Session 7

Application to Problems of Maxima and Minima (Without Calculus)

Application to Problems of Maxima and Minima (Without Calculus)

Suppose that $a_1, a_2, a_3, \ldots, a_n$ are n positive variables and k is constant, then

(i) If $a_1 + a_2 + a_3 + \ldots + a_n = k$ (constant), the value of $a_1 a_2 a_3 \ldots a_n$ is greatest when $a_1 = a_2 = a_3 = \ldots = a_n$, so that the greatest value of $a_1 a_2 a_3 \ldots a_n$ is $\left(\frac{k}{n}\right)^n$.

Proof $\because$ AM $\geq$ GM

$$\therefore \quad \frac{a_1 + a_2 + a_3 + \ldots + a_n}{n} \geq (a_1 a_2 a_3 \ldots a_n)^{1/n}$$

$$\Rightarrow \quad \frac{k}{n} \geq (a_1\ a_2\ a_3 \ldots a_n)^{1/n}$$

$$\text{or} \quad (a_1 a_2 a_3 \ldots a_n) \leq \left(\frac{k}{n}\right)^n$$

Here, $a_1 = a_2 = a_3 = \ldots = a_n$

$\therefore$ Greatest value of $a_1 a_2 a_3 \ldots a_n$ is $\left(\frac{k}{n}\right)^n$.

Example 105. Find the greatest value of xyz for positive values of x, y, z subject to the condition $yz + zx + xy = 12$.

Sol. Given, $yz + zx + xy = 12$ (constant), the value of $(yz)(zx)(xy)$ is greatest when $yz = zx = xy$

Here, $n = 3$ and $k = 12$

Hence, greatest value of $(yz)(zx)(xy)$ is $\left(\frac{12}{3}\right)^3$ i.e. 64.

$\therefore$ Greatest value of $x^2y^2z^2$ is 64.

Thus, greatest value of xyz is 8.

Aliter

Given $yz + zx + xy = 12$, the greatest value of $(yz)(zx)(xy)$ is greatest when

$$yz = zx = xy = c \qquad \text{[say]}$$

Since, $yz + zx + xy = 12$

$\therefore \quad c + c + c = 12$

$\Rightarrow \quad 3c = 12$ or $c = 4$

$\therefore \quad yz = zx = xy = 4$

Hence, greatest value of $(yz)(zx)(xy)$ is $4 \cdot 4 \cdot 4$

i.e., greatest value of $x^2y^2z^2$ is 64.

Hence, greatest value of xyz is 8.

Example 106. Find the greatest value of x^3y^4, if $2x + 3y = 7$ and $x \geq 0, y \geq 0$.

Sol. To find the greatest value of x^3y^4 or $(x)(x)(x)(y)(y)(y)(y)$

Here, x repeats 3 times and y repeats 4 times.

Given, $2x + 3y = 7$,

then multiplying and dividing coefficients of x and y by 3 and 4, respectively.

Rewrite $\quad 3\left(\frac{2x}{3}\right) + 4\left(\frac{3y}{4}\right) = 7$

$$\text{or} \quad \underset{1}{\left(\frac{2x}{3}\right)} + \underset{2}{\left(\frac{2x}{3}\right)} + \underset{3}{\left(\frac{2x}{3}\right)} + \underset{4}{\left(\frac{3y}{4}\right)} + \underset{5}{\left(\frac{3y}{4}\right)} + \underset{6}{\left(\frac{3y}{4}\right)} + \underset{7}{\left(\frac{3y}{4}\right)} = 7$$

Here, $k = 7$ and $n = 7$

Hence, greatest value of

$$\left(\frac{2x}{3}\right)\left(\frac{2x}{3}\right)\left(\frac{2x}{3}\right)\left(\frac{3y}{4}\right)\left(\frac{3y}{4}\right)\left(\frac{3y}{4}\right)\left(\frac{3y}{4}\right) \text{ is } \left(\frac{7}{7}\right)^7.$$

or greatest value of $\frac{2^3 \cdot 3^4}{3^3 \cdot 4^4}\, x^3y^4$ is 1.

Thus, greatest value of x^3y^4 is $\frac{32}{3}$.

(ii) If $a_1 a_2 a_3 \ldots a_n = k$ (constant), the value of $a_1 + a_2 + a_3 + \ldots + a_n$ is least when $a_1 = a_2 = a_3 = \ldots = a_n$, so that the least of $a_1 + a_2 + a_3 + \ldots + a_n$ is $n(k)^{1/n}$.

Proof $\because$ AM $\geq$ GM

$$\therefore \frac{a_1 + a_2 + a_3 + \ldots + a_n}{n} \geq (a_1 a_2 a_3 \ldots a_n)^{1/n} = (k)^{1/n}$$

$$\Rightarrow \quad \frac{a_1 + a_2 + a_3 + \ldots + a_n}{n} \geq (k)^{1/n}$$

$$\text{or} \quad a_1 + a_2 + a_3 + \ldots + a_n \geq n(k)^{1/n}$$

Here, $a_1 = a_2 = a_3 = \ldots = a_n$

$\therefore$ Least value of $a_1 + a_2 + a_3 + \ldots + a_n$ is $n(k)^{1/n}$

Example 107. ***Find the least value of*** $3x+4y$ ***for positive values of*** x ***and*** y***, subject to the condition*** $x^2y^3=6$**.**

Sol. Given, $x^2y^3 = 6$

or $\quad (x)(x)(y)(y)(y) = 6$

Here, x repeats 2 times and y repeats 3 times

$$\therefore \quad 3x + 4y = 2\left(\frac{3x}{2}\right) + 3\left(\frac{4y}{3}\right)$$

$$= \underset{1}{\left(\frac{3x}{2}\right)} + \underset{2}{\left(\frac{3x}{2}\right)} + \underset{3}{\left(\frac{4y}{3}\right)} + \underset{4}{\left(\frac{4y}{3}\right)} + \underset{5}{\left(\frac{4y}{3}\right)}$$

multiplying and dividing coefficient of x and y by 2 and 3 respectively and write $x^2y^3 = 6$

$$\Rightarrow \left(\frac{3x}{2}\right)\left(\frac{3x}{2}\right)\left(\frac{4y}{3}\right)\left(\frac{4y}{3}\right)\left(\frac{4y}{3}\right) = \frac{3^2}{2^2} \times \frac{4^3}{3^3} \times 6 = 32$$

Here, $n = 5$ and $k = 32$

Hence, least value of $\frac{3x}{2} + \frac{3x}{2} + \frac{4y}{3} + \frac{4y}{3} + \frac{4y}{3}$

$$= 5(32)^{1/5} = 10$$

i.e. least value of $3x + 4y = 10$

Example 108. Find the minimum value of $bcx + cay + abz$, when $xyz = abc$.

Sol. To find the minimum value of

$$bcx + cay + abz,$$

write, $\quad xyz = abc$

or $\quad (bcx)(cay)(abz) = a^3b^3c^3 = k \quad$ [constant]

Here, $\quad n = 3$

Hence, minimum value of $bcx + cay + abz = n(k)^{1/n}$

$$= 3(a^3b^3c^3)^{1/3} = 3abc$$

An Important Result

If $a_i > 0$, $i = 1, 2, 3, \ldots, n$ which are not identical, then

(i) $\dfrac{a_1^m + a_2^m + \ldots + a_n^m}{n} > \left(\dfrac{a_1 + a_2 + \ldots + a_n}{n}\right)^m$; If $m < 0$ or $m > 1$

(ii) $\dfrac{a_1^m + a_2^m + \ldots + a_n^m}{n} < \left(\dfrac{a_1 + a_2 + \ldots + a_n}{n}\right)^m$; If $0 < m < 1$

Remark

If $a_1 = a_2 = \ldots = a_n$, then use equal sign in inequalities.

Example 109. If a, b, c be positive real numbers, prove that $\dfrac{a}{b+c} + \dfrac{b}{c+a} + \dfrac{c}{a+b} \geq \dfrac{3}{2}$.

Sol. Arithmetic mean of (-1) th powers $\geq (-1)$ th power of arithmetic mean

$$\frac{\left(\frac{b+c}{a+b+c}\right)^{-1} + \left(\frac{c+a}{a+b+c}\right)^{-1} + \left(\frac{a+b}{a+b+c}\right)^{-1}}{3} \geq \left(\frac{\frac{b+c}{a+b+c} + \frac{c+a}{a+b+c} + \frac{a+b}{a+b+c}}{3}\right)^{-1}$$

$$\Rightarrow \frac{\frac{a+b+c}{b+c} + \frac{a+b+c}{c+a} + \frac{a+b+c}{a+b}}{3} \geq \left(\frac{2}{3}\right)^{-1}$$

$$\Rightarrow \frac{a}{b+c} + 1 + \frac{b}{c+a} + 1 + \frac{c}{a+b} + 1 \geq \frac{9}{2}$$

$$\Rightarrow \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{9}{2} - 3$$

$$\text{or} \quad \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}$$

Example 110. ***If*** a ***and*** b ***are positive and*** $a+b=1$**,** ***show that*** $\left(a+\dfrac{1}{a}\right)^2 + \left(b+\dfrac{1}{b}\right)^2 > \dfrac{25}{2}$**.**

Sol. Since, AM of 2nd powers > 2nd power of AM

$$\therefore \quad \frac{\left(a+\frac{1}{a}\right)^2 + \left(b+\frac{1}{b}\right)^2}{2} > \left(\frac{a+\frac{1}{a}+b+\frac{1}{b}}{2}\right)^2$$

$$= \frac{1}{4}(a+b+a^{-1}+b^{-1})^2 = \frac{1}{4}(1+a^{-1}+b^{-1})^2 \quad [\because a+b=1]$$

$$\therefore \quad \left(a+\frac{1}{a}\right)^2 + \left(b+\frac{1}{b}\right)^2 > \frac{1}{2}(1+a^{-1}+b^{-1})^2 \quad \ldots\text{(i)}$$

$$\text{Again,} \quad \frac{a^{-1}+b^{-1}}{2} > \left(\frac{a+b}{2}\right)^{-1} = \left(\frac{1}{2}\right)^{-1} = 2$$

$$\text{or} \quad \frac{a^{-1}+b^{-1}}{2} > 2$$

$$\Rightarrow \quad a^{-1}+b^{-1} > 4$$

$$\therefore \quad (1+a^{-1}+b^{-1}) > 5 \text{ or } (1+a^{-1}+b^{-1})^2 > 25$$

$$\Rightarrow \quad \frac{1}{2}(1+a^{-1}+b^{-1})^2 > \frac{25}{2} \quad \ldots\text{(ii)}$$

From Eqs. (i) and (ii), we get

$$\left(a+\frac{1}{a}\right)^2 + \left(b+\frac{1}{b}\right)^2 > \frac{25}{2}$$

Exercise for Session 7

1. The minimum value of $4^x + 4^{2-x}$, $x \in R$ is

(a) 0 (b) 2
(c) 4 (d) 8

2. If $0 < \theta < \pi$, then the minimum value of $\sin^3 \theta + \operatorname{cosec}^3 \theta + 2$, is

(a) 0 (b) 2
(c) 4 (d) 8

3. If a, b, c and d are four real numbers of the same sign, then the value of $\frac{a}{b} + \frac{b}{c} + \frac{c}{d} + \frac{d}{a}$ lies in the interval

(a) $[2, \infty)$ (b) $[3, \infty)$
(c) $(4, \infty)$ (d) $[4, \infty)$

4. If $0 < x < \frac{\pi}{2}$, then the minimum value of $2(\sin x + \cos x + \operatorname{cosec} 2x)^3$ is

(a) 27 (b) 13.5
(c) 6.75 (d) 0

5. If $a + b + c = 3$ and $a > 0, b > 0, c > 0$, then the greatest value of $a^2 b^3 c^2$ is

(a) $\frac{3^4 \cdot 2^{10}}{7^7}$ (b) $\frac{3^{10} \cdot 2^4}{7^7}$
(c) $\frac{3^2 \cdot 2^{12}}{7^7}$ (d) $\frac{3^{12} \cdot 2^2}{7^7}$

6. If $x + y + z = a$ and the minimum value of $\frac{a}{x} + \frac{a}{y} + \frac{a}{z}$ is 81^{λ}, then the value of λ is

(a) $\frac{1}{2}$ (b) 1
(c) $\frac{1}{4}$ (d) 2

7. a, b, c are three positive numbers and abc^2 has the greatest value $\frac{1}{64}$, then

(a) $a = b = \frac{1}{2}, c = \frac{1}{4}$ (b) $a = b = c = \frac{1}{3}$
(c) $a = b = \frac{1}{4}, c = \frac{1}{2}$ (d) $a = b = c = \frac{1}{4}$