# Session 4

## Differentiation of Determinant, Integration of a Determinant, Walli's Formula, Use of $\Sigma$ in Determinant

## Differentiation of Determinant

Let $\Delta(x)$ be a determinant of order *n*. If we write $\Delta(x) = [C_1\ C_2\ C_3 \ldots C_n]$, where $C_1, C_2, C_3, \ldots, C_n$ denotes 1st, 2nd, 3rd, …, *n*th columns respectively, then

$$\begin{aligned}\Delta'(x) &= [C_1'\ C_2\ C_3 \cdots C_n] + [C_1\ C_2'\ C_3 \cdots C_n] \\ &\quad + [C_1\ C_2\ C_3' \cdots C_n] + \cdots + [C_1\ C_2\ C_3 \cdots C_n'] \\ &= \sum [C_1'\ C_2\ C_3 \cdots C_n]\end{aligned}$$

where $C_i'$ denotes the column which contains the derivative of all the functions in the *i*th column $C_i$. Also, if

$$\Delta(x) = \begin{bmatrix} R_1 \\ R_2 \\ R_3 \\ \vdots \\ R_n \end{bmatrix}$$

where $R_1, R_2, R_3, \ldots, R_n$ denote 1st, 2nd, 3rd, …, *n*th rows respectively, then

$$\Delta'(x) = \begin{bmatrix} R_1' \\ R_2 \\ R_3 \\ \vdots \\ R_n \end{bmatrix} + \begin{bmatrix} R_1 \\ R_2' \\ R_3 \\ \vdots \\ R_n \end{bmatrix} + \begin{bmatrix} R_1 \\ R_2 \\ R_3' \\ \vdots \\ R_n \end{bmatrix} + \cdots + \begin{bmatrix} R_1 \\ R_2 \\ R_3 \\ \vdots \\ R_n' \end{bmatrix} = \sum \begin{bmatrix} R_1' \\ R_2 \\ R_3 \\ \vdots \\ R_n \end{bmatrix}$$

where $R_i'$ denotes the row which contains the derivative of all the functions in the *i*th row $R_i$.

***Corollary* I** For $n = 2$,

$$\Delta(x) = [C_1\ C_2], \text{ then } \Delta'(x) = [C_1'\ C_2] + [C_1\ C_2']$$

Also, if $\Delta(x) = \begin{bmatrix} R_1 \\ R_2 \end{bmatrix}$, then $\Delta'(x) = \begin{bmatrix} R_1' \\ R_2 \end{bmatrix} + \begin{bmatrix} R_1 \\ R_2' \end{bmatrix}$

*For example*, Let $\Delta(x) = \begin{vmatrix} a_1(x) & b_1(x) \\ a_2(x) & b_2(x) \end{vmatrix}$, then

$$\Delta'(x) = \begin{vmatrix} a_1'(x) & b_1'(x) \\ a_2(x) & b_2(x) \end{vmatrix} + \begin{vmatrix} a_1(x) & b_1(x) \\ a_2'(x) & b_2'(x) \end{vmatrix}$$

[derivative according to rowwise]

***Corollary* II For $n = 3$,** $\Delta(x) = [C_1\ C_2\ C_3]$, then

$$\Delta'(x) = [C_1'\ C_2\ C_3] + [C_1\ C_2'\ C_3] + [C_1\ C_2\ C_3']$$

Also, if $\Delta(x) = \begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix}$, then $\Delta'(x) = \begin{bmatrix} R_1' \\ R_2 \\ R_3 \end{bmatrix} + \begin{bmatrix} R_1 \\ R_2' \\ R_3 \end{bmatrix} + \begin{bmatrix} R_1 \\ R_2 \\ R_3' \end{bmatrix}$

*For example*, Let $\Delta(x) = \begin{vmatrix} a_1(x) & a_2(x) & a_3(x) \\ b_1(x) & b_2(x) & b_3(x) \\ c_1(x) & c_2(x) & c_3(x) \end{vmatrix}$, then

$$\Delta'(x) = \begin{vmatrix} a_1'(x) & a_2'(x) & a_3'(x) \\ b_1(x) & b_2(x) & b_3(x) \\ c_1(x) & c_2(x) & c_3(x) \end{vmatrix} + \begin{vmatrix} a_1(x) & a_2(x) & a_3(x) \\ b_1'(x) & b_2'(x) & b_3'(x) \\ c_1(x) & c_2(x) & c_3(x) \end{vmatrix} + \begin{vmatrix} a_1(x) & a_2(x) & a_3(x) \\ b_1(x) & b_2(x) & b_3(x) \\ c_1'(x) & c_2'(x) & c_3'(x) \end{vmatrix}$$

[derivative according to rowwise]

### *Remark*

1. In a third order determinant, if two rows (columns) consist functions of $x$ and third row (column) is constant, let

$$\Delta(x) = \begin{vmatrix} a_1(x) & a_2(x) & a_3(x) \\ b_1(x) & b_2(x) & b_3(x) \\ c_1 & c_2 & c_3 \end{vmatrix}, \text{ then}$$

$$\Delta'(x) = \begin{vmatrix} a_1'(x) & a_2'(x) & a_3'(x) \\ b_1(x) & b_2(x) & b_3(x) \\ c_1 & c_2 & c_3 \end{vmatrix} + \begin{vmatrix} a_1(x) & a_2(x) & a_3(x) \\ b_1'(x) & b_2'(x) & b_3'(x) \\ c_1 & c_2 & c_3 \end{vmatrix}$$

2. In a third order determinant, if only one row (column) consists functions of $x$ and other rows (columns) are constant, let

$$\Delta(x) = \begin{vmatrix} a_1(x) & a_2(x) & a_3(x) \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}, \text{ then } \Delta'(x) = \begin{vmatrix} a_1'(x) & a_2'(x) & a_3'(x) \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

and in general

$$\frac{d^n}{dx^n}\{\Delta(x)\} = \begin{vmatrix} \frac{d^n}{dx^n}\{a_1(x)\} & \frac{d^n}{dx^n}\{a_2(x)\} & \frac{d^n}{dx^n}\{a_3(x)\} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

## Important Derivatives
(Committed to Memory)

If $a$ and $b$ are constants and $n \in N$, then

1. if $y = (ax+b)^n$, then $\dfrac{d^n y}{dx^n} = n!\,a^n$
2. if $y = \sin(ax+b)$, then $\dfrac{d^n y}{dx^n} = \sin\left(\dfrac{n\pi}{2} + ax + b\right) \cdot a^n$
3. if $y = \cos(ax+b)$, then $\dfrac{d^n y}{dx^n} = \cos\left(\dfrac{n\pi}{2} + ax + b\right) \cdot a^n$

**Example 35.** ***If*** $f(x) = \begin{vmatrix} \sin x & \cos x & \sin x \\ \cos x & -\sin x & \cos x \\ x & 1 & 1 \end{vmatrix}$, ***find the value of*** $2^{f'(0)} + \{f'(1)\}^2$.

***Sol.*** $f'(x) = \begin{vmatrix} \cos x & -\sin x & \cos x \\ \cos x & -\sin x & \cos x \\ x & 1 & 1 \end{vmatrix} + \begin{vmatrix} \sin x & \cos x & \sin x \\ -\sin x & -\cos x & -\sin x \\ x & 1 & 1 \end{vmatrix}$

$+ \begin{vmatrix} \sin x & \cos x & \sin x \\ \cos x & -\sin x & \cos x \\ 1 & 0 & 0 \end{vmatrix}$ [derivative according to rowwise]

$= 0 + 0 + 1 \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1$

$\therefore\ f'(x) = 1 \Rightarrow f'(0) = 1$ and $f'(1) = 1$

$\Rightarrow\quad 2^{f'(0)} + \{f'(1)\}^2 = 2^1 + 1^2 = 3$

**Example 36.** ***Let*** $f(x) = \begin{vmatrix} \cos x & \sin x & \cos x \\ \cos 2x & \sin 2x & 2\cos 2x \\ \cos 3x & \sin 3x & 3\cos 3x \end{vmatrix}$, ***then find the value of*** $f'\left(\dfrac{\pi}{2}\right)$.

***Sol.*** Given, $f(x) = \begin{vmatrix} \cos x & \sin x & \cos x \\ \cos 2x & \sin 2x & 2\cos 2x \\ \cos 3x & \sin 3x & 3\cos 3x \end{vmatrix}$

$\therefore\quad f'(x) = \begin{vmatrix} -\sin x & \sin x & \cos x \\ -2\sin 2x & \sin 2x & 2\cos 2x \\ -3\sin 3x & \sin 3x & 3\cos 3x \end{vmatrix}$

$+ \begin{vmatrix} \cos x & \cos x & \cos x \\ \cos 2x & 2\cos 2x & 2\cos 2x \\ \cos 3x & 3\cos 3x & 3\cos 3x \end{vmatrix} + \begin{vmatrix} \cos x & \sin x & -\sin x \\ \cos 2x & \sin 2x & -4\sin 2x \\ \cos 3x & \sin 3x & -9\sin 3x \end{vmatrix}$

[derivative according to columnwise]

$\Rightarrow f'\left(\dfrac{\pi}{2}\right) = \begin{vmatrix} -1 & 1 & 0 \\ 0 & 0 & -2 \\ 3 & -1 & 0 \end{vmatrix} + 0 + \begin{vmatrix} 0 & 1 & -1 \\ -1 & 0 & 0 \\ 0 & -1 & 9 \end{vmatrix}$

[$\because C_2 = C_3$ in second determinant]

$= 2(1-3) + 1(9-1) = -4 + 8 = 4$

**Example 37.** Let $\alpha$ be a repeated root of a quadratic equation $f(x) = 0$ and $A(x)$, $B(x)$ and $C(x)$ be polynomials of degree 3, 4, and 5 respectively, show that $\begin{vmatrix} A(x) & B(x) & C(x) \\ A(\alpha) & B(\alpha) & C(\alpha) \\ A'(\alpha) & B'(\alpha) & C'(\alpha) \end{vmatrix}$ is divisible by $f(x)$, where prime ($'$) denotes the derivatives.

***Sol.*** Since, $\alpha$ is a repeated root of the quadratic equation $f(x) = 0$, then $f(x)$ can be written as $f(x) = a(x-\alpha)^2$, where $a$ is some non-zero constant.

Let $\quad g(x) = \begin{vmatrix} A(x) & B(x) & C(x) \\ A(\alpha) & B(\alpha) & C(\alpha) \\ A'(\alpha) & B'(\alpha) & C'(\alpha) \end{vmatrix}$

$g(x)$ is divisible by $f(x)$, if it is divisible by $(x-a)^2$ i.e., $g(\alpha) = 0$ and $g'(\alpha) = 0$. As $A(x)$, $B(x)$ and $C(x)$ are polynomials of degree 3, 4 and 5, respectively.

$\therefore$ Degree of $g(x) \geq 2$

Now, $\quad g(\alpha) = \begin{vmatrix} A(\alpha) & B(\alpha) & C(\alpha) \\ A(\alpha) & B(\alpha) & C(\alpha) \\ A'(\alpha) & B'(\alpha) & C'(\alpha) \end{vmatrix} = 0$

[$\because R_1$ and $R_2$ are identical]

Also, $\quad g'(x) = \begin{vmatrix} A'(x) & B'(x) & C'(x) \\ A(\alpha) & B(\alpha) & C(\alpha) \\ A'(\alpha) & B'(\alpha) & C'(\alpha) \end{vmatrix}$

$\therefore\quad g'(\alpha) = \begin{vmatrix} A'(\alpha) & B'(\alpha) & C'(\alpha) \\ A(\alpha) & B(\alpha) & C(\alpha) \\ A'(\alpha) & B'(\alpha) & C'(\alpha) \end{vmatrix} = 0$

[$\because R_1$ and $R_3$ are identical]

This implies that $f(x)$ divides $g(x)$.

**Example 38.** Find the coefficient of $x$ in the determinant

$$\begin{vmatrix} (1+x)^{a_1 b_1} & (1+x)^{a_1 b_2} & (1+x)^{a_1 b_3} \\ (1+x)^{a_2 b_1} & (1+x)^{a_2 b_2} & (1+x)^{a_2 b_3} \\ (1+x)^{a_3 b_1} & (1+x)^{a_3 b_2} & (1+x)^{a_3 b_3} \end{vmatrix}$$

***Sol.*** We know that, if $f(x)$ be a polynomial, then coefficient of $x^n$ in $f(x) = \dfrac{1}{n!} f^n(0)$.

Let $\quad f(x) = \begin{vmatrix} (1+x)^{a_1 b_1} & (1+x)^{a_1 b_2} & (1+x)^{a_1 b_3} \\ (1+x)^{a_2 b_1} & (1+x)^{a_2 b_2} & (1+x)^{a_2 b_3} \\ (1+x)^{a_3 b_1} & (1+x)^{a_3 b_2} & (1+x)^{a_3 b_3} \end{vmatrix}$

$\therefore\ f'(x) = \begin{vmatrix} a_1 b_1 (1+x)^{a_1 b_1 - 1} & a_1 b_2 (1+x)^{a_1 b_2 - 1} & a_1 b_3 (1+x)^{a_1 b_3 - 1} \\ (1+x)^{a_2 b_1} & (1+x)^{a_2 b_2} & (1+x)^{a_2 b_3} \\ (1+x)^{a_3 b_1} & (1+x)^{a_3 b_2} & (1+x)^{a_3 b_3} \end{vmatrix}$

$$+\begin{vmatrix} (1+x)^{a_1b_1} & (1+x)^{a_1b_2} & (1+x)^{a_1b_3} \\ a_2b_1(1+x)^{a_2b_1-1} & a_2b_2(1+x)^{a_2b_2-1} & a_2b_3(1+x)^{a_2b_3-1} \\ (1+x)^{a_3b_1} & (1+x)^{a_3b_2} & (1+x)^{a_3b_3} \end{vmatrix}$$

$$+\begin{vmatrix} (1+x)^{a_1b_1} & (1+x)^{a_1b_2} & (1+x)^{a_1b_3} \\ (1+x)^{a_2b_1} & (1+x)^{a_2b_2} & (1+x)^{a_2b_3} \\ a_3b_1(1+x)^{a_3b_1-1} & a_3b_2(1+x)^{a_3b_2-1} & a_3b_3(1+x)^{a_3b_3-1} \end{vmatrix}$$

$$\therefore\ f'(0)=\begin{vmatrix} a_1b_1 & a_1b_2 & a_1b_3 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix}+\begin{vmatrix} 1 & 1 & 1 \\ a_2b_1 & a_2b_2 & a_3b_3 \\ 1 & 1 & 1 \end{vmatrix}+\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ a_3b_1 & a_3b_2 & a_3b_3 \end{vmatrix}$$

$$=0+0+0=0$$

$$\therefore\ \text{Coefficient of } x \text{ in } f(x)=\frac{f'(0)}{1!}=0$$

**Aliter**

$$\text{Let}\begin{vmatrix} (1+x)^{a_1b_1} & (1+x)^{a_1b_2} & (1+x)^{a_1b_3} \\ (1+x)^{a_2b_1} & (1+x)^{a_2b_2} & (1+x)^{a_2b_3} \\ (1+x)^{a_3b_1} & (1+x)^{a_3b_2} & (1+x)^{a_3b_3} \end{vmatrix}=A+Bx+Cx^2+\cdots$$

On differentiating both sides w.r.t. $x$ and then put $x=0$ in both sides, we get

$$B=\begin{vmatrix} a_1b_1 & a_1b_2 & a_1b_3 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix}+\begin{vmatrix} 1 & 1 & 1 \\ a_2b_1 & a_2b_2 & a_2b_3 \\ 1 & 1 & 1 \end{vmatrix}+\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ a_3b_1 & a_3b_2 & a_3b_3 \end{vmatrix}$$

$$=0+0+0=0$$

Hence, coefficient of $x$ in given determinant is 0.

**Example 39.** If $\Delta(x)=\begin{vmatrix} \alpha+x & \theta+x & \lambda+x \\ \beta+x & \phi+x & \mu+x \\ \gamma+x & \psi+x & \gamma+x \end{vmatrix}$, show that $\Delta''(x)=0$ and $\Delta(x)=\Delta(0)+Sx$, where $S$ denotes the sum of all the cofactors of all elements in $\Delta(0)$ and dash denotes the derivative.

***Sol.*** We have, $$\Delta'(x)=\begin{vmatrix} 1 & \theta+x & \lambda+x \\ 1 & \phi+x & \mu+x \\ 1 & \psi+x & \nu+x \end{vmatrix}+\begin{vmatrix} \alpha+x & 1 & \lambda+x \\ \beta+x & 1 & \mu+x \\ \gamma+x & 1 & \nu+x \end{vmatrix}+\begin{vmatrix} \alpha+x & \theta+x & 1 \\ \beta+x & \phi+x & 1 \\ \gamma+x & \psi+x & 1 \end{vmatrix}$$

Applying $C_2 \to C_2 - xC_1$ and $C_3 \to C_3 - xC_1$ in first, $C_1 \to C_1 - xC_2$ and $C_3 \to C_3 - xC_2$ in second and $C_1 \to C_1 - xC_3$ and $C_2 \to C_2 - xC_3$ in third, then

$$\Delta'(x)=\begin{vmatrix} 1 & \theta & \lambda \\ 1 & \phi & \mu \\ 1 & \psi & \nu \end{vmatrix}+\begin{vmatrix} \alpha & 1 & \lambda \\ \beta & 1 & \mu \\ \gamma & 1 & \nu \end{vmatrix}+\begin{vmatrix} \alpha & \theta & 1 \\ \beta & \phi & 1 \\ \gamma & \psi & 1 \end{vmatrix}$$

$$\left\{\text{sum of all cofactors in } \Delta(0), \text{where } \Delta(0)=\begin{vmatrix} \alpha & \theta & \lambda \\ \beta & \phi & \mu \\ \gamma & \psi & \nu \end{vmatrix}\right\}$$

$\therefore\quad \Delta''(x)=0 \quad$ [$\because$ $S$ is constant]

Since, $\quad \Delta'(x)=S$

On integrating $\Delta(x)=Sx+C$

$\therefore\quad \Delta(0)=0+C$

Hence, $\quad \Delta(x)=Sx+\Delta(0)$

**Example 40.** ***If*** $f(x)=\begin{vmatrix} x^n & \sin x & \cos x \\ n! & \sin\left(\frac{n\pi}{2}\right) & \cos\left(\frac{n\pi}{2}\right) \\ \pi & \pi^2 & \pi^3 \end{vmatrix}$, ***then find the value of*** $\frac{d^n}{dx^n}\{f(x)\}$ ***at*** $x=0$, $n\in I$.

***Sol.*** $$\frac{d^n}{dx^n}\{f(x)\}=\begin{vmatrix} \frac{d^n}{dx^n}(x^n) & \frac{d^n}{dx^n}(\sin x) & \frac{d^n}{dx^n}(\cos x) \\ n! & \sin\left(\frac{n\pi}{2}\right) & \cos\left(\frac{n\pi}{2}\right) \\ \pi & \pi^2 & \pi^3 \end{vmatrix}$$

$$=\begin{vmatrix} n! & \sin\left(\frac{n\pi}{2}+x\right) & \cos\left(\frac{n\pi}{2}+x\right) \\ n! & \sin\left(\frac{n\pi}{2}\right) & \cos\left(\frac{n\pi}{2}\right) \\ \pi & \pi^2 & \pi^3 \end{vmatrix}$$

$$\therefore\ \frac{d^n}{dx^n}\{f(x)\}\text{ at}(x=0)=\begin{vmatrix} n! & \sin\left(\frac{n\pi}{2}\right) & \cos\left(\frac{n\pi}{2}\right) \\ n! & \sin\left(\frac{n\pi}{2}\right) & \cos\left(\frac{n\pi}{2}\right) \\ \pi & \pi^2 & \pi^3 \end{vmatrix}$$

$=0 \quad$ [$\because$ $R_1$ and $R_2$ are identical]

# Integration of a Determinant

Let $$\Delta(x)=\begin{vmatrix} f(x) & g(x) & h(x) \\ p & q & r \\ l & m & n \end{vmatrix}$$

where $p, q, r, l, m$ and $n$ are constants, then

$$\int_a^b \Delta(x)\,dx=\begin{vmatrix} \int_a^b f(x)dx & \int_a^b g(x)dx & \int_a^b h(x)\,dx \\ p & q & r \\ l & m & n \end{vmatrix}$$

***Remark***

If in a determinant, the elements of more than one columns or rows are functions of $x$ , then the integration can be done only after evaluation or expansion of the determinant.

## Important Integrals (Committed to Memory)

1. (i) $\int_0^{\pi/2} \frac{\sin^n x}{\sin^n x + \cos^n x} dx = \frac{\pi}{4}$

$= \int_0^{\pi/2} \frac{\cos^n x}{\sin^n x + \cos^n x} dx, \forall n \in R$

(ii) $\int_0^{\pi/2} \frac{\tan^n x}{1 + \tan^n x} dx = \frac{\pi}{4} = \int_0^{\pi/2} \frac{dx}{1 + \tan^n x}, \forall n \in R$

(iii) $\int_0^{\pi/2} \frac{dx}{1 + \cot^n x} = \frac{\pi}{4} = \int_0^{\pi/2} \frac{\cot^n x}{1 + \cot^n x} dx, \forall n \in R$

2. (i) $\int_0^{\pi/2} \ln \sin x \, dx = \int_0^{\pi/2} \ln \cos x \, dx = -\frac{\pi}{2} \ln 2$

or $\frac{\pi}{2} \ln\left(\frac{1}{2}\right)$

(ii) $\int_0^{\pi/2} \ln \tan x \, dx = \int_0^{\pi/2} \ln \cot x \, dx = 0$

(iii) $\int_0^{\pi/2} \ln \sec x \, dx = \int_0^{\pi/2} \ln \operatorname{cosec} x \, dx = \frac{\pi}{2} \ln 2$

# Walli's Formula

(An easy way to evaluate $\int_0^{\pi/2} \sin^m x \cos^n x \, dx$, where $m, n \in W$) We have, $\int_0^{\pi/2} \sin^m x \cos^n x \, dx$

$$= \frac{\{(m-1)(m-3)\ldots 2 \text{ or } 1\} \{(n-1)(n-3)\ldots 2 \text{ or } 1\}}{\{(m+n)(m+n-2)(m+n-4)\ldots 2 \text{ or } 1\}}$$

where, $p$ is $\pi / 2$, if $m$ and $n$ are both even, otherwise $p = 1$. The last factor in each of three products is either 1 or 2. In case any of $m$ or $n$ is 1, we simply write 1 as the only factor to replace its product. If any of $m$ or $n$ is zero provided, we put 1 as the only factor in its product and we regard 0 as even.

*For example,*

1. $\int_0^{\pi/2} \sin^6 x \cos^4 x \, dx = \frac{[5 \cdot 3 \cdot 1][3 \cdot 1]}{[10 \cdot 8 \cdot 6 \cdot 4 \cdot 2]} \cdot \frac{\pi}{2} = \frac{3\pi}{512}$
2. $\int_0^{\pi/2} \sin^6 x \cos^3 x \, dx = \frac{[5 \cdot 3 \cdot 1][2]}{[9 \cdot 7 \cdot 5 \cdot 3 \cdot 1]} \cdot 1 = \frac{2}{63}$
3. $\int_0^{\pi/2} \sin^5 x \cos^7 x \, dx = \frac{[4 \cdot 2][6 \cdot 4 \cdot 2]}{[12 \cdot 10 \cdot 8 \cdot 6 \cdot 4 \cdot 2]} \cdot 1 = \frac{1}{120}$
4. $\int_0^{\pi/2} \sin^8 x \, dx = \frac{[7 \cdot 5 \cdot 3 \cdot 1]}{[8 \cdot 6 \cdot 4 \cdot 2]} \cdot \frac{\pi}{2} = \frac{35\pi}{256}$
5. $\int_0^{\pi/2} \cos^7 x \, dx = \frac{[6 \cdot 4 \cdot 2]}{[7 \cdot 5 \cdot 3 \cdot 1]} \cdot 1 = \frac{16}{35}$
6. $\int_0^{\pi/2} \sin^{10} x \cos x \, dx = \frac{[9 \cdot 7 \cdot 5 \cdot 3 \cdot 1][1]}{[11 \cdot 9 \cdot 7 \cdot 5 \cdot 3 \cdot 1]} \cdot 1 = \frac{1}{11}$

**Example 41.** ***If*** $\Delta(x) = \begin{vmatrix} a & b & c \\ 6 & 4 & 3 \\ x & x^2 & x^3 \end{vmatrix}$***, then find*** $\int_0^1 \Delta(x) \, dx$.

***Sol.*** $\int_0^1 \Delta(x) \, dx = \begin{vmatrix} a & b & c \\ 6 & 4 & 3 \\ \int_0^1 x \, dx & \int_0^1 x^2 dx & \int_0^1 x^3 dx \end{vmatrix}$

$$= \begin{vmatrix} a & b & c \\ 6 & 4 & 3 \\ \left[\frac{x^2}{2}\right]_0^1 & \left[\frac{x^3}{3}\right]_0^1 & \left[\frac{x^4}{4}\right]_0^1 \end{vmatrix} = \begin{vmatrix} a & b & c \\ 6 & 4 & 3 \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \end{vmatrix}$$

Applying $R_2 \to R_2 - 12R_3$, then $\int_0^1 \Delta(x) dx = \begin{vmatrix} a & b & c \\ 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \end{vmatrix} = 0$

**Example 42.** ***If***

$$f(x) = \begin{vmatrix} \sin^5 x & \ln \sin x & \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} \\ n & \sum_{k=1}^{n} k & \prod_{k=1}^{n} k \\ \frac{8}{15} & \frac{\pi}{2} \ln\left(\frac{1}{2}\right) & \frac{\pi}{4} \end{vmatrix},$$

***then find the value of*** $\int_0^{\pi/2} f(x) \, dx$.

***Sol.*** $\int_0^{\pi/2} f(x) \, dx$

$$= \begin{vmatrix} \int_0^{\pi/2} \sin^5 x \, dx & \int_0^{\pi/2} \ln \sin x \, dx & \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} \\ n & \sum_{k=1}^{n} k & \prod_{k=1}^{n} k \\ \frac{8}{15} & \frac{\pi}{2} \ln\left(\frac{1}{2}\right) & \frac{\pi}{4} \end{vmatrix},$$

$$= \begin{vmatrix} \frac{4}{5} \cdot \frac{2}{3} & -\frac{\pi}{2} \ln 2 & \frac{\pi}{4} \\ n & \sum_{k=1}^{n} k & \prod_{k=1}^{n} k \\ \frac{8}{15} & \frac{\pi}{2} \ln\left(\frac{1}{2}\right) & \frac{\pi}{4} \end{vmatrix} \quad \text{[by Walli's formula]}$$

$$= \begin{vmatrix} \frac{8}{15} & \frac{\pi}{2} \ln\left(\frac{1}{2}\right) & \frac{\pi}{4} \\ n & \sum_{k=1}^{n} k & \prod_{k=1}^{n} k \\ \frac{8}{15} & \frac{\pi}{2} \ln\left(\frac{1}{2}\right) & \frac{\pi}{4} \end{vmatrix} = 0 \quad \text{[since } R_1 \text{ and } R_3 \text{ are identical]}$$

**Example 43.** ***Let*** $f(x)$
$$= \begin{vmatrix} \sec x & \cos x & \sec^2 x + \cot x \operatorname{cosec} x \\ \cos^2 x & \cos^2 x & \operatorname{cosec}^2 x \\ 1 & \cos^2 x & \cos^2 x \end{vmatrix}$$
***, then find the value of*** $\int_0^{\pi/2} f(x)\,dx$.

***Sol.*** Applying $C_2 \to C_2 - \cos^2 x\, C_1$, then

$$f(x) = \begin{vmatrix} \sec x & 0 & \sec^2 x + \cot x \operatorname{cosec} x \\ \cos^2 x & \cos^2 x - \cos^4 x & \operatorname{cosec}^2 x \\ 1 & 0 & \cos^2 x \end{vmatrix}$$

[expanding along $C_2$ ]

$$= (\cos^2 x - \cos^4 x) \begin{vmatrix} \sec x & \sec^2 x + \cot x \operatorname{cosec} x \\ 1 & \cos^2 x \end{vmatrix}$$
$$= (\cos^2 x - \cos^4 x)(\cos x - \sec^2 x - \cot x \operatorname{cosec} x)$$
$$= \cos^2 x (1 - \cos^2 x)\left(\cos x - \frac{1}{\cos^2 x} - \frac{\cos x}{\sin^2 x}\right)$$
$$= \cos^2 x \sin^2 x \left(\cos x - \frac{1}{\cos^2 x} - \frac{\cos x}{\sin^2 x}\right)$$
$$= \cos^3 x \sin^2 x - \sin^2 x - \cos^3 x$$
$$= -\cos^3 x (1 - \sin^2 x) - \sin^2 x$$
$$f(x) = -\cos^5 x - \sin^2 x$$
$$\therefore \int_0^{\pi/2} f(x)\,dx = -\int_0^{\pi/2} \cos^5 x\,dx - \int_0^{\pi/2} \sin^2 x\,dx$$
$$= -\left(\frac{4}{5}\cdot\frac{2}{3}\cdot 1\right) - \left(\frac{1}{2}\cdot\frac{\pi}{2}\right) = -\left(\frac{8}{15} + \frac{\pi}{4}\right)$$

[ by Walli's formula]

# Use of $\Sigma$ in Determinant

If $$\Delta(r) = \begin{vmatrix} f(r) & g(r) & h(r) \\ a & b & c \\ a_1 & b_1 & c_1 \end{vmatrix}$$

where $a, b, c, a_1, b_1$ and $c_1$ are constants, independent of $r$, then

$$\sum_{r=1}^{n} \Delta(r) = \begin{vmatrix} \sum_{r=1}^{n} f(r) & \sum_{r=1}^{n} g(r) & \sum_{r=1}^{n} h(r) \\ a & b & c \\ a_1 & b_1 & c_1 \end{vmatrix}$$

***Remark***

If in a determinant, the elements of more than one columns or rows are function of $r$, then the $\Sigma$ can be done only after evaluation or expansion of the determinant.

# Important Summation

(Committed to Memory)

**1.** $\sum_{r=1}^{n} r = \Sigma n = 1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$

**2.** $\sum_{r=1}^{n} r^2 = \Sigma n^2 = 1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$

**3.** $\sum_{r=1}^{n} r^3 = \Sigma n^3 = 1^3 + 2^3 + 3^3 + \ldots + n^3$

$= \left[\frac{n(n+1)}{2}\right]^2 = (\Sigma n)^2$

**4.** $\sum_{r=1}^{n} a = \Sigma a = \underbrace{a + a + a + \ldots + a}_{n \text{ times}} = an$

**5.** $\sum_{r=1}^{n} (\lambda - 1)\lambda^{r-1} = \lambda^n - 1, \forall\, \lambda \neq 1$ and $\lambda > 1$

**6.** $$\sum_{r=1}^{n} \sin[\alpha + (r-1)\beta] = \frac{\sin\left\{\alpha + \frac{(n-1)\beta}{2}\right\} \sin\left(\frac{n\beta}{2}\right)}{\sin\left(\frac{\beta}{2}\right)}$$

**Particular** For $\alpha = \beta = \theta$.

$$\sum_{r=1}^{n} \sin r\theta = \frac{\sin\left\{\left(\frac{n+1}{2}\right)\theta\right\} \cdot \sin\left(\frac{n\theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)}$$

**7.** $$\sum_{r=1}^{n} \cos\{\alpha + (r-1)\beta\} = \frac{\cos\left\{\alpha + \frac{(n-1)}{2}\beta\right\} \sin\left(\frac{n\beta}{2}\right)}{\sin\left(\frac{\beta}{2}\right)}$$

**Particular** For $\alpha = \beta = \theta$.

$$\sum_{r=1}^{n} \cos r\theta = \frac{\cos\left\{\left(\frac{n+1}{2}\right)\theta\right\} \sin\left(\frac{n\theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)}$$

**8.** $\sum_{r=1}^{n} \{f(r+1) - f(r)\} = f(n+1) - f(1)$

**Particular** $\sum_{r=1}^{n} \frac{1}{r(r+1)} = \sum_{r=1}^{n} \left(\frac{1}{r} - \frac{1}{r+1}\right)$

$= \frac{1}{1} - \frac{1}{n+1} = \frac{n}{n+1}$

**9.** $\sum_{r=1}^{n} {}^nC_r = 2^n$

### *Remark*

Capital pie $\Pi$ is not direct applicable in determint i.e.,

$$\prod_{r=1}^{n} \Delta(r) \neq \begin{vmatrix} \prod_{r=1}^{n} f(r) & \prod_{r=1}^{n} g(r) & \prod_{r=1}^{n} h(r) \\ a & b & c \\ a_1 & b_1 & c_1 \end{vmatrix}$$

**Explanation** $\prod_{r=1}^{n} \Delta(r) = \Delta(1) \times \Delta(2) \times \ldots \times \Delta(n)$

$$= \begin{vmatrix} f(1) & g(1) & h(1) \\ a & b & c \\ a_1 & b_1 & c_1 \end{vmatrix} \times \begin{vmatrix} f(2) & g(2) & h(2) \\ a & b & c \\ a_1 & b_1 & c_1 \end{vmatrix} \times \ldots \times \begin{vmatrix} f(n) & g(n) & h(n) \\ a & b & c \\ a_1 & b_1 & c_1 \end{vmatrix}$$

$$\neq \begin{vmatrix} \prod_{r=1}^{n} f(r) & \prod_{r=1}^{n} g(r) & \prod_{r=1}^{n} h(r) \\ a & b & c \\ a_1 & b_1 & c_1 \end{vmatrix}$$

**Example 44.** ***Let** $n$ **be a positive integer and***

$$\Delta_r = \begin{vmatrix} 2r-1 & {}^nC_r & 1 \\ n^2-1 & 2^n & n+1 \\ \cos^2 n^2 & \cos^2 n & \cos^2(n+1) \end{vmatrix}, \textit{ prove that}$$

$$\sum_{r=0}^{n} \Delta_r = 0.$$

***Sol.*** We have, $\Delta_r = \begin{vmatrix} 2r-1 & {}^nC_r & 1 \\ n^2-1 & 2^n & n+1 \\ \cos^2(n^2) & \cos^2 n & \cos^2(n+1) \end{vmatrix}$

$$\therefore \quad \sum_{r=0}^{n} \Delta_r = \begin{vmatrix} \sum_{r=0}^{n} (2r-1) & \sum_{r=0}^{n} {}^nC_r & \sum_{r=0}^{n} 1 \\ n^2-1 & 2^n & n+1 \\ \cos^2(n^2) & \cos^2 n & \cos^2(n+1) \end{vmatrix}$$

Now, $\sum_{r=0}^{n} (2r-1) = 2\sum_{r=0}^{n} r - \sum_{r=0}^{n} 1$

$$= 2(0+1+2+3+\ldots+n) - \underbrace{(1+1+1+\ldots+1)}_{(n+1)\text{ times}}$$

$$= \frac{2n(n+1)}{2} - (n+1) = (n+1)(n-1) = n^2-1$$

$$\sum_{r=0}^{n} \Delta_r = \begin{vmatrix} n^2-1 & 2^n & n+1 \\ n^2-1 & 2^n & n+1 \\ \cos^2(n^2) & \cos^2 n & \cos^2(n+1) \end{vmatrix} = 0$$

[since $R_1$ and $R_2$ are identical]

**Example 45.** Let $n$ be a positive integer and

$$\Delta_r = \begin{vmatrix} r^2+r & r+1 & r-2 \\ 2r^2+3r-1 & 3r & 3r-3 \\ r^2+2r+3 & 2r-1 & 2r-1 \end{vmatrix} \text{ and}$$

$\sum_{r=1}^{n} \Delta_r = an^2+bn+c$, find the value of $a+b+c$.

***Sol.*** We have, $\Delta_r = \begin{vmatrix} r^2+1r & r+1 & r-2 \\ 2r^2+3r-1 & 3r & 3r-3 \\ r^2+2r+3 & 2r-1 & 2r-1 \end{vmatrix}$

Applying $R_2 \to R_2 - (R_1 + R_3)$, then

$$\Delta_r = \begin{vmatrix} r^2+r & r+1 & r-2 \\ \vdots & & \\ -4 \quad \ldots & 0 \quad \ldots & 0 \\ \vdots & & \\ r^2+2r+3 & 2r-1 & 2r-1 \end{vmatrix}$$

Expanding along $R_2$, we get

$$= 4\begin{vmatrix} r+1 & r-2 \\ 2r-1 & 2r-1 \end{vmatrix}$$

$$= 4[(r+1)(2r-1) - (r-2)(2r-1)]$$

$$= 24r - 12$$

Now, $\sum_{r=1}^{n} \Delta_r = 24\sum_{r=1}^{n} r - 12\sum_{r=1}^{n} 1$

$$= 24\frac{n(n+1)}{2} - 12n = 12n(n+1-1)$$

$$= 12n^2 = an^2 + bn + c \qquad \text{[given]}$$

For $n = 1$, we have

$$a + b + c = 12$$

## Exercise for Session 4

**1.** If $f(x)=\begin{vmatrix} x & x^2 & x^3 \\ 1 & 2 & 3 \\ 0 & 1 & x \end{vmatrix}$, $f'(1)$ is equal to

(a) $-1$ (b) 0 (c) 1 (d) 2

**2.** Let $f(x)=\begin{vmatrix} \sec x & x^2 & x \\ 2\sin x & x^3 & 2x^2 \\ \tan 3x & x^2 & x \end{vmatrix}$, $\lim_{x\to 0}\frac{f(x)}{x^4}$ is equal to

(a) 0 (b) $-1$ (c) 2 (d) 3

**3.** Let $\begin{vmatrix} x & 2 & x \\ x^2 & x & 6 \\ x & x & 6 \end{vmatrix}=Ax^4+Bx^3+Cx^2+Dx+E$, the value of $5A+4B+3C+2D+E$ is equal to

(a) $-16$ (b) $-11$ (c) 0 (d) 16

**4.** Let $f(x)=\begin{vmatrix} x^3 & \sin x & \cos x \\ 6 & -1 & 0 \\ p & p^2 & p^3 \end{vmatrix}$, where $p$ is a constant. Then $\frac{d^3}{dx^3}\{f(x)\}$ at $x=0$ is

(a) $p$ (b) $p+p^2$ (c) $p+p^3$ (d) independent of $p$

**5.** If $y=\sin mx$, the value of the determinant $\begin{vmatrix} y & y_1 & y_2 \\ y_3 & y_4 & y_5 \\ y_6 & y_7 & y_8 \end{vmatrix}$, where $y_n=\frac{d^ny}{dx^n}$, is

(a) $m^2$ (b) $m^3$ (c) $m^9$ (d) None of these

**6.** Let $f(x)=\begin{vmatrix} 2\cos^2 x & \sin 2x & -\sin x \\ \sin 2x & 2\sin^2 x & \cos x \\ \sin x & -\cos x & 0 \end{vmatrix}$, the value of $\int_0^{\pi/2}\{f(x)+f'(x)\}\,dx$, is

(a) $\frac{\pi}{2}$ (b) $\pi$ (c) $\frac{3\pi}{2}$ (d) $2\pi$

**7.** If $f(x)=\begin{vmatrix} \cos x & e^{x^2} & 2x\cos^2\left(\frac{x}{2}\right) \\ x^2 & \sec x & \sin x+x^3 \\ 1 & 2 & x+\tan x \end{vmatrix}$, the value of $\int_{-\pi/2}^{\pi/2}(x^2+1)[f(x)+f''(x)]\,dx$, is

(a) $-1$ (b) 0 (c) 1 (d) 2

**8.** If $f(x)=\begin{vmatrix} \sin^2 x+ & \cos^4 x\ln\cos x & \frac{1}{1+(\tan x)^{\sqrt{2}}} \\ \pi & \pi^2 & \pi^4 \\ \frac{7}{16} & -\frac{1}{2}\ln 2 & \frac{1}{4} \end{vmatrix}$, the value of $\int_0^{\pi/2} f(x)\,dx$ is

(a) 2 (b) $-1$ (c) 0 (d) None of these

**9.** If $\Delta_k=\begin{vmatrix} 1 & n & n \\ 2k & n^2+n+1 & n^2+n \\ 2k-1 & n^2 & n^2+n+1 \end{vmatrix}$ and $\sum_{k=1}^{n}\Delta_k=56$, then $n$ is equal to

(a) 4 (b) 6 (c) 8 (d) None of these

**10.** The value of $\sum_{r=2}^{n}(-2)^r\begin{vmatrix} {}^{n-2}C_{r-2} & {}^{n-2}C_{r-1} & {}^{n-2}C_r \\ -3 & 1 & 1 \\ 2 & -1 & 0 \end{vmatrix}$ $(n>2)$ is

(a) $2n-1+(-1)^n$ (b) $2n+1+(-1)^n$ (c) $2n-3+(-1)^n$ (d) None of these

# Answers

## Exercise for Session 4

**1.** (c) **2.** (b) **3.** (b) **4.** (d) **5.** (d) **6.** (b)
**7.** (b) **8.** (c) **9.** (d) **10.** (a)