CHAPTER

04

Logarithms and Their Properties

The technique of logarithms was introduced by **John Napier** (1550-1617). The logarithm is a form of indices which is used to simplify the algebraic calculations. The operations of multiplication, division of a very large number becomes quite easy and get converted into simple operations of addition and subtraction, respectively. The results obtained are correct upto some decimal places.

Session 1

Definition, Characteristic and Mantissa

Definition

The logarithm of any positive number, whose base is a number (>0) different from 1, is the index or the power to which the base must be raised in order to obtain the given number.

i.e. if $a^x = b$ (where $a > 0, \neq 1$), then x is called the logarithm of b to the base a and we write $\log_a b = x$, clearly $b > 0$. Thus, $\log_a b = x \Leftrightarrow a^x = b, a > 0, a \neq 1$ and $b > 0$.

If $a = 10$, then we write $\log b$ rather than $\log_{10} b$. If $a = e$, we write $\ln b$ rather than $\log_e b$. Here, 'e' is called as **Napier's base** and has numerical value equal to 2.7182. Also, $\log_{10} e$ is known as **Napierian constant**.

i.e. $\log_{10} e = 0.4343$

$\therefore$ $\ln b = 2.303 \log_{10} b$

$$\left[\text{since, } \ln b = \log_{10} b \times \log_e 10 = \frac{1}{\log_{10} e} \times \log_{10} b = \frac{1}{0.4343} \log_{10} b = 2.303 \log_{10} b\right]$$

Remember

(i) $\log 2 = \log_{10} 2 = 0.3010$

(ii) $\log 3 = \log_{10} 3 = 0.4771$

(iii) $\ln 2 = 2.303 \log 2 = 0.693$

(iv) $\ln 10 = 2.303$

***Corollary* I** From the definition of the logarithm of the number b to the base a, we have an identity

$$a^{\log_a b} = b, a > 0, a \neq 1 \text{ and } b > 0$$

which is known as the **Fundamental Logarithmic Identity**.

***Corollary* II** The function defined by $f(x) = \log_a x, a > 0, a \neq 1$ is called logarithmic function. Its domain is $(0, \infty)$ and range is R (set of all real numbers).

***Corollary* III** $a^x > 0, \forall x \in R$

(i) If $a > 1$, then a^x is monotonically increasing.

For example, $5^{2.7} > 5^{2.5}, 3^{222} > 3^{111}$

(ii) If $0 < a < 1$, then a^x is monotonically decreasing.

For example, $\left(\frac{1}{5}\right)^{2.7} < \left(\frac{1}{5}\right)^{2.5}, (0.7)^{222} < (0.7)^{212}$

***Corollary* IV**

(i) If $a > 1$, then $a^{-\infty} = 0$

$\therefore$ $\log_a 0 = -\infty$ (if $a > 1$)

(ii) If $0 < a < 1$, then $a^{\infty} = 0$

$\therefore$ $\log_a 0 = +\infty$ (if $0 < a < 1$)

***Corollary* V** (i) $\log_a b \to \infty$, if $a > 1, b \to \infty$

(ii) $\log_a b \to -\infty$, if $0 < a < 1, b \to \infty$

Remark

1. 'log' is the abbreviation of the word 'logarithm'.
2. **Common logarithm** (Brigg's logarithms) The base is 10.
3. If $x < 0, a > 0$ and $a \neq 1$, then $\log_a x$ is an imaginary.
4. If $a > 1$, $\log_a x = \begin{cases} +\text{ve}, & x > 1 \\ 0, & x = 1 \\ -\text{ve}, & 0 < x < 1 \end{cases}$

   And if $0 < a < 1$, $\log_a x = \begin{cases} +\text{ve}, & 0 < x < 1 \\ 0, & x = 1 \\ -\text{ve}, & x > 1 \end{cases}$
5. $\log_a 1 = 0$ $(a > 0, a \neq 1)$

   $\log_a a = 1$ $(a > 0, a \neq 1)$ and $\log_{(1/a)} a = -1$ $(a > 0, a \neq 1)$

Example 1. Find the value of the following :

(i) $\log_9 27$ (ii) $\log_{3\sqrt{2}} 324$

(iii) $\log_{1/9}(27\sqrt{3})$ (iv) $\log_{(5+2\sqrt{6})}(5-2\sqrt{6})$

(v) $\log_{0.2} 0.008$ (vi) $2^{2\log_4 5}$

(vii) $(0.4)^{-\log_{2.5}\left\{\frac{1}{3}+\frac{1}{3^2}+\frac{1}{3^3}+\ldots\right\}}$ (viii) $(0.05)^{\log_{\sqrt{20}}(0.\bar{3})}$

Sol. (i) Let $x = \log_9 27$

$\Rightarrow 9^x = 27 \Rightarrow 3^{2x} = 3^3 \Rightarrow 2x = 3$

$\therefore x = \frac{3}{2}$

(ii) Let $x = \log_{3\sqrt{2}} 324$

$\Rightarrow (3\sqrt{2})^x = 324 = 2^2 \cdot 3^4 \Rightarrow (3\sqrt{2})^x = (3\sqrt{2})^4$

$\therefore x = 4$

(iii) Let $x = \log_{1/9}(27\sqrt{3})$

$\Rightarrow \left(\frac{1}{9}\right)^x = 27\sqrt{3} \Rightarrow 3^{-2x} = 3^{7/2} \Rightarrow -2x = 7/2$

$\therefore x = -\frac{7}{4}$

(iv) $\because (5+2\sqrt{6})(5-2\sqrt{6}) = 1$

or $5+2\sqrt{6} = \frac{1}{5-2\sqrt{6}}$ ...(i)

Now, let $x = \log_{(5+2\sqrt{6})}(5-2\sqrt{6})$

$= \log_{1/(5-2\sqrt{6})} 5-2\sqrt{6} = -1$ [from Eq. (i)]

(v) Let $x = \log_{0.2} 0.008$

$\Rightarrow (0.2)^x = 0.008 \Rightarrow (0.2)^x = (0.2)^3 \Rightarrow x = 3$

(vi) Let $x = 2^{2\log_4 5} = 4^{\log_4 5} = 5$

(vii) Let $x = (0.4)^{-\log_{2.5}\left\{\frac{1}{3}+\frac{1}{3^2}+\frac{1}{3^3}+\ldots\right\}}$

$= \left(\frac{4}{10}\right)^{-\log_{2.5}\left\{\frac{\frac{1}{3}}{1-\frac{1}{3}}\right\}} = \left(\frac{2}{5}\right)^{-\log_{2.5}\left(\frac{1}{2}\right)} = \left(\frac{5}{2}\right)^{\log_{5/2}\left(\frac{1}{2}\right)} = \frac{1}{2}$

(viii) Let $x = (0.05)^{\log_{\sqrt{20}}(0.\bar{3})} = (0.05)^{\log_{\sqrt{20}}(\lambda)}$ ...(i)

where, $\lambda = 0.\bar{3}$

Then, $\lambda = 0.33333\ldots$ ...(ii)

$\Rightarrow 10\lambda = 3.33333\ldots$ ...(iii)

On subtracting Eq. (ii) from Eq. (iii), we get

$9\lambda = 3 \Rightarrow \lambda = \frac{1}{3}$

Now, from Eq. (i), $x = (0.05)^{\log_{\sqrt{20}}\left(\frac{1}{3}\right)}$

$= \left(\frac{1}{20}\right)^{\log_{(20)^{1/2}}(3)^{-1}} = \left(\frac{1}{20}\right)^{-\frac{1}{1/2}\log_{20} 3}$

$= 20^{(2\log_{20} 3)} = 20^{\log_{20} 3^2} = 3^2 = 9$

Example 2. ***Find the value of the following:***

(i) $\log_{\tan 45°}\cot 30°$ (ii) $\log_{(\sec^2 60° - \tan^2 60°)}\cos 60°$

(iii) $\log_{(\sin^2 30° + \cos^2 30°)} 1$ (iv) $\log_{30} 1$

Sol. (i) Here, base $= \tan 45° = 1$ tan

$\therefore$ log is not defined.

(ii) Here, base $= \sec^2 60° - \tan^2 60° = 1$

$\therefore$ log is not defined.

(iii) $\because \log_{(\sin^2 30° + \cos^2 30°)} 1 = \log_1 1 \neq 1$

$\because$ Here, base = 1

$\therefore$ log is not defined.

(iv) $\log_{30} 1 = 0$

Characteristic and Mantissa

The integral part of a logarithm is called the **characteristic** and the fractional part (decimal part) is called **mantissa**.

i.e., $\log N =$ Integer + Fractional or decimal part (+ve)

$\downarrow$ Characteristic $\quad \downarrow$ Mantissa

The mantissa of log of a number is always kept positive.

i.e., if $\log 564 = 2.751279$, then 2 is the characteristic and 0.751279 is the mantissa of the given number 564.

And if $\log 0.00895 = -2.0481769$

$= -2 - 0.0481769$

$= (-2-1) + (1-0.0481769)$

$= -3 + 0.9518231$

Hence, -3 is the characteristic and 0.9518231 (not 0.0481769) is mantissa of log 0.00895.

In short, $-3 + 0.9518231$ is written as $\bar{3}.9518231$.

Remark

1. If $N > 1$, the characteristic of $\log N$ will be one less than the number of digits in the integral part of N.
   For example, If $\log 235.68 = 2.3723227$
   Here, $N = 235.68$
   $\therefore$ Number of digits in the integral part of $N = 3$
   $\Rightarrow$ Characteristic of $\log 235.68 = N - 1 = 3 - 1 = 2$
2. If $0 < N < 1$, the characteristic of $\log N$ is negative and numerically it is one greater than the number of zeroes immediately after the decimal part in N.
   For example, If $\log 0.0000279 = \bar{5}.4456042$
   Here, four zeroes immediately after the decimal point in the number 0.0000279 is $(\overline{4+1})$, i.e. $\bar{5}$.
3. If the characteristics of $\log N$ be n, then number of digits in N is $(n+1)$ (Here, $N > 1$).
4. If the characteristics of $\log N$ be $-n$, then there exists $(n-1)$ number of zeroes after decimal part of N (here, $0 < N < 1$).

Example 3. ***If*** $\log 2 = 0.301$ ***and*** $\log 3 = 0.477$, ***find the number of digits in*** 6^{20}.

Sol. Let $P = 6^{20} = (2 \times 3)^{20}$

$$\therefore \quad \log P = 20 \log(2 \times 3) = 20\{\log 2 + \log 3\}$$
$$= 20\{0.301 + 0.477\}$$
$$= 20 \times 0.778 = 15.560$$

Since, the characteristic of $\log P$ is 15, therefore the number of digits in P will be 15 + 1, i.e. 16.

Example 4. Find the number of zeroes between the decimal point and first significant digit of $(0.036)^{16}$, where $\log 2 = 0.301$ and $\log 3 = 0.477$.

Sol. Let $P = (0.036)^{16} \Rightarrow \log P = 16 \log(0.036)$

$$= 16 \log\left(\frac{36}{1000}\right) = 16 \log\left(\frac{2^2 \cdot 3^2}{1000}\right)$$
$$= 16\{\log 2^2 + \log 3^2 - \log 10^3\}$$
$$= 16\{2\log 2 + 2\log 3 - 3\}$$
$$= 16\{2 \times 0.301 + 2 \times 0.477 - 3\}$$
$$= 16\{1.556 - 3\} = 24.896 - 48$$
$$= -48 + 24 + 0.896$$
$$= -24 + 0.896 = \overline{24} + 0.896$$

$\therefore$ The required number of zeroes = 24 − 1 = 23.

Exercise for Session 1

1. The value of $\log_{2\sqrt{3}} 1728$ is

(a) 6 (b) 8
(c) 3 (d) 5

2. The value of $\log_{(8-3\sqrt{7})}(8 + 3\sqrt{7})$ is

(a) −2 (b) −1
(c) 0 (d) Not defined

3. The value of $(0.16)^{\log_{2.5}\left\{\frac{1}{3} + \frac{1}{3^2} + \dots\right\}}$ is

(a) 2 (b) 4
(c) 6 (d) 8

4. If $\log 2 = 0.301$, the number of integers in the expansion of 4^{17} is

(a) 9 (b) 11
(c) 13 (d) 15

5. If $\log 2 = 0.301$, then the number of zeroes between the decimal point and the first significant figure of 2^{-34} is

(a) 9 (b) 10
(c) 11 (d) 12

Answers

Exercise for Session 1

1. (a) **2.** (b) **3.** (b) **4.** (b) **5.** (b)