CHAPTER

06

Binomial Theorem

Session 1

Binomial Theorem for Positive Integral Index, Pascal's Triangle

An algebraic expression consisting of two dissimilar terms with positive or negative sign between them is called a binomial expressions.

For example, $x+a, x^2a-\frac{a}{x}, \frac{p}{x^2}-\frac{q}{x^4}, 5-x,$

$(x^2+1)^{1/3}-\frac{1}{\sqrt{(x^3+1)}}$, etc., are called binomial expressions.

Remarks

1. An algebraic expression consisting of three dissimilar terms is called a trinomial. e.g. $a+2b+c, x-2y+3z, 2\alpha-\frac{3}{\beta}+\gamma,$ etc. are called the trinomials.
2. In general, expressions consisting more than two dissimilar terms are known as multinomial expressions.

Binomial Theorem for Positive Integral Index

If $x, a \in C$ and $n \in N$, then

$$(x+a)^n = {}^nC_0\, x^{n-0}\, a^0 + {}^nC_1\, x^{n-1}a^1 + {}^nC_2\, x^{n-2}\, a^2 + \ldots + {}^nC_r\, x^{n-r}\, a^r + \ldots + {}^nC_{n-1}\, x^1\, a^{n-1} + {}^nC_n\, x^0\, a^n \quad \ldots\text{(i)}$$

or $$(x+a)^n = \sum_{r=0}^{n} {}^nC_r\, x^{n-r}\, a^r$$

Hence, ${}^nC_0, {}^nC_1, {}^nC_2, \ldots, {}^nC_n$ are called binomial coefficients.

Remark

1. In each term, the degree is n and the coefficient of $x^{n-r}\, a^r$ is equal to the number of ways $\underbrace{x, x, x, \ldots, x}_{(n-r)\text{ times}}, \underbrace{a, a, a, \ldots, a}_{r\text{ times}}$ can be arranged, which is given by $\frac{n!}{(n-r)!\, r!} = {}^nC_r$

   For example, $(x+a)^5 = \frac{5!}{5!\,0!}x^5a^0 + \frac{5!}{4!\,1!}x^4a + \frac{5!}{3!\,2!}x^3a^2 + \frac{5!}{2!\,3!}x^2a^3 + \frac{5!}{1!\,4!}xa^4 + \frac{5!}{0!\,5!}x^0a^5$

   $= {}^5C_0x^5 + {}^5C_1x^4a + {}^5C_2x^3a^2 + {}^5C_3x^2a^3 + {}^5C_4xa^4 + {}^5C_5a^5$

2. Let $S = (x+a)^n = \sum_{r=0}^{n} {}^nC_r\, x^{n-r}\, a^r$

   Replacing r by $n-r$, we have

   $S = (x+a)^n = \sum_{r=0}^{n} {}^nC_{n-r}\, x^{n-(n-r)}\, a^{n-r} = \sum_{r=0}^{n} {}^nC_{n-r}\, x^r\, a^{n-r}$

   $= {}^nC_n\, a^n + {}^nC_{n-1}\, a^{n-1}\, x + {}^nC_{n-2}\, a^{n-2}\, x^2 + \ldots + {}^nC_0\, x^n$

   Thus, replacing r by $n-r$, we are infact writing the binomial expansion in reverse order.

Some Important Points

1. Replacing a by $(-a)$ in Eq. (i), we get

$$(x-a)^n = {}^nC_0\, x^{n-0}\, a^0 - {}^nC_1\, x^{n-1}\, a^1 + {}^nC_2\, x^{n-2}\, a^2 - \ldots + \ldots + (-1)^r\, {}^nC_r\, x^{n-r}\, a^r + \ldots + (-1)^n\, {}^nC_n\, x^0\, a^n \quad \ldots\text{(ii)}$$

or $$(x-a)^n = \sum_{r=0}^{n} (-1)^r\, {}^nC_r\, x^{n-r}\, a^r$$

2. On adding Eqs. (i) and (ii), we get

$$(x+a)^n + (x-a)^n = 2\{{}^nC_0\, x^{n-0}\, a^0 + {}^nC_2\, x^{n-2}\, a^2 + {}^nC_4\, x^{n-4}\, a^4 + \ldots\}$$
$$= 2\ \{\text{Sum of terms at odd places}\}$$

   The last term is ${}^nC_n\, a^n$ or ${}^nC_{n-1}\, x\, a^{n-1}$, according as n is even or odd, respectively.

3. On subtracting Eq. (ii) from Eq. (i), we get

$$(x+a)^n - (x-a)^n = 2\{{}^nC_1\, x^{n-1}\, a^1 + {}^nC_3\, x^{n-3}\, a^3 + {}^nC_5\, x^{n-5}\, a^5 + \ldots\}$$
$$= 2\ \{\text{Sum of terms at even places}\}$$

   The last term is ${}^nC_{n-1}\, x\, a^{n-1}$ or ${}^nC_n\, a^n$, according as n is even or odd, respectively.

4. Replacing x by 1 and a by x in Eq. (i), we get

$$(1+x)^n = {}^nC_0\, x^0 + {}^nC_1\, x^1 + {}^nC_2\, x^2 + \ldots + {}^nC_r\, x^r + \ldots + {}^nC_{n-1}\, x^{n-1} + {}^nC_n\, x^n \quad \ldots\text{(iii)}$$

or $$(1+x)^n = \sum_{r=0}^{n} {}^nC_r\, x^r$$

5. Replacing x by $(-x)$ in Eq. (iii), we get

$$(1-x)^n = {}^nC_0\, x^0 - {}^nC_1\, x^1 + {}^nC_2\, x^2 - \dots + (-1)^r\, {}^nC_r\, x^r + \dots + {}^nC_n (-1)^n x^n$$

or $(1-x)^n = \sum_{r=0}^{n} (-1)^r\, {}^nC_r\, x^r$

Example 1. Expand $\left(2a - \frac{3}{b}\right)^5$ by binomial theorem.

Sol. Using binomial theorem, we get

$$\left(2a-\frac{3}{b}\right)^5 = {}^5C_0\,(2a)^{5-0}\left(-\frac{3}{b}\right)^0 + {}^5C_1(2a)^{5-1}\left(-\frac{3}{b}\right)^1$$

$$+\, {}^5C_2\,(2a)^{5-2}\left(-\frac{3}{b}\right)^2 + {}^5C_3\,(2a)^{5-3}\left(-\frac{3}{b}\right)^3$$

$$+\, {}^5C_4\,(2a)^{5-4}\left(-\frac{3}{b}\right)^4 + {}^5C_5\,(2a)^{5-5}\left(-\frac{3}{b}\right)^5$$

$$= {}^5C_0\,(2a)^5 - {}^5C_1\,(2a)^4\left(\frac{3}{b}\right) + {}^5C_2\,(2a)^3\left(\frac{3}{b}\right)^2$$

$$-\, {}^5C_3\,(2a)^2\left(\frac{3}{b}\right)^3 + {}^5C_4\,(2a)^1\left(\frac{3}{b}\right)^4 - {}^5C_5\left(\frac{3}{b}\right)^5$$

$$= 32a^5 - \frac{240\,a^4}{b} + \frac{720\,a^3}{b^2} - \frac{1080\,a^2}{b^3} + \frac{810\,a}{b^4} - \frac{243}{b^5}$$

Example 2. ***Simplify***

$$(x+\sqrt{(x^2-1)})^6 + (x-\sqrt{(x^2-1)})^6.$$

Sol. Let $\sqrt{(x^2-1)} = a$

Then, $(x+a)^6 + (x-a)^6 = 2\{{}^6C_0\, x^{6-0}\, a^0 + {}^6C_2\, x^{6-2}\, a^2 + {}^6C_4\, x^{6-4}\, a^4 + {}^6C_6\, x^{6-6}\, a^6\}$

$= 2\{x^6 + 15x^4a^2 + 15x^2a^4 + a^6\}$ [from point (2)]

$= 2\{x^6 + 15x^4(x^2-1) + 15x^2(x^2-1)^2 + (x^2-1)^3\}$ $[\because a = \sqrt{x^2-1}]$

$= 2(32x^6 - 48x^4 + 18x^2 - 1)$

Example 3. In the expansion of $(x+a)^n$, if sum of odd terms is P and sum of even terms is Q, prove that

(i) $P^2 - Q^2 = (x^2-a^2)^n$

(ii) $4PQ = (x+a)^{2n} - (x-a)^{2n}$

Sol. $\because$ $(x+a)^n = {}^nC_0\, x^{n-0}a^0 + {}^nC_1\, x^{n-1}a^1 + {}^nC_2\, x^{n-2}\, a^2 + {}^nC_3\, x^{n-3}\, a^3 + \dots + \dots + {}^nC_n\, x^{n-n}\, a^n$

$= ({}^nC_0\, x^n + {}^nC_2\, x^{n-2}a^2 + {}^nC_4\, x^{n-4}\, a^4 + \dots)$
$+ ({}^nC_1\, x^{n-1}\, a^1 + {}^nC_3\, x^{n-3}\, a^3 + {}^nC_5\, x^{n-5}\, a^5 + \dots)$

$= P + Q$ (given) ...(i)

and $(x-a)^n = {}^nC_0\, x^{n-0}\, a^0 - {}^nC_1\, x^{n-1}a^1 + {}^nC_2\, x^{n-2}\, a^2 - {}^nC_3\, x^{n-3}\, a^3 + \dots + {}^nC_n\, x^{n-n}\, a^n$

$= ({}^nC_0\, x^n + {}^nC_2\, x^{n-2}\, a^2 + {}^nC_4\, x^{n-4}\, a^4 + \dots)$
$- ({}^nC_1\, x^{n-1}\, a + {}^nC_3\, x^{n-3}\, a^3 + {}^nC_5\, x^{n-5}\, a^5 + \dots)$

$= P - Q$ (given) ...(ii)

(i) $P^2 - Q^2 = (P+Q)(P-Q)$

$= (x+a)^n \cdot (x-a)^n$

$= (x^2-a^2)^n$ [from Eqs. (i) and (ii)]

(ii) $(x+a)^{2n} - (x-a)^{2n} = [(x+a)^n]^2 - [(x-a)^n]^2$

$= (P+Q)^2 - (P-Q)^2$

$= 4PQ$ [from Eqs. (i) and (ii)]

Example 4. ***Show that*** $(101)^{50} > (100)^{50} + (99)^{50}$**.**

Sol. Since, $(101)^{50} - (99)^{50} = (100+1)^{50} - (100-1)^{50}$

$= 2\{{}^{50}C_1\,(100)^{49} + {}^{50}C_3\,(100)^{47} + {}^{50}C_5\,(100)^{45} + \dots\}$

$= 2\times {}^{50}C_1\,(100)^{49} + 2\{{}^{50}C_3\,(100)^{47} + {}^{50}C_5\,(100)^{45} + \dots\}$

$= (100)^{50} + (\text{a positive number}) > (100)^{50}$

Hence, $(101)^{50} - (99)^{50} > (100)^{50}$

$\Rightarrow$ $(101)^{50} > (100)^{50} + (99)^{50}$

Example 5. ***If*** $a_n = \sum_{r=0}^{n} \frac{1}{{}^nC_r}$, ***find the value of*** $\sum_{r=0}^{n} \frac{r}{{}^nC_r}$.

Sol. Let $P = \sum_{r=0}^{n} \frac{r}{{}^nC_r}$ ...(i)

Replacing r by $(n-r)$ in Eq. (i), we get

$P = \sum_{r=0}^{n} \frac{(n-r)}{{}^nC_{n-r}} = \sum_{r=0}^{n} \frac{(n-r)}{{}^nC_r}$ $[\because {}^nC_r = {}^nC_{n-r}]$ ...(ii)

On adding Eqs. (i) and (ii), we get

$$2P = \sum_{r=0}^{n} \frac{n}{{}^nC_r} = n \sum_{r=0}^{n} \frac{1}{{}^nC_r} = na_n \quad \text{[given]}$$

$\therefore$ $P = \frac{n}{2}\, a_n$

Hence, $\sum_{r=0}^{n} \frac{r}{{}^nC_r} = \frac{n}{2}\, a_n$

Properties of Binomial Expansion $(x + a)^n$

(i) This expansion has $(n+1)$ terms.

(ii) Since, ${}^nC_r = {}^nC_{n-r}$, we have

$$ {}^nC_0 = {}^nC_n = 1 $$
$$ {}^nC_1 = {}^nC_{n-1} = n $$
$$ {}^nC_2 = {}^nC_{n-2} = \frac{n(n-1)}{2!} \text{ and so on.} $$

(iii) In any term, the suffix of C is equal to the index of a and the index of $x = n -$ (suffix of C).

(iv) In each term, sum of the indices of x and a is equal to n.

Properties of Binomial Coefficient

(i) ${}^nC_r$ can also be represented by $C(n, r)$ or $\binom{n}{r}$.

(ii) ${}^nC_x = {}^nC_y$, then either $x = y$ or $n = x + y$.

So, ${}^nC_r = {}^nC_{n-r} = \dfrac{n!}{r!(n-r)!}$

(iii) ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$

(iv) $\dfrac{{}^nC_r}{{}^nC_{r-1}} = \dfrac{n-r+1}{r}$

(v) ${}^nC_r = \dfrac{n}{r} \cdot {}^{n-1}C_{r-1}$

Pascal's Triangle

Coefficients of binomial expansion can also be easily determined by Pascal's triangle.

$(x+a)^0$	1
$(x+a)^1$	1 1
$(x+a)^2$	1 2 1
$(x+a)^3$	1 3 3 1
$(x+a)^4$	1 4 6 4 1
$(x+a)^5$	1 5 10 10 5 1

Pascal triangle gives the direct binomial coefficients.

For example,

$$ (x+a)^4 = 1 \cdot x^4 + 4 \cdot x^3 \cdot a + 6 \cdot x^2 a^2 + 4 \cdot x a^3 + 1 \cdot a^4 $$
$$ = x^4 + 4x^3 a + 6x^2 a^2 + 4x a^3 + a^4 $$

How to Construct a Pascal's Triangle

Binomial coefficients in the expansion of $(x + a)^3$ are

	1	3	3	1	
	1	$(1+3)$	$(3+3)$	$(3+1)$	1
Then,	1	4	6	4	1

are the binomial coefficients in the expansion of $(x + a)^4$.

Example 6. ***Find the number of dissimilar terms in the expansion of*** $(1 - 3x + 3x^2 - x^3)^{33}$.

Sol. $(1 - 3x + 3x^2 - x^3)^{33} = [(1-x)^3]^{33} = (1-x)^{99}$

Therefore, number of dissimilar terms in the expansion of $(1 - 3x + 3x^2 - x^3)^3$ is 100.

Example 7. ***Find the value of*** $\displaystyle\sum_{r=1}^{n} \frac{r \cdot {}^nC_r}{{}^nC_{r-1}}$.

Sol. $\because \quad \dfrac{{}^nC_r}{{}^nC_{r-1}} = \dfrac{n-r+1}{r}$

$$ \therefore \quad \frac{r \cdot {}^nC_r}{{}^nC_{r-1}} = (n - r + 1) $$

$$ \therefore \quad \sum_{r=1}^{n} \frac{r \cdot {}^nC_r}{{}^nC_{r-1}} = \sum_{r=1}^{n} (n - r + 1) = \sum_{r=1}^{n} (n+1) - \sum_{r=1}^{n} r $$
$$ = (n+1) \sum_{r=1}^{n} 1 - (1 + 2 + 3 + \dots + n) $$
$$ = (n+1) \cdot n - \frac{n(n+1)}{2} = \frac{n(n+1)}{2} $$

Example 8. ***Let*** C_r ***stands for*** ${}^nC_r$, ***prove that***

$$ (C_0 + C_1)(C_1 + C_2)(C_2 + C_3) \dots (C_{n-1} + C_n) = \frac{(n+1)^n}{n!} (C_0 C_1 C_2 \dots C_{n-1}). $$

Sol. LHS $= (C_0 + C_1)(C_1 + C_2)(C_2 + C_3) \dots (C_{n-1} + C_n)$

$$ = \prod_{r=1}^{n} (C_{r-1} + C_r) = \prod_{r=1}^{n} ({}^{n+1}C_r) \quad [\because {}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r] $$
$$ = \prod_{r=1}^{n} \left(\frac{n+1}{r}\right) {}^nC_{r-1} \quad \left[\because {}^nC_r = \frac{n}{r} \cdot {}^{n-1}C_{r-1}\right] $$
$$ = \prod_{r=1}^{n} (n+1) \cdot \prod_{r=1}^{n} \frac{1}{r} \cdot \prod_{r=1}^{n} C_{r-1} $$
$$ = (n+1)^n \cdot \frac{1}{n!} \cdot (C_0 C_1 C_2 \dots C_{n-1}) $$
$$ = \frac{(n+1)^n}{n!} (C_0 \, C_1 \, C_2 \dots C_{n-1}) = \text{RHS} $$

Example 9. *Find the sum of the series*

$$\sum_{r=0}^{n} (-1)^r \, {}^nC_r \left\{ \frac{1}{2^r} + \frac{3^r}{2^{2r}} + \frac{7^r}{2^{3r}} + \frac{15^r}{2^{4r}} + \dots \text{ upto } m \text{ terms} \right\}.$$

Sol. $\because (1-x)^n = \sum_{r=0}^{n} (-1)^r \, {}^nC_r \, x^r \quad \dots\text{(i)}$

Let $$P = \sum_{r=0}^{n} (-1)^r \, {}^nC_r \left\{ \left(\frac{1}{2}\right)^r + \left(\frac{3}{4}\right)^r + \left(\frac{7}{8}\right)^r + \left(\frac{15}{16}\right)^r + \dots \text{ upto } m \text{ terms} \right\}$$

$$= \sum_{r=0}^{n} (-1)^r \, {}^nC_r \cdot \left(\frac{1}{2}\right)^r + \sum_{r=0}^{n} (-1)^r \, {}^nC_r \cdot \left(\frac{3}{4}\right)^r + \sum_{r=0}^{n} (-1)^r \, {}^nC_r \cdot \left(\frac{7}{8}\right)^r + \sum_{r=0}^{n} (-1)^r \, {}^nC_r \cdot \left(\frac{15}{16}\right)^r + \dots \text{ upto } m \text{ terms}$$

$$= \left(1 - \frac{1}{2}\right)^n + \left(1 - \frac{3}{4}\right)^n + \left(1 - \frac{7}{8}\right)^n + \left(1 - \frac{15}{16}\right)^n + \dots \text{ upto } m \text{ terms}$$

$$= \left(\frac{1}{2}\right)^n + \left(\frac{1}{2}\right)^{2n} + \left(\frac{1}{2}\right)^{3n} + \left(\frac{1}{2}\right)^{4n} + \dots \text{ upto } m \text{ terms}$$

$$= \frac{\left(\frac{1}{2}\right)^n \left[1 - \left\{ \left(\frac{1}{2}\right)^n \right\}^m \right]}{1 - \left(\frac{1}{2}\right)^n}$$

$$= \frac{(2^{mn} - 1)}{2^{mn}\,(2^n - 1)}$$

Exercise for Session 1

1. The value of $\sum_{r=0}^{10} r \cdot {}^{10}C_r \cdot 3^r \cdot (-2)^{10-r}$ is

(a) 10 (b) 20 (c) 30 (d) 300

2. The number of dissimilar terms in the expansion of $\left(x + \frac{1}{x} + x^2 + \frac{1}{x^2}\right)^{15}$ are

(a) 61 (b) 121 (c) 255 (d) 16

3. The expansion $\{x + (x^3 - 1)^{1/2}\}^5 + \{x - (x^3 - 1)^{1/2}\}^5$ is a polynomial of degree

(a) 5 (b) 6 (c) 7 (d) 8

4. $(\sqrt{2} + 1)^6 - (\sqrt{2} - 1)^6$ is equal to

(a) 101 (b) $70\sqrt{2}$ (c) $140\sqrt{2}$ (d) $120\sqrt{2}$

5. The total number of dissimilar terms in the expansion of $(x + a)^{100} + (x - a)^{100}$ after simplification will be

(a) 202 (b) 51
(c) 50 (d) 101

6. The number of non-zero terms in the expansion of $(1 + 3\sqrt{2}x)^9 + (1 - 3\sqrt{2}x)^9$, is

(a) 0 (b) 5
(c) 9 (d) 10

7. If $(1 + x)^n = \sum_{r=0}^{n} C_r \, x^r$, $\left(1 + \frac{C_1}{C_0}\right)\left(1 + \frac{C_2}{C_1}\right) \dots \left(1 + \frac{C_n}{C_{n-1}}\right)$ is equal to

(a) $\frac{n^{n-1}}{(n-1)!}$ (b) $\frac{(n+1)^{n-1}}{(n-1)!}$

(c) $\frac{(n+1)^n}{n!}$ (d) $\frac{(n+1)^{n+1}}{n!}$

8. If ${}^{n+1}C_{r+1} : {}^nC_r : {}^{n-1}C_{r-1} = 11 : 6 : 3$, nr is equal to

(a) 20 (b) 30
(c) 0 (d) 50

Answers

Exercise for Session 1

1. (c) **2.** (a) **3.** (c) **4.** (c) **5.** (b) **6.** (b)
7. (c) **8.** (d)