Session 4

Solutions of Linear Simultaneous Equations Using Matrix Method

Solutions of Linear Simultaneous Equations Using Matrix Method

Let us consider a system of n linear equations in n unknowns say $x_1, x_2, x_3, \ldots, x_n$ given as below

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots + a_{2n}x_n = b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \ldots + a_{3n}x_n = b_3 \\ \ldots \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \\ \ldots \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \\ a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \ldots + a_{nn}x_n = b_n \end{array}\right\} \quad \ldots(\text{i})$$

If $b_1 = b_2 = b_3 = \ldots = b_n = 0$, then the system of Eq.(i) is called a system of homogeneous linear equations and if atleast one of $b_1, b_2, b_3, \ldots, b_n$ is non-zero, then it is called a system of non-homogeneous linear equation. We write the above system of Eq. (i) in the matrix form as

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \ldots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \ldots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \ldots & a_{3n} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & a_{n3} & \ldots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \ldots \\ \ldots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \ldots \\ \ldots \\ b_n \end{bmatrix}$$

$$\Rightarrow \quad AX = B \quad \ldots(\text{ii})$$

where $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \ldots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \ldots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \ldots & a_{3n} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & a_{n3} & \ldots & a_{nn} \end{bmatrix}$,

$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \ldots \\ \ldots \\ x_n \end{bmatrix}$ and $B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \ldots \\ \ldots \\ b_n \end{bmatrix}$

Pre-multiplying Eq. (ii) by A^{-1}, we get

$$A^{-1}(AX) = A^{-1}B \Rightarrow (A^{-1}A)X = A^{-1}B$$

$$\Rightarrow \quad IX = A^{-1}B$$

$$\Rightarrow \quad X = A^{-1}B = \frac{(\text{adj } A)B}{|A|}$$

Types of Equations

(1) **When system of equations is non-homogeneous**

(i) If $|A| \neq 0$, then the system of equations is consistent and has a unique solution given by $X = A^{-1}B$.

(ii) If $|A| = 0$ and $(\text{adj}A) \cdot B \neq 0$, then the system of equations is inconsistent and has no solution.

(iii) If $|A| = 0$ and $(\text{adj}A) \cdot B = O$, then the system of equations is consistent and has an infinite number of solutions.

(2) **When system of equations is homogeneous**

(i) If $|A| \neq 0$, then the system of equations has only trivial solution and it has one solution.

(ii) If $|A| = 0$, then the system of equations has non-trivial solution and it has infinite solutions.

(iii) If number of equations < number of unknowns, then it has non-trivial solution.

Note

Non-homogeneous linear equations can also be solved by Cramer's rule, this method has been discussed in the chapter on determinants.

Example 45. Solve the system of equations $x + 2y + 3z = 1$, $2x + 3y + 2z = 2$ and $3x + 3y + 4z = 1$ with the help of matrix inversion.

Sol. We have,

$$x + 2y + 3z = 1,\ 2x + 3y + 2z = 2 \text{ and } 3x + 3y + 4z = 1$$

The given system of equations in the matrix form are written as below.

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

$$AX = B$$

$$\Rightarrow \quad X = A^{-1}B \quad \text{...(i)}$$

where $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

$$|A| = 1(12-6) - 2(8-6) + 3(6-9)$$
$$= 6 - 4 - 9 = -7 \neq 0$$

$\therefore$ A^{-1} exists and has unique solution.

Let C be the matrix of cofactors of elements in $|A|$.

Now, cofactors along $R_1 = 6, -2, -3$

cofactors along $R_2 = 1, -5, 3$

and cofactors along $R_3 = -5, 4, -1$

$$\therefore \quad C = \begin{bmatrix} 6 & -2 & -3 \\ 1 & -5 & 3 \\ -5 & 4 & -1 \end{bmatrix}$$

$$\therefore \quad \text{adj } A = C^T$$

$$\Rightarrow \quad \text{adj } A = \begin{bmatrix} 6 & -2 & -3 \\ 1 & -5 & 3 \\ -5 & 4 & -1 \end{bmatrix}^T = \begin{bmatrix} 6 & 1 & -5 \\ -2 & -5 & 4 \\ -3 & 3 & -1 \end{bmatrix}$$

$$\Rightarrow \quad A^{-1} = \frac{\text{adj } A}{|A|} = -\frac{1}{7}\begin{bmatrix} 6 & 1 & -5 \\ -2 & -5 & 4 \\ -3 & 3 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{6}{7} & -\frac{1}{7} & \frac{5}{7} \\ \frac{2}{7} & \frac{5}{7} & -\frac{4}{7} \\ \frac{3}{7} & -\frac{3}{7} & \frac{1}{7} \end{bmatrix}$$

From Eq. (i), $X = A^{-1}B$

$$\Rightarrow \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -\frac{6}{7} & -\frac{1}{7} & \frac{5}{7} \\ \frac{2}{7} & \frac{5}{7} & -\frac{4}{7} \\ \frac{3}{7} & -\frac{3}{7} & \frac{1}{7} \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{3}{7} \\ \frac{8}{7} \\ -\frac{2}{7} \end{bmatrix}$$

Hence, $x = -\frac{3}{7}, y = \frac{8}{7}$ and $z = -\frac{2}{7}$ is the required solution.

Example 46. Solve the system of equations $x + y + z = 6$, $x + 2y + 3z = 14$ and $x + 4y + 7z = 30$ with the help of matrix method.

Sol. We have, $x + y + z = 6,$

$$x + 2y + 3z = 14$$

and $x + 4y + 7z = 30$

The given system of equations in the matrix form are written as below :

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 7 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 14 \\ 30 \end{bmatrix}$$

$$AX = B \quad \text{...(i)}$$

where, $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 7 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 6 \\ 14 \\ 30 \end{bmatrix}$

$$|A| = 1(14-12) - 1(7-3) + 1(4-2) = 2 - 4 + 2 = 0$$

$\therefore$ The equation either has no solution or an infinite number of solutions. To decide about this, we proceed to find (adj A) B.

Let C be the matrix of cofactors of elements in $|A|$.

Now, cofactors along $R_1 = 2, -4, 2$

cofactors along $R_2 = -3, 6, -3$

and cofactors along $R_3 = 1, -2, 1$

$$\therefore \quad C = \begin{bmatrix} 2 & -4 & 2 \\ -3 & 6 & -3 \\ 1 & -2 & 1 \end{bmatrix}$$

$$\Rightarrow \quad \text{adj} = C^T = \begin{bmatrix} 2 & -3 & 1 \\ -4 & 6 & -2 \\ 2 & -3 & 1 \end{bmatrix},$$

then $$(\text{adj } A)\, B = \begin{bmatrix} 2 & -3 & 1 \\ -4 & 6 & -2 \\ 2 & -3 & 1 \end{bmatrix}\begin{bmatrix} 6 \\ 14 \\ 30 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = O$$

Hence, both conditions $|A| = 0$ and $(\text{adj } A)\, B = O$ are satisfied, then the system of equations is consistent and has an infinite number of solutions.

Proceed as follows :

$$[A : B] = \begin{bmatrix} 1 & 1 & 1 & \vdots & 6 \\ 1 & 2 & 3 & \vdots & 14 \\ 1 & 4 & 7 & \vdots & 30 \end{bmatrix}$$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, then

$$[A : B] = \begin{bmatrix} 1 & 1 & 1 & \vdots & 6 \\ 0 & 1 & 2 & \vdots & 8 \\ 0 & 2 & 4 & \vdots & 16 \end{bmatrix}$$

Applying $R_3 \to R_3 - 2R_2$, then

$$[A : B] = \begin{bmatrix} 1 & 1 & 1 & \vdots & 6 \\ 0 & 1 & 2 & \vdots & 8 \\ 0 & 0 & 0 & \vdots & 0 \end{bmatrix}$$

Then, Eq. (i) reduces to

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 8 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} x + y + z \\ y + 2z \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 8 \\ 0 \end{bmatrix}$$

On comparing $x + y + z = 6$ and $y + 2z = 8$

Taking $z = k \in R$, then $y = 8 - 2k$ and $x = k - 2$.

Since, k is arbitrary, hence the number of solutions is infinite.

Example 47. Solve the system of equations $x + 3y - 2z = 0$, $2x - y + 4z = 0$ and $x - 11y + 14z = 0$.

Sol. We have,

$$x + 3y - 2z = 0$$
$$2x - y + 4z = 0$$
$$x - 11y + 14z = 0$$

The given system of equations in the matrix form are written as below.

$$\begin{bmatrix} 1 & 3 & -2 \\ 2 & -1 & 4 \\ 1 & -11 & 14 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$AX = O \quad \text{...(i)}$$

where $A = \begin{bmatrix} 1 & 3 & -2 \\ 2 & -1 & 4 \\ 1 & -11 & 14 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $O = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\therefore \quad |A| = 1(-14 + 44) - 3(28 - 4) - 2(-22 + 1)$

$= 30 - 72 + 42 = 0$

and therefore the system has a non-trivial solution. Now, we may write first two of the given equations

$$x + 3y = 2z \text{ and } 2x - y = -4z$$

Solving these equations in terms of z, we get

$$x = -\frac{10}{7}z \text{ and } y = \frac{8}{7}z$$

Putting $x = -\frac{10}{7}z$ and $y = \frac{8}{7}z$ in third equation of the given system,

we get, $\text{LHS} = -\frac{10}{7}z - \frac{88}{7}z + 14z = 0 = \text{RHS}$

Now, if $z = 7k$, then $x = -10k$ and $y = 8k$.

Hence, $x = -10k$, $y = 8k$ and $z = 7k$ (where k is arbitrary) are the required solutions.

Example 48. Solve the system of equations

$$2x + 3y - 3z = 0,$$
$$3x - 3y + z = 0$$
and
$$3x - 2y - 3z = 0$$

Sol. We have,

$$2x + 3y - 3z = 0$$
$$3x - 3y + z = 0$$
$$3x - 2y - 3z = 0$$

The given system of equations in the matrix form are written as below.

$$\begin{bmatrix} 2 & 3 & -3 \\ 3 & -3 & 1 \\ 3 & -2 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$AX = O \quad \text{...(i)}$$

where $A = \begin{bmatrix} 2 & 3 & -3 \\ 3 & -3 & 1 \\ 3 & -2 & -3 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $O = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\therefore \quad |A| = 2(9 + 2) - 3(-9 - 3) - 3(-6 + 9)$

$= 22 + 36 - 9 = 49 \neq 0$

Hence, the equations have a unique trivial solution $x = 0$, $y = 0$ and $z = 0$ only.

Echelon Form of a Matrix

A matrix A is said to be in echelon form, if

(i) The first non-zero element in each row is 1.

(ii) Every non-zero row in A precedes every zero-row.

(iii) The number of zeroes before the first non-zero element in 1st, 2nd, 3rd, ... rows should be in increasing order.

For example,

(i) $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 4 & 5 \\ 0 & 0 & 1 & 9 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

(iii) $\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 2 & 4 & 3 \\ 0 & 0 & 1 & 2 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

Rank of Matrix

The rank of a matrix is said to be r, if

(i) It has atleast minors of order r is different from zero.

(ii) All minors of A of order higher than r are zero.
The rank of A is denoted by $\rho(A)$.

Note

1. The rank of a zero matrix is zero and the rank of an identity matrix of order n is n.
2. The rank of a matrix in echelon form is equal to the number of non-zero rows of the matrix.
3. The rank of a non-singular matrix ($|A| \neq 0$) of order n is n.

Properties of Rank of Matrices

(i) If $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{m \times n}$, then

$$\rho(A + B) \leq \rho(A) + \rho(B)$$

(ii) If $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{n \times p}$, then

$$\rho(AB) \leq \rho(A) \text{ and } \rho(AB) \leq \rho(B)$$

(iii) If $A = [a_{ij}]_{n \times n}$, then $\rho(A) = \rho(A')$

Example 49. Find the rank of $\begin{bmatrix} 3 & -1 & 2 \\ -3 & 1 & 2 \\ -6 & 2 & 4 \end{bmatrix}$.

Sol. We have,

Let $$A = \begin{bmatrix} 3 & -1 & 2 \\ -3 & 1 & 2 \\ -6 & 2 & 4 \end{bmatrix}$$

Applying $R_2 \to R_2 + R_1$ and $R_3 \to R_3 + 2R_1$, we get
$$A = \begin{bmatrix} 3 & -1 & 2 \\ 0 & 0 & 4 \\ 0 & 0 & 8 \end{bmatrix}$$

Applying $R_3 \to R_3 - 2R_2$, we get
$$A = \begin{bmatrix} 3 & -1 & 2 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix}$$

Applying $R_1 \to \left(\frac{1}{3}\right) R_1$ and $R_2 \to \left(\frac{1}{4}\right) R_2$, then
$$A = \begin{bmatrix} 1 & -\frac{1}{3} & \frac{2}{3} \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

This is Echelon form of matrix A.

$\therefore$ Rank = Number of non-zero rows $\Rightarrow \rho(A) = 2$

Aliter $$|A| = \begin{vmatrix} 3 & -1 & 2 \\ -3 & 1 & 2 \\ -6 & 2 & 4 \end{vmatrix}$$
$$= 3(4 - 4) + 1(-12 + 12) + 2(-6 + 6) = 0$$

$\therefore$ Rank of $A \neq 3$ but less than 3.

There will be ${}^3C_2 \times {}^3C_2 = 9$ square minors of order 2. Now, we consider of there minors.

(i) $\begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} = 0$ (ii) $\begin{vmatrix} 3 & 2 \\ -6 & 4 \end{vmatrix} = 24 \neq 0$

Hence, all minors are not zero.

Hence, rank of A is 2. $\Rightarrow \rho(A) = 2$

Solutions of Linear Simultaneous Equations Using Rank Method

Let us consider a system of n linear equations in n unknowns say $x_1, x_2, x_3, ..., x_n$ given as below.

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots + a_{2n}x_n &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \ldots + a_{3n}x_n &= b_3 \\ \ldots \quad \ldots \ldots \ldots \ldots \ldots \quad \ldots &\ \ldots \\ \ldots \quad \ldots \ldots \ldots \ldots \ldots \quad \ldots &\ \ldots \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \ldots + a_{mn}x_n &= b_m \end{aligned}\right\} \quad \ldots(i)$$

We write the above system of Eq. (i) in the matrix form as
$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & \cdots & a_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \cdots \\ \cdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \cdots \\ \cdots \\ b_m \end{bmatrix}$$

$$\Rightarrow \qquad AX = B \qquad \ldots(ii)$$

where $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & \cdots & a_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & \cdots & a_{mn} \end{bmatrix}$,

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \cdots \\ \cdots \\ x_n \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \cdots \\ \cdots \\ b_m \end{bmatrix}$$

The matrix A is called the coefficient matrix and the matrix
$$C = [A : B] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & \cdots & a_{1n} & \vdots & b_1 \\ a_{21} & a_{22} & a_{23} & \cdots & \cdots & a_{2n} & \vdots & b_2 \\ a_{31} & a_{32} & a_{33} & \cdots & \cdots & a_{3n} & \vdots & b_3 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots & \cdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & \cdots & a_{mn} & \vdots & b_m \end{bmatrix}$$

is called the augmented matrix of the given system of equations.

Types of Equations

1. **Consistent Equation** If $\rho(A) = \rho(C)$
   - (i) **Unique Solution** If $\rho(A) = \rho(C) = n$, where n = number of knowns.
   - (ii) **Infinite Solution** If $\rho(A) = \rho(C) = r$, where $r < n$.
2. **Inconsistent Equation** If $\rho(A) \neq \rho(C)$, then no solution.

Example 50. Determine for what values of λ and μ the following system of equations
$$x + y + z = 6,$$
$$x + 2y + 3z = 10$$

and $\quad x+2y+\lambda z=\mu$

have (i) no solution? (ii) a unique solution? (iii) an infinite number of solutions?

Sol. We can write the above system of equations in the matrix form

$$\begin{bmatrix}1 & 1 & 1\\1 & 2 & 3\\1 & 2 & \lambda\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}6\\10\\\mu\end{bmatrix}$$

$$\Rightarrow \qquad AX=B$$

where $A=\begin{bmatrix}1 & 1 & 1\\1 & 2 & 3\\1 & 2 & \lambda\end{bmatrix}$, $X=\begin{bmatrix}x\\y\\z\end{bmatrix}$ and $B=\begin{bmatrix}6\\10\\\mu\end{bmatrix}$

$\therefore$ The augmented matrix

$$C=[A:B]=\begin{bmatrix}1 & 1 & 1 & \vdots & 6\\1 & 2 & 3 & \vdots & 10\\1 & 2 & \lambda & \vdots & \mu\end{bmatrix}$$

Applying $R_2 \to R_2-R_1$ and $R_3 \to R_3-R_1$, we get

$$C=\begin{bmatrix}1 & 1 & 1 & \vdots & 6\\0 & 1 & 2 & \vdots & 4\\0 & 1 & \lambda-1 & \vdots & \mu-6\end{bmatrix}$$

Applying $R_3 \to R_3-R_2$, we get

$$C=\begin{bmatrix}1 & 1 & 1 & \vdots & 6\\0 & 1 & 2 & \vdots & 4\\0 & 0 & \lambda-3 & \vdots & \mu-10\end{bmatrix}$$

(i) No solution $\rho(A)\neq\rho(C)$

i.e. $\lambda-3=0$ and $\mu-10\neq 0$

$\therefore \qquad \lambda=3$ and $\mu\neq 10$

(ii) A unique solution $\rho(A)=\rho(C)=3$

i.e., $\qquad \lambda-3\neq 0$ and $\mu\in R$

$\therefore \qquad \lambda\neq 3$ and $\mu\in R$

(iii) Infinite number of solutions

$\rho(A)=\rho(C)\,(\angle 3)$

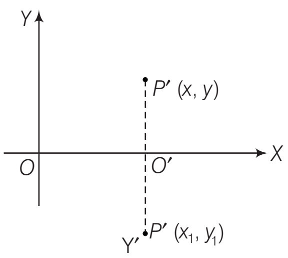
i.e. $\lambda-3=0$ and $\mu-10=0$

$\therefore \qquad \lambda=3$ and $\mu=10$

Reflection Matrix

(i) Reflection in the X-axis

Let $P(x,y)$ be any point and $P'(x_1,y_1)$ be its image after reflection in the X-axis, then



$$\begin{cases}x_1=x\\y_1=-y\end{cases} \qquad [O' \text{ is the mid-point of } P \text{ and } P']$$

These may be rewritten as

$$\begin{cases}x_1=1\cdot x+0\cdot y\\y_1=0\cdot x+(-1)\cdot y\end{cases}$$

These system of equations in the matrix form are written as below.

$$\begin{bmatrix}x_1\\y_1\end{bmatrix}=\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}$$

Thus, **the matrix $\begin{bmatrix}\mathbf{1} & \mathbf{0}\\\mathbf{0} & \mathbf{-1}\end{bmatrix}$ describes the reflection of a point $P(x,y)$ in the X-axis.**

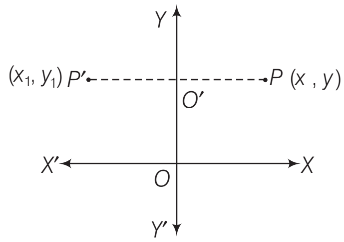
(ii) Reflection in the Y-axis

Let $P(x,y)$ be any point and $P'(x_1,y_1)$ be its image after reflection in the Y-axis, then

$$\begin{cases}x_1=-x\\y_1=y\end{cases} \qquad [O' \text{ is the mid-point of } P \text{ and } P']$$

These may be written as

$$\begin{cases}x_1=(-1)\cdot x+0\cdot y\\y_1=\ \ 0\cdot x+1\cdot y\end{cases}$$



These system of equations in the matrix form are written as below.

$$\begin{bmatrix}x_1\\y_1\end{bmatrix}=\begin{bmatrix}-1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}$$

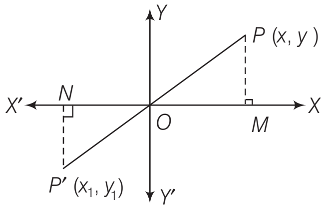
Thus, **the matrix $\begin{bmatrix}\mathbf{-1} & \mathbf{0}\\\mathbf{0} & \mathbf{1}\end{bmatrix}$ describes the reflection of a point $P(x,y)$ in the Y-axis.**

(iii) Reflection through the origin

Let $P(x,y)$ be any point and $P'(x_1,y_1)$ be its image after reflection through the origin, then

$$\begin{cases}x_1=-x\\y_1=-y\end{cases} \qquad [O' \text{ is the mid-point of } P \text{ and } P']$$

These may be written as



$$\begin{cases} x_1 = (-1)\,x + 0 \cdot y \\ y_1 = 0 \cdot x + (-1) \cdot y \end{cases}$$

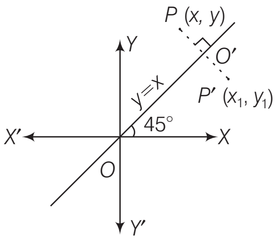
These system of equations in the matrix form are written as below.

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Thus, **the matrix $\begin{bmatrix} \mathbf{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{-1} \end{bmatrix}$ describes the reflection of a point $P(x, y)$ through the origin.**

(iv) Reflection in the line $y = x$

Let $P(x, y)$ be any point and $P'(x_1, y_1)$ be its image after reflection in the line $y = x$, then



$$\begin{cases} x_1 = y \\ y_1 = x \end{cases} \quad [O' \text{ is the mid-point of } P \text{ and } P']$$

These may be written as

$$\begin{cases} x_1 = 0 \cdot x + 1 \cdot y \\ y_1 = 1 \cdot x + 0 \cdot y \end{cases}$$

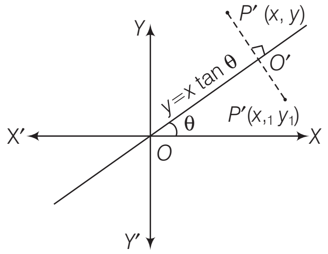
These system of equations in the matrix form are written as below.

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Thus, **the matrix $\begin{bmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{bmatrix}$ describes the reflection of a point $P(x, y)$ in the line $y = x$.**

(v) Reflection in the line $y = x \tan\theta$

Let $P(x, y)$ be any point and $P'(x_1, y_1)$ be its image after reflection in the line $y = x\tan\theta$, then



$$\begin{cases} x_1 = x\cos 2\theta + y\sin 2\theta \\ y_1 = x\sin 2\theta - y\cos 2\theta \end{cases}$$

$[O'$ is the mid-point of P and $P']$

These may be written as

$$\begin{cases} x_1 = x \cdot \cos 2\theta + y \cdot \sin 2\theta \\ y_1 = x \cdot \sin 2\theta + y \cdot (-\cos 2\theta) \end{cases}$$

These system of equations in the matrix form are written as below.

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Thus, the matrix $\begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$ describes the reflection of a point $P(x, y)$ in the line $y = x\tan\theta$.

Note

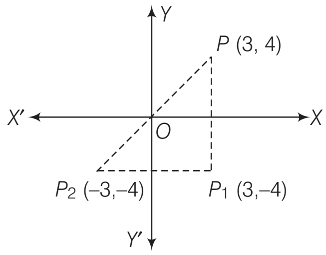
By putting $\theta = 0, \frac{\pi}{2}, \frac{\pi}{4}$, we can get the reflection matrices in the X-axis, Y-axis and the line $y = x$, respectively.

Example 51. The point $P(3, 4)$ undergoes a reflection in the X-axis followed by a reflection in the Y-axis. Show that their combined effect is the same as the single reflection of $P(3, 4)$ in the origin.

Sol. Let $P_1(x_1, y_1)$ be the image of $P(3, 4)$ after reflection in the X-axis. Then,

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ -4 \end{bmatrix}$$

Therefore, the image of $P(3, 4)$ after reflection in the X-axis is $P_1(3, -4)$.



Now, let $P_2(x_2, y_2)$ be the image of $P_1(3, -4)$ after reflection in the Y-axis, then

$$\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ -4 \end{bmatrix} = \begin{bmatrix} -3 \\ -4 \end{bmatrix}$$

Therefore, the image of P_1 (3, – 4) after reflection in the Y-axis is P_2 (– 3, – 4).

Further, let P_3 (x_3, y_3) be the image of P (3, 4) in the origin O. Then,

$$\begin{bmatrix} x_3 \\ y_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} -3 \\ -4 \end{bmatrix}$$

Therefore, the image of P (3, 4) after reflection in the origin is P_3 (– 3, – 4) . It is clear that $P_2 = P_3$

Hence, the image of P_2 of P often successive reflections in their X-axis and Y-axis is the same as P_3, which is single reflection of P in the origin.

Example 52. Find the image of the point (– 2, – 7) under the transformations $(x, y) \to (x - 2y, -3x + y)$.

Sol. Let (x_1, y_1) be the image of the point (x, y) under the given transormations, then

$$\begin{cases} x_1 = x - 2y = 1 \cdot x + (-2) \cdot y \\ y_1 = -3x + y = (-3) \cdot x + 1 \cdot y \end{cases}$$

$$\Rightarrow \quad \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -2 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ -7 \end{bmatrix} = \begin{bmatrix} -2+14 \\ 6-7 \end{bmatrix} = \begin{bmatrix} 12 \\ -1 \end{bmatrix}$$

Therefore, the required image is (12, – 1).

Example 53. The image of the point $A(2, 3)$ by the line mirror $y = x$ is the point B and the image of B by the line mirror $y = 0$ is the point (α, β). Find α and β.

Sol. Let $B(x_1, y_1)$ be the image of the point A(2,3) about the line $y = x$, then

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

Therefore, the image of A (2, 3) by the line mirror $y = x$ is B (3, 2).

Given, image of B by the line mirror $y = 0$ (X-axis) is (α, β), then

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

On comparing, we get $\alpha = 3$ and $\beta = -2$.

Example 54. Find the image of the point $(-\sqrt{2}, \sqrt{2})$ by the line mirror $y = x \tan\left(\frac{\pi}{8}\right)$.

Sol. Let (x_1, y_1) be the image of $(-\sqrt{2}, \sqrt{2})$ about the line $y = x \tan\left(\frac{\pi}{8}\right)$.

On comparing $y = x \tan\left(\frac{\pi}{8}\right)$ by $y = x \tan\theta$

$$\therefore \quad \theta = \frac{\pi}{8}$$

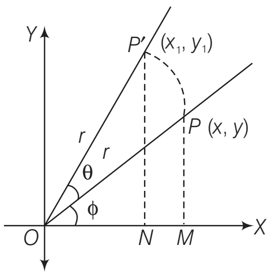
Now, $$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix} \begin{bmatrix} -\sqrt{2} \\ \sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} -\sqrt{2} \\ \sqrt{2} \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$$

On comparing $x_1 = 0$ and $y_1 = -2$.

Therefore, the required image is (0, – 2).

Rotation Through an Angle θ



Let $P(x, y)$ be any point such that $OP = r$ and $\angle POX = \phi$. Let OP rotate through an angle θ in the anti-clockwise direction such that $P'(x_1, y_1)$ is the new position.

$$\therefore \quad OP' = r, \qquad [\because OP = OP']$$

then $$\begin{cases} x_1 = x\cos\theta - y\sin\theta \\ x_1 = x\sin\theta + y\cos\theta \end{cases}$$

These system of equations in the matrix form are written as below.

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Thus, the matrix $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ describes a rotation of a line segment through an angle θ.

Remember Use of complex number

$$OP' = OP\, e^{i\theta}, i = \sqrt{-1}$$

$$(x_1 + iy_1) = (x + iy)(\cos\theta + i\sin\theta)$$

$$= (x\cos\theta - y\sin\theta) + i(x\sin\theta + y\cos\theta)$$

$$\therefore \quad x_1 = x\cos\theta - y\sin\theta$$

and $$y_1 = x\sin\theta + y\cos\theta$$

Example 55. Find the matrices of transformation T_1T_2 and T_2T_1 when T_1 is rotation through an angle 60° and T_2 is the reflection in the Y-axis. Also, verify that $T_1T_2 \neq T_2T_1$.

Sol. $T_1=\begin{bmatrix}\cos 60^\circ & -\sin 60^\circ\\ \sin 60^\circ & \cos 60^\circ\end{bmatrix}=\begin{bmatrix}\frac{1}{2} & -\frac{\sqrt{3}}{2}\\ \frac{\sqrt{3}}{2} & \frac{1}{2}\end{bmatrix}=\frac{1}{2}\begin{bmatrix}1 & -\sqrt{3}\\ \sqrt{3} & 1\end{bmatrix}$

and $T_2=\begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}$

$\therefore\quad T_1T_2=\frac{1}{2}\begin{bmatrix}1 & -\sqrt{3}\\ \sqrt{3} & 1\end{bmatrix}\times\begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}=\frac{1}{2}\begin{bmatrix}-1+0 & 0-\sqrt{3}\\ -\sqrt{3}+0 & 0+1\end{bmatrix}$

$$=\frac{1}{2}\begin{bmatrix}-1 & -\sqrt{3}\\ -\sqrt{3} & 1\end{bmatrix}=\begin{bmatrix}-\frac{1}{2} & -\frac{\sqrt{3}}{2}\\ -\frac{\sqrt{3}}{2} & \frac{1}{2}\end{bmatrix}\quad \text{...(i)}$$

and $T_2T_1=\begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}\times\frac{1}{2}\begin{bmatrix}1 & -\sqrt{3}\\ \sqrt{3} & 1\end{bmatrix}=\frac{1}{2}\begin{bmatrix}-1+0 & \sqrt{3}+0\\ 0+\sqrt{3} & 0+1\end{bmatrix}$

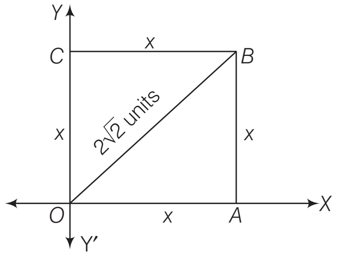
$$=\frac{1}{2}\begin{bmatrix}-1 & \sqrt{3}\\ \sqrt{3} & 1\end{bmatrix}=\begin{bmatrix}-\frac{1}{2} & \frac{\sqrt{3}}{2}\\ \frac{\sqrt{3}}{2} & \frac{1}{2}\end{bmatrix}\quad \text{...(ii)}$$

It is clear from Eqs.(i) and (ii), then

$$T_1T_2 \neq T_2T_1$$

Example 56. Write down 2×2 matrix A which corresponds to a counterclockwise rotation of 60° about the origin. In the diagram the square $OABC$ has its diagonal OB of $2\sqrt{2}$ units in length. The square is rotated counterclockwise about O through 60°. Find the coordiates of the vertices of the square after rotating.

Sol. The matrix describes a rotation through an angle 60° in counterclockwise direction is



$$\begin{bmatrix}\cos 60^\circ & -\sin 60^\circ\\ \sin 60^\circ & \cos 60^\circ\end{bmatrix}=\begin{bmatrix}\frac{1}{2} & -\frac{\sqrt{3}}{2}\\ \frac{\sqrt{3}}{2} & \frac{1}{2}\end{bmatrix}=\frac{1}{2}\begin{bmatrix}1 & -\sqrt{3}\\ \sqrt{3} & 1\end{bmatrix}$$

Since, each side of the square be x,

then $\quad x^2+x^2=(2\sqrt{2})^2$

$\Rightarrow\quad 2x^2=8 \Rightarrow x^2=4$

$\therefore\quad x=2$ units

Therefore, the coordinates of the vertices O, A, B and C are (0, 0), (2, 0),
(2, 2) and (0, 2), respectively. Let after rotation A map into A', B map into B', C map into C' but the O map into itself.

If coordinates of A', B' and C' are (x', y'), (x'', y'') and (x''', y'''), respectively.

$\therefore\quad \begin{bmatrix}x'\\ y'\end{bmatrix}=\frac{1}{2}\begin{bmatrix}1 & -\sqrt{3}\\ \sqrt{3} & 1\end{bmatrix}\begin{bmatrix}2\\ 0\end{bmatrix}=\frac{1}{2}\begin{bmatrix}2\\ 2\sqrt{3}\end{bmatrix}=\begin{bmatrix}1\\ \sqrt{3}\end{bmatrix}$

$\therefore\quad x'=1, y'=\sqrt{3} \Rightarrow A(2,0)\rightarrow A'(1,\sqrt{3})$

and $\begin{bmatrix}x''\\ y''\end{bmatrix}=\frac{1}{2}\begin{bmatrix}1 & -\sqrt{3}\\ \sqrt{3} & 1\end{bmatrix}\begin{bmatrix}2\\ 2\end{bmatrix}=\frac{1}{2}\begin{bmatrix}2-2\sqrt{3}\\ 2\sqrt{3}+2\end{bmatrix}=\begin{bmatrix}1-\sqrt{3}\\ \sqrt{3}+1\end{bmatrix}$

$\therefore\quad x''=1-\sqrt{3}, y''=\sqrt{3}+1$

$\Rightarrow\quad B(2,2)\rightarrow B'(1-\sqrt{3}, \sqrt{3}+1)$

$$\begin{bmatrix}x'''\\ y'''\end{bmatrix}=\frac{1}{2}\begin{bmatrix}1 & -\sqrt{3}\\ \sqrt{3} & 1\end{bmatrix}\begin{bmatrix}0\\ 2\end{bmatrix}=\frac{1}{2}\begin{bmatrix}-2\sqrt{3}\\ 2\end{bmatrix}=\begin{bmatrix}-\sqrt{3}\\ 1\end{bmatrix}$$

$\therefore\quad x'''=-\sqrt{3}, y'''=1$

$\Rightarrow\quad C(0,2)\rightarrow C'(-\sqrt{3},1)$

Eigen Values or Characteristic roots and Characteristic Vectors of a square matrix

Let X be any non-zero vector satisfying

$$AX=\lambda X \quad \text{...(i)}$$

where λ is any scalar, then λ is said to be eigen value or characteristic root of square matrix A and the vector X is called eigen vector or characteristic vector of matrix A.

Now, from Eq. (i), we have

$$(A-\lambda I)X=O$$

Since, $X\neq O$, we deduce that the matrix $(A-\lambda I)$ is singular, so that its determinant is 0

i.e.

$$|A-\lambda I|=0 \quad \text{...(ii)}$$

is called characteristic equation of matrix A.

If A be $n\times n$ matrix, then equation $|A-\lambda I|=0$ reduces to polynomial equation of nth from degree in λ, which given n values of λ i.e., matrix A will have n characteristic roots or eigen values.

Important Properties of Eigen Values

(i) Any square matrix A and its transpose A^T have the same eigen values.

(ii) The sum of the eigen values of a matrix is equal to the trace of the matrix.

(iii) The product the eigen values of a matrix A is equal to the determinant of A.

(iv) If $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \ldots, \lambda_n$ are the eigen values of A, then the eigen values of

(a) kA are $k\lambda_1, k\lambda_2, k\lambda_3, k\lambda_4, \ldots, k\lambda_n$.

(b) A^m are $\lambda_1^m, \lambda_2^m, \lambda_3^m, \lambda_4^m, \ldots, \lambda_n^m$.

(c) A^{-1} are $\dfrac{1}{\lambda_1}, \dfrac{1}{\lambda_2}, \dfrac{1}{\lambda_3}, \dfrac{1}{\lambda_4}, \ldots, \dfrac{1}{\lambda_n}$.

Remark

1. All the eigen values of a real symmetric matrix are real and the eigen vectors corresponding to two distinct eigen values are orthogonal.
2. All the eigen values of a real skew-symmetric matrix are purely imaginary or zero. An odd order skew-symmetric matrix is singular and hence has zero as an eigen value.

Example 57. Let matrix $A = \begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -4 & -3 \end{bmatrix}$, find the non-zero column vector X such that $AX = \lambda X$ for some scalar λ.

Sol. The characteristic equation is $|A - \lambda I| = 0$

$$\Rightarrow \quad \begin{bmatrix} 4-\lambda & 6 & 6 \\ 1 & 3-\lambda & 2 \\ -1 & -4 & -3-\lambda \end{bmatrix} = 0$$

$$\Rightarrow \quad \lambda^3 - 4\lambda^2 - \lambda + 4 = 0$$

$$\text{or} \quad (\lambda + 1)(\lambda - 1)(\lambda - 4) = 0$$

The eigen values are $\lambda = -1, 1, 4$

If $\lambda = -1$, we get $5x + 6y + 6z = 0$, $x + 4y + 2z = 0$

and $\quad -x - 4y - 2z = 0$

Giving $\quad \dfrac{x}{6} = \dfrac{y}{2} = \dfrac{z}{-7}, X = \begin{bmatrix} 6 \\ 2 \\ -7 \end{bmatrix}$

If $\lambda = 1$, we get $3x + 6y + 6z = 0$, $x + 2y + 2z = 0$

and $\quad -x - 4y - 4z = 0$

Giving, $\quad \dfrac{x}{0} = \dfrac{y}{1} = \dfrac{z}{-1}, X = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$

If $\lambda = 4$, we get $0 \cdot x + 6y + 6z = 0$, $x - y + 2z = 0$

and $-x - 4y - 7z = 0$

Giving, $\quad \dfrac{x}{3} = \dfrac{y}{1} = \dfrac{3}{-1}, x = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}$

Hence, vector are $X = \begin{bmatrix} 6 \\ 2 \\ -7 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}$.

Example 58. If A and P are the square matrices of the same order and if P be invertible, show that the matrices A and $P^{-1}AP$ have the same characteristic roots.

Sol. Let $\quad P^{-1}AP = B$

$$\therefore \quad |B - \lambda I| = |P^{-1}AP - \lambda I|$$

$$= |P^{-1}AP - P^{-1}\lambda P| \qquad [\because P^{-1}P = I]$$

$$= |P^{-1}(A - \lambda I)P|$$

$$= |P^{-1}||A - \lambda I||P|$$

$$= \frac{1}{|P|}|A - \lambda I||P| = |A - \lambda I|$$

Example 59. Show that the characteristic roots of an idempotent matrix are either zero or unity.

Sol. Let A be an idempotent matrix, then

$$A^2 = A \qquad \ldots(\text{i})$$

If λ be an eigen value of the matrix A corresponding to eigen vector X, so that

$$AX = \lambda X \qquad \ldots(\text{ii})$$

where $\quad X \neq 0$

From Eq. (ii), $\quad A(AX) = A(\lambda X)$

$\Rightarrow \quad (AA)X = \lambda(AX)$

$\Rightarrow \quad A^2X = \lambda(\lambda X) \qquad$ [from Eq. (ii)]

$\Rightarrow \quad AX = \lambda^2 X \qquad$ [from Eq. (i)]

$\Rightarrow \quad \lambda X = \lambda^2 X \qquad$ [from Eq. (ii)]

$\Rightarrow \quad (\lambda - \lambda^2)X = 0$

$\Rightarrow \quad \lambda - \lambda^2 = 0 \qquad [\because X \neq 0]$

$\therefore \quad \lambda = 0$

or $\quad \lambda = 1$

Example 60. If $3, -2$ are the eigen values of a non-singular matrix A and $|A| = 4$, find the eigen values of adj (A).

Sol. $\because A^{-1} = \dfrac{\text{adj } A}{|A|}$, if λ is eigen value of A, then λ^{-1} is eigen value of A^{-1}.

Thus, for adj $(A)X = (A^{-1}X)|A| = |A|\lambda^{-1}I$

Thus, eigen value corresponding to $\lambda = 3$ is $\dfrac{4}{3}$ and

corresponding to $\lambda = -2$ is $\dfrac{4}{-2} = -2$

Cayley-Hamilton Theorem

Every square matrix A satisfies its characteristic equation $|A - \lambda I| = 0$

i.e., $\quad a_0\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \ldots + a_n = 0$

$\therefore$ By Cayley-Hamilton theorem

$$a_0 A^n + a_1 A^{n-1} + a_2 A^{n-2} + \ldots + a_n I = O$$

$$\Rightarrow A^{-1} = -\left\{\frac{a_0}{a_n} A^{n-1} + \frac{a_1}{a_n} A^{n-2} + \frac{a_2}{a_n} A^{n-3} + \ldots + \frac{a_{n-1}}{a_n} I\right\}$$

Example 61. Find the characteristic equation of the matrix $A = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$ and hence find its inverse using Cayley-hamilton theorem.

Sol. Characteristic equation is

$$|A - \lambda I| = 0 \Rightarrow \begin{bmatrix} 2-\lambda & 1 \\ 3 & 2-\lambda \end{bmatrix} = 0$$

$$\Rightarrow \quad (2-\lambda)^2 - 3 = 0$$

$$\Rightarrow \quad \lambda^2 - 4\lambda + 1 = 0$$

$\therefore$ By Cayley-hamilton theorem,

$$A^2 - 4A + I = O \text{ or } I = 4A - A^2$$

Multiplying by A^{-1}, we get

$$A^{-1} = 4A^{-1}A - A^{-1}AA$$

$$= 4I - IA = 4I - A$$

$$= 4\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$$

Exercise for Session 4

1 If the system of equations $ax + y = 1, x + 2y = 3, 2x + 3y = 5$ are consistent, then a is given by

(a) 0 (b) 1 (c) 2 (d) None of these

2 The system of equations $x + y + z = 2, 2x + y - z = 3, 3x + 2y + \lambda z = 4$ has unique solution if

(a) $\lambda \neq 0$ (b) $-1 < \lambda < 1$ (c) $\lambda = 0$ (d) $-2 < \lambda < 2$

3 The value of a for which the following system of equations $a^3 x + (a+1)^3 y + (a+2)^3 z = 0$, $ax + (a+1)y + (a+2)z = 0, x + y + z = 0$ has a non-trivial solution is equal to

(a) 2 (b) 1 (c) 0 (d) –1

4 The number of solutions of the set of equations

$\frac{2x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0, -\frac{x^2}{a^2} + \frac{2y^2}{b^2} - \frac{z^2}{c^2} = 0, -\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{2z^2}{c^2} = 0$ is

(a) 6 (b) 7 (c) 8 (d) 9

5 The matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is the matrix reflection in the line

(a) $x = 1$ (b) $x + y = 1$ (c) $y = 1$ (d) $x = y$

6 The matrix S is rotation through an angle 45° and G is the reflection about the line $y = 2x$, then $(SG)^2$ is equal to

(a) $7I$ (b) $5I$ (c) $3I$ (d) I

7 If $A = \begin{bmatrix} -2 & 3 \\ -1 & 1 \end{bmatrix}$, then A^3 is equal to

(a) $2A$ (b) A (c) $2I$. (d) I

8 If $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$ and the sum of eigen values of A is m and product of eigen values of A is n, then $m + n$ is equal to

(a) 10 (b) 12 (c) 14 (d) 16

9 If $A = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}$ and θ be the angle between the two non-zero column vectors X such that $AX = \lambda X$ for some scalar λ, then $9\sec^2\theta$ is equal to

(a) 13 (b) 12 (c) 11 (d) 10

Answers

Exercise for Session 4

1. (a) **2.** (a) **3.** (d) **4.** (d) **5.** (d) **6.** (d)
7. (d) **8.** (b) **9.** (d)