3 such that $|A| = 5$, find $|A.\text{ adj. } A|$. *(Assam B. 2016)*

(*c*) Write the inverse of the matrix :

(*i*) $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

(*ii*) $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$. *(Karnataka B. 2013)*

6. (*a*) If A is non-singular matrix such that $A^2 + A - I = O$, what is A^{-1} ? *(Assam B. 2015)*

(*b*) If A is an invertible matrix of order 3 and $|A| = 7$, then find $|\text{adj } A|$. *(A.I.C.B.S.E. 2010)*

Very Short Answer Type Questions

VSATQ

7. (*a*) Verify A (adj. A) = (adj. A) A = $|A|$ I :

(*i*) $\begin{bmatrix} 2 & 3 \\ -4 & -6 \end{bmatrix}$ (*ii*) $\begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & -2 \\ 1 & 0 & 3 \end{bmatrix}$. *(N.C.E.R.T.)*

(*b*) Verify that A(adj A) = I when :

$$A = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$ *(H.B. 2011)*

8. Find the inverse of each of the following matrices :

(*i*) $\begin{bmatrix} -1 & 5 \\ -3 & 2 \end{bmatrix}$ *(N.C.E.R.T.)*

(*ii*) $\begin{bmatrix} 2 & -2 \\ 4 & 3 \end{bmatrix}$. *(N.C.E.R.T.)*

9. (*i*) If $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$, verify $A^2 - 4A + 3I = O$, where $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Hence find A^{-1}.

(W. Bengal B. 2016)

(*ii*) If $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$, show that $A^2 - 5A + 7I = O$. Hence, find A^{-1}. *(N.C.E.R.T. ; Rajasthan B. 2013)*

(*iii*) Consider the matrix $A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$.

(*a*) Show that $A^2 - 7A - 2I = O$.

(*b*) Hence, find A^{-1}. *(Kerala B. 2014)*

10. If $A = \begin{bmatrix} 2 & 3 \\ 5 & -2 \end{bmatrix}$, write A^{-1} in terms of A.

(A.I.C.B.S.E. 2011)

11. Verify $(AB)^{-1} = B^{-1}A^{-1}$ for the matrices A and B, where :

(*i*) $A = \begin{bmatrix} 2 & 3 \\ 1 & -4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$

(N.C.E.R.T. ; Kashmir B. 2016; Rajasthan B. 2013)

(*ii*) $A = \begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 6 & 8 \\ 7 & 9 \end{bmatrix}$

(N.C.E.R.T.; H.B. 2013; Jammu B. 2012)

(*iii*) $A = \begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 6 & 7 \\ 8 & 9 \end{bmatrix}$ *(Jharkhand B. 2013)*

(*iv*) $A = \begin{bmatrix} 4 & 1 \\ 6 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 5 \\ 1 & 6 \end{bmatrix}$. *(Nagaland B. 2013)*

12. (*a*) Show that the matrix $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$ satisfies the equation $A^2 - 4A + I = O$ and hence, find A^{-1}. *(H.B. 2012)*

(*b*) If $A = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$, then show that $A^2 - 5A + 7I_2 = O$; hence find A^{-1}. *(W. Bengal B. 2017)*

(*c*) If $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$, show that $A^2 - 5A + 7I = O$. Hence, find A^{-1}. *(N.C.E.R.T.; Assam B. 2016)*

Short Answer Type Questions

15. Find the inverse of each of the following :

(*i*) $\begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix}$ *(N.C.E.R.T.)*

(*ii*) $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 4 \\ 0 & 0 & 5 \end{bmatrix}$ *(N.C.E.R.T.)*

(*iii*) $\begin{bmatrix} 2 & 1 & 3 \\ 4 & -1 & 0 \\ -7 & 2 & 1 \end{bmatrix}$ *(N.C.E.R.T.)*

(*iv*) $\begin{bmatrix} 1 & 0 & 0 \\ 3 & 3 & 0 \\ 5 & 2 & -1 \end{bmatrix}$. *(N.C.E.R.T.)*

16. If $P = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$, find P^{-1}. Verify that $PP^{-1} = I$.

17. Given $A = \begin{bmatrix} 5 & 0 & 4 \\ 2 & 3 & 2 \\ 1 & 2 & 1 \end{bmatrix}$, $B^{-1} = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$. Compute $(AB)^{-1}$.

18. If $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$, prove that $A^2 - 4A - 5I = O$ and hence, obtain A^{-1}. *(A.I.C.B.S.E. 2015; Assam B. 2017)*

19. If $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$, verify that :

$A^3 - 6A^2 + 9A - 4I = O$ and hence, find A^{-1}.

(Assam B. 2017; N.C.E.R.T. B. 2017)

13. (*a*) For the matrix $A = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$, find the numbers '*a*' and '*b*' such that $A^2 + aA + bI = O$. Hence, find A^{-1}. *(N.C.E.R.T.)*

(*b*) If $A = \begin{bmatrix} 2 & -3 \\ -4 & 7 \end{bmatrix}$, compute A^{-1} and show that $2A^{-1} + A - 9I = O$. *(J. & K.B. 2011)*

14. Find the inverse of the matrix $\begin{bmatrix} -3 & 2 \\ 5 & -3 \end{bmatrix}$.

Hence, find the matrix P satisfying the matrix equation :

$$P\begin{bmatrix} -3 & 2 \\ 5 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}.$$

(C.B.S.E. Sample Paper 2018)

SATQ

20. If $A^{-1} = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$, find $(AB)^{-1}$. *(N.C.E.R.T. ; C.B.S.E. (F) 2012)*

21. If $A = \begin{bmatrix} 1 & -2 & 1 \\ -2 & 3 & 1 \\ 1 & 1 & 5 \end{bmatrix}$, verify that :

(*i*) $(\text{adj. } A)^{-1} = \text{adj. } (A^{-1})$

(*ii*) $(A^{-1})^{-1} = A$. *(N.C.E.R.T.)*

22. Verify : $A(\text{adj. } A) = (\text{adj. } A)A = |A|\, I$ when :

(*i*) $A = \begin{bmatrix} 1 & -2 & 2 \\ 2 & 3 & 5 \\ -2 & 0 & 1 \end{bmatrix}$ *(Jammu B. 2015)*

(*ii*) $A = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & -2 \\ 1 & 0 & 3 \end{bmatrix}$ *(Jammu B. 2018, 14)*

(*iii*) $A = \begin{bmatrix} 2 & 1 & 5 \\ 3 & -2 & -4 \\ -3 & 1 & -2 \end{bmatrix}$ *(Meghalaya B. 2013)*

(*iv*) $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 0 \\ 2 & 4 & 3 \end{bmatrix}$. *(Nagaland B. 2018)*

23. Find the inverse of the matrix $A = \begin{bmatrix} a & b \\ c & \dfrac{1+bc}{a} \end{bmatrix}$ and show that :

$$a A^{-1} = (a^2 + bc + 1)I - aA.$$

Answers

1. (*a*) True (*b*) Invertible. **2.** ± 15.

3. (*a*) (*i*) $x = 1$ (*ii*) $x = 3$ (*b*) $x = 17$.

4. (*i*) $\begin{bmatrix} 3 & 1 \\ -4 & 2 \end{bmatrix}$ (*ii*) $\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$ (*iii*) $\begin{bmatrix} 4 & -3 \\ -1 & 2 \end{bmatrix}$

(*iv*) $\begin{bmatrix} 3 & 1 & -11 \\ -12 & 5 & -1 \\ 6 & 2 & 5 \end{bmatrix}$.

5. (*a*) -11 (*b*) 125 (*c*) (*i*) $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$.

(*ii*) $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$.

6. (*a*) $A + I$ (*b*) 49.

8. (*i*) $\begin{bmatrix} 2/13 & -5/13 \\ 3/13 & -1/13 \end{bmatrix}$ (*ii*) $\begin{bmatrix} 3/14 & 1/7 \\ -2/7 & 1/7 \end{bmatrix}$.

9. (*i*) $\begin{bmatrix} 1/3 & -2/3 \\ -2/3 & 1 \end{bmatrix}$ (*ii*) $\begin{bmatrix} 2/7 & -1/7 \\ 1/7 & 3/7 \end{bmatrix}$.

(*iii*) (*b*) $\begin{bmatrix} -5/2 & 3/2 \\ 2 & -1 \end{bmatrix}$.

10. $A^{-1} = \frac{1}{19} A$.

12. (*a*) $\begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$ (*b*) $\frac{1}{7}\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$ (*c*) $\frac{1}{7}\begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$.

13. (*a*) $a = -4, b = 1$; $\begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$.

14. $\begin{bmatrix} 3 & 2 \\ 5 & 3 \end{bmatrix}; \begin{bmatrix} 13 & 8 \\ 1 & 1 \end{bmatrix}$.

15. (*i*) $\begin{bmatrix} -2 & 0 & 1 \\ 9 & 2 & -3 \\ 6 & 1 & -2 \end{bmatrix}$

(*ii*) $\begin{bmatrix} 1 & -1 & 1/5 \\ 0 & 1/2 & -2/5 \\ 0 & 0 & 1/5 \end{bmatrix}$

(*iii*) $\begin{bmatrix} 1/3 & -5/3 & -1 \\ 4/3 & -23/3 & -4 \\ -1/3 & 11/3 & 2 \end{bmatrix}$

(*iv*) $\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1/3 & 0 \\ 3 & 2/3 & -1 \end{bmatrix}$.

17. $\begin{bmatrix} -2 & 20 & -29 \\ -2 & 18 & -25 \\ -3 & 29 & -42 \end{bmatrix}$. **18.** $\begin{bmatrix} -3/5 & 2/5 & 2/5 \\ 2/5 & -3/5 & 2/5 \\ 2/5 & 2/5 & -3/5 \end{bmatrix}$.

19. $\begin{bmatrix} 3/4 & 1/4 & -1/4 \\ 1/4 & 3/4 & 1/4 \\ -1/4 & 1/4 & 3/4 \end{bmatrix}$.

20. $\begin{bmatrix} 9 & -3 & 5 \\ -2 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}$.

Hints to Selected Questions

2. Use $|\text{adj } A| = |A|^{n-1}$.

3. (*a*) (*i*) The given matrix is singular

if $\begin{vmatrix} 3-2x & x+1 \\ 2 & 4 \end{vmatrix} = 0$ if $4(3-2x) - 2(x+1) = 0$

if $10x = 10$ if $x = 1$.

9. (*i*) $A^2 - 4A + 3I = O \Rightarrow A^2 - 4A = -3I$

$\Rightarrow AA - 4A = -3I \Rightarrow AAA^{-1} - 4AA^{-1} = -3IA^{-1}$

$\Rightarrow A(AA^{-1}) - 4AA^{-1} = (-3I)\, IA^{-1}$

$\Rightarrow AI - 4I = (-3I)\, IA^{-1}$

$\Rightarrow A - 4I = -3A^{-1}$

$\Rightarrow A^{-1} = -\frac{1}{3}(A - 4I)$.

10. $\text{adj } A = \begin{bmatrix} -2 & -5 \\ -3 & 2 \end{bmatrix}' = \begin{bmatrix} -2 & -3 \\ -5 & 2 \end{bmatrix}$.

$\therefore A^{-1} = \frac{1}{-19}\begin{bmatrix} -2 & -3 \\ -5 & 2 \end{bmatrix} = \frac{1}{19}\begin{bmatrix} 2 & 3 \\ 5 & -2 \end{bmatrix} = \frac{1}{19} A$.

23. Here $\text{adj. } A = \begin{bmatrix} \frac{1+bc}{a} & -c \\ -b & a \end{bmatrix}' = \begin{bmatrix} \frac{1+bc}{a} & -b \\ -c & a \end{bmatrix}$.

$\therefore A^{-1} = \frac{\text{adj. } A}{|A|} = \frac{1}{I}\begin{bmatrix} \frac{1+bc}{a} & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} \frac{1+bc}{a} & -b \\ -c & a \end{bmatrix}$

$\Rightarrow aA^{-1} = \begin{bmatrix} 1+bc & -ab \\ -ac & a^2 \end{bmatrix}$; etc.

4.13. LINEAR EQUATIONS (SOLUTION OF EQUATIONS BY FINDING INVERSE)

(*a*) Consider the following system of linear equations :

$$a_{11}\,x_1 + a_{12}\,x_2 + \ldots\ldots + a_{1n}\,x_n = b_1 \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$a_{21}\,x_1 + a_{22}\,x_2 + \ldots\ldots + a_{2n}\,x_n = b_2 \qquad a_{m1}\,x_1 + a_{m2}\,x_2 + \ldots\ldots + a_{mn}\,x_n = b_m.$$

These equations can be put in the form AX = B ....(1), where :

$$A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix}, \text{ called the } \textbf{co-efficient matrix},$$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \text{ and } B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}.$$

Let A be non-singular matrix *i.e.* $|A| \neq 0$.

Multiplying (1) by A^{-1}, we get $A^{-1}(AX) = A^{-1}B$

$\Rightarrow (A^{-1}A)X = A^{-1}B \Rightarrow IX = A^{-1}B \Rightarrow X = A^{-1}B.$

$$\text{Hence, } \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \frac{1}{|A|}\begin{bmatrix} A_{11} & A_{21} & \ldots\ldots & A_{m1} \\ A_{12} & A_{22} & \ldots\ldots & A_{m2} \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ A_{1n} & A_{2n} & \ldots\ldots & A_{mn} \end{bmatrix}\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}.$$

Note. The above method is applicable provided $|A| \neq 0$ *i.e.* when the system has a unique solution.

Definition

(*i*) **Consistent Equations.** *A system of equations is said to be consistent if its solution exists.*

(*ii*) **Inconsistent Equations.** *A system of equations is said to be inconsistent if its solution does not exist.*

(*iii*) **Dependent Equations.** *If a system of linear equations has an infinite number of solutions, then the equations are said to be dependent.*

Criterion of Consistency.

We state (without proof) for the consistency or inconsistency of a system of linear equations given by AX = B, where A is a square matrix.

(I) If $|A| \neq 0$, then the system is *consistent* and has a *unique solution*.

(II) If $|A| = 0$ and (adj. A) B = O, then the system is *consistent* and has *infinitely many solutions*.

(III) If $|A| = 0$ and (adj. A) B $\neq$ O, then the system is *inconsistent*.

(*b*) Consider the following system of **Homogeneous* Linear Equations :**

$$a_1\,x + b_1\,y + c_1\,z = 0$$

$$a_2\,x + b_2\,y + c_2\,z = 0$$

$$a_3\,x + b_3\,y + c_3\,z = 0.$$

$$\text{These can be written as } AX = O, \text{ where } A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \text{ and } X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

$x = y = z = 0$ always satisfy the given equations.

---

*A linear equation is said to be **homogeneous** if the constant term is zero.

Hence, the system is always consistent and the solution $x = y = z = 0$ is called the **trivial solution.**

Here we want to examine whether the system has non-trivial solutions.

Suppose A is non-singular. Then A^{-1} exists. Here $A^{-1}(AX) = O \Rightarrow IX = O \Rightarrow X = O$.

Hence, *if A is non-singular, the system has only trivial solution.*

For non–trivial solution, $|A| = 0$.

Thus we have the following theorem :

Theorem. *A system of n homogeneous linear equations in n unknowns has non–trivial solutions if and only if the coefficient matrix is singular.*

Proof is beyond the scope of the present volume.

Note. In view of the criteria of consistency, mentioned earlier, we have the following regarding AX = O, where A is a square matrix.

(*i*) If $|A| \neq 0$, the system has only *trivial solution.*

(*ii*) If $|A| = 0$, the system has *infinitely many solutions.*

KEY POINT

If $|A| = 0$, then (adj. A) B = O as B = O.

Frequently Asked Questions

FAQs

Example 1. Use matrix method to show that the system of equations x + 2y = 9, 2x + 4y = 7 is inconsistent.

Solution. The given system of equations is :

$$x + 2y = 9 \quad ...(1)$$

and $$2x + 4y = 7 \quad ...(2)$$

These equations can be expressed in the matrix form as :

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 9 \\ 7 \end{bmatrix} \text{ or } AX = B,$$

where $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$ and $B = \begin{bmatrix} 9 \\ 7 \end{bmatrix}$.

Now $|A| = \begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} = 4 - 4 = 0.$

$\therefore$ A is singular.

Either the given system has no solution or an infinite number of solutions.

Now $A_{11} = 4, A_{12} = -2, A_{21} = -2, A_{22} = 1.$

$$\therefore \text{ adj. } A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}' = \begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix}' = \begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix}.$$

$$\therefore \quad (\text{adj.}A)\,B = \begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 9 \\ 7 \end{bmatrix}$$

$$= \begin{bmatrix} 36 - 14 \\ -18 + 7 \end{bmatrix} = \begin{bmatrix} 22 \\ -11 \end{bmatrix} \neq O.$$

Hence, the given system has no solution, and is, therefore, inconsistent.

Example 2. Solve the following system of equations by matrix method :

3 x – 4 y = 5 and 4 x + 2 y = 3.

Solution. The given system of equations is :

$$3x - 4y = 5 \quad ...(1)$$

and $$4x + 2y = 3 \quad ...(2)$$

These equations can be expressed in the matrix form as :

$$AX = B \quad ...(3),$$

where $A = \begin{bmatrix} 3 & -4 \\ 4 & 2 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$ and $B = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$.

Now $|A| = \begin{vmatrix} 3 & -4 \\ 4 & 2 \end{vmatrix} = 6 + 16 = 22 \neq 0.$

$\therefore$ A is non–singular and as such the given system of equations has a unique solution.

Now $A_{11} = 2, A_{12} = -4, A_{21} = 4, A_{22} = 3.$

$$\therefore \text{ adj. } A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}' = \begin{bmatrix} 2 & -4 \\ 4 & 3 \end{bmatrix}' = \begin{bmatrix} 2 & 4 \\ -4 & 3 \end{bmatrix}.$$

$$\therefore \quad A^{-1} = \frac{1}{|A|}\text{ adj. } A = \frac{1}{22}\begin{bmatrix} 2 & 4 \\ -4 & 3 \end{bmatrix}.$$

From (3), $AX = B$

$\Rightarrow A^{-1}(AX) = A^{-1}B \Rightarrow (A^{-1}A)X = A^{-1}B$

$\Rightarrow IX = A^{-1}B.$

$$\therefore\ X = A^{-1}B = \frac{1}{22}\begin{bmatrix} 2 & 4 \\ -4 & 3 \end{bmatrix}\begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

$$= \frac{1}{22}\begin{bmatrix} 10+12 \\ -20+9 \end{bmatrix} = \frac{1}{22}\begin{bmatrix} 22 \\ -11 \end{bmatrix}.$$

$$\therefore \quad \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ -1/2 \end{bmatrix}.$$

Hence, $x = 1$ and $y = -\frac{1}{2}$.

Example 3. Using matrices, solve the following system of equations :

$4x + 3y + 2z = 60,\ x + 2y + 3z = 45,\ 6x + 2y + 3z = 70.$

(A.I.C.B.S.E. 2011)

Solution. The given system of equations can be written as $AX = B$ ...(1),

$$\text{where } A = \begin{bmatrix} 4 & 3 & 2 \\ 1 & 2 & 3 \\ 6 & 2 & 3 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 60 \\ 45 \\ 70 \end{bmatrix}.$$

$$\text{Now } |A| = \begin{vmatrix} 4 & 3 & 2 \\ 1 & 2 & 3 \\ 6 & 2 & 3 \end{vmatrix}$$

$$= 4(6-6) - 3(3-18) + 2(2-12)$$

$$= 0 + 45 - 20 = 25 \neq 0.$$

$\therefore$ A is non-singular and as such A^{-1} exists.

$$\text{Now } A_{11} = \begin{vmatrix} 2 & 3 \\ 2 & 3 \end{vmatrix} = 6 - 6 = 0;$$

$$A_{12} = -\begin{vmatrix} 1 & 3 \\ 6 & 3 \end{vmatrix} = -(3-18) = 15;$$

$$A_{13} = \begin{vmatrix} 1 & 2 \\ 6 & 2 \end{vmatrix} = 2 - 12 = -10.$$

$$A_{21} = -\begin{vmatrix} 3 & 2 \\ 2 & 3 \end{vmatrix} = -(9-4) = -5;$$

$$A_{22} = \begin{vmatrix} 4 & 2 \\ 6 & 3 \end{vmatrix} = 12 - 12 = 0;$$

$$A_{23} = -\begin{vmatrix} 4 & 3 \\ 6 & 2 \end{vmatrix} = -(8-18) = 10.$$

$$A_{31} = \begin{vmatrix} 3 & 2 \\ 2 & 3 \end{vmatrix} = 9 - 4 = 5;$$

$$A_{32} = -\begin{vmatrix} 4 & 2 \\ 1 & 3 \end{vmatrix} = -(12-2) = -10;$$

$$A_{33} = \begin{vmatrix} 4 & 3 \\ 1 & 2 \end{vmatrix} = 8 - 3 = 5.$$

$$\therefore \quad \text{adj } A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}'$$

$$= \begin{bmatrix} 0 & 15 & -10 \\ -5 & 0 & 10 \\ 5 & -10 & 5 \end{bmatrix}'$$

$$= \begin{bmatrix} 0 & -5 & 5 \\ 15 & 0 & -10 \\ -10 & 10 & 5 \end{bmatrix}.$$

$$\therefore \quad A^{-1} = \frac{\text{adj } A}{|A|} = \frac{1}{25}\begin{bmatrix} 0 & -5 & 5 \\ 15 & 0 & -10 \\ -10 & 10 & 5 \end{bmatrix}$$

$$= \frac{1}{5}\begin{bmatrix} 0 & -1 & 1 \\ 3 & 0 & -2 \\ -2 & 2 & 1 \end{bmatrix}.$$

From (1), $AX = B \Rightarrow A^{-1}(AX) = A^{-1}B$

$\Rightarrow (A^{-1}A)X = A^{-1}B \Rightarrow IX = A^{-1}B.$

$$\therefore \quad X = A^{-1}B = \frac{1}{5}\begin{bmatrix} 0 & -1 & 1 \\ 3 & 0 & -2 \\ -2 & 2 & 1 \end{bmatrix}\begin{bmatrix} 60 \\ 45 \\ 70 \end{bmatrix}$$

$$= \frac{1}{5}\begin{bmatrix} 0 - 45 + 70 \\ 180 + 0 - 140 \\ -120 + 90 + 70 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 25 \\ 40 \\ 40 \end{bmatrix}.$$

$$\therefore \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \\ 8 \end{bmatrix}.$$

Hence, $x = 5$, $y = 8$ and $z = 8$.

Example 4. Solve the following system of equations by matrix method, where $x \neq 0, y \neq 0, z \neq 0$:

$$\frac{2}{x} - \frac{3}{y} + \frac{3}{z} = 10;\ \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 10;\ \frac{3}{x} - \frac{1}{y} + \frac{2}{z} = 13.$$

Solution. The given system of equations is :

$$\frac{2}{x} - \frac{3}{y} + \frac{3}{z} = 10 \quad ...(1)$$

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 10 \quad ...(2)$$

and $$\frac{3}{x} - \frac{1}{y} + \frac{2}{z} = 13 \quad ...(3)$$

These equations can be written as $AX = B$ ...(4),

$$\text{where } A = \begin{bmatrix} 2 & -3 & 3 \\ 1 & 1 & 1 \\ 3 & -1 & 2 \end{bmatrix}, X = \begin{bmatrix} 1/x \\ 1/y \\ 1/z \end{bmatrix}, B = \begin{bmatrix} 10 \\ 10 \\ 13 \end{bmatrix}.$$

Now $\quad |A| = \begin{vmatrix} 2 & -3 & 3 \\ 1 & 1 & 1 \\ 3 & -1 & 2 \end{vmatrix}$

$$= 2(2+1) + 3(2-3) + 3(-1-3)$$
$$= 6 - 3 - 12 = -9 \neq 0.$$

$\therefore$ A is non–singular and as such A^{-1} exists.

Now $\quad A_{11} = \begin{vmatrix} 1 & 1 \\ -1 & 2 \end{vmatrix} = 2 + 1 = 3$;

$$A_{12} = -\begin{vmatrix} 1 & 1 \\ 3 & 2 \end{vmatrix} = -(2-3) = 1;$$

$$A_{13} = \begin{vmatrix} 1 & 1 \\ 3 & -1 \end{vmatrix} = -1 - 3 = -4.$$

$$A_{21} = -\begin{vmatrix} -3 & 3 \\ -1 & 2 \end{vmatrix} = -(-6+3) = 3;$$

$$A_{22} = \begin{vmatrix} 2 & 3 \\ 3 & 2 \end{vmatrix} = 4 - 9 = -5;$$

$$A_{23} = -\begin{vmatrix} 2 & -3 \\ 3 & -1 \end{vmatrix} = -(-2+9) = -7.$$

$$A_{31} = \begin{vmatrix} -3 & 3 \\ 1 & 1 \end{vmatrix} = -3 - 3 = -6;$$

$$A_{32} = -\begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} = -(2-3) = 1;$$

$$A_{33} = \begin{vmatrix} 2 & -3 \\ 1 & 1 \end{vmatrix} = 2 + 3 = 5.$$

$$\therefore \quad \text{adj. } A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}' = \begin{bmatrix} 3 & 1 & -4 \\ 3 & -5 & -7 \\ -6 & 1 & 5 \end{bmatrix}'$$

$$= \begin{bmatrix} 3 & 3 & -6 \\ 1 & -5 & 1 \\ -4 & -7 & 5 \end{bmatrix}.$$

$$\therefore \quad A^{-1} = \frac{\text{adj. } A}{|A|} = \frac{1}{-9}\begin{bmatrix} 3 & 3 & -6 \\ 1 & -5 & 1 \\ -4 & -7 & 5 \end{bmatrix}.$$

From (4), $AX = B \Rightarrow A^{-1}(AX) = A^{-1}B$

$\Rightarrow (A^{-1}A)X = A^{-1}B \Rightarrow IX = A^{-1}B.$

$$\therefore \quad X = A^{-1}B = \frac{1}{-9}\begin{bmatrix} 3 & 3 & -6 \\ 1 & -5 & 1 \\ -4 & -7 & 5 \end{bmatrix}\begin{bmatrix} 10 \\ 10 \\ 13 \end{bmatrix}$$

$$= \frac{1}{-9}\begin{bmatrix} 30+30-78 \\ 10-50+13 \\ -40-70+65 \end{bmatrix} = \frac{1}{-9}\begin{bmatrix} -18 \\ -27 \\ -45 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}.$$

$$\therefore \quad \begin{bmatrix} 1/x \\ 1/y \\ 1/z \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}$$

$$\Rightarrow \quad \frac{1}{x} = 2, \frac{1}{y} = 3 \text{ and } \frac{1}{z} = 5.$$

Hence, $\quad x = \frac{1}{2}, y = \frac{1}{3}$ and $z = \frac{1}{5}$.

Example 5. Using elementary transformations, find the inverse of the matrix :

$$A = \begin{bmatrix} 8 & 4 & 3 \\ 2 & 1 & 1 \\ 1 & 2 & 2 \end{bmatrix}$$

and use it to solve the following system of equations :

$$8x + 4y + 3z = 19$$
$$2x + y + z = 5$$
$$x + 2y + 2z = 7.$$

(C.B.S.E. 2016)

Solution. (I) We know that $AA^{-1} = I$

$$\Rightarrow \quad \begin{bmatrix} 8 & 4 & 3 \\ 2 & 1 & 1 \\ 1 & 2 & 2 \end{bmatrix} A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 1 \\ 8 & 4 & 3 \end{bmatrix} A^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

[*Applying* $R_1 \leftrightarrow R_3$]

$$\Rightarrow \quad \begin{bmatrix} 1 & 2 & 2 \\ 0 & -3 & -3 \\ 0 & -12 & -13 \end{bmatrix} A^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -2 \\ 1 & 0 & -8 \end{bmatrix}$$

[*Applying* $R_2 \to R_2 - 2R_1$ *and* $R_3 \leftrightarrow R_3 - 8R_1$]

$$\Rightarrow \quad \begin{bmatrix} 1 & 2 & 2 \\ 0 & -3 & -3 \\ 0 & 0 & 1 \end{bmatrix} A^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -2 \\ -1 & 4 & 0 \end{bmatrix}$$

[*Applying* $R_3 \to -R_3 + 4R_2$]

$$\Rightarrow \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A^{-1} = \begin{bmatrix} 2 & -8 & 1 \\ 1 & -13/3 & 2/3 \\ -1 & 4 & 0 \end{bmatrix}$$

[*Applying* $R_2 \to (-1/3)\, R_2 - R_3$ *and* $R_1 \to R_1 - 2R_3$]

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A^{-1} = \begin{bmatrix} 0 & 2/3 & -1/3 \\ 1 & -13/3 & 2/3 \\ -1 & 4 & 0 \end{bmatrix}.$$

[*Applying* $R_1 \to R_1 - 2R_2$]

$$\text{Hence, } A^{-1} = \begin{bmatrix} 0 & 2/3 & -1/3 \\ 1 & -13/3 & 2/3 \\ -1 & 4 & 0 \end{bmatrix}.$$

(II) The given system of equations can be written as

$AX = B$ ...(1),

$$\text{where } A = \begin{bmatrix} 8 & 4 & 3 \\ 2 & 1 & 1 \\ 1 & 2 & 2 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 19 \\ 5 \\ 7 \end{bmatrix}.$$

From (1), $AX = B \quad \Rightarrow \quad A^{-1}(AX) = A^{-1}B$

$\Rightarrow (A^{-1}A)X = A^{-1}B \quad \Rightarrow \quad IX = A^{-1}B$

$\Rightarrow \quad X = A^{-1}B.$

$$\therefore \quad X = \begin{bmatrix} 0 & 2/3 & -1/3 \\ 1 & -13/3 & 2/3 \\ -1 & 4 & 0 \end{bmatrix} \begin{bmatrix} 19 \\ 5 \\ 7 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 + \frac{10}{3} - \frac{7}{3} \\ 19 - \frac{65}{3} + \frac{14}{3} \\ -19 + 20 + 0 \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}.$$

Hence, $x = 1$, $y = 2$ and $z = 1$.

Example 6. If $A = \begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}$, **find** A^{-1}.

Using A^{-1}, solve the following system of equations :

$2x - 3y + 5z = 11$, $3x + 2y - 4z = -5$, $x + y - 2z = -3$.

(Assam B. 2017; Meghalaya B. 2014 ; Kashmir B. 2011)

Solution. (I) $|A| = \begin{vmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{vmatrix}$

$= 2(-4+4) + 3(-6+4) + 5(3-2)$

$= 2(0) + 3(-2) + 5(1)$

$= 0 - 6 + 5 = -1 \neq 0.$

$\therefore$ A is non-singular and as such A^{-1} exists.

Now $A_{11} = \begin{vmatrix} 2 & -4 \\ 1 & -2 \end{vmatrix} = -4 + 4 = 0;$

$A_{12} = -\begin{vmatrix} 3 & -4 \\ 1 & -2 \end{vmatrix} = -(-6+4) = 2;$

$A_{13} = \begin{vmatrix} 3 & 2 \\ 1 & 1 \end{vmatrix} = 3 - 2 = 1.$

$A_{21} = -\begin{vmatrix} -3 & 5 \\ 1 & -2 \end{vmatrix} = -(6-5) = -1;$

$A_{22} = \begin{vmatrix} 2 & 5 \\ 1 & -2 \end{vmatrix} = -4 - 5 = -9;$

$A_{23} = -\begin{vmatrix} 2 & -3 \\ 1 & 1 \end{vmatrix} = -(2+3) = -5.$

$A_{31} = \begin{vmatrix} -3 & 5 \\ 2 & -4 \end{vmatrix} = 12 - 10 = 2;$

$A_{32} = -\begin{vmatrix} 2 & 5 \\ 3 & -4 \end{vmatrix} = -(-8-15) = 23;$

$A_{33} = \begin{vmatrix} 2 & -3 \\ 3 & 2 \end{vmatrix} = 4 + 9 = 13.$

$$\therefore \quad \text{adj } A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}'$$

$$= \begin{bmatrix} 0 & 2 & 1 \\ -1 & -9 & -5 \\ 2 & 23 & 13 \end{bmatrix}' = \begin{bmatrix} 0 & -1 & 2 \\ 2 & -9 & 23 \\ 1 & -5 & 13 \end{bmatrix}.$$

$$\therefore \quad A^{-1} = \frac{\text{adj } A}{|A|} = \frac{1}{-1} \begin{bmatrix} 0 & -1 & 2 \\ 2 & -9 & 23 \\ 1 & -5 & 13 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & -2 \\ -2 & 9 & -23 \\ -1 & 5 & -13 \end{bmatrix} \quad ...(1)$$

(II) The given system of equations is :

$$2x - 3y + 5z = 11$$
$$3x + 2y - 4z = -5$$
$$x + y - 2z = -3.$$

These equations can be written as AX = B,

where $A = \begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

and $B = \begin{bmatrix} 11 \\ -5 \\ -3 \end{bmatrix}$.

Since A is non-singular, $[\because |A| \neq 0]$

$\therefore$ the given system has a unique solution given by $X = A^{-1}B$ $[\because AX = B \Rightarrow A^{-1}(AX) = A^{-1}B$
$\Rightarrow (A^{-1}A)X = A^{-1}B \Rightarrow IX = A^{-1}B \Rightarrow X = A^{-1}B]$

i.e. $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 & 1 & -2 \\ -2 & 9 & -23 \\ -1 & 5 & -13 \end{bmatrix} \begin{bmatrix} 11 \\ -5 \\ -3 \end{bmatrix}$ [*Using (1)*]

$$= \begin{bmatrix} 0-5+6 \\ -22-45+69 \\ -11-25+39 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

Hence, $x = 1$, $y = 2$ and $z = 3$.

Example 7. If $A = \begin{bmatrix} 2 & 3 & 10 \\ 4 & -6 & 5 \\ 6 & 9 & -26 \end{bmatrix}$, find A^{-1}.

Using A^{-1}, solve the system of equations :

$$\frac{2}{x} + \frac{3}{y} + \frac{10}{z} = 2$$

$$\frac{4}{x} - \frac{6}{y} + \frac{5}{z} = 5$$

$$\frac{6}{x} + \frac{9}{y} - \frac{20}{z} = -4.$$ *(C.B.S.E. 2017; Jammu B. 2017)*

Solution. The given system of equatons is :

$$\frac{2}{x} + \frac{3}{y} + \frac{10}{z} = 2 \quad ...(1)$$

$$\frac{4}{x} - \frac{6}{y} + \frac{5}{z} = 5 \quad ...(2)$$

and $$\frac{6}{x} + \frac{9}{y} - \frac{20}{z} = -4 \quad ...(3)$$

These equations can be written as AX = B ...(4),

where $A = \begin{bmatrix} 2 & 3 & 10 \\ 4 & -6 & 5 \\ 6 & 9 & -20 \end{bmatrix}$, $X = \begin{bmatrix} 1/x \\ 1/y \\ 1/z \end{bmatrix}$ and $B = \begin{bmatrix} 2 \\ 5 \\ -4 \end{bmatrix}$.

Now $$|A| = \begin{vmatrix} 2 & 3 & 10 \\ 4 & -6 & 5 \\ 6 & 9 & -20 \end{vmatrix}$$

$$= 2(120 - 45) - 3(-80 - 30) + 10(36 + 36)$$

$$= 150 + 330 + 720 = 1200 \neq 0.$$

$\therefore$ A is non-singular and as such A^{-1} exists.

Now $$A_{11} = \begin{vmatrix} -6 & 5 \\ 9 & -20 \end{vmatrix} = 120 - 45 = 75;$$

$$A_{12} = -\begin{vmatrix} 4 & 5 \\ 6 & -20 \end{vmatrix} = -(-80 - 30) = 110;$$

$$A_{13} = \begin{vmatrix} 4 & -6 \\ 6 & 9 \end{vmatrix} = 36 + 36 = 72.$$

$$A_{21} = -\begin{vmatrix} 3 & 10 \\ 9 & -20 \end{vmatrix} = -(-60 - 90) = 150;$$

$$A_{22} = \begin{vmatrix} 2 & 10 \\ 6 & -20 \end{vmatrix} = -40 - 60 = -100;$$

$$A_{23} = -\begin{vmatrix} 2 & 3 \\ 6 & 9 \end{vmatrix} = -(18 - 18) = 0.$$

$$A_{31} = \begin{vmatrix} 3 & 10 \\ -6 & 5 \end{vmatrix} = 15 + 60 = 75;$$

$$A_{32} = -\begin{vmatrix} 2 & 10 \\ 4 & 5 \end{vmatrix} = -(10 - 40) = 30;$$

$$A_{33} = \begin{vmatrix} 2 & 3 \\ 4 & -6 \end{vmatrix} = -12 - 12 = -24.$$

$$\therefore \text{adj } A = \begin{bmatrix} 75 & 110 & 72 \\ 150 & -100 & 0 \\ 75 & 30 & -24 \end{bmatrix}' = \begin{bmatrix} 75 & 150 & 75 \\ 110 & -100 & 30 \\ 72 & 0 & -24 \end{bmatrix}.$$

$$\therefore \quad A^{-1} = \frac{\text{adj } A}{|A|} = \frac{1}{1200}\begin{bmatrix} 75 & 150 & 75 \\ 110 & -100 & 30 \\ 72 & 0 & -24 \end{bmatrix}.$$

From (4), $A^{-1}(AX) = A^{-1}B \Rightarrow (A^{-1}A)X = A^{-1}B$

$\Rightarrow \quad IX = A^{-1}B \Rightarrow X = A^{-1}B.$

$$\therefore \quad X = \frac{1}{1200}\begin{bmatrix} 75 & 150 & 75 \\ 110 & -100 & 30 \\ 72 & 0 & -24 \end{bmatrix}\begin{bmatrix} 2 \\ 5 \\ -4 \end{bmatrix}$$

$$= \frac{1}{1200}\begin{bmatrix} 150+750-300 \\ 220-500-120 \\ 144+0+96 \end{bmatrix} = \frac{1}{1200}\begin{bmatrix} 600 \\ -400 \\ 240 \end{bmatrix}.$$

$$\therefore \begin{bmatrix} 1/x \\ 1/y \\ -1/z \end{bmatrix} = \begin{bmatrix} 1/2 \\ -1/3 \\ 1/5 \end{bmatrix}$$

$$\Rightarrow \quad \frac{1}{x} = \frac{1}{2}, \ \frac{1}{y} = -\frac{1}{3} \text{ and } \frac{1}{z} = \frac{1}{5}.$$

Hence, $x = 2, y = -3$ and $z = 5$.

Example 8. Use product :

$$\begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix}\begin{bmatrix} -2 & 0 & 1 \\ 9 & 2 & -3 \\ 6 & 1 & -2 \end{bmatrix}$$

to solve the system of equations :

$x - 3z = 9, -x + 2y - 2z = 4, 2x - 3y + 4z = -3.$

(C.B.S.E. 2017)

Solution. Here $\begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix}\begin{bmatrix} -2 & 0 & 1 \\ 9 & 2 & -3 \\ 6 & 1 & -2 \end{bmatrix}$

$$= \begin{bmatrix} -2-9+12 & 0-2+2 & 1+3-4 \\ -0+18-18 & 0+4-3 & 0-6+6 \\ -6-18+24 & 0-4+4 & 3+6-8 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

$$\Rightarrow \quad \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix}^{-1} = \begin{bmatrix} -2 & 0 & 1 \\ 9 & 2 & -3 \\ 6 & 1 & -2 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} 1 & 0 & 3 \\ -1 & 2 & -2 \\ 2 & -3 & 4 \end{bmatrix}' = \begin{bmatrix} -2 & 9 & 6 \\ 0 & 2 & 1 \\ 1 & -3 & -2 \end{bmatrix} \quad ...(1)$$

The given system of equations is :

$$x + 3z = 9$$
$$-x + 2y - 2z = 4$$
$$2x - 3y + 4z = -3.$$

The system of equations can be written as $AX = B$ ...(2),

where $A = \begin{bmatrix} 1 & 0 & 3 \\ -1 & 2 & -2 \\ 2 & -3 & 4 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 9 \\ 4 \\ -3 \end{bmatrix}$.

From (2), $AX = B \Rightarrow A^{-1}(AX) = A^{-1}B$

$\Rightarrow (A^{-1}A)X = A^{-1}B \Rightarrow IX = A^{-1}B \Rightarrow X = A^{-1}B.$

$$\therefore \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 & 9 & 6 \\ 0 & 2 & 1 \\ 1 & -3 & -2 \end{bmatrix}\begin{bmatrix} 9 \\ 4 \\ -3 \end{bmatrix} \quad [\textit{Using (1)}]$$

$$= \begin{bmatrix} -18+36-18 \\ 0+8-3 \\ 9-12+6 \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \\ 3 \end{bmatrix}.$$

Hence, $x = 0$, $y = 5$ and $z = 3$.

Example 9. Solve, using matrices :

$2x - y + 3z = 5; 3x + 2y - z = 7$

and $4x + 5y - 5z = 9$.

Solution. The given system of equations can be written as :

$$AX = B, \text{ where : } A = \begin{bmatrix} 2 & -1 & 3 \\ 3 & 2 & -1 \\ 4 & 5 & -5 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 5 \\ 7 \\ 9 \end{bmatrix}.$$

Now $\quad |A| = \begin{vmatrix} 2 & -1 & 3 \\ 3 & 2 & -1 \\ 4 & 5 & -5 \end{vmatrix}$

$$= \begin{vmatrix} 0 & -1 & 0 \\ 7 & 2 & 5 \\ 14 & 5 & 10 \end{vmatrix}$$

$[\textit{Operating } C_1 \to C_1 + 2C_2 \ \& \ C_3 \to C_3 + 3C_2]$

$= (1)(70 - 70) = 0$

$\Rightarrow$ A is singular

$\Rightarrow$ Either the given system has no solution or an infinite number of solutions.

Now $A_{11} = \begin{vmatrix} 2 & -1 \\ 5 & -5 \end{vmatrix} = -10 + 5 = -5;$

$$A_{12} = -\begin{vmatrix} 3 & -1 \\ 4 & -5 \end{vmatrix} = -(-15+4) = 11;$$

$$A_{13} = \begin{vmatrix} 3 & 2 \\ 4 & 5 \end{vmatrix} = 15 - 8 = 7.$$

$$A_{21} = -\begin{vmatrix} -1 & 3 \\ 5 & -5 \end{vmatrix} = -(5-15) = 10;$$

$$A_{22} = \begin{vmatrix} 2 & 3 \\ 4 & -5 \end{vmatrix} = -10 - 12 = -22;$$

$$A_{23} = -\begin{vmatrix} 2 & -1 \\ 4 & 5 \end{vmatrix} = -(10+4) = -14.$$

$$A_{31} = \begin{vmatrix} -1 & 3 \\ 2 & -1 \end{vmatrix} = 1 - 6 = -5;$$

$$A_{32} = -\begin{vmatrix} 2 & 3 \\ 3 & -1 \end{vmatrix} = -(-2-9) = 11;$$

$$A_{33} = \begin{vmatrix} 2 & -1 \\ 3 & 2 \end{vmatrix} = 4 + 3 = 7.$$

$$\therefore \text{ adj. } A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}' = \begin{bmatrix} -5 & 11 & 7 \\ 10 & -22 & -14 \\ -5 & 11 & 7 \end{bmatrix}'$$

$$= \begin{bmatrix} -5 & 10 & -5 \\ 11 & -22 & 11 \\ 7 & -14 & 7 \end{bmatrix}.$$

$$\therefore \text{ (adj. A) B} = \begin{bmatrix} -5 & 10 & -5 \\ 11 & -22 & 11 \\ 7 & -14 & 7 \end{bmatrix} \begin{bmatrix} 5 \\ 7 \\ 9 \end{bmatrix}$$

$$= \begin{bmatrix} -25+70-45 \\ 55-154+99 \\ 35-98+63 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = O$$

$\Rightarrow$ the system is consistent and has an infinite number of solutions.

Now $\quad 2x - y = 5 - 3z \quad$ ...(1)

and $\quad 3x + 2y = 7 + z \quad$ ...(2)

Multiplying (1) by 2, $\quad 4x - 2y = 10 - 6z \quad$ ...(3)

Adding (2) and (3), $7x = 17 - 5z \Rightarrow x = \text{The}\frac{17}{7} - \frac{5}{7}z.$

Putting in (1), $\frac{34}{7} - \frac{10}{7}z - y = 5 - 3z$

$$\Rightarrow y = \frac{34}{7} - 5 + 3z - \frac{10}{7}z = -\frac{1}{7} + \frac{11}{7}z.$$

For any k, let $x = \frac{17}{7} - \frac{5}{7}k$, $y = -\frac{1}{7} + \frac{11}{7}k$, $z = k$.

These also satisfy $4x + 5y - 5z = 9$.

Example 10. Solve the following systems of linear homogeneous equations :

(i) $2x + y - 3z = 0$, $x + 3y + z = 0$ and $3x - 2y + z = 0$

(ii) $3x + 2y + 7z = 0$, $4x - 3y - 2z = 0$ and $5x + 9y + 23z = 0$.

Solution. (*i*) The given system of linear homogeneous equations can be written as :

AX = O, where :

$$A = \begin{bmatrix} 2 & 1 & -3 \\ 1 & 3 & 1 \\ 3 & -2 & 1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } O = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

$$\text{Now } |A| = \begin{vmatrix} 2 & 1 & -3 \\ 1 & 3 & 1 \\ 3 & -2 & 1 \end{vmatrix}$$

$$= 2(3+2) - 1.(1-3) - 3(-2-9)$$

$$= 10 + 2 + 33 = 45 \neq 0.$$

Thus the given system has no non-trivial solution.

Hence, the system has only trivial solution *viz.*

$x = y = z = 0$.

(*ii*) The given system of linear homogeneous equations can be written as :

AX = O,

$$\text{where } A = \begin{bmatrix} 3 & 2 & 7 \\ 4 & -3 & -2 \\ 5 & 9 & 23 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } O = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

$$\text{Now } |A| = \begin{vmatrix} 3 & 2 & 7 \\ 4 & -3 & -2 \\ 5 & 9 & 23 \end{vmatrix}$$

$$= 3(-69+18) - 2(92+10) + 7(36+15)$$

$$= -153 - 204 + 357 = 0.$$

Thus the given system has non-trivial solutions. Actually it has infinitely many solutions.

From first two equations, $\dfrac{x}{-4+21}=\dfrac{y}{28+6}=\dfrac{z}{-9-8}$

$$\Rightarrow \frac{x}{17}=\frac{y}{34}=\frac{z}{-17} \Rightarrow \frac{x}{1}=\frac{y}{2}=\frac{z}{-1}=k\,(say).$$

Hence, $x=k$, $y=2k$ and $z=-k$; k being any real number.

Example 11. The sum of three numbers is – 1. If we multiply second number by 2, third number by 3 and add them, we get 5. If we subtract the third number from the sum of first and second numbers, we get – 1. Represent it by a system of equations. Find the numbers, using inverse of a matrix. *(H.B. 2012)*

Solution. Let first, second and third numbers be x, y and z respectively.

By the question, $x+y+z=-1$ ...(1)

$2y+3z=5$ ...(2)

$x+y-z=-1$ ...(3)

These equations can be written AX = B ...(4),

where $A=\begin{bmatrix}1&1&1\\0&2&3\\1&1&-1\end{bmatrix}, X=\begin{bmatrix}x\\y\\z\end{bmatrix}, B=\begin{bmatrix}-1\\5\\-1\end{bmatrix}.$

Now $|A|=\begin{vmatrix}1&1&1\\0&2&3\\1&1&-1\end{vmatrix}$

$=(1)(-2-3)-(1)(-0-3)+(1)(0-2)$

$=-5+3-2=-4\neq 0.$

$\therefore$ A is non-singular and as such A^{-1} exists.

Now $A_{11}=\begin{vmatrix}2&3\\1&-1\end{vmatrix}=-2-3=-5;$

$A_{12}=-\begin{vmatrix}0&3\\1&-1\end{vmatrix}=-(-0-3)=3;$

$A_{13}=\begin{vmatrix}0&2\\1&1\end{vmatrix}=0-2=-2.$

$A_{21}=-\begin{vmatrix}1&1\\1&-1\end{vmatrix}=-(-1-1)=2;$

$A_{22}=\begin{vmatrix}1&1\\1&-1\end{vmatrix}=-1-1=-2;$

$A_{23}=-\begin{vmatrix}1&1\\1&1\end{vmatrix}=-(1-1)=0.$

$A_{31}=\begin{vmatrix}1&1\\2&3\end{vmatrix}=3-2=1;$

$A_{32}=-\begin{vmatrix}1&1\\0&3\end{vmatrix}=-(3-0)=-3;$

$A_{33}=\begin{vmatrix}1&1\\0&2\end{vmatrix}=2-0=2.$

$$\therefore \text{adj. A}=\begin{bmatrix}A_{11}&A_{12}&A_{13}\\A_{21}&A_{22}&A_{23}\\A_{31}&A_{32}&A_{33}\end{bmatrix}'=\begin{bmatrix}-5&3&-2\\2&-2&0\\1&-3&2\end{bmatrix}'$$

$$=\begin{bmatrix}-5&2&1\\3&-2&-3\\-2&0&2\end{bmatrix}.$$

$$\therefore \quad A^{-1}=\frac{\text{adj. A}}{|A|}=\frac{1}{-4}\begin{bmatrix}-5&2&1\\3&-2&-3\\-2&0&2\end{bmatrix}.$$

From (4), $AX=B \Rightarrow A^{-1}(AX)=A^{-1}B$

$\Rightarrow (A^{-1}A)X=A^{-1}B \Rightarrow IX=A^{-1}B.$

$$\therefore \quad X=A^{-1}B=\frac{1}{-4}\begin{bmatrix}-5&2&1\\3&-2&-3\\-2&0&2\end{bmatrix}\begin{bmatrix}-1\\5\\-1\end{bmatrix}$$

$$=\frac{1}{-4}\begin{bmatrix}5+10-1\\-3-10+3\\2+0-2\end{bmatrix}=-\frac{1}{4}\begin{bmatrix}14\\-10\\0\end{bmatrix}=\begin{bmatrix}-7/2\\5/2\\0\end{bmatrix}.$$

$$\therefore \quad \begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}-7/2\\5/2\\0\end{bmatrix}.$$

Thus $x=-\dfrac{7}{2},\ y=\dfrac{5}{2},\ z=0.$

Hence, the reqd. numbers are $-\dfrac{7}{2}, \dfrac{5}{2}$ and 0.

Example 12. The monthly incomes of Aryan and Babban are in the ratio 3 : 4 and their monthly expenditures are in the ratio 5 : 7. If each saves ₹ 15,000 per month, find their monthly incomes, using matrix method. *(C.B.S.E. 2016)*

Solution. Let ₹ $3x$ and ₹ $4x$ be the monthly income of Aryan and Babban respectively.

Let ₹ $5y$ and ₹ $7y$ be the monthly expenditure of Aryan and Babban respectively.

By the question, $3x-5y=15000$ ...(1)

and $4x-7y=15000$ ...(2)

These equations can be written as AX = B ...(3),

where $A = \begin{bmatrix} 3 & -5 \\ 4 & -7 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$ and $B = \begin{bmatrix} 15000 \\ 15000 \end{bmatrix}$.

Now $|A| = \begin{vmatrix} 3 & -5 \\ 4 & -7 \end{vmatrix} = -21 + 20 = -1 \neq 0$.

$\therefore$ A is non-singular and as such A^{-1} exists.

Now adj $A = \begin{bmatrix} -7 & -4 \\ 5 & 3 \end{bmatrix}' = \begin{bmatrix} -7 & 5 \\ -4 & 3 \end{bmatrix}$.

$$\therefore \quad A^{-1} = \frac{\text{adj } A}{|A|} = \frac{1}{-1}\begin{bmatrix} -7 & 5 \\ -4 & 3 \end{bmatrix} = \begin{bmatrix} 7 & -5 \\ 4 & -3 \end{bmatrix}.$$

From (3), $A^{-1}(AX) = A^{-1}B$

$\Rightarrow \quad (A^{-1}A)X = A^{-1}B$

$\Rightarrow IX = A^{-1}B \Rightarrow X = A^{-1}B$

$$\therefore \quad X = \begin{bmatrix} 7 & -5 \\ 4 & -3 \end{bmatrix}\begin{bmatrix} 15000 \\ 15000 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 105000 - 75000 \\ 60000 - 45000 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 30000 \\ 15000 \end{bmatrix} \Rightarrow x = 30{,}000.$$

Hence, monthly income of Aryan = 3(30,000)

= ₹ 90,000

and monthly income of Babban = 4(30,000)

= ₹ 1,20,000.

Example 13. A typist charges ₹ 145 for typing 10 English and 3 Hindi pages, while charges for typing 3 English and 10 Hindi pages are ₹ 180. Using matrices, find the charges of typing one English and one Hindi page separately. However, typist charged only ₹ 2 per page from a poor student Shyam for 5 Hindi pages. How much less was charged from this poor boy ? *(A.I.C.B.S.E. 2016)*

Solution. Let ₹ x and ₹ y be the charges for 1 page of English and Hindi respectively.

By the question, $10x + 3y = 145$ ...(1)

and $3x + 10y = 180$ ...(2)

These can be written as $AX = B$ ...(3),

where $A = \begin{bmatrix} 10 & 3 \\ 3 & 10 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$ and $B = \begin{bmatrix} 145 \\ 180 \end{bmatrix}$.

Now $|A| = \begin{vmatrix} 10 & 3 \\ 3 & 10 \end{vmatrix} = 100 - 9 = 91 \neq 0$.

$\therefore$ A is non-singular and as such A^{-1} exists.

Now adj $A = \begin{bmatrix} 10 & -3 \\ -3 & 10 \end{bmatrix}' = \begin{bmatrix} 10 & -3 \\ -3 & 10 \end{bmatrix}$.

$$\therefore \quad A^{-1} = \frac{\text{adj. } A}{|A|} = \frac{1}{91}\begin{bmatrix} 10 & -3 \\ -3 & 10 \end{bmatrix}.$$

From (3), $A^{-1}(AX) = A^{-1}B$

$\Rightarrow \quad (A^{-1}A)X = A^{-1}B \Rightarrow IX = A^{-1}B$.

$$\therefore \quad X = \frac{1}{91}\begin{bmatrix} 10 & -3 \\ -3 & 10 \end{bmatrix}\begin{bmatrix} 145 \\ 180 \end{bmatrix}$$

$$= \frac{1}{91}\begin{bmatrix} 1450 - 540 \\ -435 + 1800 \end{bmatrix} = \frac{1}{91}\begin{bmatrix} 910 \\ 1365 \end{bmatrix}.$$

$$\therefore \quad \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 10 \\ 15 \end{bmatrix}.$$

Thus $x = 10$ and $y = 15$.

Hence, the charges are ₹ 10 and ₹ 15 per page of English and Hindi respectively.

(*ii*) From Shyam he charges ₹ 2 per Hindi page.

$\therefore$ Less charges = 13 × 5 = ₹ 65.

Example 14. Two schools A and B decided to award prizes to their students for three values honesty (x), punctuality (y) and obedience (z). School A decided to award a total of ₹ 11,000 for the three values to 5, 4 and 3 students respectively while school B decided to award ₹ 10,700 for the three values to 4, 3 and 5 students respectively. If all the three prizes together amount to ₹ 2,700, then :

(*i*) Represent the above situation by a matrix equation and form linear equations, using matrix multiplication.

(*ii*) Is it possible to solve the system of equations so obtained, using matrix multiplication ?

Solution. (*i*) Here x, y, z refer to honesty, punctuality and obedience respectively.

By the question,

$$\begin{pmatrix} 5 & 4 & 3 \\ 4 & 3 & 5 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 11000 \\ 10700 \\ 2700 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 5x + 4y + 3z \\ 4x + 3y + 5z \\ x + y + z \end{pmatrix} = \begin{pmatrix} 11000 \\ 10700 \\ 2700 \end{pmatrix}.$$

Then $\quad 5x+4y+3z = 11000$

$4x+3y+5y = 10700$

and $\quad x+y+z = 2700.$

(*ii*) Let $A = \begin{pmatrix} 5 & 4 & 3 \\ 4 & 3 & 5 \\ 1 & 1 & 1 \end{pmatrix}$.

$$\therefore \quad |A| = \begin{vmatrix} 5 & 4 & 3 \\ 4 & 3 & 5 \\ 1 & 1 & 1 \end{vmatrix}$$

$$= 5(3-5) - 4(4-5) + 3(4-3)$$

$$= 5(-2) - 4(-1) + 3(1)$$

$$= -10 + 4 + 3 = -3 \neq 0.$$

$\therefore$ A^{-1} exists $\Rightarrow$ equations have a unique solution.

Yes, it is possible to solve the system of equations, using matrix multiplication.

Example 15. The management committee of a residential colony decided to award some of its members (say x) for honesty, some (say y) for helping others and some others (say z) for supervising the workers to keep the colony neat and clean. The sum of all the awardees is 12. Three times the sum of the awardees for cooperation and supervision added to two times the number of awardees for honesty is 33. If the sum of the number of awardees for honesty and supervision is twice the number of awardees for helping others. Using matrix method, find the number of awardees of each category. *(A.I.C.B.S.E. 2013)*

Solution. Let x, y and z refer to honesty, co-operation and supervision respectively.

$\therefore \quad x+y+z = 12 \quad$ ...(1)

$3(y+z)+2x = 33 \quad i.e. \quad 2x+3y+3z = 33 \quad$ ...(2)

and $\quad x+z = 2y \quad i.e. \quad x-2y+z = 0 \quad$ ...(3)

These can be written as $AX = B \quad$ ...(4),

$$\text{where}: A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 3 \\ 1 & -2 & 1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 12 \\ 33 \\ 0 \end{bmatrix}.$$

$$\text{Now} \quad |A| = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 3 \\ 1 & -2 & 1 \end{vmatrix}$$

$$= (1)(3+6) - (1)(2-3) + (1)(-4-3)$$

$$= 9+1-7 = 3 \neq 0.$$

$\therefore$ A is non-singular and as such A^{-1} exists.

$$\text{Now} \quad A_{11} = \begin{vmatrix} 3 & 3 \\ -2 & 1 \end{vmatrix} = 3+6 = 9;$$

$$A_{12} = -\begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} = -(2-3) = 1;$$

$$A_{13} = \begin{vmatrix} 2 & 3 \\ 1 & -2 \end{vmatrix} = -4-3 = -7.$$

$$A_{21} = -\begin{vmatrix} 1 & 1 \\ -2 & 1 \end{vmatrix} = -(1+2) = -3;$$

$$A_{22} = \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 1-1 = 0;$$

$$A_{23} = -\begin{vmatrix} 1 & 1 \\ 1 & -2 \end{vmatrix} = -(-2-1) = 3.$$

$$A_{31} = \begin{vmatrix} 1 & 1 \\ 3 & 3 \end{vmatrix} = 3-3 = 0;$$

$$A_{32} = -\begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} = -(3-2) = -1;$$

$$A_{33} = \begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} = 3-2 = 1.$$

$$\therefore \quad \text{adj } A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}'$$

$$= \begin{bmatrix} 9 & 1 & -7 \\ -3 & 0 & 3 \\ 0 & -1 & 1 \end{bmatrix}' = \begin{bmatrix} 9 & -3 & 0 \\ 1 & 0 & -1 \\ -7 & 3 & 1 \end{bmatrix}.$$

$$\therefore \quad A^{-1} = \frac{\text{adj } A}{|A|} = \frac{1}{3}\begin{bmatrix} 9 & -3 & 0 \\ 1 & 0 & -1 \\ -7 & 3 & 1 \end{bmatrix}.$$

From (4), $AX = B \Rightarrow A^{-1}(AX) = A^{-1}B$

$\Rightarrow (A^{-1}A)X = A^{-1}B \Rightarrow IX = A^{-1}B.$

$$\therefore \quad X = A^{-1}B = \frac{1}{3}\begin{bmatrix} 9 & -3 & 0 \\ 1 & 0 & -1 \\ -7 & 3 & 1 \end{bmatrix}\begin{bmatrix} 12 \\ 33 \\ 0 \end{bmatrix}.$$

$$= \frac{1}{3}\begin{bmatrix} 108-99+0 \\ 12+0-0 \\ -84+99+0 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 9 \\ 12 \\ 15 \end{bmatrix}.$$

$$\therefore \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}.$$

Thus $x = 3$, $y = 4$ and $z = 5$.

Hence, the number of awardees are 3, 4 and 5 in three categories.

EXERCISE 4 (h)

Very Short Answer Type Questions

VSATQ

1. Classify the following systems of equations as consistent or inconsistent :

(*i*) $x + 2y = 2$, $2x + 3y = 3$ (*ii*) $2x - y = 5$, $x + y = 4$ (*iii*) $x + 3y = 5$, $2x + 6y = 8$.

(N.C.E.R.T.; H.P.B. 2014)

Solve the following equations (2 – 3), using inverse of a matrix :

2. (*i*) $2x - y = -2$
$3x + 4y = 3$ *(N.C.E.R.T.)*

(*ii*) $5x + 2y = 3$
$3x + 2y = 5$

(N.C.E.R.T. ; Uttarakhand B. 2013)

(*iii*) $2x + 3y = 4$
$4x + 5y = 6$ *(Kerala B. 2014)*

(*iv*) $2x + 5y = 1$
$3x + 2y = 7$ *(N.C.E.R.T.)*

(*v*) $5x + 2y = 4$
$7x + 3y = 5$ *(N.C.E.R.T.)*

(*vi*) $4x - 3y = 3$
$3x - 5y = 7$. *(N.C.E.R.T.)*

3. $x + y = 5; y + z = 3; z + x = 4$. *(P.B. 2010)*

Short Answer Type Questions

SATQ

4. Classify the following system of equations as consistent or inconsistent :

(*i*) $x + y + z = 1$
$2x + 3y + 2z = 2$
$ax + ay + 2az = 4$

(*ii*) $3x - y - 2z = 2$
$2y - z = -1$
$3x - 5y = 3$

(*iii*) $5x - y + 4z = 5$
$2x + 3y + 5z = 2$
$5x - 2y + 6z = -1$ *(N.C.E.R.T.)*

(*iv*) $5x - 6y + 4z = 15$
$7x + 4y - 3z = 19$
$2x + y + 6z = 46$. *(Kerala B. 2013)*

5. Using matrices, solve the following system of equations for x, y and z :

(*i*) $x + y - z = 3$
$2x + 3y + z = 10$
$3x - y - 7z = 1$ *(A.I.C.B.S.E. 2012)*

(*ii*) $x - y + z = 4$
$2x + y - 3z = 0$
$x + y + z = 2$ *(H.P.B. 2017; P.B. 2015)*

(*iii*) $x + 2y + z = 7$
$x + 3z = 11$
$2x - 3y = 1$ *(A.I.C.B.S.E. 2011)*

(*iv*) $x + 2y + z = 8$
$2x + y - z = 1$
$x - y + z = 2$ *(Kerala B. 2015)*

(*v*) $2x - 3y + 5z = 11$
$3x + 2y - 4z = -5$
$x + y - 2z = -3$

(H.P.B. 2016; Nagaland B. 2016; H.B. 2015, 13; C.B.S.E. 2009)

(*vi*) $x + y + z = 3$; $x - 2y + 3z = 2$; $2x - y + z = 2$

(N.C.E.R.T.)

(*vii*) $x + y + z = 3$; $y + 3z = 4$; $x - 2y + z = 0$

(P.B. 2017)

(*viii*) $x + y - z = -1$; $3x + y - 2z = 3$; $x - y - z = -1$

(Mizoram B. 2017; P.B. 2013)

(*ix*) $2x + 3y + 3z = 5$; $x - 2y + z = -4$; $3x - y - 2z = 3$

(N.C.E.R.T.; H.B. 2018, 17, 14; Kashmir B. 2017; H.P.B. 2016, 10; Jammu B. 2013)

(*x*) $3x - 2y + 3z = 8$; $2x + y - z = 1$; $4x - 3y + 2z = 4$

(N.C.E.R.T.; H.P.B. 2017, 15, 12 S; Kashmir 2015, 11)

(*xi*) $2x + y + z = 1$; $x - 2y - z = \frac{3}{2}$; $3y - 5z = 9$.

(N.C.E.R.T. ; H.P.B. Model Paper 2018; H.B. 2017; H.P.B. 2015, 13, 12)

6. Solve the following equations, using inverse of a matrix :

(*i*) $x - y + 2z = 7$
$3x + 4y - 5z = -5$
$2x - y + 3z = 12$

(N.C.E.R.T.; H.P.B. 2018, 14, 13s, 13, 12, 11; H.B. 2017; P.B. 2014; H.B. 2013; Jammu B. 2013 ; C.B.S.E. 2012)

(*ii*) $x - 2y + 3z = -5$
$3x + y + z = 8$
$2x - y + 2z = 1$ *(P.B. 2018)*

(*iii*) $x + 2y = 5$
$y + 2z = 8$
$z + 2x = 5$ *(P.B. 2011)*

(*iv*) $x - y + z = 4$
$2x + y - 3z = 0$
$x + y + z = 2$

(N.C.E.R.T.; P.B. 2015 ; Karnataka B. 2014; H.P.B. 2013 S, 11, 10; Jammu B. 2013 ; H.B. 2011)

(*v*) $x - y + z = 4$
$2x + y - 3z = 0$
$x + y + z = 2$

(H.P.B. 2018, 17, 14. 11; Kerala B. 2017)

(*vi*) $x + 2y - 3z = -4$
$2x + 3y + 2z = 2$
$3x - 3y - 4z = 11$
(H.B. 2018; Mizoram B. 2016; A.I.C.B.S.E. 2011, 10)

(*vii*) $2x - 3y + 5z = 16$
$3x + 2y - 4z = -4$
$x + y - 2z = -3$
(Meghalaya B. 2018; P. B. 2013)

(*viii*) $2x + y + z = 1$
$2x - 4y - 2z = 3$
$3y - 5z = 9$ *(H.B. 2016)*

(*ix*) $2x + 3y + 3z = 5$
$x - 2y + z = -4$
$3x - y - 2z = 3$
(H.P.B. 2013 S ; A.I.C.B.S.E. 2012)

(*x*) $3x + 4y + 7z = 4$
$2x - y + 3z = -3$
$x + 2y - 3z = 8$ *(A.I.C.B.S.E. 2012)*

(*xi*) $8x + 4y + 3z = 18$
$2x + y + z = 5$
$x + 2y + z = 5$
(H.B. 2014 ; A.I.C.B.S.E. 2009 C)

(*xii*) $x + y + z = 6$
$y + 3z = 11$
$x - 2y + z = 0$ *(H.B. 2018)*

(*xiii*) $5x - y + z = 4$
$3x + 2y - 5z = 2$
$x + 3y - 2z = 5$ *(Maghalaya B. 2016)*

(*xiv*) $x + 2y + z = 7$
$x + 3z = 11$
$2x - 3y = 11.$ *(W. Bengal B. 2016)*

(*xv*) $3x - 2y + 3z = 8$
$2x + y - z = 1$
$4x - 3y + 2z = 4.$ *(Assam B. 2018)*

(*xvi*) $3x - y + z = 5$
$2x - 2y + 3z = 7$
$x + y - z = -1.$ *(Nagaland B. 2018)*

7. Solve the following system of equations :

(*i*) $\frac{2}{x} + \frac{3}{y} + \frac{10}{z} = 4$

$\frac{4}{x} - \frac{6}{y} + \frac{5}{z} = 1$

$\frac{6}{x} + \frac{9}{y} - \frac{20}{z} = 2; x, y, z \neq 0$

(N.C.E.R.T.; Jammu B. 2016; Kashmir B. 2013; H.P.B. 2013 ; C.B.S.E. 2011)

(*ii*) $\frac{1}{x} - \frac{1}{y} + \frac{2}{z} = 7$

$\frac{3}{x} + \frac{4}{y} - \frac{5}{z} = -5$

$\frac{2}{x} - \frac{1}{y} + \frac{3}{z} = 12$ *(P.B. 2014 S)*

(*iii*) $\frac{1}{x} - \frac{1}{y} + \frac{2}{z} = 4$

$\frac{2}{x} + \frac{1}{y} - \frac{3}{z} = 0$

$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 2 ; x, y, z \neq 0.$ *(H.B. 2015)*

8. If $A = \begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}$, find A^{-1}.

Using A^{-1}, solve the system of linear equations :

(*i*) $2x - 3y + 5z = 16, 3x + 2y - 4z = -4, x + y - 2z = -3$
(N.C.E.R.T. ; A.I.C.B.S.E. 2017; H.P.B. 2010 S)

(*ii*) $2x - 3y + 5z = 11, 3x + 2y - 4z = -5, x + y - 2z = -3.$
(C.B.S.E. 2018; A.I.C.B.S.E. 2017)

9. (*a*) If $A = \begin{bmatrix} 3 & 2 & 1 \\ 4 & -1 & 2 \\ 7 & 3 & -3 \end{bmatrix}$, then find A^{-1} and hence solve the following system of equations :

$3x + 4y + 7z = 14, 2x - y + 3z = 4, x + 2y - 3z = 0.$
(C.B.S.E. Sample Paper 2018)

(*b*) If $A = \begin{bmatrix} 3 & 1 & 2 \\ 3 & 2 & -3 \\ 2 & 0 & -1 \end{bmatrix}$, find A^{-1}.

Hence, solve the system of equations :

$3x + 3y + 2z = 1, x + 2y = 4, 2x - 3y - z = 5.$
(C.B.S.E. Sample Paper 2019)

10. (*i*) Given that $A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{bmatrix}$

and $B = \begin{bmatrix} 2 & 2 & -4 \\ -4 & 2 & -4 \\ 2 & -1 & 5 \end{bmatrix}$. Find AB.

Use this to solve that following system of equations :

$x - y = 3, 2x + 3y + 4z = 17, y + 2z = 7.$
(Assam B. 2013)

(*ii*) Given that $A = \begin{bmatrix} -4 & 4 & 4 \\ -7 & 1 & 3 \\ 5 & -3 & -1 \end{bmatrix}$

and $B = \begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & -2 \\ 2 & 1 & 3 \end{bmatrix}$. Find AB.

Use this to solve the following system of equations :
$x-y+z=4, x-2y-2z=9, 2x+y+3z=1.$

(C.B.S.E. 2010 C)

11. (*i*) Determine the product :

$$\begin{bmatrix} -4 & 4 & 4 \\ -7 & 1 & 3 \\ 5 & -3 & -1 \end{bmatrix}\begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & -2 \\ 2 & 1 & 3 \end{bmatrix}$$ and use it

to solve the system of equations :

$$x-y+z=4,\ x-2y-2z=9,\ 2x+y+3z=1.$$

(A.I.C.B.S.E. 2017)

(*ii*) Use product $\begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix}\begin{bmatrix} -2 & 0 & 1 \\ 9 & 2 & -3 \\ 6 & 1 & -2 \end{bmatrix}$ to solve the system of equations :

$$x-y+2z=1,\ 2y-3z=1,\ 3x-2y+4z=2.$$

(Kerala B. 2018; C.B.S.E. (F) 2011)

12. Solve the following systems of homogeneous equations :

(*i*) $\begin{aligned} 2x+3y-z&=0 \\ x-y-2z&=0 \\ 3x+y+3z&=0 \end{aligned}$

(*ii*) $\begin{aligned} 3x+y-2z&=0 \\ x+y+z&=0 \\ x-2y+z&=0 \end{aligned}$

(*iii*) $\begin{aligned} x+y-2z&=0 \\ 2x+y-3z&=0 \\ 5x+4y-9z&=0 \end{aligned}$

(*iv*) $\begin{aligned} x+y-z&=0 \\ x-2y+z&=0 \\ 3x+6y-5z&=0. \end{aligned}$

13. A school wants to award its students for the values of Honesty, Regularity and Hardwork with a total cash award of ₹ 6,000. Three times the award money for Hard work added to that given for honesty amounts to ₹ 11,000. The award money given for Honesty and Hard work together is double the one given for Regularity. Represent the above situation algebraically and find the award money for each value, using matrix method. *(C.B.S.E. 2013)*

14. Two schools A and B want to award their selected students on the values of sincerity, truthfulness and helpfulness. The school A wants to award ₹ x each, ₹ y each and ₹ z each for the three respective values to 3, 2 and 1 students respectively with a total award money of ₹ 1,600. School B wants to spend ₹ 2,300 to award its 4, 1 and 3 students on the respective values (by giving the same award money to the three values as before). If the total amount of award for one prize on each value is ₹ 900, using matrices, find the award money for each value.

(A.I.C.B.S.E. 2014)

15. Two schools P and Q want to award their selected students on the values of Discipline, Politeness and Punctuality. The school P wants to award ₹ x each, ₹ y each and ₹ z each for the three respective values to its 3, 2 and 1 students with a total award money of ₹ 1,000. School Q wants to spend ₹ 1,500 to award its 4, 1 and 3 students on the respective values (by giving the same award money for the three values as before). If the total amount of awards for one prize on each value is ₹ 600, using matrices, find the award money for each value. *(C.B.S.E. 2014)*

16. (*i*)The sum of three numbers is 6. If we multiply third number by 3 and add second number to it, we get 11. By adding first and third numbers, we get double of the second number. Represent it algebraically and find the numbers, using matrix method. *(N.C.E.R.T.; H.B. 2012)*

(*ii*) The sum of three numbers is 6. Twice the third number when added to the first number gives 7. On adding the sum of the second and the third numbers to thrice the first number, we get 12. Find the numbers by using matrix method.

(Meghalaya B. 2015)

17. The cost of 4 kg onion ; 3 kg wheat and 2 kg rice is ₹ 60. The cost of 2 kg onion, 4 kg wheat and 6 kg rice is ₹ 90. The cost of 6 kg onion, 2 kg wheat and 3 kg rice is ₹ 70. Find cost of each item per kg by matrix method.

(N.C.E.R.T. ; H.B. 2012)

Answers

1. (*i*) – (*ii*) Consistent (*iii*) Inconsistent.

2. (*i*) $x=-\frac{5}{11}, y=\frac{12}{11}$

(*ii*) $x=-1,\ y=4$

(*iii*) $x=-1,\ y=2$

(*iv*) $x=3,\ y=-1$

(*v*) $x=2,\ y=-3$

(*vi*) $x=\frac{-6}{11}, y=\frac{-19}{11}.$

3. $x=3, y=2, z=1.$

4. (*i*) Consistent when $a \neq 0$; Inconsistent when $a=0$

(*ii*) Inconsistent (*iii*) – (*iv*) Consistent.

5. (*i*) $x=3, y=1, z=1$

(*ii*) $x=2, y=-1, z=1$

(*iii*) $x=2, y=1, z=3$

(*iv*) $x = 1, y = 2, z = 3$

(*v*) $x = 1, y = 2, z = 3$

(*vi*) $x = 1, y = 1, z = 1$

(*vii*) $x = 1, y = 1, z = 1$

(*viii*) $x = 2, y = 1, z = 2$

(*ix*) $x = 1, y = 2, z = -1$

(*x*) $x = 1, y = 2, z = 3$

(*xi*) $x = 1, y = \frac{1}{2}, z = -\frac{3}{2}$.

6. (*i*) $x = 2, y = 1, z = 3$

(*ii*) $x = 3, y = 1, z = -2$

(*iii*) $x = 1, y = 2, z = 3$

(*iv*) $x = 2, y = -1, z = 1$

(*v*) $x = 2, y = -1, z = 1$

(*vi*) $x = 3, y = -2, z = 1$

(*vii*) $x = 2, y = 1, z = 3$

(*viii*) $x = 1, y = \frac{1}{2}, z = -\frac{3}{2}$

(*ix*) $x = 1, y = 2, z = -1$

(*x*) $x = 1, y = 2, z = -1$

(*xi*) $x = 1, y = 1, z = 2$

(*xii*) $x = 1, y = 2, z = 3$

(*xiii*) $x = 1, y = 2, z = 1$

(*xiv*) $x = 2, y = 1, z = 3$

(*xv*) $x = 1, y = 2, z = 3$

(*xvi*) $x = 1, y = -1, z = 1$.

7. (*i*) $x = 2, y = 3, z = 5$

(*ii*) $x = \frac{1}{2}, y = 1, z = \frac{1}{3}$

(*iii*) $x = \frac{11}{18}, y = \frac{-11}{6}, z = \frac{11}{10}$.

8. (*i*) $\begin{bmatrix} 0 & 1 & -2 \\ -2 & 9 & -23 \\ -1 & 5 & -13 \end{bmatrix}$;

$x = 2, y = 1, z = 3$.

(*ii*) $x = 1, y = 2, z = 3$.

9. (*a*) $x = 1, y = 1, z = 1$

(*b*) $x = 2, y = 1, z = 4$.

10. (*i*) $AB = 6I$; $x = 2, y = -1, z = 4$

(*ii*) $AB = 8I$; $x = 3, y = -2, z = -1$.

11. (*i*) $8I_3$; $x = 3, y = -2, z = -1$

(*ii*) $x = 0, y = 5, z = 3$.

12. (*i*) – (*ii*) $x = y = z = 0$

(*iii*) $x = y = z = k$, for any real k

(*iv*) $x = k, y = 2k, z = 3k$ for any real k.

13. Honesty : ₹ 500, Regularity : ₹ 2,000, Hard work : ₹ 3,500.

14. ₹ 200, ₹ 300 and ₹ 400.

15. ₹ 100, ₹ 200 and ₹ 300.

16. (*i*) 1, 2, 3. (*ii*) 3, 1, 2.

17. Onion : ₹ 5, Wheat : ₹ 8, Rice : ₹ 8.

Hints to Selected Questions

10. (*i*) $AB = 6I \Rightarrow A^{-1} = \frac{1}{6}B$.

Now $X = A^{-1}\begin{bmatrix} 3 \\ 17 \\ 7 \end{bmatrix} = \frac{1}{6}B\begin{bmatrix} 3 \\ 17 \\ 7 \end{bmatrix}$; etc.

14. Here system of equations is :

$$x + y + z = 65000$$

$$0.06x + 0.08y + 0.09z = 4800$$

$$\Rightarrow \quad 6x + 8y + 9z = 480000$$

$0x - 0.08y + 0.09z = 600 \Rightarrow 0x - 8y + 9z = 60000$.

17. Let the price of 1 kg onion = ₹ x, of 1 kg wheat = ₹ y and of 1 kg rice = ₹ z.

Then $\quad 4x + 3y + 2z = 60$

$2x + 4y + 6z = 90$

and $\quad 6x + 2y + 3z = 70$; etc.

NCERT-FILE

Questions from NCERT Book

(For each unsolved question, refer : "Solution of Modern's abc of Mathematics")

Exercise 4.1

Evaluate the determinants in Exercises 1 and 2.

1. $\begin{vmatrix} 2 & 4 \\ -5 & -1 \end{vmatrix}$.

Solution : $\begin{vmatrix} 2 & 4 \\ -5 & -1 \end{vmatrix} = 2(-1) - (-5)(4)$

$= -2 + 20 = 18.$

2. (*i*) $\begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix}$ (*ii*) $\begin{vmatrix} x^2 - x + 1 & x - 1 \\ x + 1 & x + 1 \end{vmatrix}$.

[**Solution :** Refer Q. 5 (*v*)-(*vi*); Ex. 4(*a*)]

3. If $A = \begin{bmatrix} 1 & 2 \\ 4 & 2 \end{bmatrix}$, then show that $|2A| = 4|A|$.

[**Solution :** Refer Q. 4 (*ii*); Ex. 4(*a*)]

4. If $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{bmatrix}$, then show that $|3A| = 27|A|$.

Solution : We have : $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{bmatrix}$.

$\therefore \quad |A| = \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{vmatrix}$

$= (1)\begin{vmatrix} 1 & 2 \\ 0 & 4 \end{vmatrix}$ [*Expanding by* C_1]

$= (1)(4 - 0) = 4.$

$\therefore \quad 27|A| = 27(4) = 108 \quad ...(1)$

Now $\quad 3A = \begin{bmatrix} 3 & 0 & 3 \\ 0 & 3 & 6 \\ 0 & 0 & 12 \end{bmatrix}$.

$\therefore \quad |3A| = \begin{vmatrix} 3 & 0 & 3 \\ 0 & 3 & 6 \\ 0 & 0 & 12 \end{vmatrix}$

$= 3\begin{vmatrix} 3 & 6 \\ 0 & 12 \end{vmatrix}$

[*Expanding by* C_1]

$= 3(36 - 0) = 108 \quad ...(2)$

From (1) and (2), $|3A| = 27|A|$, which is true.

5. Evaluate the determinants

(*i*) $\begin{vmatrix} 3 & -1 & -2 \\ 0 & 0 & -1 \\ 3 & -5 & 0 \end{vmatrix}$ (*ii*) $\begin{vmatrix} 3 & -4 & 5 \\ 1 & 1 & -2 \\ 2 & 3 & 1 \end{vmatrix}$

(*iii*) $\begin{vmatrix} 0 & 1 & 2 \\ -1 & 0 & -3 \\ -2 & 3 & 0 \end{vmatrix}$ (*iv*) $\begin{vmatrix} 2 & -1 & -2 \\ 0 & 2 & -1 \\ 3 & -5 & 0 \end{vmatrix}$.

[**Solution :** Refer Q. 9 ; Ex. 4(*a*)]

6. If $A = \begin{bmatrix} 1 & 1 & -2 \\ 2 & 1 & -3 \\ 5 & 4 & -9 \end{bmatrix}$, find $|A|$.

[**Solution :** Refer Q. 11 ; Ex. 4(*a*)]

7. Find values of '*x*', if :

(*i*) $\begin{vmatrix} 2 & 4 \\ 5 & 1 \end{vmatrix} = \begin{vmatrix} 2x & 4 \\ 6 & x \end{vmatrix}$ (*ii*) $\begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} = \begin{vmatrix} x & 3 \\ 2x & 5 \end{vmatrix}$.

[**Solution :** Refer Q. 6 (*a*) (*i*)-(*ii*); Ex. 4(*a*)]

8. If $\begin{vmatrix} x & 2 \\ 18 & x \end{vmatrix} = \begin{vmatrix} 6 & 2 \\ 18 & 6 \end{vmatrix}$, then '*x*' is equal to :

(A) 6 (B) ± 6 (C) – 6 (D) 0.

[**Ans.** (B)]

Exercise 4.2

Using the property of determinants and without expanding in Exercises 1 to 5, prove that :

1. $\begin{vmatrix} x & a & x+a \\ y & b & y+b \\ z & c & z+c \end{vmatrix} = 0$.

[**Solution :** Refer Q. 9 (*iii*); Ex. 4(*b*)]

2. $\begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix} = 0$.

[**Solution :** Refer Q. 5 (*a*) (*i*); Ex. 4(*b*)]

3. $\begin{vmatrix} 2 & 7 & 65 \\ 3 & 8 & 75 \\ 5 & 9 & 86 \end{vmatrix} = 0$.

[**Solution :** Refer Q. 9 (*i*); Ex. 4(*b*)]

4. $\begin{vmatrix} 1 & bc & a(b+c) \\ 1 & ca & b(c+a) \\ 1 & ab & c(a+b) \end{vmatrix} = 0$.

[**Solution :** Refer Q. 10 (*ii*); Ex. 4(*b*)]

5. $\begin{vmatrix} b+c & q+r & y+z \\ c+a & r+p & z+x \\ a+b & p+q & x+y \end{vmatrix} = 2\begin{vmatrix} a & p & x \\ b & q & y \\ c & r & z \end{vmatrix}$.

[**Solution :** Refer Q. 17 (*ii*); Ex. 4(*b*)]

By using properties of determinants, in Exercises 6 to 14, show that :

6. $\begin{vmatrix} 0 & a & -b \\ -a & 0 & -c \\ b & c & 0 \end{vmatrix} = 0$.

[**Solution :** Refer Q. 9 (*v*); Ex. 4(*b*)]

7. $\begin{vmatrix} -a^2 & ab & ac \\ ba & -b^2 & bc \\ ca & cb & -c^2 \end{vmatrix} = 4a^2b^2c^2$.

Solution : LHS = $\begin{vmatrix} -a^2 & ab & ac \\ ba & -b^2 & bc \\ ca & cb & -c^2 \end{vmatrix}$

$= abc\begin{vmatrix} -a & b & c \\ a & -b & c \\ a & b & -c \end{vmatrix}$

[*Taking a, b and c common from R_1, R_2 and R_3 respectively*]

$= a^2b^2c^2\begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$

[*Taking a, b and c common from C_1, C_2 and C_3 respectively*]

$= a^2b^2c^2\begin{vmatrix} 0 & 0 & 1 \\ 2 & -2 & 1 \\ 0 & 2 & -1 \end{vmatrix}$

[*Operating $C_1 \to C_1 + C_3$ & $C_2 \to C_2 - C_3$*]

$= a^2b^2c^2\,(1)\begin{vmatrix} 2 & -2 \\ 0 & 2 \end{vmatrix}$ [*Expanding by R_1*]

$= a^2b^2c^2\,(4 + 0) = 4a^2b^2c^2$ **= RHS.**

8. (*i*) $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$.

[**Solution :** Refer Q. 18 (*i*); Ex. 4(*b*)]

(*ii*) $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c)$.

Solution : $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix}$

$= \begin{vmatrix} 1 & 0 & 0 \\ a & b-a & c-b \\ a^3 & b^3-a^3 & c^3-b^3 \end{vmatrix}$

[*Operating $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_2$*]

$= \begin{vmatrix} 1 & 0 & 0 \\ a & (b-a) & (c-b) \\ a^3 & (b-a)(b^2+ba+a^2) & (c-b)(c^2+cb+b^2) \end{vmatrix}$

$= (b-a)(c-b)$

$\begin{vmatrix} 1 & 0 & 0 \\ a & 1 & 1 \\ a^3 & a^2+ab+b^2 & b^2+bc+c^2 \end{vmatrix}$

$= (a-b)(b-c)$

$\begin{vmatrix} 1 & 1 \\ a^2+ab+b^2 & b^2+bc+c^2 \end{vmatrix}$

[*Expanding by R_1*]

$= (a-b)(b-c)\,[(b^2+bc+c^2) - (a^2+ab+b^2)]$

$= (a-b)(b-c)\,[(c^2-a^2) + b(c-a)]$

$= (a-b)(b-c)(c-a)(a+b+c)$,

which is true.

9. $\begin{vmatrix} x & x^2 & yz \\ y & y^2 & zx \\ z & z^2 & xy \end{vmatrix} = (x-y)(y-z)(z-x)(xy+yz+zx)$.

[**Solution :** Refer Q. 24 (*ii*); Ex. 4(*b*)]

10. (*i*) $\begin{vmatrix} x+4 & 2x & 2x \\ 2x & x+4 & 2x \\ 2x & 2x & x+4 \end{vmatrix} = (5x+4)(4-x)^2$.

[**Solution :** Refer Q. 17 (*b*) (*i*); Ex. 4(*b*)]

(*ii*) $\begin{vmatrix} y+k & y & y \\ y & y+k & y \\ y & y & y+k \end{vmatrix} = k^2(3y+k)$.

[**Solution :** Refer Q. 17 (*b*) (*iii*); Ex. 4(*b*)]

11. (*i*) $\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} = (a+b+c)^3$.

Solution :

Operating "$R_1 \to R_1 + R_2 + R_3$", we get :

$$\Delta = \begin{vmatrix} a+b+c & a+b+c & a+b+c \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

$$= (a+b+c) \begin{vmatrix} 1 & 1 & 1 \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

[*Taking* $(a + b + c)$ *common from* R_1]

Operating "$C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$", we get :

$$\Delta = (a+b+c) \begin{vmatrix} 1 & 0 & 0 \\ 2b & -(a+b+c) & 0 \\ 2c & 0 & -(a+b+c) \end{vmatrix}$$

$$= (a+b+c) \begin{vmatrix} -(a+b+c) & 0 \\ 0 & -(a+b+c) \end{vmatrix}$$

[*Expanding by* R_1]

$= (a + b + c) \times (a + b + c)^2$

$= (a + b + c)^3$, which is true.

(*ii*) $\begin{vmatrix} x+y+2z & x & y \\ z & y+z+2x & y \\ z & x & z+x+2y \end{vmatrix} = 2(x+y+z)^3$.

[**Solution :** Refer Q. 25; Ex. 4(*b*)]

12. $\begin{vmatrix} 1 & x & x^2 \\ x^2 & 1 & x \\ x & x^2 & 1 \end{vmatrix} = (1-x^3)^2$. *(Kerala B. 2016)*

Solution : $\begin{vmatrix} 1 & x & x^2 \\ x^2 & 1 & x \\ x & x^2 & 1 \end{vmatrix}$

$$= \begin{vmatrix} 1+x+x^2 & x & x^2 \\ 1+x+x^2 & 1 & x \\ 1+x+x^2 & x^2 & 1 \end{vmatrix}$$

[*Operating* $C_1 \to C_1 + C_2 + C_3$]

$$= (1+x+x^2) \begin{vmatrix} 1 & x & x^2 \\ 1 & 1 & x \\ 1 & x^2 & 1 \end{vmatrix}$$

[*Taking* $(1 + x + x^2)$ *common from* C_1]

$$= (1+x+x^2) \begin{vmatrix} 1 & x & x^2 \\ 0 & 1-x & x-x^2 \\ 0 & x^2-1 & 1-x \end{vmatrix}$$

[*Operating* $R_2 \to R_2 - R_1$ & $R_3 \to R_3 - R_2$]

$$= (1+x+x^2) \begin{vmatrix} 1-x & x-x^2 \\ x^2-1 & 1-x \end{vmatrix}$$

[*Expanding by* C_1]

$$= (1+x+x^2)(1-x)^2 \begin{vmatrix} 1 & x \\ -1-x & 1 \end{vmatrix}$$

[*Taking* $(1 - x)$ *common from* C_1 & C_2]

$= (1 - x)^2 (1 + x + x^2)(1 + x + x^2)$

$= (1 - x)^2 (1 + x + x^2)^2$

$= (1 - x^3)^2$, which is true.

13. $\begin{vmatrix} 1+a^2-b^2 & 2ab & -2b \\ 2ab & 1-a^2+b^2 & 2a \\ 2b & -2a & 1-a^2-b^2 \end{vmatrix} = (1+a^2+b^2)^3$.

[**Solution :** Refer Q. 23 (*iii*) Ex. 4(*b*)]

14. $\begin{vmatrix} a^2+1 & ab & ac \\ ab & b^2+1 & bc \\ ca & cb & c^2+1 \end{vmatrix} = 1+a^2+b^2+c^2$.

Solution : LHS $= \begin{vmatrix} a^2+1 & ab & ac \\ ab & b^2+1 & bc \\ ac & bc & c^2+1 \end{vmatrix}$

$$= abc \begin{vmatrix} a+\frac{1}{a} & b & c \\ a & b+\frac{1}{b} & c \\ a & b & c+\frac{1}{c} \end{vmatrix}$$

[*Taking a common from* R_1, *b from* R_2 & *c from* R_3]

$$= \begin{vmatrix} a^2+1 & b^2 & c^2 \\ a^2 & b^2+1 & c^2 \\ a^2 & b^2 & c^2+1 \end{vmatrix}$$

[*Multiplying* C_1 *by a,* C_2 *by b* & C_3 *by c*]

$$= \begin{vmatrix} 1+a^2+b^2+c^2 & b^2 & c^2 \\ 1+a^2+b^2+c^2 & b^2+1 & c^2 \\ 1+a^2+b^2+c^2 & b^2 & c^2+1 \end{vmatrix}$$

[*Operating* $C_1 \to C_1 + C_2 + C_3$]

$$= (1+a^2+b^2+c^2) \begin{vmatrix} 1 & b^2 & c^2 \\ 1 & b^2+1 & c^2 \\ 1 & b^2 & c^2+1 \end{vmatrix}$$

[*Taking* $(1 + a^2 + b^2 + c^2)$ *common from* C_1]

$$= (1+a^2+b^2+c^2) \begin{vmatrix} 1 & b^2 & c^2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

[*Operating* $R_2 \to R_2 - R_1$ & $R_3 \to R_3 - R_1$]

$= (1 + a^2 + b^2 + c^2)\,(1)\,(1 - 0)$ [*Expanding by* C_1]

$= 1 + a^2 + b^2 + c^2 =$ **RHS.**

Choose the correct answer in Exercises 15 and 16.

15. Let A be a square matrix of order 3×3, then $|kA|$ is equal to :

(A) $k|A|$ (B) $k^2|A|$ (C) $k^3|A|$ (D) $3k|A|$.

[**Ans.** (C)]

16. Which of the following is correct ?

(A) Determinant is a square matrix

(B) Determinant is a number associated to a matrix

(C) Determinant is a number associated to a square matrix

(D) None of these. [**Ans.** (C)]

Exercise 4.3

1. Find area of the triangle with vertices at the points given in each of the following :

(*i*) (1, 0), (6, 0), (4, 3) (*ii*) (2, 7), (1, 1), (10, 8)

(*iii*) (– 2, – 3), (3, 2), (– 1, – 8).

[**Solution :** Refer Q. 1 (*i*), Q. 2 (*ii*) and (*iii*); Ex. 4(*d*)]

2. Show that points :

A $(a, b + c)$, B $(b, c + a)$, C $(c, a + b)$ are collinear.

[**Solution :** Refer Q. 11; Ex. 4(*d*)]

3. Find values of '*k*' if area of the triangle is 4 sq. units and vertices are :

(*i*) $(k, 0)$, (4, 0), (0, 2).

Solution : We have : $\frac{1}{2}\begin{vmatrix} k & 0 & 1 \\ 4 & 0 & 1 \\ 0 & 2 & 1 \end{vmatrix} = \pm 4.$

Expanding by R_1, $\frac{1}{2}(k)\begin{vmatrix} 0 & 1 \\ 2 & 1 \end{vmatrix} + \frac{1}{2}(1)\begin{vmatrix} 4 & 0 \\ 0 & 2 \end{vmatrix} = \pm 4$

$\Rightarrow \quad \frac{k}{2}(0-2) + \frac{1}{2}(8-0) = \pm 4$

$\Rightarrow \quad -k + 4 = \pm 4.$

Hence, $k = 0, 8.$

(*ii*) (– 2, 0), (0, 4), (0, *k*).

[**Solution :** Refer Q. 6 (*i*); Ex. 4(*d*)]

4. (*i*) Find equation of line joining (1, 2) and (3, 6), using determinants.

[**Solution :** Refer Q. 8(*i*); Ex. 4(*d*)]

(*ii*) Find equation of line joining (3, 1) and (9, 3), using determinants.

[**Solution :** Refer Q. 8(*ii*); Ex. 4(*d*)]

5. If area of triangle is 35 sq. units with vertices (2, – 6), (5, 4) and (*k*, 4). Then '*k*' is :

(A) 12 (B) – 2 (C) – 12, – 2 (D) 12, – 2.

[**Ans.** (D)]

Exercise 4.4

Write Minors and Cofactors of the elements of following determinants :

1. (*i*) $\begin{vmatrix} 2 & -4 \\ 0 & 3 \end{vmatrix}$ (*ii*) $\begin{vmatrix} a & c \\ b & d \end{vmatrix}$.

[**Solution :** Refer Q. 5; Ex. 4(*e*)]

2. (*i*) $\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$ (*ii*) $\begin{vmatrix} 1 & 0 & 4 \\ 3 & 5 & -1 \\ 0 & 1 & 2 \end{vmatrix}$.

[**Solution :** Refer Q. 7(*i*); Ex. 4(*e*)]

[**Solution :** Refer Q. 8(*iii*); Ex. 4(*e*)]

3. Using Cofactors of elements of second row, evaluate

$$\Delta = \begin{vmatrix} 5 & 3 & 8 \\ 2 & 0 & 1 \\ 1 & 2 & 3 \end{vmatrix}.$$

Solution : (*i*) Here $a_{21} = 2$, $a_{22} = 0$ and $a_{23} = 1$.

$\therefore \quad M_{21} = \begin{vmatrix} 3 & 8 \\ 2 & 3 \end{vmatrix} = 9 - 16 = -7$;

$M_{22} = \begin{vmatrix} 5 & 8 \\ 1 & 3 \end{vmatrix} = 15 - 8 = 7$;

$M_{23} = \begin{vmatrix} 5 & 3 \\ 1 & 2 \end{vmatrix} = 10 - 3 = 7.$

And $A_{21} = (-1)^{2+1} M_{21} = (-1)(-7) = 7$

$A_{22} = (-1)^{2+2} M_{22} = (1)(7) = 7$

$A_{23} = (-1)^{2+3} M_{23} = (-1)(7) = -7.$

(*ii*) $\Delta = a_{21} A_{21} + a_{22} A_{22} + a_{23} A_{23}$

$= (2)(7) + (0)(7) + (1)(-7)$

$= 14 + 0 - 7 = 7.$

4. Using Cofactors of elements of third column, evaluate :

$$\Delta = \begin{vmatrix} 1 & x & yz \\ 1 & y & zx \\ 1 & z & xy \end{vmatrix}.$$

[**Solution :** Refer Q. 9; Ex. 4(*e*)]

5. If $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ and A_{ij} is Cofactors of a_{ij}, then value of Δ is given by :

(A) $a_{11} A_{31} + a_{12} A_{32} + a_{13} A_{33}$

(B) $a_{11} A_{11} + a_{12} A_{21} + a_{13} A_{31}$

(C) $a_{21} A_{11} + a_{22} A_{12} + a_{23} A_{13}$

(D) $a_{11} A_{11} + a_{21} A_{21} + a_{31} A_{31}$.

[**Ans.** (D)]

Exercise 4.5

Find adjoint of each of the matrices in Exercises 1 and 2 :

1. $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

[**Solution :** Refer Q. 4(*ii*); Ex. 4(*g*)]

2. $\begin{bmatrix} 1 & -1 & 2 \\ 2 & 3 & 5 \\ -2 & 0 & 1 \end{bmatrix}$.

[**Solution :** Refer Q. 4(*iv*); Ex. 4(*g*)]

Verify A (*adj* A) = (*adj* A) A = | A | I in Exercises 3 and 4 :

3. $\begin{bmatrix} 2 & 3 \\ -4 & -6 \end{bmatrix}$.

4. $\begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & -2 \\ 1 & 0 & 3 \end{bmatrix}$.

[**Solutions :** (3-4) : Refer Q. 7(*a*); Ex. 4(*g*)]

Find the inverse of each of the matrices (if it exists) given in Exercises 5 to 11 :

5. $\begin{bmatrix} 2 & -2 \\ 4 & 3 \end{bmatrix}$. **6.** $\begin{bmatrix} -1 & 5 \\ -3 & 2 \end{bmatrix}$.

[**Solution :** (5-6) : Refer Q. 8; Ex. 4(*g*)]

7. $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 4 \\ 0 & 0 & 5 \end{bmatrix}$.

8. $\begin{bmatrix} 1 & 0 & 0 \\ 3 & 3 & 0 \\ 5 & 2 & -1 \end{bmatrix}$.

9. $\begin{bmatrix} 2 & 1 & 3 \\ 4 & -1 & 0 \\ -7 & 2 & 1 \end{bmatrix}$.

10. $\begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix}$.

[**Solutions :** (7-10) : Refer Q. 15; Ex. 4(*g*)]

11. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & \sin\alpha & -\cos\alpha \end{bmatrix}$.

Solution : Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & \sin\alpha & -\cos\alpha \end{bmatrix}$.

$\therefore \quad |A| = \begin{vmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & \sin\alpha & -\cos\alpha \end{vmatrix}$

$= (1)(-\cos^2\alpha - \sin^2\alpha) = -1 \neq 0.$

$\therefore$ A is non-singular $\Rightarrow A^{-1}$ exists.

Now $\quad A_{11} = (-1)^{1+1} \begin{vmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & -\cos\alpha \end{vmatrix}$

$= (+1)(-\cos^2\alpha - \sin^2\alpha)$

$= (1)(-1) = -1$;

$A_{12} = (-1)^{1+2} \begin{vmatrix} 0 & \sin\alpha \\ 0 & -\cos\alpha \end{vmatrix} = (-1)(0) = 0$;

$A_{13} = (-1)^{1+3} \begin{vmatrix} 0 & \cos\alpha \\ 0 & \sin\alpha \end{vmatrix} = (+1)(0) = 0.$

$A_{21} = (-1)^{2+1} \begin{vmatrix} 0 & 0 \\ \sin\alpha & -\cos\alpha \end{vmatrix} = (-1)(0) = 0$;

$A_{22} = (-1)^{2+2} \begin{vmatrix} 1 & 0 \\ 0 & -\cos\alpha \end{vmatrix} = (+1)(-\cos\alpha) = -\cos\alpha$;

$A_{23} = (-1)^{2+3} \begin{vmatrix} 1 & 0 \\ 0 & \sin\alpha \end{vmatrix} = (-1)(\sin\alpha) = -\sin\alpha.$

$$A_{31} = (-1)^{3+1} \begin{vmatrix} 0 & 0 \\ \cos\alpha & \sin\alpha \end{vmatrix} = (+1)(0) = 0 ;$$

$$A_{32} = (-1)^{3+2} \begin{vmatrix} 1 & 0 \\ 0 & \sin\alpha \end{vmatrix} = (-1)(\sin\alpha) = -\sin\alpha ;$$

$$A_{33} = (-1)^{3+3} \begin{vmatrix} 1 & 0 \\ 0 & \cos\alpha \end{vmatrix} = (+1)(\cos\alpha) = \cos\alpha.$$

$$\therefore \quad \text{adj } A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -\cos\alpha & -\sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{bmatrix}'$$

$$= \begin{bmatrix} -1 & 0 & 0 \\ 0 & -\cos\alpha & -\sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{bmatrix}.$$

$$\therefore \quad A^{-1} = \frac{1}{|A|} (\text{adj } A)$$

$$= \frac{1}{-1} \begin{bmatrix} -1 & 0 & 0 \\ 0 & -\cos\alpha & -\sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & \sin\alpha & -\cos\alpha \end{bmatrix}.$$

12. Let $A = \begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 6 & 8 \\ 7 & 9 \end{bmatrix}$.

Verify that $(AB)^{-1} = B^{-1} A^{-1}$.

[**Solution :** Refer Q. 11(*ii*); Ex. 4(*g*)]

13. Let $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$, show that $A^2 - 5A + 7I = O$.

Hence, find A^{-1}.

[**Solution :** Refer Q. 12(*c*); Ex. 4(*g*)]

14. For the matrix $A = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$, find the numbers '*a*' and '*b*' such that $A^2 + aA + bI = O$.

[**Solution :** Refer Q. 13(*a*); Ex. 4(*g*)]

15. For the matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}$.

Show that $A^3 - 6A^2 + 5A + 11 I = O$. Hence, find A^{-1} .

Solution. We have :

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}.$$

$$\therefore \quad A^2 = A\,A$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1+1+2 & 1+2-1 & 1-3+3 \\ 1+2-6 & 1+4+3 & 1-6-9 \\ 2-1+6 & 2-2-3 & 2+3+9 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 2 & 1 \\ -3 & 8 & -14 \\ 7 & -3 & 14 \end{bmatrix}.$$

And $\quad A^3 = A^2 A =$

$$= \begin{bmatrix} 4 & 2 & 1 \\ -3 & 8 & -14 \\ 7 & -3 & 14 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 4+2+2 & 4+4-1 & 4-6+3 \\ -3+8-28 & -3+16+14 & -3-24-42 \\ 7-3+28 & 7-6-14 & 7+9+42 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 7 & 1 \\ -23 & 27 & -69 \\ 32 & -13 & 58 \end{bmatrix}.$$

Now $A^3 - 6A^2 + 5A + 11 I$

$$= \begin{bmatrix} 8 & 7 & 1 \\ -23 & 27 & -69 \\ 32 & -13 & 58 \end{bmatrix} - 6 \begin{bmatrix} 4 & 2 & 1 \\ -3 & 8 & -14 \\ 7 & -3 & 14 \end{bmatrix}$$

$$+ 5 \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix} + 11 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 7 & 1 \\ -23 & 27 & -69 \\ 32 & -13 & 58 \end{bmatrix} + \begin{bmatrix} -24 & -12 & -6 \\ 18 & -48 & 84 \\ -42 & 18 & -84 \end{bmatrix}$$

$$+ \begin{bmatrix} 5 & 5 & 5 \\ 5 & 10 & -15 \\ 10 & -5 & 15 \end{bmatrix} + \begin{bmatrix} 11 & 0 & 0 \\ 0 & 11 & 0 \\ 0 & 0 & 11 \end{bmatrix}$$

$$= \begin{bmatrix} 8-24+5+11 & 7-12+5+0 & 1-6+5+0 \\ -23+18+5+0 & 27-48+10+11 & -69+84-15+0 \\ 32-42+10+0 & -13+18-5+0 & 58-84+15+11 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O, \text{ which is true.}$$

Now $A^3 - 6A^2 + 5A + 11\,I = O$.

Multiplying by A^{-1}, we get :

$(A^{-1}A)\,A^2 - 6\,(A^{-1}A)\,A + 5\,(A^{-1}A) + 11\,(A^{-1}I) = O$

$\Rightarrow IA^2 - 6\,IA + 5\,I + 11\,A^{-1} = O$

$\Rightarrow 11A^{-1} = -A^2 + 6A - 5I$

$$= -\begin{bmatrix} 4 & 2 & 1 \\ -3 & 8 & -14 \\ 7 & -3 & 14 \end{bmatrix} + 6\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -4 & -2 & -1 \\ 3 & -8 & 14 \\ -7 & 3 & -14 \end{bmatrix} + \begin{bmatrix} 6 & 6 & 6 \\ 6 & 12 & -18 \\ 12 & -6 & 18 \end{bmatrix} + \begin{bmatrix} -5 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & -5 \end{bmatrix}$$

$$= \begin{bmatrix} -4+6-5 & -2+6+0 & -1+6+0 \\ 3+6+0 & -8+12-5 & 14-18+0 \\ -7+12+0 & 3-6+0 & -14+18-5 \end{bmatrix}$$

$$= \begin{bmatrix} -3 & 4 & 5 \\ 9 & -1 & -4 \\ 5 & -3 & -1 \end{bmatrix}.$$

$$\text{Hence, } A^{-1} = \begin{bmatrix} \frac{-3}{11} & \frac{4}{11} & \frac{5}{11} \\ \frac{9}{11} & \frac{-1}{11} & \frac{-4}{11} \\ \frac{5}{11} & \frac{-3}{11} & \frac{-1}{11} \end{bmatrix}.$$

16. If $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$.

Verify that $A^3 - 6A^2 + 9A - 4I = O$ and hence, find A^{-1}.

[**Solution :** Refer Q. 19; Ex. 4(*g*)]

17. Let A be a non-singular square matrix of order 3×3. Then $|adj\,A|$ is equal to :

(A) $|A|$ (B) $|A|^2$ (C) $|A|^3$ (D) $3\,|A|$. [**Ans.** (B)]

18. If A is an invertible matrix of order 2, then det (A^{-1}) is equal to :

(A) det (A) (B) $\frac{1}{\det(A)}$ (C) 1 (D) 0. [**Ans.** (B)]

Exercise 4.6

Examine the consistency of the system of equations in Exercise 1 to 6 :

1. $x + 2y = 2$, $2x + 3y = 3$. **2.** $2x - y = 5$, $x + y = 4$. **3.** $x + 3y = 5$, $2x + 6y = 8$.

[**Solutions :** (1-3) : Refer Q. 1; Ex. 4(*h*)]

4. $x + y + z = 1$
$2x + 3y + 2z = 2$
$ax + ay + 2az = 4$.

3. $3x - y + 2z = 2$
$2y - z = -1$
$3x - 5y = 3$.

6. $5x - y - 4z = 5$
$2x + 3y + 5z = 2$
$5x - 2y + 6z = -1$.

[**Solutions :** (4-6) : Refer Q. 4; Ex. 4(*h*)]

Solve system of linear equations, using matrix method, in Exercises 7 to 14 :

7. $5x + 2y = 4$, $7x + 3y = 5$. **8.** $2x - y = -2$, $3x + 4y = 3$. **9.** $4x - 3y = 3$, $3x - 5y = 7$.

10. $5x + 2y = 3$
$3x + 2y = 5$.

[**Solution :** (7-10) : Refer Q. 2; Ex. 4(*h*)]

11. $2x + y + z = 1$
$x - 2y - z = \frac{3}{2}$
$3y - 5z = 9$.

[**Solution :** Refer Q. 5(*xi*); Ex. 4(*h*)]

12. $x - y + z = 4$
$2x + y - 3z = 0$
$x + y + z = 2$.

[**Solution :** Refer Q. 6(*iv*); Ex. 4(*h*)]

13. $2x + 3y + 3z = 5$
$x - 2y + z = -4$
$3x - y - 2z = 3$.

[**Solution :** Refer Q. 5(*ix*); Ex. 4(*h*)]

14. $x - y + 2z = 7$
$3x + 4y - 5z = -5$
$2x - y + 3z = 12$.

[**Solution :** Refer Q. 6(*i*); Ex. 4(*h*)]

15. If $A = \begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}$, find A^{-1}.

Using A^{-1} solve the system of equations :

$2x - 3y + 5z = 11$

$3x + 2y - 4z = -5$

$x + y - 2z = -3.$

[**Solution :** Refer Q. 8(*ii*); Ex. 4(*h*)]

16. The cost of 4 kg onion, 3 kg wheat and 2 kg rice is ₹ 60. The cost of 2 kg onion, 4 kg wheat and 6 kg rice is ₹ 90. The cost of 6 kg onion, 2 kg wheat and 3 kg rice is ₹ 70. Find cost of each item per kg by matrix method.

[**Solution :** Refer Q. 17; Ex. 4(*h*)]

Miscellaneous Exercise on Chapter 4

1. Prove that the determinant $\begin{vmatrix} x & \sin\theta & \cos\theta \\ -\sin\theta & -x & 1 \\ \cos\theta & 1 & x \end{vmatrix}$ is independent of θ.

[**Solution :** Refer Q. 34; Ex. 4(*b*)]

2. Without expanding the determinant, prove that

$$\begin{vmatrix} a & a^2 & bc \\ b & b^2 & ca \\ c & c^2 & ab \end{vmatrix} = \begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix}.$$

[**Solution :** Refer Q. 13; Ex. 4(*b*)]

3. Evaluate $\begin{vmatrix} \cos\alpha\cos\beta & \cos\alpha\sin\beta & -\sin\alpha \\ -\sin\beta & \cos\beta & 0 \\ \sin\alpha\cos\beta & \sin\alpha\sin\beta & \cos\alpha \end{vmatrix}$.

[**Solution :** Refer Q. 11; Rev. Ex.]

4. If a, b and c are real numbers, and

$$\Delta = \begin{vmatrix} b+c & c+a & a+b \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix} = 0,$$

show that either $a + b + c = 0$ or $a = b = c$.

Solution : $\begin{vmatrix} b+c & c+a & a+b \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix} = 0$

$$\Rightarrow \begin{vmatrix} 2(a+b+c) & c+a & a+b \\ 2(a+b+c) & a+b & b+c \\ 2(a+b+c) & b+c & c+a \end{vmatrix} = 0$$

[*Operating* $C_1 \to C_1 + C_2 + C_3$]

$$\Rightarrow 2(a+b+c) \begin{vmatrix} 1 & c+a & a+b \\ 1 & a+b & b+c \\ 1 & b+c & c+a \end{vmatrix} = 0$$

[*Taking* $2(a + b + c)$ *common from* C_1]

$$\Rightarrow 2(a+b+c) \begin{vmatrix} 1 & c+a & a+b \\ 0 & b-c & c-a \\ 0 & c-a & a-b \end{vmatrix} = 0$$

[*Operating* $R_2 \to R_2 - R_1$ & $R_3 \to R_3 - R_2$]

$$\Rightarrow 2(a+b+c)(1) \begin{vmatrix} b-c & c-a \\ c-a & a-b \end{vmatrix} = 0$$

[*Expanding by* C_1]

$\Rightarrow 2(a + b + c)[(b - c)(a - b) - (c - a)^2] = 0$

$\Rightarrow 2(a + b + c)(ab - b^2 - ca + bc - c^2 - a^2 + 2ca) = 0$

$\Rightarrow -2(a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca) = 0$

$\Rightarrow -(a + b + c)[2a^2 + 2b^2 + 2c^2 - 2ab - 2bc - 2ca] = 0$

$\Rightarrow -(a + b + c)[(a^2 + b^2 - 2ab) + (b^2 + c^2 - 2bc) + (c^2 + a^2 - 2ca)] = 0$

$\Rightarrow -(a + b + c)[(a - b)^2 + (b - c)^2 + (c - a)^2] = 0$

$\Rightarrow$ either $a + b + c = 0$

or $[(a - b)^2 + (b - c)^2 + (c - a)^2] = 0$

i.e. $a - b = 0$, $b - c = 0$, $c - a = 0$

i.e. $a = b$, $b = c$, $c = a$

i.e. $a = b = c$,

which is true.

5. Solve the equation $\begin{vmatrix} x+a & x & x \\ x & x+a & x \\ x & x & x+a \end{vmatrix} = 0, a \neq 0$.

Solution : We have : $\begin{vmatrix} x+a & x & x \\ x & x+a & x \\ x & x & x+a \end{vmatrix} = 0$

$$\Rightarrow \begin{vmatrix} 3x+a & x & x \\ 3x+a & x+a & x \\ 3x+a & x & x+a \end{vmatrix} = 0$$

[*Operating* $C_1 \to C_1 + C_2 + C_3$]

$$\Rightarrow (3x+a)\begin{vmatrix} 1 & x & x \\ 1 & x+a & x \\ 1 & x & x+a \end{vmatrix} = 0$$

[*Taking* $(3x + a)$ *common from* C_1]

$$\Rightarrow (3x+a)\begin{vmatrix} 1 & x & x \\ 0 & a & 0 \\ 0 & 0 & a \end{vmatrix} = 0$$

[*Operating* $R_2 \to R_2 - R_1$ & $R_3 \to R_3 - R_1$]

$$\Rightarrow (3x + a)\,(1)\begin{vmatrix} a & 0 \\ 0 & a \end{vmatrix} = 0 \qquad [\textit{Expanding by } C_1]$$

$$\Rightarrow \quad (3x + a)\,(a^2 - 0) = 0$$

$$\Rightarrow \quad 3x + a = 0. \qquad [\because a^2 \neq 0]$$

Hence, $\quad x = -\dfrac{a}{3}.$

6. Prove that $\begin{vmatrix} a^2 & bc & ac+c^2 \\ a^2+ab & b^2 & ac \\ ab & b^2+bc & c^2 \end{vmatrix} = 4a^2b^2c^2$.

Solution : $\begin{vmatrix} a^2 & bc & ac+c^2 \\ a^2+ab & b^2 & ac \\ ab & b^2+bc & c^2 \end{vmatrix}$

$$= abc\begin{vmatrix} a & c & a+c \\ a+b & b & a \\ b & b+c & c \end{vmatrix}$$

[*Taking a, b, c common from* C_1, C_2, C_3 *respectively*]

$$= abc\begin{vmatrix} 2(a+c) & c & a+c \\ 2(a+b) & b & a \\ 2(b+c) & b+c & c \end{vmatrix}$$

[*Operating* $C_1 \to C_1 + C_2 + C_3$]

$$= 2abc\begin{vmatrix} a+c & c & a+c \\ a+b & b & a \\ b+c & b+c & c \end{vmatrix}$$

[*Taking 2 common from* C_1]

$$= 2abc\begin{vmatrix} a+c & -a & 0 \\ a+b & -a & -b \\ b+c & 0 & -b \end{vmatrix}$$

[*Operating* $C_2 \to C_2 - C_1$ *and* $C_3 \to C_3 - C_1$]

$$= 2a^2b^2c\begin{vmatrix} a+c & 1 & 0 \\ a+b & 1 & 1 \\ b+c & 0 & 1 \end{vmatrix}$$

[*Taking* $(-a)$ *common from* C_2 *and* $(-b)$ *common from* C_3]

$$= 2a^2b^2c\,[(-1)(-b-c) + (1)(a+c-a-b)]$$

[*Expanding by* C_3]

$$= 2a^2b^2c\,[b+c+a+c-a-b]$$

$$= 2a^2b^2c\,(2c) = 4\,a^2b^2c^2, \text{ which is true.}$$

7. If $A^{-1} = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$

and $B = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$, find $(AB)^{-1}$.

[**Solution :** Refer Q. 20; Ex. 4(*g*)]

8. Let $A = \begin{bmatrix} 1 & -2 & 1 \\ -2 & 3 & 1 \\ 1 & 1 & 5 \end{bmatrix}$. Verify that :

(*i*) $[adj\ A]^{-1} = adj\ (A)^{-1}$ (*ii*) $(A^{-1})^{-1} = A$.

[**Solution :** Refer Q. 21; Ex. 4(*g*)]

9. Evaluate $\begin{vmatrix} x & y & x+y \\ y & x+y & x \\ x+y & x & y \end{vmatrix}$.

[**Solution :** Refer Q. 16(*iv*); Ex. 4(*b*)]

10. Evaluate $\begin{vmatrix} 1 & x & y \\ 1 & x+y & y \\ 1 & x & x+y \end{vmatrix}$.

[**Solution :** Refer Q. 16(*v*); Ex. 4(*b*)]

Using properties of determinants in Exercises 11 to 15, prove that :

11. $\begin{vmatrix} \alpha & \alpha^2 & \beta+\gamma \\ \beta & \beta^2 & \gamma+\alpha \\ \gamma & \gamma^2 & \alpha+\beta \end{vmatrix} = (\beta-\gamma)(\gamma-\alpha)(\alpha-\beta)(\alpha+\beta+\gamma)$.

[**Solution :** Refer Q. 22(*iv*); Ex. 4(*b*)]

12. $\begin{vmatrix} x & x^2 & 1+px^3 \\ y & y^2 & 1+py^3 \\ z & z^2 & 1+pz^3 \end{vmatrix} = (1+pxyz)(x-y)(y-z)(z-x)$.

[**Solution :** Refer Q. 29; Ex. 4(*b*)]

13. $\begin{vmatrix} 3a & -a+b & -a+c \\ -b+a & 3b & -b+c \\ -c+a & c+b & 3c \end{vmatrix}$

$$= 3(a+b+c)(ab+bc+ca).$$

Solution : $\begin{vmatrix} 3a & -a+b & -a+c \\ -b+a & 3b & -b+c \\ -c+a & -c+b & 3c \end{vmatrix}$

$$= \begin{vmatrix} a+b+c & -a+b & -a+c \\ a+b+c & 3b & -b+c \\ a+b+c & -c+b & 3c \end{vmatrix}$$

[*Operating* $C_1 \longrightarrow C_1 + C_2 + C_3$]

$$= (a+b+c) \begin{vmatrix} 1 & -a+b & -a+c \\ 1 & 3b & -b+c \\ 1 & -c+b & 3c \end{vmatrix}$$

$$= (a+b+c) \begin{vmatrix} 1 & -a+b & -a+c \\ 0 & 2b+a & -b+a \\ 0 & -c+a & 2c+a \end{vmatrix}$$

[*Operating* $R_2 \longrightarrow R_2 - R_1$ *and* $R_3 \longrightarrow R_3 - R_1$]

$= (a + b + c)\ [(2b + a)\ (2c + a) - (a - b)\ (a - c)]$

$= (a + b + c)\ [4bc + 2ab + 2ca + a^2 - a^2 + ac + ab - bc]$

$= (a + b + c)\ [3bc + 3ca + 3ab]$

$= 3\ (a + b + c)\ (ab + bc + ca)$, which is true.

14. $\begin{vmatrix} 1 & 1+p & 1+p+q \\ 2 & 3+2p & 4+3p+2q \\ 3 & 6+3p & 10+6p+3q \end{vmatrix} = 1.$

Solution : Operating $R_2 \to R_2 - 2R_1$ and $R_3 \to R_3 - 3R_1$, we get :

$$\Delta = \begin{vmatrix} 1 & 1+p & 1+p+q \\ 0 & 1 & 2+p \\ 0 & 3 & 7+3p \end{vmatrix}$$

$$= (1) \begin{vmatrix} 1 & 2+p \\ 3 & 7+3p \end{vmatrix}$$

[*Expanding by* C_1]

$= (7 + 3p) - (6 + 3p) = 7 - 6 = 1.$

15. $\begin{vmatrix} \sin\alpha & \cos\alpha & \cos(\alpha+\delta) \\ \sin\beta & \cos\beta & \cos(\beta+\delta) \\ \sin\gamma & \cos\gamma & \cos(\gamma+\delta) \end{vmatrix} = 0.$

Solution : $\begin{vmatrix} \sin\alpha & \cos\alpha & \cos(\alpha+\delta) \\ \sin\beta & \cos\beta & \cos(\beta+\delta) \\ \sin\gamma & \cos\gamma & \cos(\gamma+\delta) \end{vmatrix}$

$$= \begin{vmatrix} \sin\alpha & \cos a & 0 \\ \sin\beta & \cos\beta & 0 \\ \sin\gamma & \cos\gamma & 0 \end{vmatrix}$$

[*Operating* $C_3 \longrightarrow C_3 + \sin\delta C_1 - \cos\delta\, C_2$]

$= 0.$

16. Solve the system of the following equations :

$$\frac{2}{x} + \frac{3}{y} + \frac{10}{z} = 4$$

$$\frac{4}{x} - \frac{6}{y} + \frac{5}{z} = 1$$

$$\frac{6}{x} + \frac{9}{y} - \frac{20}{z} = 2.$$

[**Solution :** Refer Q. 7(*i*); Ex. 4(*h*)]

Choose the correct answer in Exercises 17 to 19 :

17. If *a, b, c* are in A.P., then the determinant

$\begin{vmatrix} x+2 & x+3 & x+2a \\ x+3 & x+4 & x+2b \\ x+4 & x+5 & x+2c \end{vmatrix}$ is :

(A) 0 (B) 1 (C) x (D) $2x$.

[**Ans.** (A)]

18. If *x, y, z* are non-zero real numbers, then the inverse of matrix $A = \begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix}$ is :

(A) $\begin{bmatrix} x^{-1} & 0 & 0 \\ 0 & y^{-1} & 0 \\ 0 & 0 & z^{-1} \end{bmatrix}$ (B) $xyz \begin{bmatrix} x^{-1} & 0 & 0 \\ 0 & y^{-1} & 0 \\ 0 & 0 & z^{-1} \end{bmatrix}$

(C) $\frac{1}{xyz} \begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix}$ (D) $\frac{1}{xyz} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

[**Ans.** (A)]

19. Let $A = \begin{bmatrix} 1 & \sin\theta & 1 \\ -\sin\theta & 1 & \sin\theta \\ -1 & -\sin\theta & 1 \end{bmatrix}$,

where $0 \le \theta \le 2\pi$. Then :

(A) Det (A) = 0 (B) Det (A) $\in (2, \infty)$

(C) Det (A) $\in (2, 4)$ (D) Det (A) $\in [2, 4]$.

[**Ans.** (D)]

Questions From NCERT Exemplar

Example 1. If $\Delta = \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$, $\Delta_1 = \begin{vmatrix} 1 & 1 & 1 \\ yz & zx & xy \\ x & y & z \end{vmatrix}$,

then prove that $\Delta + \Delta_1 = 0$.

Solution. $\Delta_1 = \begin{vmatrix} 1 & 1 & 1 \\ yz & zx & xy \\ x & y & z \end{vmatrix}$

$= \begin{vmatrix} 1 & yz & x \\ 1 & zx & y \\ 1 & xy & z \end{vmatrix}$ [*Inter-changing rows and columns*]

$= \dfrac{1}{xyz}\begin{vmatrix} x & xyz & x^2 \\ y & xyz & y^2 \\ z & xyz & z^2 \end{vmatrix}$

[*Multiplying* R_1 *by x,* R_2 *by y and* R_3 *by z and taking* $\dfrac{1}{xyz}$ *outside*]

$= \dfrac{xyz}{xyz}\begin{vmatrix} x & 1 & x^2 \\ y & 1 & y^2 \\ z & 1 & z^2 \end{vmatrix}$ [*Taking xyz common from* C_2]

$= (-1)\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$ [*Operating* $C_1 \leftrightarrow C_2$]

$= -\Delta.$

Hence, $\Delta + \Delta_1 = 0$.

Example 2. Without expanding, show that :

$$\Delta = \begin{vmatrix} \operatorname{cosec}^2\theta & \cot^2\theta & 1 \\ \cot^2\theta & \operatorname{cosec}^2\theta & -1 \\ 42 & 40 & 2 \end{vmatrix} = 0.$$

Solution. Operating $C_1 \to C_1 - C_2 - C_3$, we get :

$$\Delta = \begin{vmatrix} \operatorname{cosec}^2\theta - \cot^2\theta - 1 & \cot^2\theta & 1 \\ \cot^2\theta - \operatorname{cosec}^2\theta + 1 & \operatorname{cosec}^2\theta & -1 \\ 42 - 40 - 2 & 40 & 2 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & \cot^2\theta & 1 \\ 0 & \operatorname{cosec}^2\theta & -1 \\ 0 & 40 & 2 \end{vmatrix} = 0.$$

Example 3. In a triangle ABC, if :

$$\begin{vmatrix} 1 & 1 & 1 \\ 1+\sin A & 1+\sin B & 1+\sin C \\ \sin A + \sin^2 A & \sin B + \sin^2 B & \sin C + \sin^2 C \end{vmatrix} = 0,$$

then prove that ΔABC is an isosceles triangle.

Solution. Let

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1+\sin A & 1+\sin B & 1+\sin C \\ \sin A + \sin^2 A & \sin B + \sin^2 B & \sin C + \sin^2 C \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ 1+\sin A & 1+\sin B & 1+\sin C \\ -\cos^2 A & -\cos^2 B & -\cos^2 C \end{vmatrix}$$

[*Operating* $R_3 \to R_3 - R_2$]

$$= \begin{vmatrix} 1 & 0 & 0 \\ 1+\sin A & \sin B - \sin A & \sin C - \sin B \\ -\cos^2 A & \cos^2 A - \cos^2 B & \cos^2 B - \cos^2 C \end{vmatrix}$$

[*Operating* $C_2 \to C_2 - C_1$ & $C_3 \to C_3 - C_2$]

$= (1)(\sin B - \sin A)(\cos^2 B - \cos^2 C)$
$\quad - (\cos^2 A - \cos^2 B)(\sin C - \sin B)$

[*Expanding by* R_1]

$= (\sin B - \sin A)(1 - \sin^2 B - 1 + \sin^2 C)$
$\quad - (1 - \sin^2 A - 1 + \sin^2 B)(\sin C - \sin B)$

$= (\sin B - \sin A)(\sin^2 C - \sin^2 B)$
$\quad - (\sin C - \sin B)(\sin^2 B - \sin^2 A)$

$= (\sin B - \sin A)(\sin C - \sin B)$
$\quad (\sin C + \sin B - \sin B - \sin A)$

$= (\sin B - \sin A)(\sin C - \sin B)(\sin C - \sin A).$

Now $\Delta = 0 \Rightarrow \sin B - \sin A = 0$

or $\sin C - \sin B = 0$ or $\sin C - \sin A = 0$

$\Rightarrow$ A = B or B = C or C = A

$\Rightarrow$ Triangle ABC is isosceles.

Example 4. Show that if the determinant :

$$\Delta = \begin{vmatrix} 3 & -2 & \sin 3\theta \\ -7 & 8 & \cos 2\theta \\ -11 & 14 & 2 \end{vmatrix} = 0, \text{ then } \sin\theta = 0 \text{ or } \frac{1}{2}.$$

Solution : We have : $\begin{vmatrix} 3 & -2 & \sin 3\theta \\ -7 & 8 & \cos 2\theta \\ -11 & 14 & 2 \end{vmatrix}$

$\Rightarrow \begin{vmatrix} 3 & -2 & \sin 3\theta \\ 5 & 0 & \cos 2\theta + 4\sin 3\theta \\ 10 & 0 & 2 + 7\sin 3\theta \end{vmatrix} = 0$

[*Operating* $R_2 \to R_2 + 4R_1$ *and* $R_3 \to R_3 + 7R_1$]

$\Rightarrow 2[5(2 + 7\sin 3\theta) - 10(\cos 2\theta + 4\sin 3\theta)] = 0$

[*Expanding by* C_2]

$\Rightarrow 10[(2 + 7 \sin 3\theta) - (2 \cos 2\theta + 8 \sin 3\theta)] = 0$

$\Rightarrow 2 + 7 \sin 3\theta - 2 \cos 2\theta - 8 \sin 3\theta = 0$

$\Rightarrow 2 - 2 \cos 2\theta - \sin 3\theta = 0$

$\Rightarrow 2 - 2(1 - 2 \sin^2\theta) - (3 \sin\theta - 4 \sin^3\theta) = 0$

$\Rightarrow 4 \sin^3\theta - 3 \sin\theta + 4 \sin^2\theta = 0$

$\Rightarrow \sin\theta (4 \sin^2\theta + 4 \sin\theta - 3) = 0$

$\Rightarrow \sin\theta (2 \sin\theta - 1)(2 \sin\theta + 3) = 0$

$\Rightarrow \sin\theta = 0$ or $\sin\theta = \frac{1}{2}$, which is true.

$\left[\because \sin\theta = -\frac{3}{2} \text{ is not true}\right]$

Exercise

1. Show that $\Delta = \begin{vmatrix} x & p & q \\ p & x & q \\ q & q & x \end{vmatrix} = (x - p)(x^2 + px - 2q^2)$.

2. If $\Delta = \begin{vmatrix} 0 & b-a & c-a \\ a-b & 0 & c-b \\ a-c & b-c & 0 \end{vmatrix}$, then show that Δ is equal to zero.

3. If $x = -4$ is a root of $\Delta = \begin{vmatrix} x & 2 & 3 \\ 1 & x & 1 \\ 3 & 2 & 2 \end{vmatrix} = 0$, then find the other two roots.

4. Show that the points $(a + 5, a - 4)$, $(a - 1, a + 3)$ and (a, a) do not lie on a straight line for any value of a.

Answers

3. 1 and 3.

Revision Exercise

1. For the matrix $A = \begin{bmatrix} 3 & 1 \\ 7 & 5 \end{bmatrix}$, find '$x$' and '$y$' so that $A^2 + xI = yA$. Hence find A^{-1}.

2. Without expanding, show that the following determinants vanish :

(*i*) $\begin{vmatrix} 1 & \cos(\beta-\alpha) & \cos(\gamma-\alpha) \\ \cos(\alpha-\beta) & 1 & \cos(\gamma-\beta) \\ \cos(\alpha-\gamma) & \cos(\beta-\gamma) & 1 \end{vmatrix}$

(*ii*) $\begin{vmatrix} 2b_1c_1 & b_1c_2+b_2c_1 & b_1c_3+b_3c_1 \\ b_1c_2+b_2c_1 & 2b_2c_2 & b_2c_3+b_3c_2 \\ b_1c_3+b_3c_1 & b_2c_3+b_3c_2 & 2b_3c_3 \end{vmatrix}$.

Prove the following (3 – 8) :

3. $\begin{vmatrix} a+b+nc & na-a & nb-b \\ nc-c & b+c+na & nb-b \\ nc-c & na-a & c+a+nb \end{vmatrix} = n(a + b + c)^3$.

4. $\begin{vmatrix} \frac{a^2+b^2}{c} & c & c \\ a & \frac{b^2+c^2}{a} & a \\ b & b & \frac{c^2+a^2}{b} \end{vmatrix} = 4abc$.

5. $\begin{vmatrix} a^2 & a^2-(b-c)^2 & bc \\ b^2 & b^2-(c-a)^2 & ca \\ c^2 & c^2-(a-b)^2 & ab \end{vmatrix}$

$= (a - b)(b - c)(c - a)(a + b + c)(a^2 + b^2 + c^2)$.

6. $\begin{vmatrix} -a(b^2+c^2-a^2) & 2b^3 & 2c^3 \\ 2a^3 & -b(c^2+a^2-b^2) & 2c^3 \\ 2a^3 & 2b^3 & -c(a^2+b^2-c^2) \end{vmatrix} = abc\,(a^2+b^2+c^2)^3$.

7. $\begin{vmatrix} 1 & bc+ad & b^2c^2+a^2d^2 \\ 1 & ca+bd & c^2a^2+b^2d^2 \\ 1 & ab+cd & a^2b^2+c^2d^2 \end{vmatrix}$

$= (a - b)(a - c)(a - d)(b - c)(b - d)(c - d)$.

8. $\begin{vmatrix} (a+1)(a+2) & a+2 & 1 \\ (a+2)(a+3) & a+3 & 1 \\ (a+3)(a+4) & a+4 & 1 \end{vmatrix} = -2$.

9. Evaluate : $\begin{vmatrix} {}^xC_1 & {}^xC_2 & {}^xC_3 \\ {}^yC_1 & {}^yC_2 & {}^yC_3 \\ {}^zC_1 & {}^zC_2 & {}^zC_3 \end{vmatrix}$. *(N.C.E.R.T.)*

10. Prove that :

$$\Delta = \begin{vmatrix} a+bx & c+dx & p+qx \\ ax+b & cx+d & px+q \\ u & v & w \end{vmatrix} = (1-x^2)\begin{vmatrix} a & c & p \\ b & d & q \\ u & v & w \end{vmatrix}.$$

(N.C.E.R.T.)

11. Evaluate : $\begin{vmatrix} \cos\alpha\cos\beta & \cos\alpha\sin\beta & -\sin\alpha \\ -\sin\beta & \cos\beta & 0 \\ \sin\alpha\cos\beta & \sin\alpha\sin\beta & \cos\alpha \end{vmatrix}$.

(N.C.E.R.T.)

12. If a, b, c are different, then the determinant :

$$\begin{vmatrix} 1 & 1 & 1 \\ (x-a)^2 & (x-b)^2 & (x-c)^2 \\ (x-b)(x-c) & (x-c)(x-a) & (x-a)(x-b) \end{vmatrix}$$

vanishes when $x = \frac{1}{3}(a+b+c)$.

13. If $\begin{vmatrix} x^2-a^2 & x^2-b^2 & x^2-c^2 \\ (x-a)^3 & (x-b)^3 & (x-c)^3 \\ (x+a)^3 & (x+b)^3 & (x+c)^3 \end{vmatrix} = 0$, and a, b, c are all distinct, then prove that $x = 0, \pm\sqrt{\frac{1}{3}(\Sigma\, a\, b)}$.

14. If a, b, c are in A.P., find the value of :

$$\begin{vmatrix} \mathbf{2y+4} & \mathbf{5y+7} & \mathbf{8y+a} \\ \mathbf{3y+5} & \mathbf{6y+8} & \mathbf{9y+b} \\ \mathbf{4y+6} & \mathbf{7y+9} & \mathbf{10y+c} \end{vmatrix}.$$

(N.C.E.R.T.; Jammu B. 2017)

Solution. Operating $R_1 \to R_1 + R_3 - 2R_2$, we get :

$$\Delta = \begin{vmatrix} 0 & 0 & 0 \\ 3y+5 & 6y+8 & 9y+b \\ 4y+6 & 79y+9 & 10y+c \end{vmatrix} = 0.$$

[$\because a, b, c$ *are in A.P.* $\Rightarrow 2b = a + c$]

15. If $ax^2 + 2\,hxy + by^2 + 2\,gx + 2\,fy + c$ $\equiv (lx + my + n)(l'x + m'y + n')$, then prove that :

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0.$$

16. If $a + b + c = 0$ and $\begin{vmatrix} a-x & c & b \\ c & b-x & a \\ b & a & c-x \end{vmatrix} = 0$, then show that :

$$x = 0 \text{ or } x = \sqrt{\left(\frac{3}{2}\right)(a^2+b^2+c^2)}.$$

17. If A + B + C = π, find the value of :

$$\begin{vmatrix} \mathbf{\sin(A+B+C)} & \mathbf{\sin B} & \mathbf{\cos C} \\ \mathbf{-\sin B} & \mathbf{0} & \mathbf{\tan A} \\ \mathbf{\cos(A+B)} & \mathbf{-\tan A} & \mathbf{0} \end{vmatrix}.$$

Solution. Here given det.

$$= \begin{vmatrix} 0 & \sin B & \cos C \\ -\sin B & 0 & \tan A \\ \cos(A+B) & -\tan A & 0 \end{vmatrix}$$

[$\because \sin(A+B+C) = \sin\pi = 0$]

$$= \begin{vmatrix} 0 & \sin B & \cos C \\ -\sin B & 0 & \tan A \\ -\cos C & -\tan A & 0 \end{vmatrix},$$

[$\because \cos(A+B) = \cos(\pi - C) = -\cos C$]

which vanishes as it is skew–symmetric matrix and of an odd order.

18. Prove that :

$$\begin{vmatrix} x & x^2 & 1+px^3 \\ y & y^2 & 1+py^3 \\ z & z^2 & 1+pz^3 \end{vmatrix}$$

$$= (1 + pxyz)(x-y)(y-z)(z-x),$$

where p is a scalar.

19. If $A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$, then $A^3 = A^{-1}$. What is adj. A ?

20. If $A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$, show that $A^4 = I$, Hence find A^{-1}.

21. If $A = \begin{bmatrix} 1 & \tan x \\ -\tan x & 1 \end{bmatrix}$, show that :

$$A'A^{-1} = \begin{bmatrix} \cos 2x & -\sin 2x \\ \sin 2x & \cos 2x \end{bmatrix}.$$

22. If $A = \begin{bmatrix} 2 & -3 \\ 4 & 6 \end{bmatrix}$, verify that :

$$(\text{adj. } A)^{-1} = (\text{adj. } A^{-1}).$$

23. Prove that :

(*i*) adj. $I_n = I_n$ (*ii*) adj. $O = O$

(*iii*) $I_n^{-1} = I_n$.

24. Let D = diag. $[d_1, d_2, d_3]$, where none of d_1, d_2, d_3 is 0 ; prove that :

D^{-1} = diag. $[d_1^{-1}, d_2^{-1}, d_3^{-1}]$.

25. Let $F(\alpha) = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$

and $G(\beta) = \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix}$. Show that

$[F(\alpha).G(\beta)]^{-1} = G(-\beta) . F(-\alpha)$.

26. Obtain the inverses of the matrices $\begin{bmatrix} 1 & p & 0 \\ 0 & 1 & p \\ 0 & 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 & 0 \\ q & 1 & 0 \\ 0 & q & 1 \end{bmatrix}$. And, hence find the inverse of the matrix $\begin{bmatrix} 1+pq & p & 0 \\ q & 1+pq & p \\ 0 & q & 1 \end{bmatrix}$.

27. Use product $\begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix}\begin{bmatrix} -2 & 0 & 1 \\ 9 & 2 & -3 \\ 6 & 1 & -2 \end{bmatrix}$ **to solve the equations :**

$x - y + 2z = 1, 2y - 3z = 1, 3x - 2y + 4z = 2.$

(N.C.E.R.T.)

Solution. (*i*) Consider the product :

$$\begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix}\begin{bmatrix} -2 & 0 & 1 \\ 9 & 2 & -3 \\ 6 & 1 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} -2-9+12 & 0-2+2 & 1+3-4 \\ -0+18-18 & 0+4-3 & 0-6+6 \\ -6-18+24 & 0-4+4 & 3+6-8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Hence, $\begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix}^{-1} = \begin{bmatrix} -2 & 0 & 1 \\ 9 & 2 & -3 \\ 6 & 1 & -2 \end{bmatrix}$.

(*ii*) Given equations can be written as :

$$\begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

i.e. AX = B

$\Rightarrow \quad A^{-1}(AX) = A^{-1}B \quad \Rightarrow \quad (A^{-1}A)X = A^{-1}B$

$\Rightarrow \quad IX = A^{-1}B \quad \Rightarrow \quad X = A^{-1}B$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 & 0 & 1 \\ 9 & 2 & -3 \\ 6 & 1 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} -2+0+2 \\ 9+2-6 \\ 6+1-4 \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \\ 3 \end{bmatrix}.$$

Hence, $x = 0, y = 5, z = 3$.

28. If $a \neq p$, $b \neq q$, $c \neq r$ and $\begin{vmatrix} p & b & c \\ a & q & c \\ a & b & r \end{vmatrix} = 0$, then find the value of $\dfrac{p}{p-a} + \dfrac{q}{q-b} + \dfrac{r}{r-c}$.

Solution. $\begin{vmatrix} p & b & c \\ a & q & c \\ a & b & r \end{vmatrix} = 0$

$$\Rightarrow \begin{vmatrix} p-a & b-q & 0 \\ 0 & q-b & c-r \\ a & b & r \end{vmatrix} = 0$$

[*Operating* $R_1 \to R_1 - R_2$, $R_2 \to R_2 - R_3$]

$\Rightarrow (p-a)(q-b)r + (p-a)(r-c)b + a(q-b)(r-c) = 0$ [*Expanding by* C_1]

$$\Rightarrow \frac{r}{r-c} + \frac{b}{q-b} + \frac{a}{p-a} = 0$$

[*Dividing by* $(p-a)(q-b)(r-c)$]

$$\Rightarrow \frac{r}{r-c} + \frac{b-q+q}{q-b} + \frac{a-p+p}{p-a} = 0 \quad \textbf{(\textit{Note this step})}$$

$$\Rightarrow \frac{r}{r-c} - 1 + \frac{q}{q-b} - 1 + \frac{p}{p-a} = 0.$$

Hence, $\dfrac{p}{p-a} + \dfrac{q}{q-b} + \dfrac{r}{r-c} = 2$.

29. Let the three-digit numbers A28, 3B9 and 62C, where A, B and C are integers between 0 and 9, be divisible by fixed integer k. Show that the determinant $\begin{vmatrix} A & 3 & 6 \\ 8 & 9 & C \\ 2 & B & 2 \end{vmatrix}$ is divisible by k.

Solution. Since A28, 3B9 and 62C are divisible by *k*,

$\therefore \quad A28 = 100A + 20 + 8 = lk$ ...(1)

$3B9 = 300 + 10B + 9 = mk$ ...(2)

and $62C = 600 + 20 + C = nk$ ...(3),

where $l, m, n \in \mathbf{I}$.

Now $\Delta = \begin{vmatrix} A & 3 & 6 \\ 8 & 9 & C \\ 2 & B & 2 \end{vmatrix}$

$$= \begin{vmatrix} A & 3 & 6 \\ 100A + 28 & 309 + 10B & 620 + C \\ 2 & B & 2 \end{vmatrix}$$

[*Operating* $R_2 \to R_2 + 100R_1 + 10R_3$]

$$= \begin{vmatrix} A & 3 & 6 \\ lk & mk & nk \\ 2 & B & 2 \end{vmatrix}$$ [*Using (1), (2) and (3)*]

$$= k \begin{vmatrix} A & 3 & 6 \\ l & m & n \\ 2 & B & 2 \end{vmatrix}$$, which is divisible by k.

30. Let a > 0, d > 0. Find the value of the determinant :

$$\begin{vmatrix} \frac{1}{a} & \frac{1}{a(a+d)} & \frac{1}{(a+d)(a+2d)} \\ \frac{1}{a+d} & \frac{1}{(a+d)(a+2d)} & \frac{1}{(a+2d)(a+3d)} \\ \frac{1}{a+2d} & \frac{1}{(a+2d)(a+3d)} & \frac{1}{(a+3d)(a+4d)} \end{vmatrix}.$$

Solution. $$\Delta = \begin{vmatrix} \frac{1}{a} & \frac{1}{a(a+d)} & \frac{1}{(a+d)(a+2d)} \\ \frac{1}{a+d} & \frac{1}{(a+d)(a+2d)} & \frac{1}{(a+2d)(a+3d)} \\ \frac{1}{a+2d} & \frac{1}{(a+2d)(a+3d)} & \frac{1}{(a+3d)(a+4d)} \end{vmatrix}$$

$$= \frac{1}{a(a+d)^2(a+2d)^3(a+3d)^2(a+4d)}$$

$$\begin{vmatrix} (a+d)(a+2d) & a+2d & a \\ (a+2d)(a+3d) & a+3d & a+d \\ (a+3d)(a+4d) & a+4d & a+2d \end{vmatrix}$$

$$= \frac{1}{a(a+d)^2(a+2d)^3(a+3d)^2(a+4d)}$$

$$\begin{vmatrix} (a+d)(a+2d) & 2d & a \\ (a+2d)(a+3d) & 2d & a+d \\ (a+3d)(a+4d) & 2d & a+2d \end{vmatrix}$$

[*Operating* $C_2 \to C_2 - C_3$]

$$= \frac{1}{a(a+d)^2(a+2d)^3(a+3d)^2(a+4d)}$$

$$\begin{vmatrix} (a+d)(a+2d) & 2d & a \\ 2d(a+2d) & 0 & d \\ 2d(a+3d) & 0 & d \end{vmatrix}$$

[*Operating* $R_2 \to R_2 - R_1$ & $R_3 \to R_3 - R_2$]

$$= \frac{1}{a(a+d)^2(a+2d)^3(a+3d)^2(a+4d)}.$$

$$(-2d)\,[2d^2(a+2d) - 2d^2(a+3d)]$$

[*Expanding by* C_2]

$$= \frac{-2d}{a(a+d)^2(a+2d)^3(a+3d)^2(a+4d)}\,[-2d]^3$$

$$= \frac{4d^4}{a(a+d)^2(a+2d)^3(a+3d)^2(a+4d)}.$$

Answers

1. $x = 8, y = 8$; $A^{-1} = \begin{pmatrix} 5/8 & -1/8 \\ -7/8 & 3/8 \end{pmatrix}$.

9. $\frac{1}{12}\,xyz\,(x-y)(y-z)(z-x)$.

11. 1.

19. $\begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix}$.

20. $\begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix}$.

26. $\begin{bmatrix} 1 & -p & p^2 \\ 0 & 1 & -p \\ 0 & 0 & 1 \end{bmatrix}$; $\begin{bmatrix} 1 & 0 & 0 \\ -q & 1 & 0 \\ q^2 & -q & 1 \end{bmatrix}$ and

$$\begin{bmatrix} 1 & -p & p^2 \\ -q & pq+1 & -p^2q-p \\ q^2 & -pq^2-q & p^2q^2+pq+1 \end{bmatrix}.$$

Hints to Selected Questions

1. $A^2 + x\,I = y\,A$

$\Rightarrow A^{-1}A^2 + x\,A^{-1}I = y\,A^{-1}A$

$\Rightarrow (A^{-1}A)\,A + xA^{-1}I = yA^{-1}A$

$\Rightarrow IA + xA^{-1}I = y\,I$

$\Rightarrow A + x\,A^{-1} = y\,I$

$\Rightarrow A^{-1} = \dfrac{y\,I - A}{x}$.

2. (*i*) Given det. $= \begin{vmatrix} \cos\alpha & \sin\alpha & 0 \\ \cos\beta & \sin\beta & 0 \\ \cos\gamma & \sin\gamma & 0 \end{vmatrix}$

$\begin{vmatrix} \cos\alpha & \cos\beta & \cos\gamma \\ \sin\alpha & \sin\beta & \sin\gamma \\ 0 & 0 & 1 \end{vmatrix}$.

(*ii*) Given det. $= \begin{vmatrix} b_1 & c_1 & 0 \\ b_2 & c_2 & 0 \\ b_3 & c_3 & 0 \end{vmatrix} \begin{vmatrix} c_1 & c_2 & c_3 \\ b_1 & b_2 & b_3 \\ 0 & 0 & 0 \end{vmatrix}$.

3. Operate $R_1 \to R_1 - R_2$ & $R_2 \to R_2 - R_3$.

4. Multiply R_1, R_2 and R_3 by c, a and b respectively and dividing by abc.

5. Oparate $C_2 \to C_2 - C_1$, then take (-1) common from C_2 and then operate $C_2 \to C_2 + C_1 + 2C_3$.

6. Operate $R_1 \to R_1 - R_2$ & $R_2 \to R_2 - R_3$.

7. Operate $R_2 \to R_2 - R_3$ & $R_3 \to R_3 - R_2$.

19. adj. $A = |A|\,A^{-1}$.

24. $D = \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix}$. Find D^{-1}.

25. Prove that $F(\alpha)\,F(-\alpha) = I_3$

$G(\beta).\,G(-\beta) = I_3$

$\Rightarrow [F(\alpha)]^{-1} = F(-\alpha);\ [G(\beta)]^{-1} = G(-\beta)$.

Now $[F(\alpha)G(\beta)]^{-1} = [G(\beta)]^{-1}[F(\alpha)]^{-1}$

$= G(-\beta)\,F(-\alpha)$.

26. If A and B are given matrices, then $(AB)^{-1}$ is required.

CHECK YOUR UNDERSTANDING

1. Let $A = \begin{bmatrix} 1 & 2 \\ 4 & 2 \end{bmatrix}$. Is $|2A| = 4\,|A|$?

Ans. Yes.

2. If $\begin{vmatrix} x & 2 \\ 8 & x \end{vmatrix} = \begin{vmatrix} 3 & 2 \\ 9 & 6 \end{vmatrix}$, then the value of '$x$' is .....

(Fill in the blank)

Ans. $x = \pm 4$.

(Jammu B. 2016)

3. Answer in one word :

$\begin{vmatrix} 3 & 1 & 9 \\ 5 & 2 & 15 \\ 7 & 4 & 21 \end{vmatrix}$.

Ans. 0.

4. If any two rows (or columns) of a determinant are identical, then the value of the determinant is zero.

(True/False)

Ans. True.

5. Is $\left(\dfrac{1}{a} + \dfrac{1}{b} + \dfrac{1}{c} - 1\right)$ a factor of $\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix}$?

Ans. No.

6. If A is an invertible square matrix of order 4, then | adj A | =.....

(Fill in blank)

Ans. $|A|^3$.

7. If A is an invertible square matrix of order n, then adj (adj A) $= |A|^{n-1}\,A$.

(True/False) Ans. False.

8. Find the inverse of $A = \begin{bmatrix} 2 & -6 \\ 1 & -2 \end{bmatrix}$.

(Kerala B. 2016)

Ans. $\dfrac{1}{2}\begin{bmatrix} -2 & 6 \\ -1 & 2 \end{bmatrix}$.

9. If A is a square matrix satisfying $A^2 = I$, then what is the inverse of A ?

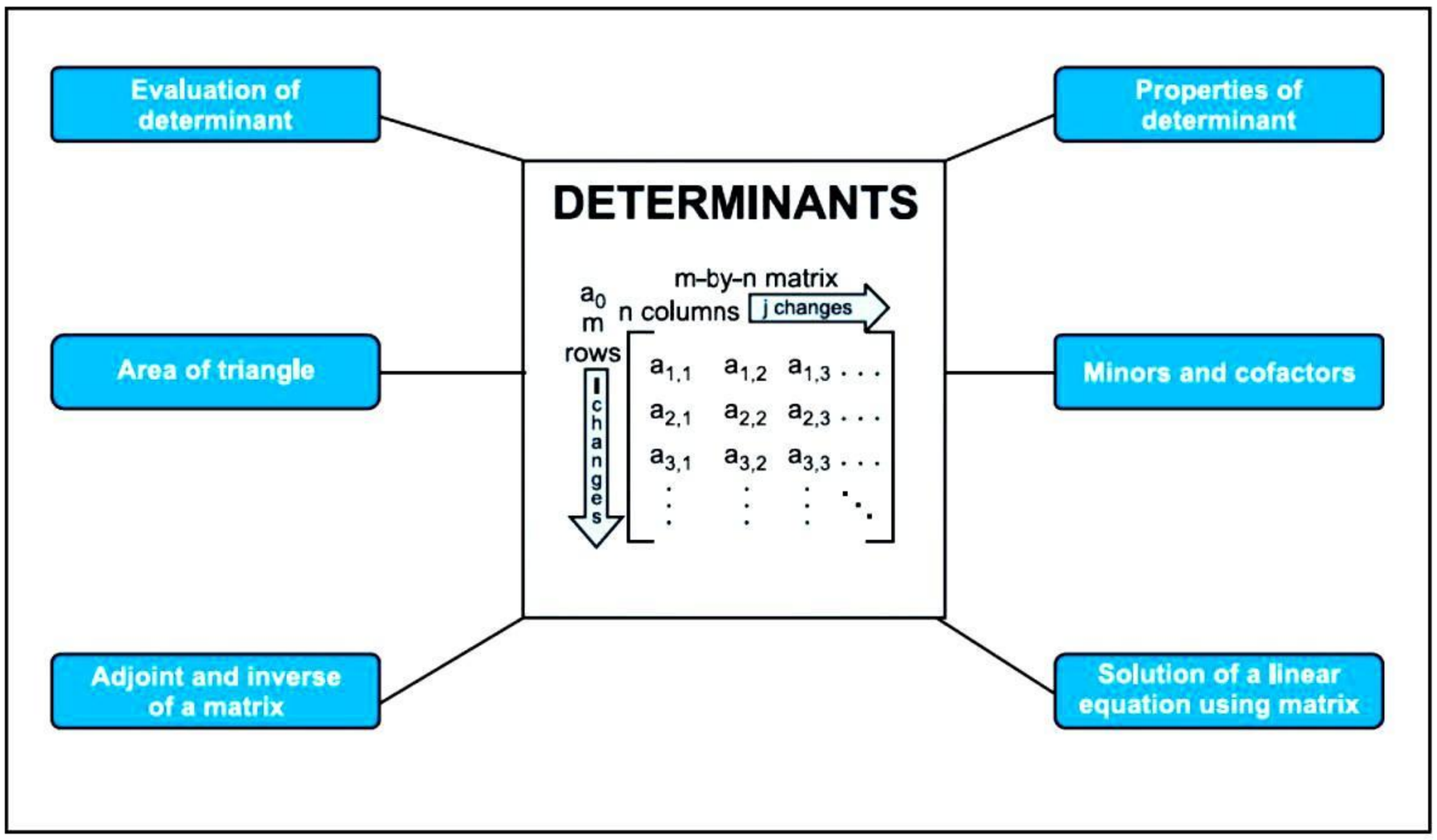
Ans. A.

10. Is the matrix $\begin{bmatrix} \pi & 22 \\ \dfrac{1}{7} & 1 \end{bmatrix}$ singular ?

Ans. Yes.

SUMMARY

DETERMINANTS



DEFINITIONS AND IMPORTANT RESULTS

1. DETERMINANT OF A SQUARE MATRIX

(*i*) If $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, then det. $A = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$

$= a_{11}a_{22} - a_{21}a_{12}$.

(*ii*) If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$,

then det. $A = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$

$- a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$

$= a_{11}a_{22}a_{33} - a_{11}a_{32}a_{23} - a_{12}a_{21}a_{33} + a_{12}a_{31}a_{23}$
$+ a_{13}a_{21}a_{32} - a_{13}a_{31}a_{22}$.

2. MINOR AND CO–FACTOR

(*i*) The **minor** of an element a_{ij} is a determinant, which is obtained by supressing the *i*th row and *j*th column. The minor of an element a_{ij} is denoted by $\mathbf{M_{ij}}$.

(*ii*) The **co–factor** of an element is its minor with proper sign. The co–factor of an element a_{ij} is denoted by $\mathbf{A_{ij}}$.

$$\boxed{\mathbf{A_{ij} = (-1)^{i+j} M_{ij}}}.$$

3. PROPERTIES

(I) Reflection Property. The value of the determinant remains unaltered by interchanging its rows and columns.

(II) Switching Property. If two adjacent rows (or columns) of a determinant are interchanged, then the sign of the determinant is changed.

(III) Repetition Property. If two rows (or columns) of a determinant are identical, then its value is zero.

(IV) Scalar Multiple Property. If each element of a row (or column) of a determinant is multiplied by a constant '*k*', then its value gets multiplied by the scalar '*k*'.

(V) Sum Property. If each element of a row (or column) of a determinant is expressed as the sum of two or more terms, then the determinant can be expressed as the sum of two or more determinants.

(VI) Invariance Property. If to any row (or column) of a determinant, a multiple of another row or column is added, the value of the determinant remains the same.

(VII) Factor Property. If a determinant Δ vanishes when for x is put a in those elements of Δ, which are polynomials in x, then $(x - a)$ is a factor of Δ.

4. AREA OF A TRIANGLE

Area of a triangle whose vertices are $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ is given by :

$$\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}.$$

When the area of the triangle is **zero**, then the points are **collinear.**

5. CRAMER'S RULE

To solve the equations :

$a_{11}x_1 + a_{12}x_2 + \ldots\ldots\ldots + a_{1n}x_n = b_1$

$a_{21}x_1 + a_{22}x_2 + \ldots\ldots\ldots + a_{2n}x_n = b_2$

.....

$a_{n1}x_1 + a_{n2}x_2 + \ldots\ldots\ldots + a_{nn}x_n = b_n.$

Let $\quad D = \begin{vmatrix} a_{11} & a_{12} & & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ a_{n1} & a_{n2} & & a_{nn} \end{vmatrix}.$

Let D_j be the determinant obtained from D after replacing jth column by $\begin{bmatrix} b_1 \\ b_2 \\ \ldots \\ b_n \end{bmatrix}$.

Then $x_1 = \dfrac{D_1}{D}, x_2 = \dfrac{D_2}{D}, \ldots\ldots, x_n = \dfrac{D_n}{D}$, where $D \neq 0$.

(*i*) If $b_1 = b_2 = \ldots\ldots = b_n = 0$, then $D_i = 0$ and $D \neq 0$, then the system has only **trivial solution.**

(*ii*) If $D = 0$ but $D_i = 0$ (for at least one i), then the system has **no solution.**

(*iii*) If $D = 0$, $D_1 = D_2 = \ldots\ldots = 0$, then the system has **infinite number of solutions or no solution.**

6. ADJOINT OF A MATRIX

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, then adj. A = $\begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$, where capital letters are co–factors of corresponding small letters.

7. INVERSE OF A MATRIX

Invertible Matrix. Any n–rowed square matrix A is said to be invertible if there exists an n–rowed matrix B such that

$AB = BA = I_n$.

B is called the inverse of A and is denoted as A^{-1}.

Theorems.

(*i*) Inverse of every square matrix, if it exists, is unique.

(*ii*) A is invertible iff $|A| \neq 0$ (*iii*) $A^{-1} = \dfrac{\text{adj. A}}{|A|}$, if $|A| \neq 0$.

Properties :

(*i*) $(AB)^{-1} = B^{-1}A^{-1}$ (*ii*) $(A')^{-1} = (A^{-1})'$

(*iii*) $(A^k)^{-1} = (A^{-1})^k$, where k is any positive integer.

8. SINGULAR AND NON–SINGULAR MATRICES

A square matrix is said to be singular if $|A| = 0$ and non–singular if $|A| \neq 0$.

9. SOLUTIONS OF EQUATIONS BY MATRIX METHOD

To solve the equations :

$a_{11}x_1 + a_{12}x_2 + \ldots\ldots + a_{1n}x_n = b_1$

$a_{21}x_1 + a_{22}x_2 + \ldots\ldots + a_{2n}x_n = b_2$

.....

$a_{m1}x_1 + a_{m2}x_2 + \ldots\ldots + a_{mn}x_n = b_m.$

Here $X = A^{-1}B$,

where $A = \begin{bmatrix} a_{11} & a_{12} \ldots\ldots\ldots a_{1n} \\ a_{21} & a_{22} \ldots\ldots\ldots a_{2n} \\ \ldots & \ldots\ldots\ldots \\ a_{m1} & a_{m2} \ldots\ldots\ldots a_{mn} \end{bmatrix}$, $A = \begin{bmatrix} x_1 \\ x_2 \\ \ldots \\ x_n \end{bmatrix}$ and $B \begin{bmatrix} b_1 \\ b_2 \\ \ldots \\ b_m \end{bmatrix}$.

(I) If $|A| \neq 0$, then the system is **consistent** and has a **unique solution.**

(II) If $|A| = 0$ and (adj. A) B = O, then the system is **consistent** and has **infinitely many solutions.**

(III) If $|A| = 0$ and (adj. A) B $\neq$ O, then the system is **inconsistent and has no solution.**

10. SOLUTION OF HOMOGENEOUS EQUATIONS

To solve the equations :

$a_1x + b_1y + c_1z = 0$

$a_2x + b_2y + c_2z = 0$

$a_3x + b_3y + c_3z = 0.$

Here AX = O, where $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$ and $A = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$.

(I) If $|A| \neq 0$, then system has only **trivial solution.**

(II) If $|A| = 0$, the system has **infinitely many solutions.**

MULTIPLE CHOICE QUESTIONS

For Board Examinations

1. If A is a matrix of order 3×3 and $|A| = 10$. then $|\text{adj } A|$ is :
(A) 0 (B) 10
(C) 100 (D) 1000. *(P.B. 2018)*

2. If A is an invertible matrix of order 2, then det (A^{-1}) is equal to :
(A) det (A) (B) $\frac{1}{\det(A)}$
(C) 1 (D) 0. *(H.P.B. 2018)*

3. If $\begin{vmatrix} 2 & 3-x \\ 1 & 4 \end{vmatrix} = 0$, then value of x is :
(A) 3 (B) – 3
(C) 5 (D) – 5. *(H.B. 2018)*

4. If $\begin{vmatrix} x & 12 \\ 3 & x \end{vmatrix} = \begin{vmatrix} 6 & 18 \\ 2 & 6 \end{vmatrix}$, then value of x is :
(A) ± 4 (B) ± 6
(C) ± 8 (D) None of these.
(H.B. 2018)

5. Let A be a non-singular square matrix of order 3×3. Then $|\text{adj } A|$ is :
(A) $|A|^3$ (B) $|A|$
(C) $3|A|$ (D) $|A|^2$. *(H.B. 2017)*

6. If A and B are invertible matrices of the same order, then $(AB)'$ is equal to :
(A) $A'B'$ (B) $B'A'$
(C) $-A'B'$ (D) $-B'A'$. *(Jammu B. 2017)*

7. If determinant A is of order 2×2 and $|A| = 3$, then the value of $|2A|$ is :
(A) 6 (B) 12
(C) – 6 (D) None of these.
(Kerala B. 2016)

8. If matric A = $\begin{vmatrix} 3-2x & x+1 \\ 2 & 4 \end{vmatrix}$ is singular, then x is equal to :
(A) 0 (B) 1
(C) – 1 (D) – 2. *(Mizoram B. 2016)*

RCQ Pocket

(Single Correct Answer Type)

(JEE-Main and Advanced)

9. Let A be a square matrix all of whose entries are integers. Then which of the following is true ?
(A) If det A = ± 1, then A^{-1} need not exist
(B) If det A = ± 1, then A^{-1} exists but all entries are not necesserily integers.
(C) If det A ≠ ± 1, then A^{-1} exists and all its entries are non-integers
(D) If det A = ± 1, then A^{-1} exists and all its entries are integers. *(A.I.E.E.E. 2008)*

10. Let a, b, c be such that $b(a + c) \neq 0$. If :

$$\begin{vmatrix} a & a+1 & a-1 \\ -b & b+1 & b-1 \\ c & c-1 & c+1 \end{vmatrix} + \begin{vmatrix} a+1 & b+1 & c-1 \\ a-1 & b-1 & c+1 \\ (-1)^{n+2}a & (-1)^{n+1}b & (-1)^n c \end{vmatrix} = 0,$$

then the value of n is :
(A) zero
(B) any even integer
(C) any odd integer
(D) any integer. *(A.I.E.E.E. 2009)*

11. Let $\omega \neq 1$ be a cube root of unity and S be the set of all non-singular matrices of the form :

$$\begin{bmatrix} 1 & a & b \\ \omega & 1 & c \\ \omega^2 & \omega & 1 \end{bmatrix},$$

where each of a, b and c is either 1 or 2. Then the number of distinct matrices in the set S is :
(A) 2 (B) 6
(C) 4 (D) 8. *(I.I.T. 2011)*

12. Let M and N be two 3×3 non-singular skew-symmetric matrices such that MN = NM. Let P^T denote the transpose of P, then $M^2N^2 (M^TN)^{-1} (MN^{-1})^T$ is equal to :
(A) M^2 (B) $-N^2$
(C) $-M^2$ (D) MN. *(I.I.T. 2011)*

13. The number of values of 'k' for which the linear equations :
$4x + ky + 2z = 0$
$kx + 4y + z = 0$
$2x + 2y + z = 0$
possesses a non-zero solution is :
(A) 3 (B) 2
(C) 1 (D) zero. *(A.I.E.E.E. 2011)*

14. The number of values of 'k', for which the system of equations :
$(k + 1)x + 8y = 4k$
$kx + (k + 3)y = 3k - 1$
has no solution is :
(A) 1 (B) 2
(C) 3 (D) infinite.
(J.E.E. (Main) 2013)

15. If $P = \begin{bmatrix} 1 & \alpha & 3 \\ 1 & 3 & 3 \\ 2 & 4 & 4 \end{bmatrix}$ is the adjoint of a 3×3 matrix A and $|A| = 4$, then α is equal to :
(A) 11 (B) 5
(C) 0 (D) 4. *(J.E.E. (Main) 2013)*

16. If $\alpha, \beta \neq 0$ and $f(x) = \alpha^n + \beta^n$ and

$$\begin{vmatrix} 3 & 1+f(1) & 1+f(2) \\ 1+f(1) & 1+f(2) & 1+f(3) \\ 1+f(2) & 1+f(3) & 1+f(4) \end{vmatrix}$$

$= k\,(1-\alpha)^2(1-\beta)^2(\alpha-\beta)^2$, then '$k$' is equal to :

(A) $\dfrac{1}{\alpha\beta}$ (B) 1

(C) -1 (D) $\alpha\beta$. *(J.E.E. (Main) 2014)*

17. The set of all values of 'λ' for which the system of linear equations :

$$2x_1 - 2x_2 + x_3 = \lambda x_1$$
$$2x_1 - 3x_2 + 2x_3 = \lambda x_2$$
$$-x_1 + 2x_2 = \lambda x_3$$

has a non-trivial solution :

(A) is an empty set
(B) is a singleton
(C) contains 2 elements
(D) contains more than 2 elements.

(J.E.E. (Main) 2015)

18. The system of linear equations :

$x + \lambda y - z = 0$
$\lambda x - y - z = 0$
$x + y - \lambda z = 0$

has a non-trivial solution for :

(A) Exactly one value of λ
(B) Exactly two values of λ
(C) Exactly three values of λ
(D) Infinitely many values of λ. *(J.E.E. (Main) 2016)*

19. Let ω be a complex number such that $2\omega + 1 = z$, where $z = \sqrt{-3}$.

If $\begin{vmatrix} 1 & 1 & 1 \\ 1 & -\omega^2-1 & \omega^2 \\ 1 & \omega^2 & \omega \end{vmatrix} = 3k$, then k is equal to :

(A) -1 (B) 1
(C) $-z$ (D) z. *(J.E.E. (Main) 2017)*

20. If S is the set of distinct values of 'b' for which the following system of linear equations :

$x + y + z = 1,\ x + ay + z = 1,\ ax + by + z = 1$

has no solution, then S is :

(A) a finite set containing two or more elements
(B) a singleton
(C) an empty set
(D) an infinite set. *(J.E.E. (Main) 2017)*

21. If $A = \begin{bmatrix} 2 & -3 \\ -4 & 1 \end{bmatrix}$, then adj. $(3A^2 + 12A)$ is equal to :

(A) $\begin{bmatrix} 51 & 84 \\ 63 & 72 \end{bmatrix}$ (B) $\begin{bmatrix} 72 & -63 \\ -84 & 51 \end{bmatrix}$

(C) $\begin{bmatrix} 72 & -84 \\ -63 & 51 \end{bmatrix}$ (D) $\begin{bmatrix} 51 & 63 \\ 84 & 72 \end{bmatrix}$.

(J.E.E. (Main) 2017)

22. If $\begin{vmatrix} x-4 & 2x & 2x \\ 2x & x-4 & 2x \\ 2x & 2x & x-4 \end{vmatrix} = (A + Bx)(x - A)^2$,

then the ordered pair (A, B) is equal to :

(A) $(-4, -5)$ (B) $(-4, 3)$
(C) $(-4, 5)$ (D) $(4, 5)$.

(J.E.E. (Main) 2018)

23. If the system of linear equations :

$x + ky + 3z = 0$
$3x + ky - 2z = 0$
$2x + 4y - 3z = 0$

has a non-zero solution (x, y, z), then $\dfrac{xz}{y^2}$ is equal to :

(A) -10 (B) 10
(C) -30 (D) 30.

(J.E.E. (Main) 2018)

Answers

1. (C)	**2.** (A)	**3.** (D)	**4.** (B)	**5.** (D)	**6.** (B)	**7.** (B)	**8.** (B)	**9.** (D)
10. (C)	**11.** (A)	**12.** (C)	**13.** (B)	**14.** (A)	**15.** (A)	**16.** (B)	**17.** (C)	**18.** (C)
19. (C)	**20.** (B)	**21.** (D)	**22.** (C)	**23.** (B).				

Hints/Solutions

RCQ Pocket

9. (D) Since each entry of A is an integer,
$\therefore$ co-factor of each entry is also an integer.
Hence, each entry of the adjoint is an integer.

Also det A = ± 1 and $A^{-1} = \dfrac{1}{(\det A)}$ (adj A).

Hence, all entries of A^{-1} are integers.

10. (C) $\begin{vmatrix} a & a+1 & a-1 \\ -b & b+1 & b-1 \\ c & c-1 & c+1 \end{vmatrix}$

$$+\begin{vmatrix} a+1 & b+1 & c-1 \\ a-1 & b-1 & c+1 \\ (-1)^{n+2}a & (-1)^{n+1}b & (-1)^n c \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} a & a+1 & a-1 \\ -b & b+1 & b-1 \\ c & c-1 & c+1 \end{vmatrix}$$

$$+(-1)^n \begin{vmatrix} a+1 & b+1 & c-1 \\ a-1 & b-1 & c+1 \\ a & -b & c \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} a & a+1 & a-1 \\ -b & b+1 & b-1 \\ c & c-1 & c+1 \end{vmatrix}$$

$$+(-1)^n \begin{vmatrix} a+1 & a-1 & a \\ b+1 & b-1 & -b \\ c-1 & c+1 & c \end{vmatrix} = 0$$

[*Interchanging rows and columns in IInd det.*]

$$\Rightarrow \begin{vmatrix} a & a+1 & a-1 \\ -b & b+1 & b-1 \\ c & c-1 & c+1 \end{vmatrix}$$

$$+(-1)^n \begin{vmatrix} a & a+1 & a-1 \\ -b & b+1 & b-1 \\ c & c-1 & c+1 \end{vmatrix} = 0$$

$$\Rightarrow (1+(-1)^n) \begin{vmatrix} a & a+1 & a-1 \\ -b & b+1 & b-1 \\ c & c-1 & c+1 \end{vmatrix} = 0 \quad ...(1)$$

$$\text{Now} \begin{vmatrix} a & a+1 & a-1 \\ -b & b+1 & b-1 \\ c & c-1 & c+1 \end{vmatrix} = \begin{vmatrix} a & 2 & a-1 \\ -b & 2 & b-1 \\ c & -2 & c+1 \end{vmatrix}$$

[*Operating* $C_2 \to C_2 - C_3$]

$$= \begin{vmatrix} a+c & 0 & a+c \\ -b+c & 0 & b+c \\ c & -2 & c+1 \end{vmatrix}$$

[*Operating* $R_1 \to R_1 + R_3$ & $R_2 \to R_2 + R_3$]

$$= (2) \begin{vmatrix} a+c & a+c \\ -b+c & b+c \end{vmatrix}$$ [*Expanding by* C_2]

$= 2\,[(a+c)(b+c) - (a+c)(c-b)]$
$= 2\,[ab + ac + bc + c^2 - ac + ab - c^2 + bc]$
$= 2\,[2ab + 2bc] = 4b\,(a+c) \neq 0.$

[*By hypothesis*]

From (1), Given $= (1+(-1)^n)\,D = 0$

$\Rightarrow \quad 1 + (-1)^n = 0,$ [$\because D \neq 0$]

which is true when n is any odd integer.

11. (A) For non-singular matrices,

$$\begin{vmatrix} 1 & a & b \\ \omega & 1 & c \\ \omega^2 & \omega & 1 \end{vmatrix} \neq 0$$

$\Rightarrow \quad 1.(1-\omega c) - a\,(\omega - \omega^2 c) + b\,(\omega^2 - \omega^2) \neq 0$
$\Rightarrow \quad 1 - \omega c - a\omega + ac\omega^2 \neq 0$
$\Rightarrow \quad 1 - (a+c)\,\omega + ac\omega^2 \neq 0$
$\Rightarrow \quad a + c \neq -1$ and $ac \neq 1$ [$\because$ *Otherwise* $1 + \omega + \omega^2 = 0$]
$\Rightarrow \quad a = \omega$ or ω^2 or $c = \omega$ or ω^2.
b can take any value ω and ω^2.
But $c \neq \omega^2$. [$\because$ *Otherwise determinant vanishes*]
Also $a \neq \omega^2$.
$\therefore \quad a = \omega, b = \omega^2$ or $\omega, c = \omega$.
$\therefore$ Number of such matrices $= 2$.

12. (C) Since M and N are skew-symmetric,
$\therefore \quad M^T = -M$ and $N^T = -T$.
Now $M^2 N^2 (M^T N)^{-1} (MN^{-1})^T$
$= M^2 N^2 (-MN)^{-1} (MN^{-1})^T$
$= M^2 N^2 (N^{-1} (-M)^{-1}) (N^{-1})^T . M^T)$
$= M^2 N^2 (-M)^{-1} . (-N)^{-1} (-M)$
$= M\,(MN)\,(MN)^{-1}\,(-M) = -M^2.$

13. (B) The system possesses non-zero solution

if $\begin{vmatrix} 4 & k & 2 \\ k & 4 & 1 \\ 2 & 2 & 1 \end{vmatrix} = 0$

if $4\,(4-2) - k\,(k-2) + 2\,(2k-8) = 0$
if $8 - k^2 + 2k + 4k - 16 = 0$
if $k^2 - 6k + 8 = 0$
if $(k-2)\,(k-4) = 0$ if $k = 2$ or 4
if $k = 2$.

14. (A) For no solution, $\dfrac{k+1}{k} = \dfrac{8}{k+3} \neq \dfrac{4k}{3k-1}$ ...(1)

Taking first two, $(k+1)\,(k+3) = 8k$
$\Rightarrow \quad k^2 + 4k + 3 = 8k \quad \Rightarrow k^2 - 4k + 3 = 0$
$\Rightarrow \quad (k-1)\,(k-3) = 0 \Rightarrow \quad k = 1, 3.$
But for $k = 1$, (1) is not satisfied.
Hence, $k = 3$.

15. (A) Here $|\text{adj } A| = |A|^{3-1} = |A|^2 = 4^2 = 16$
$\Rightarrow \quad 1\,.\,(12-12) - \alpha(4-6) + 3(4-6) = 16$
$\Rightarrow \quad 2\alpha - 6 = 16 \quad \Rightarrow \quad 2\alpha = 22.$
Hence, $\alpha = 11$.

16. (B) $\begin{vmatrix} 3 & 1+f(1) & 1+f(2) \\ 1+f(1) & 1+f(2) & 1+f(3) \\ 1+f(2) & 1+f(3) & 1+f(4) \end{vmatrix}$

$$= \begin{vmatrix} 3 & 1+\alpha+\beta & 1+\alpha^2+\beta^2 \\ 1+\alpha+\beta & 1+\alpha^2+\beta^2 & 1+\alpha^3+\beta^3 \\ 1+\alpha^2+\beta^2 & 1+\alpha^3+\beta^3 & 1+\alpha^4+\beta^4 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ 1 & \alpha & \beta \\ 1 & \alpha^2 & \beta^2 \end{vmatrix} \begin{vmatrix} 1 & 1 & 1 \\ 1 & \alpha & \alpha^2 \\ 1 & \beta & \beta^2 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ 1 & \alpha-1 & \beta-1 \\ 1 & \alpha^2-1 & \beta^2-1 \end{vmatrix}^2$$

$= [(\alpha-1)\,(\beta^2-1) - (\alpha^2-1)\,(\beta-1)]^2$
$= (\alpha-1)^2\,(\beta-1)^2\,(\alpha-\beta)^2.$
Hence, $k = 1$.

17. (C) The given equations are :
$(2-\lambda)\,x_1 - 2x_2 + x_3 = 0$
$2x_1 - (3+\lambda)\,x_2 + 2\,x_3 = 0$
and $-x_1 + 2x_2 - \lambda x_3 = 0.$

For non-trivial solution, $\begin{vmatrix} 2-\lambda & -2 & 1 \\ 2 & -3-\lambda & 2 \\ -1 & 2 & -\lambda \end{vmatrix} = 0$

$\Rightarrow$ $(2-\lambda)(3\lambda+\lambda^2-4)+2(-2\lambda+2)+1.(4-3-\lambda)=0$

$\Rightarrow$ $6\lambda+2\lambda^2-8-3\lambda^2-\lambda^3+4\lambda-4\lambda+4+4-3-\lambda=0$

$\Rightarrow$ $-\lambda^3-\lambda^2+5\lambda-3=0$

$\Rightarrow$ $\lambda^3+\lambda^2-5\lambda+3=0$

$\Rightarrow$ $(\lambda-1)(\lambda^2+2\lambda-3)=0 \Rightarrow (\lambda-1)(\lambda-1)(\lambda+3)=0$

$\Rightarrow$ $(\lambda-1)^2(\lambda+3)=0 \Rightarrow \lambda=1, 1, -3.$

Hence, $\{\lambda\}=\{1,-3\}$, which contains 2 elements.

18. (C) The system AX = O has non-trivial solution if det A = 0

i.e. if $\begin{vmatrix} 1 & \lambda & -1 \\ \lambda & -1 & -1 \\ 1 & 1 & -\lambda \end{vmatrix} = 0$

$\Rightarrow$ $(1)(\lambda+1)-\lambda(-\lambda^2+1)+(-1)(\lambda+1)=0$

$\Rightarrow$ $\lambda+1+\lambda^3-\lambda-\lambda-1=0$

$\Rightarrow$ $\lambda^3-\lambda=0$

$\Rightarrow$ $\lambda(\lambda^2-1)=0$

$\Rightarrow$ $\lambda=0, 1, -1.$

Hence, $\lambda=-1, 0, 1.$

19. (C) Here $2\omega+1=z$, $\omega=\dfrac{\sqrt{3}i-1}{2}$.

Now $\begin{vmatrix} 1 & 1 & 1 \\ 1-\omega^2 & -1 & \omega^2 \\ 1 & \omega^2 & \omega \end{vmatrix}$

$= \begin{vmatrix} 3 & 0 & 0 \\ 1 & -\omega^2-1 & \omega^2 \\ 1 & \omega^2 & \omega \end{vmatrix}$

[*Operating* $R_1 \to R_1+R_2+R_3$]

$= 3[-\omega(\omega^2+1)-\omega^4]$

$= 3[-\omega^3-\omega-\omega] = 3[-1-2\omega]$

$= -3(1+2\omega) = -3z.$

Thus $3k = -3z.$

Hence, $k = -z.$

20. (B) $D = \begin{vmatrix} 1 & 1 & 1 \\ 1 & a & 1 \\ a & b & 1 \end{vmatrix} = 0$

$\Rightarrow$ $a = 1.$

$\therefore$ $x+y+z=1$

and $x+by+z=0.$

The planes are parallel $\Rightarrow$ $b=1.$

Hence, S is a singleton.

21. (D) Here $A = \begin{bmatrix} 2 & -3 \\ -4 & 1 \end{bmatrix}.$

$\therefore$ $A^2 = \begin{bmatrix} 2 & -3 \\ -4 & 1 \end{bmatrix}\begin{bmatrix} 2 & -3 \\ -4 & 1 \end{bmatrix}$

$= \begin{bmatrix} 4+12 & -6-3 \\ -8-4 & 12+1 \end{bmatrix} = \begin{bmatrix} 16 & -9 \\ -12 & 13 \end{bmatrix}$

and $12A = \begin{bmatrix} 24 & -36 \\ -48 & 12 \end{bmatrix}.$

$\therefore$ $3A^2+12A = \begin{bmatrix} 72 & -63 \\ -84 & 51 \end{bmatrix}.$

Hence, adj $(3A^2+12A) = \begin{bmatrix} 51 & 63 \\ 84 & 72 \end{bmatrix}.$

22. (C) $\begin{vmatrix} x-4 & 2x & 2x \\ 2x & x-4 & 2x \\ 2x & 2x & x-4 \end{vmatrix} = (A+Bx)(x-A)^2$

$\Rightarrow \begin{vmatrix} 5x-4 & 2x & 2x \\ 5x-4 & x-4 & 2x \\ 5x-4 & 2x & x-4 \end{vmatrix} = (A+Bx)(x-A)^2$

[*Operating* $C_1 \to C_1+C_2+C_3$]

$\Rightarrow (5x-4)\begin{vmatrix} 1 & 2x & 2x \\ 1 & x-4 & 2x \\ 1 & 2x & x-4 \end{vmatrix} = (A+Bx)(x-A)^2$

[*Taking* $(5x-4)$ *common from* C_1]

$\Rightarrow (5x-4)\begin{vmatrix} 1 & 2x & 2x \\ 0 & -x-4 & 0 \\ 0 & 0 & -x-4 \end{vmatrix} = (A+Bx)(x-A)^2$

[*Operating* $R_2 \to R_2-R_1$ & $R_3 \to R_3-R_1$]

$\Rightarrow (5x-4)(x+4)^2 = (A+Bx)(x-A)^2.$

Comparing, A = – 4 and B = 5.

Hence, (A, B) = (– 4, 5).

23. (B) The given system has non-zero solution

$\Rightarrow \begin{vmatrix} 1 & k & 3 \\ 3 & k & -2 \\ 2 & 4 & -3 \end{vmatrix} = 0$

$\Rightarrow 1.(-3k+8)-k(-9+4)+3(12-2k)=0$

$\Rightarrow 44-4k=0 \Rightarrow k=11.$

Let $z=\lambda$

Then, $x+11y=-3\lambda$

$3x+11y=2\lambda.$

Solving, $x=\dfrac{5\lambda}{2}$, $y=-\dfrac{\lambda}{2}$ and $z=\lambda.$

Hence, $\dfrac{xz}{y^2} = \dfrac{\left(\frac{5\lambda}{2}\right)(\lambda)}{\left(\frac{-\lambda}{2}\right)^2} = \dfrac{5/2}{1/4} = 10.$

CHAPTER TEST 4

Time Allowed : 1 Hour **Max. Marks : 34**

Notes : **1.** *All questions are compulsory.*
2. *Marks have been indicated against each question.*

1. If $\begin{vmatrix} x & x \\ 1 & x \end{vmatrix} = \begin{vmatrix} 3 & 4 \\ 1 & 2 \end{vmatrix}$, write the positive value of 'x'. **(1)**

2. If A is a square matrix of order 3 and $|3A| = k\,|A|$, then write the value of 'k'. **(1)**

3. If A is a square matrix of order 3 such that $|\text{adj } A| = 225$, find $|A'|$. **(2)**

4. Prove that $\begin{vmatrix} a+b+2c & a & b \\ c & b+c+2a & b \\ c & a & c+a+2b \end{vmatrix} = 2\,(a+b+c)^3$. **(2)**

5. Prove that $\begin{vmatrix} a^2+1 & ab & ac \\ ab & b^2+1 & bc \\ ac & bc & c^2+1 \end{vmatrix} = 1 + a^2 + b^2 + c^2$. **(4)**

6. Using co-factors of third column, evaluate :

$$\Delta = \begin{vmatrix} 1 & x & yz \\ 1 & y & zx \\ 1 & z & xy \end{vmatrix}.$$ **(4)**

7. Find the value of 'k' if the area of the triangle is 4 square units and vertices are $(-2, 0)$, $(0, 4)$, $(0, k)$. **(4)**

8. Find the inverse of the matrix $A = \begin{bmatrix} a & b \\ c & \dfrac{1+bc}{a} \end{bmatrix}$ and show that :

$aA^{-1} = (a^2 + bc + 1)\,I - aA$. **(4)**

9. Prove that

$$\begin{vmatrix} a & a+b & a+b+c \\ 2a & 3a+2b & 4a+3b+2c \\ 3a & 6a+3b & 10a+6b+3c \end{vmatrix} = a^3.$$ **(6)**

10. If $A = \begin{vmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{vmatrix}$, find A^{-1}.

Using A^{-1}, solve the system of linear equations :
$2x - 3y + 5z = 16$, $3x + 2y - 4z = -4$, $x + y - 2z = -3$. **(6)**

Answers

1. $x = 2$. **2.** $k = 27$. **3.** ± 15.

6. $A_{13} = z - y$, $A_{23} = x - z$, $A_{33} = y - x$;
$\Delta = xy^2 - z^2x - x^2y + zx^2 + yz^2 - y^2z$.

7. $k = 0, 8$. **10.** $x = 2$, $y = 1$, $z = 3$.