JEE Type Solved Examples :
Single Option Correct Type Questions

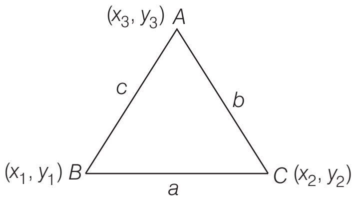
- This section contains **10 multiple choice examples.** Each example has four choices (a), (b), (c) and (d) out of which **ONLY ONE** is correct.

● **Ex. 1** *If* $(x_1 - x_2)^2 + (y_1 - y_2)^2 = a^2$, $(x_2 - x_3)^2 + (y_2 - y_3)^2 = b^2$ *and* $(x_3 - x_1)^2 + (y_3 - y_1)^2 = c^2$ *and* $k\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}^2$ $= (a+b+c)(b+c-a)(c+a-b)(a+b-c)$, *the value of k is*

(a) 1 (b) 2 (c) 4 (d) 8

Sol. (*c*) Consider the triangle with vertices $B(x_1, y_1), C(x_2, y_2)$ and $A(x_3, y_3)$ and

$AB = c$, $BC = a$ and $CA = b$



$\therefore$ Area of $\Delta ABC = \dfrac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$ ...(i)

Also, area of $\Delta ABC = \sqrt{s(s-a)(s-b)(s-c)}$,

where $2s = a + b + c$ ...(ii)

From Eqs. (i) and (ii), we get

$$\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \sqrt{s(s-a)(s-b)(s-c)}$$

On squaring and simplifying, we get

$$4\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}^2 = (a+b+c)(b+c-a)(c+a-b)(a+b-c)$$

Hence, the value of k is 4.

● **Ex. 2** *If a, b and c are complex numbers, the determinant*

$$\Delta = \begin{vmatrix} 0 & -b & -c \\ \bar{b} & 0 & -a \\ \bar{c} & \bar{a} & 0 \end{vmatrix} \text{ is}$$

(a) a non-zero real number (b) purely imaginary
(c) 0 (d) None of these

Sol. (*b*) We observe that,

$$\bar{\Delta} = \begin{vmatrix} 0 & -\bar{b} & -\bar{c} \\ b & 0 & -\bar{a} \\ c & a & 0 \end{vmatrix}$$

$$\Rightarrow \quad \bar{\Delta} = \begin{vmatrix} 0 & b & c \\ -\bar{b} & 0 & a \\ -\bar{c} & -\bar{a} & 0 \end{vmatrix}$$

[interchanging rows and columns]

$$= -\begin{vmatrix} 0 & -b & -c \\ \bar{b} & 0 & -a \\ \bar{c} & \bar{a} & 0 \end{vmatrix}$$

[taking (– 1) common from each row]

$\Rightarrow \quad \bar{\Delta} = -\Delta$

Hence, Δ is purely imaginary.

● **Ex. 3** *The equation*

$$\begin{vmatrix} (1+x)^2 & (1-x)^2 & -(2+x^2) \\ 2x+1 & 3x & 1-5x \\ x+1 & 2x & 2-3x \end{vmatrix} + \begin{vmatrix} (1+x)^2 & 2x+1 & x+1 \\ (1-x)^2 & 3x & 2x \\ 1-2x & 3x-2 & 2x-3 \end{vmatrix} = 0 \text{ has}$$

(a) no real solution
(b) 4 real solutions
(c) two real and two non-real solutions
(d) infinite number of solutions, real or non-real

Sol. (*d*) Interchanging rows and columns in first determinant, then

$$\begin{vmatrix} (1+x)^2 & 2x+1 & x+1 \\ (1-x)^2 & 3x & 2x \\ -(2+x^2) & 1-5x & 2-3x \end{vmatrix} + \begin{vmatrix} (1+x)^2 & 2x+1 & x+1 \\ (1-x)^2 & 3x & 2x \\ 1-2x & 3x-2 & 2x-3 \end{vmatrix} = 0$$

$$\Rightarrow \quad \begin{vmatrix} (1+x)^2 & 2x+1 & x+1 \\ (1-x)^2 & 3x & 2x \\ -(1+x)^2 & -2x-1 & -x-1 \end{vmatrix} = 0$$

Applying $R_3 \to R_3 + R_1$, then

$$\Rightarrow \begin{vmatrix} (1+x)^2 & 2x+1 & x+1 \\ (1-x)^2 & 3x & 2x \\ 0 & 0 & 0 \end{vmatrix} = 0$$

$$\Rightarrow \qquad 0 = 0$$

which is true for all values of x.

Hence, given equation has infinite number of solutions, real or non-real.

Ex. 4 *If X, Y and Z are positive numbers such that Y and Z have respectively 1 and 0 at their unit's place and*

$$\Delta = \begin{vmatrix} X & 4 & 1 \\ Y & 0 & 1 \\ Z & 1 & 0 \end{vmatrix}$$

If $(\Delta + 1)$ is divisible by 10, then X has at its unit's place

(a) 0 (b) 1

(c) 2 (d) 3

Sol. (c) Let $X = 10x + \lambda, Y = 10y + 1$ and $Z = 10z$, where $x, y, z \in N$, then

$$\Delta = \begin{vmatrix} X & 4 & 1 \\ Y & 0 & 1 \\ Z & 1 & 0 \end{vmatrix} = \begin{vmatrix} 10x+\lambda & 4 & 1 \\ 10y+1 & 0 & 1 \\ 10z & 1 & 0 \end{vmatrix}$$

$$= \begin{vmatrix} 10x & 4 & 1 \\ 10y & 0 & 1 \\ 10z & 1 & 0 \end{vmatrix} + \begin{vmatrix} \lambda & 4 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix}$$

$$= 10 \begin{vmatrix} x & 4 & 1 \\ y & 0 & 1 \\ z & 1 & 0 \end{vmatrix} + (1 - \lambda)$$

$\Rightarrow \Delta + 1 = 10k + (2 - \lambda)$,

where $k = \begin{vmatrix} x & 4 & 1 \\ y & 0 & 1 \\ z & 1 & 0 \end{vmatrix}$.

It is given that $(\Delta + 1)$ is divisible by 10. Therefore, $2 - \lambda = 0$ i.e., $\lambda = 2$

$$\therefore \qquad X = 10x + 2$$

$\Rightarrow 2$ is at unit's place of X.

Ex. 5 *The number of distinct values of a 2×2 determinant whose entries are from the set $\{-1, 0, 1\}$, is*

(a) 3 (b) 4 (c) 5 (d) 6

Sol. (c) Possible values are $-2, -1, 0, 1, 2$

i.e., $\begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix} = 1, \begin{vmatrix} 1 & -1 \\ 0 & 0 \end{vmatrix} = 0, \begin{vmatrix} 0 & 1 \\ 1 & -1 \end{vmatrix} = -1,$

$\begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = 2, \begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} = -2$

Ex. 6 *If $f(x) = \begin{vmatrix} (1+x)^a & (1+2x)^b & 1 \\ 1 & (1+x)^a & (1+2x)^b \\ (1+2x)^b & 1 & (1+x)^a \end{vmatrix}$; a, b being positive integers, then*

(a) constant term in $f(x)$ is 4

(b) coefficient of x in $f(x)$ is 0

(c) constant term in $f(x)$ is $(a - b)$

(d) constant term in $f(x)$ is $(a + b)$

Sol. (b) Let $\begin{vmatrix} (1+x)^a & (1+2x)^b & 1 \\ 1 & (1+x)^a & (1+2x)^b \\ (1+2x)^b & 1 & (1+x)^a \end{vmatrix}$

$$= A + Bx + Cx^2 + \dots$$

Put $x = 0$, then $A = 0$

On differentiating both sides w.r.t. x and then put $x = 0$

$$\begin{vmatrix} a & 2b & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 1 & 1 \\ 0 & a & 2b \\ 1 & 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 2b & 0 & a \end{vmatrix} = B$$

$$\therefore \qquad 0 + 0 + 0 = B$$

$$\Rightarrow \qquad B = 0$$

Hence, constant term in $f(x)$ is zero and coefficient of x in $f(x)$ is 0.

Ex. 7 *If $f_j = \sum_{i=0}^{2} a_{ij} x^i$, $j = 1, 2, 3$ and f_j' and f_j'' are denoted by $\frac{df_j}{dx}$ and $\frac{d^2 f_j}{dx^2}$ respectively, then $g(x) = \begin{vmatrix} f_1 & f_2 & f_3 \\ f_1' & f_2' & f_3' \\ f_1'' & f_2'' & f_3'' \end{vmatrix}$ is*

(a) a constant (b) a linear in x

(c) a quadratic in x (d) a cubic in x

Sol. (a) $\because g'(x) = \begin{vmatrix} f_1' & f_2' & f_3' \\ f_1' & f_2' & f_3' \\ f_1'' & f_2'' & f_3'' \end{vmatrix} + \begin{vmatrix} f_1 & f_2 & f_3 \\ f_1'' & f_2'' & f_3'' \\ f_1'' & f_2'' & f_3'' \end{vmatrix} + \begin{vmatrix} f_1 & f_2 & f_3 \\ f_1' & f_2' & f_3' \\ f_1''' & f_2''' & f_3''' \end{vmatrix}$

$$= 0 + 0 + 0 \qquad [\because f_j \text{ is a quadratic function}]$$

$$\therefore \qquad g(x) = c = \text{constant}$$

Ex. 8 *Let $\Delta_a = \begin{vmatrix} (a-1) & n & 6 \\ (a-1)^2 & 2n^2 & 4n-2 \\ (a-1)^3 & 3n^3 & 3n^2-3n \end{vmatrix}$, the value of $\sum_{a=1}^{n} \Delta_a$ is*

(a) 0 (b) $\frac{(n-1)n}{2}$

(c) $\frac{(n-1)n^2}{2}$ (d) $\frac{(n-1)n(2n-1)}{3}$

Sol. (a) $\sum_{a=1}^{n} \Delta_a = \begin{vmatrix} \sum_{a=1}^{n}(a-1) & n & 6 \\ \sum_{a=1}^{n}(a-1)^2 & 2n^2 & 4n-2 \\ \sum_{a=1}^{n}(a-1)^3 & 3n^3 & 3n^2-3n \end{vmatrix}$

$$= \begin{vmatrix} \frac{(n-1)n}{2} & n & 6 \\ \frac{(n-1)n(2n-1)}{6} & 2n^2 & 4n-2 \\ \frac{(n-1)^2n^2}{4} & 3n^3 & 3n^2-3n \end{vmatrix}$$

$$= \frac{(n-1)n^2}{2} \begin{vmatrix} 1 & 1 & 6 \\ \frac{2n-1}{3} & 2n & 4n-2 \\ \frac{(n-1)n}{2} & 3n^2 & 3n^2-3n \end{vmatrix}$$

[taking $\frac{(n-1)n}{2}$ and n common from C_1 and C_2]

Applying $C_3 \to C_3 - 6C_1$, then

$$\sum_{a=1}^{n} \Delta_a = \frac{(n-1)n^2}{2} \begin{vmatrix} 1 & 1 & 0 \\ \frac{2n-1}{3} & 2n & 0 \\ \frac{(n-1)n}{2} & 3n^2 & 0 \end{vmatrix} = 0$$

● ***Ex. 9*** *If* $\Delta(x) = \begin{vmatrix} 1 & \cos x & 1-\cos x \\ 1+\sin x & \cos x & 1+\sin x-\cos x \\ \sin x & \sin x & 1 \end{vmatrix}$, *then* $\int_0^{\pi/2} \Delta(x)\,dx$ *is equal to*

(a) $-\frac{1}{2}$ (b) 0 (c) $\frac{1}{4}$ (d) $\frac{1}{2}$

Sol. (a) Applying $C_3 \to C_3 + C_2 - C_1$, then

$$\Delta(x) = \begin{vmatrix} 1 & \cos x & 0 \\ \vdots & & \vdots \\ 1+\sin x & \cos x & 0 \\ \vdots & & \vdots \\ \sin x \;\ldots & \sin x \;\ldots & 1 \end{vmatrix}$$

$$= 1(\cos x - \cos x - \cos x \sin x) = -\frac{1}{2}\sin 2x$$

$$\therefore \int_0^{\pi/2} \Delta(x)dx = -\frac{1}{2}\int_0^{\pi/2} \sin 2x\,dx$$

$$= \frac{1}{4}[\cos 2x]_0^{\pi/2} = \frac{1}{4}(-1-1) = -\frac{1}{2}$$

● ***Ex. 10*** *Number of values of a for which the system of equations* $a^2x + (2-a)y = 4 + a^2$ *and* $ax + (2a-1)y = a^5 - 2$ *possess no solution, is*

(a) 0 (b) 1

(c) 2 (d) infinite

Sol. (c) $\because \Delta = \begin{vmatrix} a^2 & 2-a \\ a & 2a-1 \end{vmatrix} = a^2(2a-1) - a(2-a)$

$= 2a(a+1)(a-1)$

For no solution, $\Delta = 0$

$\therefore \quad a = -1, 0, 1$

$\Rightarrow \quad \Delta_1 = \begin{vmatrix} 4+a^2 & 2-a \\ a^5-2 & 2a-1 \end{vmatrix}$

Values of Δ_1 at $a = -1, 0, 1$ are $-6, 0, 6$ respectively and

$$\Delta_2 = \begin{vmatrix} a^2 & 4+a^2 \\ a & a^5-2 \end{vmatrix}$$

Values of Δ_2 at $a = -1, 0, 1$ are $2, 0, -6$, respectively.

For no solution,

$\Delta = 0$ and atleast one of Δ_1, Δ_2 is non-zero.

$\therefore \quad a = -1, 1$

JEE Type Solved Examples : More than One Correct Option Type Questions

- This section contains **5 multiple choice examples.** Each example has four choices (a), (b), (c) and (d) out of **which more than one** may be correct.

● ***Ex. 11*** *The determinant* $\begin{vmatrix} a^2 & a^2-(b-c)^2 & bc \\ b^2 & b^2-(c-a)^2 & ca \\ c^2 & c^2-(a-b)^2 & ab \end{vmatrix}$ *is divisible by*

(a) $a+b+c$ (b) $(a+b)(b+c)(c+a)$

(c) $a^2+b^2+c^2$ (d) $(a-b)(b-c)(c-a)$

Sol. (a,c,d) Applying $C_2 \to C_2 - C_1 - 2C_3$, then

$$\begin{vmatrix} a^2 & -(b^2+c^2) & bc \\ b^2 & -(c^2+a^2) & ca \\ c^2 & -(a^2+b^2) & ab \end{vmatrix} = -\begin{vmatrix} a^2 & b^2+c^2 & bc \\ b^2 & c^2+a^2 & ca \\ c^2 & a^2+b^2 & ab \end{vmatrix}$$

Applying $C_2 \to C_2 + C_1$, then

$$= -\begin{vmatrix} a^2 & a^2+b^2+c^2 & bc \\ b^2 & a^2+b^2+c^2 & ca \\ c^2 & a^2+b^2+c^2 & ab \end{vmatrix}$$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, then

$$= -\begin{vmatrix} a^2 & a^2+b^2+c^2 & bc \\ b^2-a^2 & 0 & c(a-b) \\ c^2-a^2 & 0 & -b(c-a) \end{vmatrix}$$

$$= (a^2+b^2+c^2)\begin{vmatrix} -(a+b)(a-b) & c(a-b) \\ (c+a)(c-a) & -b(c-a) \end{vmatrix}$$

$$= (a-b)(c-a)(a^2+b^2+c^2)\begin{vmatrix} -(a+b) & c \\ c+a & -b \end{vmatrix}$$

Applying $C_1 \to C_1 - C_2$, then

$$= (a-b)(c-a)(a^2+b^2+c^2)\begin{vmatrix} -(a+b+c) & c \\ (a+b+c) & -b \end{vmatrix}$$

$$= (a-b)(b-c)(c-a)(a+b+c)(a^2+b^2+c^2)$$

● **Ex. 12** *The value of θ lying between $-\frac{\pi}{4}$ and $\frac{\pi}{2}$ and $0 \le A \le \frac{\pi}{2}$ and satisfying the equation*

$$\begin{vmatrix} 1+\sin^2 A & \cos^2 A & 2\sin 4\theta \\ \sin^2 A & 1+\cos^2 A & 2\sin 4\theta \\ \sin^2 A & \cos^2 A & 1+2\sin 4\theta \end{vmatrix} = 0, \text{ are}$$

(a) $A = \frac{\pi}{4}, \theta = -\frac{\pi}{8}$ (b) $A = \frac{3\pi}{8} = \theta$

(c) $A = \frac{\pi}{5}, \theta = -\frac{\pi}{8}$ (d) $A = \frac{\pi}{6}, \theta = \frac{3\pi}{8}$

Sol. (a, b, c, d)

$$\because \begin{vmatrix} 1+\sin^2 A & \cos^2 A & 2\sin 4\theta \\ \sin^2 A & 1+\cos^2 A & 2\sin 4\theta \\ \sin^2 A & \cos^2 A & 1+2\sin 4\theta \end{vmatrix} = 0$$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, then

$$\begin{vmatrix} 1+\sin^2 A & \cos^2 A & 2\sin 4\theta \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix} = 0$$

Applying $C_1 \to C_1 + C_2$, then

$$\begin{vmatrix} 2 & \cos^2 A & 2\sin 4\theta \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix} = 0$$

$\Rightarrow \quad 1(2+2\sin 4\theta) = 0$

$\therefore \quad \sin 4\theta = -1$

$\Rightarrow \quad 4\theta = (2n-1)\frac{\pi}{2} \Rightarrow \theta = (2n-1)\frac{\pi}{8}$

For $n = 0, 2$, then $\theta = -\frac{\pi}{8}, \frac{3\pi}{8}$ and $A \in R$

● **Ex. 13** *The digits A, B, C are such that the three digit numbers $A88, 6B8, 86C$ are divisible by 72, the determinant $\begin{vmatrix} A & 6 & 8 \\ 8 & B & 6 \\ 8 & 8 & C \end{vmatrix}$ is divisible by*

(a) 72 (b) 144 (c) 288 (d) 216

Sol. (a, b, c)

$\because A88, 6B8, 86C$ are divisible by 72.

$\therefore A88, 6B8, 86C$ are also divisible by 9.

$\Rightarrow \quad A+8+8, 6+B+8, 8+6+C$

are divisible by 9, then $A = 2, B = 4, C = 4$

$$\text{Let} \quad \Delta = \begin{vmatrix} A & 6 & 8 \\ 8 & B & 6 \\ 8 & 8 & C \end{vmatrix} = \begin{vmatrix} 2 & 6 & 8 \\ 8 & 4 & 6 \\ 8 & 8 & 4 \end{vmatrix}$$

$$= 2(16-48) - 6(32-48) + 8(64-32) = 288$$

Hence, Δ is divisible by 72, 144 and 288.

● **Ex. 14** *If p, q, r and s are in AP and*

$$f(x) = \begin{vmatrix} p+\sin x & q+\sin x & p-r+\sin x \\ q+\sin x & r+\sin x & -1+\sin x \\ r+\sin x & s+\sin x & s-q+\sin x \end{vmatrix}$$

such that $\int_0^1 f(x)\,dx = -2$, the common difference of the AP can be

(a) -1 (b) 1/2 (c) 1 (d) 2

Sol. (a, c)

$$\because f(x) = \frac{1}{2}\begin{vmatrix} p+\sin x & q+\sin x & p-r+\sin x \\ 2q+2\sin x & 2r+2\sin x & -2+2\sin x \\ r+\sin x & s+\sin x & s-q+\sin x \end{vmatrix}$$

Applying $R_2 \to R_2 - (R_1 + R_3)$, then

$$f(x) = \frac{1}{2}\begin{vmatrix} p+\sin x & q+\sin x & p-r+\sin x \\ 0 & 0 & -2 \\ r+\sin x & s+\sin x & s-q+\sin x \end{vmatrix}$$

[$\because 2q = p+r, 2r = q+s$ and $p+s = q+r$]

$$= -\frac{(-2)}{2}\begin{vmatrix} p+\sin x & q+\sin x \\ r+\sin x & s+\sin x \end{vmatrix}$$

Applying $C_2 \to C_2 - C_1$, then

$$f(x) = \begin{vmatrix} p + \sin x & D \\ p + 2D + \sin x & D \end{vmatrix}$$

[where D = common difference]

$$= D[p + \sin x - p - 2D - \sin x] = -2D^2$$

and $\int_0^1 f(x)dx = -4$

$\Rightarrow \int_0^1 (-2D^2)dx = -4 \Rightarrow -2D^2 = -2$

$\therefore D^2 = 1 \Rightarrow D = \pm 1$

- **Ex. 15** *If the system of equations $a^2x - by = a^2 - b$ and $bx - b^2y = 2 + 4b$ possess an infinite number of solutions, the possible values of a and b are*

(a) $a = 1, b = -1$ (b) $a = 1, b = -2$
(c) $a = -1, b = -1$ (d) $a = -1, b = -2$

Sol. (a, b, c, d)

Here, $\Delta = \begin{vmatrix} a^2 & -b \\ b & -b^2 \end{vmatrix} = -a^2b^2 + b^2 = -(a^2 - 1)b^2$

If $\Delta = 0$, then $a^2 = 1, b = 0$

Now, $\Delta_1 = \begin{vmatrix} a^2 - b & -b \\ 2 + 4b & -b^2 \end{vmatrix}$

For $a^2 = 1, \Delta_1 = \begin{vmatrix} 1 - b & -b \\ 2 + 4b & -b^2 \end{vmatrix} = b(b+1)(b+2)$

and $\Delta_2 = \begin{vmatrix} a^2 & a^2 - b \\ b & 2 + 4b \end{vmatrix}$

For $a^2 = 1, \Delta_2 = \begin{vmatrix} 1 & 1 - b \\ b & 2 + 4b \end{vmatrix} = (b+1)(b+2)$

For infinite number of solutions, $\Delta = \Delta_1 = \Delta_2 = 0$

$\therefore a^2 = 1, b = -1, -2 \Rightarrow a = \pm 1, b = -1, b = -2$

JEE Type Solved Examples : Passage Based Questions

- This section contains **2 solved passages** based upon each of the passage **3 multiple choice** examples have to be answered. Each of these examples has four choices (a), (b), (c) and (d) out of which **ONLY ONE** is correct.

Passage I

(Ex. Nos. 16 to 18)

Consider the system of equations

$$2x + ay + 6z = 8; \quad x + 2y + bz = 5$$
$$x + y + 3z = 4$$

16. The system has unique solution, if

(a) $a = 2, b = 3$ (b) $a = 2, b \neq 3$
(c) $a \neq 2, b = 3$ (d) $a \neq 2, b \neq 3$

17. The system has infinite solutions, if

(a) $a = 2, b \in R$ (b) $a = 3, b \in R$
(c) $a \in R, b = 2$ (d) $a \in R, b = 3$

18. The system has no solution, if

(a) $a = 2, b = 3$ (b) $a = 2, b \neq 3$
(c) $a \neq 2, b = 3$ (d) $a \neq 2, b \neq 3$

Sol. $\Delta = \begin{vmatrix} 2 & a & 6 \\ 1 & 2 & b \\ 1 & 1 & 3 \end{vmatrix} = 2(6 - b) - a(3 - b) + 6(1 - 2)$

$= ab - 3a - 2b + 6 = (a - 2)(b - 3)$

$$\Delta_1 = \begin{vmatrix} 8 & a & 6 \\ 5 & 2 & b \\ 4 & 1 & 3 \end{vmatrix} = 8(6 - b) - a(15 - 4b) + 6(5 - 8)$$

$= 4ab - 15a - 8b + 30 = (a - 2)(4b - 15)$

$$\Delta_2 = \begin{vmatrix} 2 & 8 & 6 \\ 1 & 5 & b \\ 1 & 4 & 3 \end{vmatrix} = 0 \quad [\because R_1 = 2R_3]$$

$$\Delta_3 = \begin{vmatrix} 2 & a & 8 \\ 1 & 2 & 5 \\ 1 & 1 & 4 \end{vmatrix} = 2(8 - 5) - a(4 - 5) + 8(1 - 2)$$

$= 6 + a - 8 = a - 2$

16. (*d*) The system has unique solution, if

$\Delta \neq 0$

$\Rightarrow (a - 2)(b - 3) \neq 0$

$\Rightarrow a \neq 2, b \neq 3$

17. (*a*) The system has infinite solution, if

$\Delta = \Delta_1 = \Delta_2 = \Delta_3 = 0$

$\Rightarrow a - 2 = 0$

or $a = 2, b \in R$

18. (*c*) The system has no solution, if

$\Delta = 0$ and atleast one of Δ_1, Δ_2 and Δ_3 is non-zero.

$\Rightarrow a \neq 2, b = 3$

Passage II

(Ex. Nos. 19 to 20)

Let ${}^{x}C_i$, ${}^{x^2}C_i$ and ${}^{x^3}C_i$ $(i = 1, 2, 3)$ be Binomial coefficients, where $x \in N$

and $f(x) = 12 \begin{vmatrix} {}^{x}C_1 & {}^{x}C_2 & {}^{x}C_3 \\ {}^{x^2}C_1 & {}^{x^2}C_2 & {}^{x^2}C_3 \\ {}^{x^3}C_1 & {}^{x^3}C_2 & {}^{x^3}C_3 \end{vmatrix}$, *then*

19. $f(x)$ is a polynomial of degree

(a) 6 (b) 10
(c) 14 (d) 18

20. If $f(x) = (x-1)^m x^n (x+1)^p$, where $m, n, p \in N$, then the value of $\sum mn$ is

(a) 32 (b) 43
(c) 44 (d) 56

Sol.

$$\because f(x) = 12 \begin{vmatrix} {}^{x}C_1 & {}^{x}C_2 & {}^{x}C_3 \\ {}^{x^2}C_1 & {}^{x^2}C_2 & {}^{x^2}C_3 \\ {}^{x^3}C_1 & {}^{x^3}C_2 & {}^{x^3}C_3 \end{vmatrix}$$

$$= 12 \begin{vmatrix} x & \dfrac{x(x-1)}{2} & \dfrac{x(x-1)(x-2)}{6} \\ x^2 & \dfrac{x^2(x^2-1)}{2} & \dfrac{x^2(x^2-1)(x^2-2)}{6} \\ x^3 & \dfrac{x^3(x^3-1)}{2} & \dfrac{x^3(x^3-1)(x^3-2)}{6} \end{vmatrix}$$

Taking x, x^2 and x^3 common from R_1, R_2 and R_3, then

$$f(x) = x \cdot x^2 \cdot x^3 \begin{vmatrix} 1 & (x-1) & (x-1)(x-2) \\ 1 & (x^2-1) & (x^2-1)(x^2-2) \\ 1 & (x^3-1) & (x^3-1)(x^3-2) \end{vmatrix}$$

$$= x^6(x-1)^2 \begin{vmatrix} 1 & 1 & x-2 \\ 1 & x+1 & (x+1)(x^2-2) \\ 1 & x^2+x+1 & (x^2+x+1)(x^3-2) \end{vmatrix}$$

Applying $R_1 \to R_1 - R_2$ and $R_3 \to R_3 - R_2$, then

$$f(x) = x^6(x-1)^2 \begin{vmatrix} 0 & -x & x(3-x-x^2) \\ \vdots & & \\ 1 & \cdots \; x+1 \; \cdots & (x+1)(x^2-2) \\ \vdots & & \\ 0 & x^2 & x^2(x^2+x^3-3) \end{vmatrix}$$

Expanding along C_1, then

$$= -x^6(x-1)^2 \begin{vmatrix} -x & x(3-x-x^2) \\ x^2 & x^2(x^2+x^3-3) \end{vmatrix}$$

$$= -x^9(x-1)^2 \begin{vmatrix} -1 & 3-x-x^2 \\ 1 & x^2+x^3-3 \end{vmatrix}$$

$$= -x^9(x-1)^2(-x^2-x^3+3-3+x+x^2)$$

$$= x^{10}(x-1)^2(x^2-1) = x^{10}(x-1)^3(x+1)$$

19. (*c*) $f(x)$ is a polynomial of degree 14.

20. (*b*) Here, $m = 3, n = 10$ and $p = 1$

$\therefore \sum mn = mn + np + pm = 30 + 10 + 3 = 43$

JEE Type Solved Examples :
Single Integer Answer Type Questions

- This section contains **2 examples.** The answer to each example is a **single digit integer** ranging from **0** to **9** (both inclusive).

● ***Ex. 21*** *If* $\Delta_r = \begin{vmatrix} r & r-1 \\ r-1 & r \end{vmatrix}$, *where r is a natural number, the value of* $\sqrt[10]{\sum_{r=1}^{1024} \Delta_r}$ *is*

Sol. (4) $\because$ $\Delta_r = r^2 - (r-1)^2$

$$\therefore \sum_{r=1}^{1024} \Delta_r = (1024)^2 - (1-1)^2 = (1024)^2 = 2^{20}$$

$$\Rightarrow \sqrt[10]{\sum_{r=1}^{1024} \Delta_r} = 2^2 = 4$$

● ***Ex. 22*** *If P, Q and R are the angles of a triangle, the value of* $\begin{vmatrix} \tan P & 1 & 1 \\ 1 & \tan Q & 1 \\ 1 & 1 & \tan R \end{vmatrix}$ *is*

Sol. (2) $\begin{vmatrix} \tan P & 1 & 1 \\ 1 & \tan Q & 1 \\ 1 & 1 & \tan R \end{vmatrix} = \tan P(\tan Q \tan R - 1)$

$$-1(\tan R - 1) + 1(1 - \tan Q)$$

$= \tan P \tan Q \tan R - (\tan P + \tan Q + \tan R) + 2$

$= 0 + 2$

[$\because$ In ΔPQR, $\tan P + \tan Q + \tan R = \tan P \tan Q \tan R$]

$= 2$

JEE Type Solved Examples :
Matching Type Questions

- This section contains 2 eamples. Example 23 have four statements (A, B, C and D) given in **Column I** and four statement (p, q, r and s) in **Column II** and example 24 have three statements (A, B and C) given in **Column I** and four statements (p, q, r and s) in **Column II**. Any given statement in **Column I** can have correct matching with one or more statements(s) given in **Column II.**

Ex. 23 *Let $f(x)$ denotes the determinant*

$$f(x) = \begin{vmatrix} x^2 & 2x & 1+x^2 \\ x^2+1 & x+1 & 1 \\ x & -1 & x-1 \end{vmatrix}.$$

On expansion $f(x)$ is seen to be a 4th *degree polynomial given by $f(x) = a_0 x^4 + a_1 x^3 + a_2 x^2 + a_3 x + a_4$.*
Using differentiation of determinant or otherwise match the entries in Column I *with one or more entries of the elements of Column* II.

	Column I		Column II
(A)	$a_0^2 + a_1$ is divisible by	(p)	2
(B)	$a_2^2 + a_4$ is divisible by	(q)	3
(C)	$a_0^2 + a_2$ is divisible by	(r)	4
(D)	$a_4^2 + a_3^2 + a_1^2$ is divisible by	(s)	5

Sol. (A) $\to$ (p, s); (B) $\to$ (p, r); (C) $\to$ (p, q); (D) $\to$ (q)

$$\because \quad f(x) = \begin{vmatrix} x^2 & 2x & 1+x^2 \\ x^2+1 & x+1 & 1 \\ x & -1 & x-1 \end{vmatrix}$$

Applying $C_1 \to C_1 - C_3$, then

$$f(x) = \begin{vmatrix} -1 & 2x & 1+x^2 \\ x^2 & x+1 & 1 \\ 1 & -1 & x-1 \end{vmatrix}$$

Expanding along R_1, then

$$f(x) = -(x^2 - 1 + 1) - 2x(x^3 - x^2 - 1) + (1 + x^2)(-x^2 - x - 1)$$

$$= -3x^4 + x^3 - 3x^2 + x - 1 \quad \text{...(i)}$$

According to the question, we get

$$f(x) = a_0 x^4 + a_1 x^3 + a_2 x^2 + a_3 x + a_4 \quad \text{...(ii)}$$

From Eqs. (i) and (ii), we get

$$a_0 = -3, a_1 = 1, a_2 = -3, a_3 = 1, a_4 = -1$$

(A) $a_0^2 + a_1 = (-3)^2 + 1 = 9 + 1 = 10 = 2 \times 5$

(B) $a_2^2 + a_4 = (-3)^2 - 1 = 9 - 1 = 8 = 2 \times 4$

(C) $a_0^2 + a_2 = (-3)^2 - 3 = 9 - 3 = 6 = 2 \times 3$

(D) $a_4^2 + a_3^2 + a_1^2 = (-1)^2 + (1)^2 + (1)^2 = 1 + 1 + 1 = 3$

Ex. 24 *Suppose a, b and c are distinct and x, y and z are connected by the system of equations $x + ay + a^2 z = a^3$, $x + by + b^2 z = b^3$ and $x + cy + c^2 z = c^3$.*

	Column I		Column II
(A)	For $x = 1$, $y = 2$ and $z = 3$, $(a + b + c)^{-(ab+bc+ca)}$ is divisible by	(p)	3
(B)	For $x = 4$, $y = 3$ and $z = 2$, $(ab + bc + ca)^{abc}$ is divisible by	(q)	6
(C)	For $x = 6$, $y = 4$ and $z = 2$, $(abc)^{a+b+c}$ is divisible by	(r)	9
		(s)	12

Sol. (A) $\to$ (p, r) (B) $\to$ (p, r); (C) $\to$ (p, q, r, s)

$$a \neq b \neq c$$

$$\Delta = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$$

$$\Delta_1 = \begin{vmatrix} a^3 & a & a^2 \\ b^3 & b & b^2 \\ c^3 & c & c^2 \end{vmatrix} = \begin{vmatrix} a & a^2 & a^3 \\ b & b^2 & b^3 \\ c & c^2 & c^3 \end{vmatrix} = abc \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

$$= abc(a-b)(b-c)(c-a)$$

$$\Delta_2 = \begin{vmatrix} 1 & a^3 & a^2 \\ 1 & b^3 & b^2 \\ 1 & c^3 & c^2 \end{vmatrix} = -\begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix}$$

$$= -(a-b)(b-c)(c-a)(ab+bc+ca)$$

$$\text{and } \Delta_3 = \begin{vmatrix} 1 & a & a^3 \\ 1 & b & b^3 \\ 1 & c & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c)$$

By Cramer's rule, we get

$$x = \frac{\Delta_1}{\Delta} = abc$$

$$y = \frac{\Delta^2}{\Delta} = -(ab + bc + ca), \quad z = \frac{\Delta_3}{\Delta} = a + b + c$$

(A) $(a + b + c)^{-(ab+bc+ca)} = z^y = 3^2 = 9$, which is divisible by 3 and 9.

(B) $(ab + bc + ca)^{abc} = (-y)^x = (-3)^4 = 81$, which is divisible by 3 and 9.

(C) $(abc)^{a+b+c} = x^3 = 6^2 = 36$, which is divisible by 3, 6, 9 and 12.

JEE Type Solved Examples : Statement I and II Type Questions

- **Directions** Example numbers 25 and 26 are Assertion-Reason type examples. Each of these examples contains two statements:
  Statement-1 (Assertion) and **Statement-2** (Reason)
  Each of these examples also has four alternative choices, only one of which is the correct answer. You have to select the correct choice as given below.
  (a) Statement-1 is true, Statement-2 is true; Statement-2 is a correct explanation for Statement-1
  (b) Statement-1 is true, Statement-2 is true; Statement-2 is not a correct explanation for Statement-1
  (c) Statement-1 is true, Statement-2 is false
  (d) Statement-1 is false, Statement-2 is true

Ex. 25 **Statement-1** *Let*

$$\Delta_r = \begin{vmatrix} (r-1) & n! & 6 \\ (r-1)^2 & (n!)^2 & 4n-2 \\ (r-1)^3 & (n!)^3 & 3n^2-2n \end{vmatrix}, \text{ then } \prod_{r=2}^{n+1} \Delta_r = 0.$$

Statement-2 $\prod_{r=2}^{n+1} \Delta_r = \Delta_2 \cdot \Delta_3 \cdot \Delta_4 \ldots \Delta_{n+1}$

Sol. (*d*) $\because \prod_{r=2}^{n+1} \Delta_r = \Delta_2 \cdot \Delta_3 \cdot \Delta_4 \ldots \Delta_{n+1}$

$$= \begin{vmatrix} 1 & n! & 6 \\ 1 & (n!)^2 & 4n-2 \\ 1 & (n!)^3 & 3n^3-2n \end{vmatrix} \times \begin{vmatrix} 2 & n! & 6 \\ 4 & (n!)^2 & 4n-2 \\ 8 & (n!)^3 & 3n^2-2n \end{vmatrix} \times \ldots \times \begin{vmatrix} n & n! & 6 \\ n^2 & (n!)^2 & 4n-2 \\ n^3 & (n!)^3 & (3n^2-2n) \end{vmatrix} \neq 0$$

$\therefore$ Statement-1 is false and Statement-2 is true.

Ex. 26 *Consider the determinant*

$$f(x) = \begin{vmatrix} 0 & x^2-a & x^3-b \\ x^2+a & 0 & x^2+c \\ x^4+b & x-c & 0 \end{vmatrix}.$$

Statement-1 $f(x)=0$ *has one root* $x=0$.

Statement-2 *The value of skew-symmetric determinant of odd order is always zero.*

Sol. (a) For $x=0$, the determinant reduces to the determinant of a skew-symmetric of odd order which is always zero. Hence, $x=0$ is the solution of given equation $f(x)=0$.

Subjective Type Examples

- In this section, there are **20 subjective** solved examples.

Ex. 27 *A determinant of second order is made with the elements* 0 *and* 1. *Find the number of determinants with non-negative values.*

Sol. The number of determinants that can be made with 0 and 1

$$= 2 \times 2 \times 2 \times 2 = 16$$

and there are only three determinants of second order with negative values

i.e., $\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix}, \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix}, \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}$

Therefore, number of determinants with non-negative values $=16-3=13$

Ex. 28 *Prove that* $\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix}$

$= abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$, *hence find the value of the determinant, if* a, b *and* c *are the roots of the equation* $px^3-qx^2+rx-s=0$.

Sol. Let $\Delta = \begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix}$

Since, the answer contain abc, then taking a, b and c common from R_1, R_2 and R_3 respectively, then

$$\Delta = abc \begin{vmatrix} \frac{1}{a}+1 & \frac{1}{a} & \frac{1}{a} \\ \frac{1}{b} & \frac{1}{b}+1 & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & \frac{1}{c}+1 \end{vmatrix}$$

But answer also contains $\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$,

then applying $R_1 \to R_1 + R_2 + R_3$

$\therefore \quad \Delta = abc$

$$\begin{vmatrix} 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} & 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} & 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \\ \frac{1}{b} & \frac{1}{b}+1 & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & \frac{1}{c}+1 \end{vmatrix}$$

Taking $\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$ common from R_1, then

$$\Delta = abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\begin{vmatrix} 1 & 1 & 1 \\ \frac{1}{b} & \frac{1}{b}+1 & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & \frac{1}{c}+1 \end{vmatrix}$$

Applying $C_2 \to C_2 - C_1$, then

$$\Delta = abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\begin{vmatrix} 1 & 0 & 1 \\ \frac{1}{b} & 1 & \frac{1}{b} \\ \frac{1}{c} & 0 & \frac{1}{c}+1 \end{vmatrix}$$

Expanding along C_2, then

$$\Delta = abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\begin{vmatrix} 1 & 1 \\ \frac{1}{c} & \frac{1}{c}+1 \end{vmatrix}$$

Hence, $\Delta = abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$

2nd Part $\Delta = abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) = abc + bc + ca + ab$

$$= \frac{s}{p} + \frac{r}{p} = \left(\frac{s+r}{p}\right)$$

Ex. 29 *If a, b and c are positive and are the pth, qth and rth terms, respectively of a GP. Show without expanding that*

$$\begin{vmatrix} \log a & p & 1 \\ \log b & q & 1 \\ \log c & r & 1 \end{vmatrix} = 0.$$

Sol. Let A be the first term and R be the common ratio of GP, then

$$a = p\text{th term} = AR^{p-1}$$

$$b = q\text{th term} = AR^{q-1}$$

$$c = r\text{th term} = AR^{r-1}$$

$\therefore$ $\log a = \log A + (p-1)\log R$,

$\log b = \log A + (q-1)\log R$ and

$\log c = \log A + (r-1)\log R$

$$\therefore \quad \text{LHS} = \begin{vmatrix} \log a & p & 1 \\ \log b & q & 1 \\ \log c & r & 1 \end{vmatrix}$$

$$= \begin{vmatrix} \log A + (p-1)\log R & p & 1 \\ \log A + (q-1)\log R & q & 1 \\ \log A + (r-1)\log R & r & 1 \end{vmatrix}$$

Applying $C_1 \to C_1 - (\log A)C_3$, then

$$= \begin{vmatrix} (p-1)\log R & p & 1 \\ (q-1)\log R & q & 1 \\ (r-1)\log R & r & 1 \end{vmatrix} = \log R \begin{vmatrix} (p-1) & p & 1 \\ (q-1) & q & 1 \\ (r-1) & r & 1 \end{vmatrix}$$

Applying $C_1 \to C_1 + C_3$, then

$$= \log R \begin{vmatrix} p & p & 1 \\ q & q & 1 \\ r & r & 1 \end{vmatrix} = 0 \text{ [since } C_1 \text{ and } C_2 \text{ are identical]}$$

= RHS

Ex. 30 *Prove that*

$$\begin{vmatrix} -2a & a+b & a+c \\ b+a & -2b & b+c \\ c+a & c+b & -2c \end{vmatrix} = 4(b+c)(c+a)(a+b).$$

Sol. Let $\Delta = \begin{vmatrix} -2a & a+b & a+c \\ b+a & -2b & b+c \\ c+a & c+b & -2c \end{vmatrix}$

On putting $a+b=0$, $b=-a$

Then, $\quad \Delta = \begin{vmatrix} -2a & 0 & a+c \\ 0 & 2a & c-a \\ c+a & c-a & -2c \end{vmatrix}$

Expanding along R_1, then

$\Delta = -2a\{-4ac - (c-a)^2\} - 0 + (a+c)\{0 - 2a(c+a)\}$

$= 2a(c+a)^2 - 2a(c+a)^2 = 0$

Hence, $(a+b)$ is a factor of Δ, similarly $(b+c)$ and $(c+a)$ are the factors of Δ.

On expansion of determinant we can see that each term of the determinant is a homogeneous expression in a, b and c of degree 3 and also RHS is a homogeneous expression of degree 3.

Let $\quad \Delta = k(a+b)(b+c)(c+a)$

or $\begin{vmatrix} -2a & a+b & a+c \\ b+a & -2b & b+c \\ c+a & c+b & -2c \end{vmatrix} = k(a+b)(b+c)(c+a)$

On putting $a=0$, $b=1$ and $c=2$, we get

$$\begin{vmatrix} 0 & 1 & 2 \\ 1 & -2 & 3 \\ 2 & 3 & -4 \end{vmatrix} = k(0+1)(1+2)(2+0)$$

$\Rightarrow \quad 0 - 1(-4-6) + 2(3+4) = 6k$

$\Rightarrow \quad 24 = 6k$

$\therefore \quad k = 4$

Hence, $\begin{vmatrix} -2a & a+b & a+c \\ b+a & -2b & b+c \\ c+a & c+b & -2c \end{vmatrix} = 4(a+b)(b+c)(c+a)$

Ex. 31 *If* $bc + qr = ca + rp = ab + pq = -1$,

show that $\begin{vmatrix} ap & bp & cr \\ a & b & c \\ p & q & r \end{vmatrix} = 0.$

Sol. Given equations can be rewritten as

$$bc + qr + 1 = 0 \quad \text{...(i)}$$
$$ca + rp + 1 = 0 \quad \text{...(ii)}$$
$$ab + pq + 1 = 0 \quad \text{...(iii)}$$

On multiplying Eqs. (i), (ii) and (iii) by ap, bq and cr respectively, we get

$$(abc)p + (pqr)a + ap = 0$$
$$(abc)q + (pqr)b + bq = 0$$
$$(abc)r + (pqr)c + cr = 0$$

These equations are consistent, given equations three but abc and pqr are two.

Hence, $$\begin{vmatrix} p & a & ap \\ q & b & bq \\ r & c & cr \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} p & q & r \\ a & b & c \\ ap & bq & cr \end{vmatrix} = 0$$

[interchanging rows into columns]

$$\Rightarrow (-1)\begin{vmatrix} ap & bq & cr \\ a & b & c \\ p & q & r \end{vmatrix} = 0 \quad [R_1 \leftrightarrow R_3]$$

Hence, $$\begin{vmatrix} ap & bq & cr \\ a & b & c \\ p & q & r \end{vmatrix} = 0$$

● **Ex. 32** *If α and β are the roots of the equations $x^2 - 2x + 4 = 0$, find the value of* $\begin{vmatrix} \Sigma\alpha & \Sigma\alpha^2 & \Sigma\alpha^3 \\ \Sigma\alpha^2 & \Sigma\alpha^3 & \Sigma\alpha^4 \\ \Sigma\alpha^3 & \Sigma\alpha^4 & \Sigma\alpha^5 \end{vmatrix}$.

Sol. Given, $x^2 - 2x + 4 = 0$

$\therefore \quad x = 1 \pm i\sqrt{3}$

$\therefore \quad \alpha = 1 + i\sqrt{3}$

and $\quad \beta = 1 - i\sqrt{3}$

$$\Rightarrow \quad \alpha = -2\left(\frac{-1 - i\sqrt{3}}{2}\right) \text{ and } \beta = -2\left(\frac{-1 + i\sqrt{3}}{2}\right)$$

$\alpha = -2\omega^2$ and $\beta = -2\omega$, where ω is the cube root of unity.

$$\Sigma\alpha = \alpha + \beta = -2(\omega + \omega)^2 = -2(-1) = 2$$
$$\Sigma\alpha^2 = \alpha^2 + \beta^2 = 4\omega^4 + 4\omega^2 = 4(\omega + \omega)^2 = 4(-1) = -4$$
$$\Sigma\alpha^3 = \alpha^3 + \beta^3 = -8\omega^6 - 8\omega^3 = -8 - 8 = -16$$
$$\Sigma\alpha^4 = \alpha^4 + \beta^4 = 16\omega^8 + 16\omega^4 = 16(\omega^2 + \omega)$$
$$= 16(-1) = -16$$

and $\quad \Sigma\alpha^5 = \alpha^5 + \beta^5 = -32\omega^{10} - 32\omega^5$

$$= -32(\omega + \omega^2) = -32(-1) = 32$$

Let $$\Delta = \begin{vmatrix} \Sigma\alpha & \Sigma\alpha^2 & \Sigma\alpha^3 \\ \Sigma\alpha^2 & \Sigma\alpha^3 & \Sigma\alpha^4 \\ \Sigma\alpha^3 & \Sigma\alpha^4 & \Sigma\alpha^5 \end{vmatrix} = \begin{vmatrix} 2 & -4 & -16 \\ -4 & -16 & -16 \\ -16 & -16 & 32 \end{vmatrix}$$

$$= 2(-4)(-16)\begin{vmatrix} 1 & -2 & -8 \\ 1 & 4 & 4 \\ 1 & 1 & -2 \end{vmatrix} = 128\begin{vmatrix} 1 & -2 & -8 \\ 1 & 4 & 4 \\ 1 & 1 & -2 \end{vmatrix}$$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, then

$$\Delta = 128\begin{vmatrix} 1 & -2 & -8 \\ 0 & 6 & 12 \\ 0 & 3 & 6 \end{vmatrix}$$

Expanding along C_1, we get

$$\Delta = 128 \cdot 1 \cdot \begin{vmatrix} 6 & 12 \\ 3 & 6 \end{vmatrix} = 128(36 - 36) = 0$$

● **Ex. 33** *If $a^2 + b^2 + c^2 = 1$, prove that*

$$\begin{vmatrix} a^2 + (b^2 + c^2)\cos\phi & ab(1 - \cos\phi) & ac(1 - \cos\phi) \\ ba(1 - \cos\phi) & b^2 + (c^2 + a^2)\cos\phi & bc(1 - \cos\phi) \\ ca(1 - \cos\phi) & cb(1 - \cos\phi) & c^2 + (a^2 + b^2)\cos\phi \end{vmatrix}$$

is independent of a, b and c.

Sol. Let

$$\Delta = \begin{vmatrix} a^2 + (b^2 + c^2)\cos\phi & ab(1 - \cos\phi) & ac(1 - \cos\phi) \\ ba(1 - \cos\phi) & b^2 + (c^2 + a^2)\cos\phi & bc(1 - \cos\phi) \\ ca(1 - \cos\phi) & cb(1 - \cos\phi) & c^2 + (a^2 + b^2)\cos\phi \end{vmatrix}$$

On multiplying C_1, C_2 and C_3 by a, b and c respectively and taking a, b and c common from R_1, R_2 and R_3 respectively, we get

$$\Delta = \frac{abc}{abc}\begin{vmatrix} a^2 + (b^2 + c^2)\cos\phi & b^2(1 - \cos\phi) & c^2(1 - \cos\phi) \\ a^2(1 - \cos\phi) & b^2 + (c^2 + a^2)\cos\phi & c^2(1 - \cos\phi) \\ a^2(1 - \cos\phi) & b^2(1 - \cos\phi) & c^2 + (a^2 + b^2)\cos\phi \end{vmatrix}$$

Applying $C_1 \to C_1 + C_2 + C_3$, then

$$\Delta = \begin{vmatrix} a^2+b^2+c^2 & b^2 1-\cos\phi) & c^2(1-\cos\phi) \\ a^2+b^2+c^2 & b^2+(c^2+a^2)\cos\phi & c^2(1-\cos\phi) \\ a^2+b^2+c^2 & b^2(1-\cos\phi) & c^2+(a^2+b^2)\cos\phi \end{vmatrix}$$

Taking $a^2+b^2+c^2$ common from C_1, then

$$\Delta = (a^2+b^2+c^2)$$

$$\begin{vmatrix} 1 & b^2(1-\cos\phi) & c^2(1-\cos\phi) \\ 1 & b^2+(c^2+a^2)\cos\phi & c^2(1-\cos\phi) \\ 1 & b^2(1-\cos\phi) & c^2+(a^2+b^2)\cos\phi \end{vmatrix}$$

Applying $R_1 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, then

$$\Delta = 1$$

$$\begin{vmatrix} 1 & b^2(1-\cos\phi) & c^2(1-\cos\phi) \\ 0 & (a^2+b^2+c^2)\cos\phi & 0 \\ 0 & 0 & (a^2+b^2+c^2)\cos\phi \end{vmatrix}$$

$$= (a^2+b^2+c^2)^2\cos^2\phi$$

[by property, since all elements zero below leading diagonal]

$$= 1^2\cdot\cos\phi = \cos^2\phi \qquad [\because a^2+b^2+c^2=1]$$

which is independent of a, b and c.

Ex. 34 *If $a \neq 0$ and $a \neq 1$, show that*

$$\begin{vmatrix} x+1 & x & x \\ x & x+a & x \\ x & x & x+a^2 \end{vmatrix} = a^3\left[1+\frac{x(a^3-1)}{a^2(a-1)}\right].$$

Sol. Let

$$\text{LHS} = \Delta = \begin{vmatrix} x+1 & x & x \\ x & x+a & x \\ x & x & x+a^2 \end{vmatrix} = \begin{vmatrix} x+1 & x & x \\ x+0 & x+a & x \\ x+0 & x & x+a^2 \end{vmatrix}$$

$$= \begin{vmatrix} x & x & x \\ x & x+a & x \\ x & x & x+a^2 \end{vmatrix} + \begin{vmatrix} 1 & x & x \\ 0 & x+a & x \\ 0 & x & x+a^2 \end{vmatrix}$$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$ in first determinant, then

$$\Delta = \begin{vmatrix} x & x & x \\ 0 & a & 0 \\ 0 & 0 & a^2 \end{vmatrix} + \begin{vmatrix} 1 & x & x \\ 0 & x+a & x \\ 0 & x & x+a^2 \end{vmatrix}$$

Expanding first determinant by property, since all elements below leading diagonal are zero and expanding second determinant along C_1, then

$$\Delta = x\cdot a\cdot a^2 + 1\cdot\begin{vmatrix} x+a & x \\ x & x+a^2 \end{vmatrix}$$

$$= xa^3 + \{(x+a)(x+a^2) - x^2\}$$

$$= xa^3 + (x^2 + a^2x + ax + a^3 - x^2)$$

$$= xa^3 + a^2x + ax + a^3 = a^3 + x(a^3+a^2+a)$$

$$= a^3 + \frac{x\cdot a(a^3-1)}{(a-1)} = a^3\left[1+\frac{x(a^3-1)}{a^2(a-1)}\right] = \text{RHS}$$

Ex. 35 (i) *Prove that* $\begin{vmatrix} bc-a^2 & ca-b^2 & ab-c^2 \\ ca-b^2 & ab-c^2 & bc-a^2 \\ ab-c^2 & bc-a^2 & ca-b^2 \end{vmatrix}$

$$= \begin{vmatrix} \alpha^2 & \beta^2 & \beta^2 \\ \beta^2 & \alpha^2 & \beta^2 \\ \beta^2 & \beta^2 & \alpha^2 \end{vmatrix},$$

where $\alpha^2 = a^2+b^2+c^2$ and $\beta^2 = ab+bc+ca$.

(ii) *Prove that* $\begin{vmatrix} bc-a^2 & ca-b^2 & ab-c^2 \\ ca-b^2 & ab-c^2 & bc-a^2 \\ ab-c^2 & bc-a^2 & ca-b^2 \end{vmatrix}$ *is divisible by $(a+b+c)^2$. Find the quotient.*

(iii) *Prove that* $\begin{vmatrix} bc-a^2 & ca-b^2 & ab-c^2 \\ ca-b^2 & ab-c^2 & bc-a^2 \\ ab-c^2 & bc-a^2 & ca-b^2 \end{vmatrix}$

$$= \begin{vmatrix} a^2 & c^2 & 2ac-b^2 \\ 2ab-c^2 & b^2 & a^2 \\ b^2 & 2bc-a^2 & c^2 \end{vmatrix}.$$

(iv) *Prove that* $\begin{vmatrix} 2bc-a^2 & c^2 & b^2 \\ c^2 & 2ca-b^2 & a^2 \\ b^2 & a^2 & 2ab-c^2 \end{vmatrix}$

$$= (a^3+b^3+c^3-3abc)^2.$$

Sol. (i) Let $\Delta = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$

$\therefore$ Determinant of cofactors of Δ is

$$\Delta^c = \begin{vmatrix} bc-a^2 & ca-b^2 & ab-c^2 \\ ca-b^2 & ab-c^2 & bc-a^2 \\ ab-c^2 & bc-a^2 & ca-b^2 \end{vmatrix} = \Delta^{3-1} = \Delta^2$$

$$= \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}^2 \qquad \text{...(i)}$$

$$= \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \times \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$$

$$= \begin{vmatrix} a^2+b^2+c^2 & ab+bc+ca & ab+bc+ca \\ ab+bc+ca & a^2+b^2+c^2 & ab+bc+ca \\ ab+bc+ca & ab+bc+ca & a^2+b^2+c^2 \end{vmatrix} \quad \text{[row by row]}$$

$$= \begin{vmatrix} \alpha^2 & \beta^2 & \beta^2 \\ \beta^2 & \alpha^2 & \beta^2 \\ \beta^2 & \beta^2 & \alpha^2 \end{vmatrix}$$

Hence, $\begin{vmatrix} bc-a^2 & ca-b^2 & ab-c^2 \\ ca-b^2 & ab-c^2 & bc-a^2 \\ ab-c^2 & bc-a^2 & ca-b^2 \end{vmatrix} = \begin{vmatrix} \alpha^2 & \beta^2 & \beta^2 \\ \beta^2 & \alpha^2 & \beta^2 \\ \beta^2 & \beta^2 & \alpha^2 \end{vmatrix}$

(ii) From Eq. (i), we get

$$\begin{vmatrix} bc-a^2 & ca-b^2 & ab-c^2 \\ ca-b^2 & ab-c^2 & bc-a^2 \\ ab-c^2 & bc-a^2 & ca-b^2 \end{vmatrix} = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}^2$$

$$= (a^3+b^3+c^3-3abc)^2$$

$$= (a+b+c)^2(a^2+b^2+c^2-ab-bc-ca)^2$$

Therefore, $\begin{vmatrix} bc-a^2 & ca-b^2 & ab-c^2 \\ ca-b^2 & ab-c^2 & bc-a^2 \\ ab-c^2 & bc-a^2 & ca-b^2 \end{vmatrix}$ is divisible by $(a+b+c)^2$.

Hence, the quotient is $(a^2+b^2+c^2-ab-bc-ca)^2$.

(iii) From Eq. (i), we get

$$\begin{vmatrix} bc-a^2 & ca-b^2 & ab-c^2 \\ ca-b^2 & ab-c^2 & bc-a^2 \\ ab-c^2 & bc-a^2 & ca-b^2 \end{vmatrix} = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}^2$$

Let $\Delta = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}^2 = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \times \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$

$$= \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \times \begin{vmatrix} a & -c & b \\ b & -a & c \\ c & -b & a \end{vmatrix}$$

$$= \begin{vmatrix} a^2 & c^2 & 2ac-b^2 \\ 2ab-c^2 & b^2 & a^2 \\ b^2 & 2bc-a^2 & c^2 \end{vmatrix} \quad \text{[row by row]}$$

Hence, $\begin{vmatrix} bc-a^2 & ca-b^2 & ab-c^2 \\ ca-b^2 & ab-c^2 & bc-a^2 \\ ab-c^2 & bc-a^2 & ca-b^2 \end{vmatrix}$

$$= \begin{vmatrix} a^2 & c^2 & 2ac-b^2 \\ 2ab-c^2 & b^2 & a^2 \\ b^2 & 2bc-a^2 & c^2 \end{vmatrix}$$

(iv) LHS $= \begin{vmatrix} 2bc-a^2 & c^2 & b^2 \\ c^2 & 2ca-b^2 & a^2 \\ b^2 & a^2 & 2ab-c^2 \end{vmatrix}$

$$= \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \times \begin{vmatrix} -a & c & b \\ -b & a & c \\ -c & b & a \end{vmatrix} \quad \text{[row by row]}$$

$$= \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \times \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}^2$$

$= (a+b+c)^2(a^2+b^2+c^2-ab-bc-ca)^2$ [from Eq. (ii)]

$= [-(a^3+b^3+c^3-3abc)]^2$

$= (a^3+b^3+c^3-3abc)^2 =$ RHS

● ***Ex. 36*** *Let α and β be the roots of the equation $ax^2+bx+c=0$. Let $S_n = \alpha^n + \beta^n$ for $n \geq 1$. Evaluate the determinant* $\begin{vmatrix} 3 & 1+S_1 & 1+S_2 \\ 1+S_1 & 1+S_2 & 1+S_3 \\ 1+S_2 & 1+S_3 & 1+S_4 \end{vmatrix}$.

Sol. Since, α and β are the roots of the equation $ax^2+bx+c=0$.

$\therefore \ \alpha+\beta = -\dfrac{b}{a}, \alpha\beta = \dfrac{c}{a}$ and $\alpha - \beta = \dfrac{\sqrt{D}}{a}$

Let $\quad \Delta = \begin{vmatrix} 3 & 1+S_1 & 1+S_2 \\ 1+S_1 & 1+S_2 & 1+S_3 \\ 1+S_2 & 1+S_3 & 1+S_4 \end{vmatrix}$

$$= \begin{vmatrix} 3 & 1+\alpha+\beta & 1+\alpha^2+\beta^2 \\ 1+\alpha+\beta & 1+\alpha^2+\beta^2 & 1+\alpha^3+\beta^3 \\ 1+\alpha^2+\beta^2 & 1+\alpha^3+\beta^3 & 1+\alpha^4+\beta^4 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ 1 & \alpha & \beta \\ 1 & \alpha^2 & \beta^2 \end{vmatrix} \times \begin{vmatrix} 1 & 1 & 1 \\ 1 & \alpha & \beta \\ 1 & \alpha^2 & \beta^2 \end{vmatrix} = \Delta_1 \times \Delta_1 \quad \text{[say]}$$

$\therefore \quad \Delta = \Delta_1^2 \quad$ ...(i)

$\therefore \quad \Delta_1 = \begin{vmatrix} 1 & 1 & 1 \\ 1 & \alpha & \beta \\ 1 & \alpha^2 & \beta^2 \end{vmatrix}$

Applying $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$, then

$$\Delta_1 = \begin{vmatrix} 1 & 0 & 0 \\ 1 & \alpha-1 & \beta-1 \\ 1 & \alpha^2-1 & \beta^2-1 \end{vmatrix}$$

Expanding along R_1, then

$$\Delta_1 = \begin{vmatrix} \alpha - 1 & \beta - 1 \\ \alpha^2 - 1 & \beta^2 - 1 \end{vmatrix} = (\alpha - 1)(\beta - 1)\begin{vmatrix} 1 & 1 \\ \alpha + 1 & \beta + 1 \end{vmatrix}$$

$$= \{\alpha\beta - (\alpha + \beta) + 1\}(\beta - \alpha)$$

$$\therefore \quad \Delta = \Delta_1^2 = [\alpha\beta - (\alpha + \beta) + 1]^2 (\beta - \alpha)^2$$

$$= \left(\frac{c}{a} + \frac{b}{a} + 1\right)^2 \cdot \frac{D}{a^2} = \frac{(a + b + c)^2 (b^2 - 4ac)}{a^4}$$

● ***Ex. 37*** *If A, B and C are the angles of a triangle, show that*

(i) $\begin{vmatrix} \sin 2A & \sin C & \sin B \\ \sin C & \sin 2B & \sin A \\ \sin B & \sin A & \sin 2C \end{vmatrix} = 0.$

(ii) $\begin{vmatrix} -1 + \cos B & \cos C + \cos B & \cos B \\ \cos C + \cos A & -1 + \cos A & \cos A \\ -1 + \cos B & -1 + \cos A & -1 \end{vmatrix} = 0.$

Sol. (i) LHS $= \begin{vmatrix} \sin 2A & \sin C & \sin B \\ \sin C & \sin 2B & \sin A \\ \sin B & \sin A & \sin 2C \end{vmatrix}$

$$= \begin{vmatrix} 2ka \cos A & kc & kb \\ kc & 2kb \cos B & ka \\ kb & ka & 2kc \cos C \end{vmatrix} \text{[from sine rule]}$$

$$= k^3 \begin{vmatrix} 2a \cos A & c & b \\ c & 2b \cos B & a \\ b & a & 2c \cos C \end{vmatrix}$$

$$= k^3 \begin{vmatrix} a \cos A + a \cos A & a \cos B + b \cos A & a \cos C + c \cos A \\ a \cos B + b \cos A & b \cos B + b \cos B & b \cos C + c \cos B \\ a \cos C + c \cos A & c \cos B + b \cos C & c \cos C + c \cos C \end{vmatrix}$$

$$= k^3 \begin{vmatrix} \cos A & a & 0 \\ \cos B & b & 0 \\ \cos C & c & 0 \end{vmatrix} \times \begin{vmatrix} a & \cos A & 0 \\ b & \cos B & 0 \\ c & \cos C & 0 \end{vmatrix} = 0 \times 0 = 0 = \text{RHS}$$

(ii) LHS $= \begin{vmatrix} -1 + \cos B & \cos C + \cos B & \cos B \\ \cos C + \cos A & -1 + \cos A & \cos A \\ -1 + \cos B & -1 + \cos A & -1 \end{vmatrix}$

Applying $C_1 \to C_1 - C_3$ and $C_2 \to C_2 - C_3$, then

$$= \begin{vmatrix} -1 & \cos C & \cos B \\ \cos C & -1 & \cos A \\ \cos B & \cos A & -1 \end{vmatrix}$$

$$= \frac{1}{a} \begin{vmatrix} -a & \cos C & \cos B \\ a \cos C & -1 & \cos A \\ a \cos B & \cos A & -1 \end{vmatrix}$$

Applying $C_1 \to C_1 + bC_2 + cC_3$, then

$$= \frac{1}{a} \begin{vmatrix} 0 & \cos C & \cos B \\ 0 & -1 & \cos A \\ 0 & \cos A & -1 \end{vmatrix} = 0 = \text{RHS}$$

● ***Ex. 38*** *Without expanding at any stage, evaluate the value of the determinant*

$$\begin{vmatrix} 2 & \tan A \cot B + \cot A \tan B & \tan A \cot C + \cot A \tan C \\ \tan B \cot A + \cot B \tan A & 2 & \tan B \cot C + \cot B \tan C \\ \tan C \cot A + \cot C \tan A & \tan C \cot B + \cot C \tan B & 2 \end{vmatrix}.$$

Sol. The given determinant can be written as the product of two determinants

$$\begin{vmatrix} \tan A & \cot A & 0 \\ \tan B & \cot B & 0 \\ \tan C & \cot C & 0 \end{vmatrix} \times \begin{vmatrix} \cot A & \tan A & 0 \\ \cot B & \tan B & 0 \\ \cot C & \tan C & 0 \end{vmatrix} = 0 \times 0 = 0$$

● ***Ex. 39*** *Suppose that digit numbers $A28, 3B9$ and $62C$, where A, B and C are integers between 0 and 9 are divisible by a fixed integer k, prove that the determinant $\begin{vmatrix} A & 3 & 6 \\ 8 & 9 & C \\ 2 & B & 2 \end{vmatrix}$ is also divisible by k.*

Sol. Given, $A28, 3B9$ and $62C$ are divisible by k, then

$$A28 = 100A + 20 + 8 = n_1 k \quad \text{...(i)}$$

$$3B9 = 300 + 10B + 9 = n_2 k \quad \text{...(ii)}$$

and $$62C = 600 + 20 + C = n_3 k \quad \text{...(iii)}$$

where $n_1, n_2, n_3 \in I$ (integers).

Let $$\Delta = \begin{vmatrix} A & 3 & 6 \\ 8 & 9 & C \\ 2 & B & 2 \end{vmatrix}$$

Applying $R_2 \to R_2 + 10R_3 + 100 R_1$, then

$$A = \begin{vmatrix} A & 3 & 6 \\ 100A + 20 + 8 & 300 + 10B + 9 & 600 + 20 + C \\ 2 & B & 2 \end{vmatrix}$$

$$= \begin{vmatrix} A & 3 & 6 \\ n_1k & n_2k & n_3k \\ 2 & B & 2 \end{vmatrix} \quad \text{[using Eqs. (i), (ii) and (iii)]}$$

$$= k \begin{vmatrix} A & 3 & 6 \\ n_1 & n_2 & n_3 \\ 2 & B & 2 \end{vmatrix}$$

Hence, Δ is divisible by k.

Ex. 40 *If* $\Delta = \begin{vmatrix} \sin x & \sin(x+h) & \sin(x+2h) \\ \sin(x+2h) & \sin x & \sin(x+h) \\ \sin(x+h) & \sin(x+2h) & \sin x \end{vmatrix}$, *find* $\lim_{h \to 0} \left(\frac{\Delta}{h^2} \right)$.

Sol. Let $a = \sin x$, $b = \sin(x+h)$ and $c = \sin(x+2h)$

$$\Delta = \begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix} = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = (a^3 + b^3 + c^3 - 3abc)$$

$$= \frac{1}{2}(a+b+c)[(a-b)^2 + (b-c)^2 + (c-a)^2]$$

Now, $a - b = \sin x - \sin(x+h) = -2\cos\left(x + \frac{h}{2}\right)\sin\frac{h}{2}$

$b - c = \sin(x+h) - \sin(x+2h) = -2\cos\left(x + \frac{3h}{2}\right)\sin\frac{h}{2}$

and $c - a = \sin(x+2h) - \sin x = 2\cos(x+h)\sin h$

$$\therefore \quad \frac{\Delta}{h^2} = \frac{1}{2}(a+b+c)\left[\left(\frac{a-b}{h}\right)^2 + \left(\frac{b-c}{h}\right)^2 + \left(\frac{c-a}{h}\right)^2\right]$$

$$= \frac{1}{2}[\sin x + \sin(x+h) + \sin(x+2h)] \times \left[\left(\frac{-2\cos\left(x+\frac{h}{2}\right)\sin\frac{h}{2}}{h}\right)^2 + \left(\frac{-2\cos\left(x+\frac{3h}{2}\right)\sin\frac{h}{2}}{h}\right)^2 + \left(\frac{2\cos(x+h)\sin h}{h}\right)^2\right]$$

$$\therefore \quad \lim_{h \to 0} \frac{\Delta}{h^2} = \frac{1}{2}(3\sin x)(\cos^2 x + \cos^2 x + 4\cos^2 x)$$

$$\therefore \quad = 9\sin x \cos^2 x$$

Ex. 41 *If* $f(x) = \begin{vmatrix} x+c_1 & x+a & x+a \\ x+b & x+c_2 & x+a \\ x+b & x+b & x+c_3 \end{vmatrix}$, *show that* $f(x)$ *is linear in x. Hence, deduce that* $f(0) = \frac{bg(a) - ag(b)}{(b-a)}$, *where* $g(x) = (c_1 - x)(c_2 - x)(c_3 - x)$.

Sol. Since, $f(x) = \begin{vmatrix} x+c_1 & x+a & x+a \\ x+b & x+c_2 & x+a \\ x+b & x+b & x+c_3 \end{vmatrix}$

Applying $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_2$, then

$$f(x) = \begin{vmatrix} x+c_1 & a-c_1 & 0 \\ x+b & c_2-b & a-c_2 \\ x+b & 0 & c_3-b \end{vmatrix}$$

$$f(x) = x\begin{vmatrix} 1 & a-c_1 & 0 \\ 1 & c_2-b & a-c_2 \\ 1 & 0 & c_3-b \end{vmatrix} + \begin{vmatrix} c_1 & a-c_1 & 0 \\ b & c_2-b & a-c_2 \\ b & 0 & c_3-b \end{vmatrix}$$

So, $f(x)$ is linear.

Let $\quad f(x) = Px + Q$

Then, $\quad f(-a) = -aP + Q$, $f(-b) = -bP + Q$

$$f(0) = 0 \cdot P + Q = Q$$

$$= \frac{bf(-a) - af(-b)}{(b-a)} \qquad \text{...(ii)}$$

From Eq. (i), we get

$$f(-a) = \begin{vmatrix} c_1 - a & 0 & 0 \\ b-a & c_2-a & 0 \\ b-a & b-a & c_3-a \end{vmatrix}$$

$$= (c_1 - a)(c_2 - a)(c_3 - a)$$

Similarly, $f(-b) = (c_1 - b)(c_2 - b)(c_3 - b)$

and $\quad g(x) = (c_1 - x)(c_2 - x)(c_3 - x)$

$$g(a) = f(-a)$$

$$\therefore \quad g(b) = f(-b)$$

Now, from Eq. (ii), we get

$$f(0) = \frac{bg(a) - ag(b)}{(b-a)}$$

Ex. 42 *If* $f(x)$ *is a polynomial of degree* < 3, *prove that*

$$\begin{vmatrix} 1 & a & f(a)/(x-a) \\ 1 & b & f(b)/(x-b) \\ 1 & c & f(c)/(x-c) \end{vmatrix} \div \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = \frac{f(x)}{(x-a)(x-b)(x-c)}.$$

Sol. $$\frac{f(x)}{(x-a)(x-b)(x-c)} = \frac{A}{(x-a)} + \frac{B}{(x-b)} + \frac{C}{(x-c)} \quad \text{[let]} \qquad \text{...(i)}$$

On comparing the various powers of x, we get

$$\Rightarrow \begin{cases} A = \dfrac{f(a)}{(a-b)(a-c)} = -\dfrac{f(a)}{(a-b)(c-a)} \\ B = \dfrac{f(b)}{(b-a)(b-c)} = -\dfrac{f(b)}{(a-b)(b-c)} \\ C = \dfrac{f(c)}{(c-a)(c-b)} = -\dfrac{f(c)}{(b-c)(c-a)} \end{cases}$$

Now, from Eq. (i), we get

$$\frac{f(x)}{(x-a)(x-b)(x-c)}$$

$$=\frac{(c-b)\dfrac{f(a)}{(x-a)}-(c-a)\dfrac{f(b)}{(x-b)}+(b-a)\dfrac{f(c)}{(x-c)}}{(a-b)(b-c)(c-a)}$$

$$=\frac{\begin{vmatrix}1 & a & f(a)/(x-a)\\ 1 & b & f(b)/(x-b)\\ 1 & c & f(c)/(x-c)\end{vmatrix}}{\begin{vmatrix}1 & a & a^2\\ 1 & b & b^2\\ 1 & c & c^2\end{vmatrix}}$$

● ***Ex. 43*** *If* $f(a,b)=\dfrac{f(b)-f(a)}{b-a}$ *and*

$f(a,b,c)=\dfrac{f(b,c)-f(a,b)}{(c-a)}$, *prove that*

$$f(a,b,c)=\begin{vmatrix}f(a) & f(b) & f(c)\\ 1 & 1 & 1\\ a & b & c\end{vmatrix}\div\begin{vmatrix}1 & 1 & 1\\ a & b & c\\ a^2 & b^2 & c^2\end{vmatrix}.$$

Sol. LHS $=f(a,b,c)=\dfrac{f(b,c)-f(a,b)}{(c-a)}$

$$=\frac{\dfrac{f(c)-f(b)}{(c-b)}-\dfrac{f(b)-f(a)}{(b-a)}}{(c-a)}$$

$$=\frac{(b-a)\{f(c)-f(b)\}-(c-b)\{f(b)-f(a)\}}{(b-a)(c-b)(c-a)}$$

$$=\frac{(f(a)\cdot(c-b)-f(b)\cdot(c-a)+f(c)\cdot(b-a))}{(a-b)(b-c)(c-a)}$$

$$=\frac{f(a)\begin{vmatrix}1 & 1\\ b & c\end{vmatrix}-f(b)\begin{vmatrix}1 & 1\\ a & c\end{vmatrix}+f(c)\begin{vmatrix}1 & 1\\ a & b\end{vmatrix}}{\begin{vmatrix}1 & 1 & 1\\ a & b & c\\ a^2 & b^2 & c^2\end{vmatrix}}$$

$$=\frac{\begin{vmatrix}f(a) & f(b) & f(c)\\ 1 & 1 & 1\\ a & b & c\end{vmatrix}}{\begin{vmatrix}1 & 1 & 1\\ a & b & c\\ a^2 & b^2 & c^2\end{vmatrix}}$$

$$=\begin{vmatrix}f(a) & f(b) & f(c)\\ 1 & 1 & 1\\ a & b & c\end{vmatrix}\div\begin{vmatrix}1 & 1 & 1\\ a & b & c\\ a^2 & b^2 & c^2\end{vmatrix}=\text{RHS}$$

● ***Ex. 44*** *Let S be the sum of all possible determinants of order 2 having 0, 1, 2 and 3 as their elements. Find the common root* α *of the equations*

$$x^2+ax+[m+1]=0,$$
$$x^2+bx+[m+4]=0$$

and $\quad x^2-cx+[m+15]=0,$

such that $\alpha>S$, *where* $a+b+c=0$ *and*

$$m=\lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{2n}\frac{r}{\sqrt{(n^2+r^2)}}$$

and [.] denotes the greatest integer function.

Sol. Let α be a common root of the given equations, then

$$\alpha^2+a\alpha+[m+1]=0$$

$$\Rightarrow \quad \alpha^2+a\alpha+[m]+1=0 \quad \text{...(i)}$$

$$\alpha^2+b\alpha+[m+4]=0$$

$$\Rightarrow \quad \alpha^2+b\alpha+[m]+4=0 \quad \text{...(ii)}$$

and $\quad \alpha^2-c\alpha+[m+15]=0$

$$\Rightarrow \quad \alpha^2-c\alpha+[m]+15=0 \quad \text{...(iii)}$$

On adding Eqs. (i) and (ii) and subtracting Eq. (iii), we get

$$a^2+(a+b+c)\alpha+[m]-10=0$$

$$\alpha^2+0+[m]-10=0 \qquad [\because a+b+c=0]$$

$$\Rightarrow \quad \alpha^2+[m]-10=0 \quad \text{...(iv)}$$

Also, $\quad m=\lim_{n\to\infty}\dfrac{1}{n}\sum_{r=1}^{2n}\dfrac{r}{\sqrt{n^2+r^2}}$

$$=\lim_{n\to\infty}\sum_{r=1}^{2n}\frac{1}{n}\cdot\frac{r/n}{\sqrt{1+(r/n)^2}}=\int_0^2\frac{x}{\sqrt{(1+x^2)}}\,dx$$

$$[\sqrt{(1+x^2)}]_0^2=\sqrt{5}-1$$

Now, $\quad [m]=[\sqrt{5}-1]=1$

From Eq. (iv), we get

$$\alpha^2+1-10=0 \Rightarrow \alpha^2=9$$

$\therefore \quad \alpha=\pm 3$

Now, number of determinants of order 2 having

0, 1, 2, 3 = 4! = 24

Let $\Delta_1=\begin{vmatrix}a_1 & a_2\\ a_3 & a_4\end{vmatrix}$ be one such determinant and their exists another determinant.

Let $\quad \Delta_2=\begin{vmatrix}a_3 & a_4\\ a_1 & a_2\end{vmatrix}$ [obtained on interchanging R_1 and R_2]

such that $\Delta_1+\Delta_2=0$

$\because S=$ Sum of all the 24 determinants = 0

Since, $\quad \alpha>S \Rightarrow \alpha>0$

$\therefore \quad \alpha=3$

Ex. 45 *If a_1, a_2, a_3 and $b_1, b_2, b_3 \in R$ and are such that $a_i b_j \neq 1$ for $1 \le i, j \le 3$,*

$$\begin{vmatrix} \frac{1-a_1^3b_1^3}{1-a_1b_1} & \frac{1-a_1^3b_2^3}{1-a_1b_2} & \frac{1-a_1^3b_3^3}{1-a_1b_3} \\ \frac{1-a_2^3b_1^3}{1-a_2b_1} & \frac{1-a_2^3b_2^3}{1-a_2b_2} & \frac{1-a_2^3b_3^3}{1-a_2b_3} \\ \frac{1-a_3^3b_1^3}{1-a_3b_1} & \frac{1-a_3^3b_2^3}{1-a_3b_2} & \frac{1-a_3^3b_3^3}{1-a_3b_3} \end{vmatrix} > 0 \text{ provided either}$$

$a_1 < a_2 < a_3$ and $b_1 < b_2 < b_3$ or $a_1 > a_2 > a_3$ and $b_1 > b_2 > b_3$.

Sol. Since, $\dfrac{x^3 - y^3}{x - y} = \dfrac{(x-y)(x^2+xy+y^2)}{(x-y)} = x^2 + xy + y^2$

Hence, the given determinant becomes

$$\begin{vmatrix} 1 + a_1b_1 + a_1^2 b_1^2 & 1 + a_1b_2 + a_1^2 b_2^2 & 1 + a_1 b_3 + a_1^2 b_3^2 \\ 1 + a_2 b_1 + a_2^2 b_1^2 & 1 + a_2 b_2 + a_2^2 b_2^2 & 1 + a_2 b_3 + a_2^2 b_3^2 \\ 1 + a_3b_1 + a_3^2 b_1^2 & 1 + a_3b_2 + a_3^2 b_2^2 & 1 + a_3 b_3 + a_3^2 b_3^2 \end{vmatrix} > 0$$

$$\Rightarrow \quad \begin{vmatrix} 1 & a_1 & a_1^2 \\ 1 & a_2 & a_2^2 \\ 1 & a_3 & a_3^2 \end{vmatrix} \times \begin{vmatrix} 1 & b_1 & a_1^2 \\ 1 & b_2 & b_2^2 \\ 1 & b_3 & b_3^2 \end{vmatrix} > 0$$

$$\Rightarrow \quad (a_1 - a_2)(a_2 - a_3)(a_3 - a_1)(b_1 - b_2)(b_2 - b_3)(b_3 - b_1) > 0$$

$$\left\{\because \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)\right\}$$

***Case* I** If $a_1 < a_2 < a_3$ and $b_1 < b_2 < b_3$, then

$(a_1 - a_2) < 0, (a_2 - a_3) < 0$

and $(b_1 - b_2) < 0, (b_2 - b_3) < 0$

$(a_1 - a_3) < 0$

and $(b_1 - b_3) < 0$

$\therefore$ $(a_3 - a_1) > 0$

$\therefore$ $(b_3 - b_1) > 0$

Then, $(a_1 - a_2)(a_2 - a_3)(a_3 - a_1) > 0$

and $(b_1 - b_2)(b_2 - b_3)(b_3 - b_1) > 0$

$\therefore$ $(a_1 - a_2)(a_2 - a_3)(a_3 - a_1)(b_1 - b_2)(b_2 - b_3)(b_3 - b_1) > 0$

which is true.

***Case* II** If $a_1 > a_2 > a_3$ and $b_1 > b_2 > b_3$

$\therefore$ $a_1 - a_2 > 0, a_2 - a_3 > 0$

and $b_1 - b_2 > 0, b_2 - b_3 > 0$

$a_1 - a_3 > 0 \Rightarrow a_3 - a_1 < 0$

and $b_1 - b_3 > 0 \Rightarrow b_3 - b_1 < 0$

Hence, $(a_1 - a_2)(a_2 - a_3)(a_3 - a_1) < 0$

and $(b_1 - b_2)(b_2 - b_3)(b_3 - b_1) < 0$

$\therefore$ $(a_1 - a_2)(a_2 - a_3)(a_3 - a_1)(b_1 - b_2)(b_2 - b_3)(b_3 - b_1) > 0$

which is true.

Ex. 46 *Show that a six-digit number abcdef is divisible by 11, if and only if $ab + cd + ef$ is divisible by 11. Hence or otherwise, find one set of values of two-digit numbers x, y and z, so that the value of the determinant $\begin{vmatrix} x & 23 & 42 \\ 13 & 37 & y \\ 19 & z & 34 \end{vmatrix}$ is divisible by 99 (without expanding the determinant).*

Sol. Since, $abcdef = ab0000 + cd00 + ef$

$= (9999 + 1)\, ab + (99 + 1)\, cd + ef$

$= 9999\, ab + 99\, cd + ab + cd + ef$

Given, $abcdef$ is divisible by 11, if and only if $ab + cd + ef$ is divisible by 11. Now, let $x = ab$, $y = cd$ and $z = ef$.
[each being a two-digit number]

Again, let $\Delta = \begin{vmatrix} x & 23 & 42 \\ 13 & 37 & y \\ 19 & z & 34 \end{vmatrix} = \begin{vmatrix} ab & 23 & 42 \\ 13 & 37 & cd \\ 19 & ef & 34 \end{vmatrix}$

Applying $R_1 \to R_1 + 100\, R_2 + 10000\, R_3$, we get

$$\Delta = \begin{vmatrix} 1913ab & ef3723 & 34cd42 \\ 13 & 37 & cd \\ 19 & ef & 34 \end{vmatrix}$$

Now, $1913ab$ is divisible by 11, if and only if
$19 + 13 + ab = 32 + ab$ is divisible by $11 \Rightarrow ab = 01, 12, 23, \ldots$

Again, $1913ab$ is divisible by 9, if and only if

$1 + 9 + 1 + 3 + a + b = 14 + a + b$ is divisibe by 9.

The above two conditions are satisfied for $a = 6$, $b = 7$. Thus, $x = 67$. Similarly, $y = 23$ and $z = 39$.