JEE Type Solved Examples :
Single Option Correct Type Questions

- This section contains **8 multiple choice examples.** Each example has four choices (a), (b), (c) and (d) out of which **ONLY ONE** is correct.

Ex. 1 *The expression* $\log_2 5 - \sum_{k=1}^{4} \log_2 \left(\sin\left(\frac{k\pi}{5}\right)\right)$ *reduces to* $\frac{p}{q}$, *where p and q are co-prime, the value of* $p^2 + q^2$ *is*

(a) 13 (b) 17 (c) 26 (d) 29

Sol. (*b*) Let $p = \log_2 5 - \sum_{k=1}^{4} \log_2 \left(\sin\left(\frac{k\pi}{5}\right)\right)$

$$= \log_2 5 - \left\{ \log_2\left(\sin\left(\frac{\pi}{5}\right)\right) + \log_2\left(\sin\left(\frac{2\pi}{5}\right)\right) + \log_2\left(\sin\left(\frac{3\pi}{5}\right)\right) + \log_2\left(\sin\left(\frac{4\pi}{5}\right)\right)\right\}$$

$$= \log_2 5 - \log_2\left\{\sin\frac{\pi}{5}\cdot\sin\frac{2\pi}{5}\cdot\sin\frac{3\pi}{5}\cdot\sin\frac{4\pi}{5}\right\}$$

$$= \log_2 5 - \log_2\left\{\sin^2\left(\frac{\pi}{5}\right)\cdot\sin^2\left(\frac{2\pi}{5}\right)\right\}$$

$$= \log_2 5 - \log_2\left\{\frac{(1-\cos 72^\circ)(1-\cos 144^\circ)}{4}\right\}$$

$$= \log_2 5 - \log_2\left\{\frac{(1-\sin 18^\circ)(1+\cos 36^\circ)}{4}\right\}$$

$$= \log_2 5 - \log_2\left\{\frac{\left(1-\frac{\sqrt{5}-1}{4}\right)\left(1+\frac{\sqrt{5}+1}{4}\right)}{4}\right\}$$

$$= \log_2 5 - \log_2\left\{\frac{(5-\sqrt{5})(5+\sqrt{5})}{64}\right\} = \log_2 5 - \log_2\left(\frac{5}{16}\right)$$

$$= \log_2\left(5\times\frac{16}{5}\right) = \log_2 2^4 = \frac{4}{1} = \frac{p}{q} \quad \text{[given]}$$

$\therefore \quad p = 4, q = 1$

Hence, $p^2 + q^2 = 4^2 + 1^2 = 17$

Ex. 2 *If* $3 \le a \le 2015, 3 \le b \le 2015$ *such that* $\log_a b + 6\log_b a = 5$, *the number of ordered pairs (a, b) of integers is*

(a) 48 (b) 50 (c) 52 (d) 54

Sol. (*c*) Let $x = \log_a b$ ...(i)

$\Rightarrow \quad x + \frac{6}{x} = 5 \Rightarrow x^2 - 5x + 6 = 0 \Rightarrow x = 2, 3$

From Eq. (i), we get $\log_a b = 2, 3$

$\Rightarrow \quad b = a^2$ or a^3

The pairs (*a*, *b*) are

$(3, 3^2), (4, 4^2), (5, 5^2), (6, 6^2), \ldots, (44, 44^2)$ and $(3, 3^3), (4, 4^3), (5, 5^3), \ldots, (12, 12^3)$.

Hence, there are $42 + 10 = 52$ pairs.

Ex. 3 *The lengths of the sides of a triangle are* $\log_{10} 12$, $\log_{10} 75$ *and* $\log_{10} n$, *where* $n \in N$. *If a and b are the least and greatest values of n respectively, the value of* $b - a$ *is divisible by*

(a) 221 (b) 222 (c) 223 (d) 224

Sol. (*c*) In a triangle,

$$\log_{10} 12 + \log_{10} 75 > \log_{10} n \Rightarrow n < 12 \times 75 = 900$$

$\therefore \quad n < 900$ ...(i)

and $\quad \log_{10} 12 + \log_{10} n > \log_{10} 75$

$\Rightarrow \quad n > \frac{75}{12} = \frac{25}{4}$

$\therefore \quad n > \frac{25}{4}$ ...(ii)

From Eqs. (i) and (ii), we get $\frac{25}{4} < n < 900$

$\therefore \quad n = 7, 8, 9, 10, \ldots, 899$

Hence, $\quad a = 7, b = 899$

$\therefore \quad b - a = 892 = 4 \times 223$

Hence, $b - a$ is divisible by 223.

Ex. 4 *If* $5\log_{abc}(a^3 + b^3 + c^3) = 3\lambda\left(\frac{1+\log_3(abc)}{\log_3(abc)}\right)$ *and* $(abc)^{a+b+c} = 1$ *and* $\lambda = \frac{m}{n}$, *where m and n are relative primes, the value of* $|m+n| + |m-n|$ *is*

(a) 8 (b) 10 (c) 12 (d) 14

Sol. (*b*) $\because \quad (abc)^{a+b+c} = 1 = (abc)^0$

$\therefore \quad a + b + c = 0 \Rightarrow a^3 + b^3 + c^3 = 3abc$

Now, LHS $= 5\log_{abc}(a^3 + b^3 + c^3) = 5\log_{abc}(3abc)$ ...(i)

and RHS $= 3\lambda\left(\frac{1+\log_3(abc)}{\log_3(abc)}\right) = 3\lambda\left(\frac{\log_3(3abc)}{\log_3(abc)}\right)$

$= 3\lambda \log_{abc}(3abc)$ ...(ii)

From Eqs. (i) and (ii), we get

$5\log_{abc}(3abc) = 3\lambda\log_{abc}(3abc)$

$\therefore \quad \lambda = \frac{5}{3} = \frac{m}{n}$ [given]

$\Rightarrow \quad m = 5, n = 3$

Hence, $|m+n| + |m-n| = 8 + 2 = 10$

● **Ex. 5** *If* $a^{\log_b c} = 3 \cdot 3^{\log_4 3} \cdot 3^{\log_4 3^{\log_4 3}} \cdot 3^{\log_4 3^{\log_4 3^{\log_4 3}}} \ldots \infty$, *where* $a, b, c \in Q$, *the value of abc is*

(a) 9 (b) 12 (c) 16 (d) 20

Sol. (*c*) $a^{\log_b c} = 3^{1+\log_4 3 + (\log_4 3)^2 + (\log_4 3)^3 + \ldots \infty}$

$$= 3^{1/(1-\log_4 3)} = 3^{1/\log_4 (4/3)} = 3^{\log_{4/3} 4}$$

$\therefore$ $a = 3, b = \frac{4}{3}, c = 4$

Hence, $abc = 3 \cdot \frac{4}{3} \cdot 4 = 16$

● **Ex. 6** *Number of real roots of equation*

$$3^{\log_3 (x^2 - 4x + 3)} = (x - 3) \text{ is}$$

(a) 0 (b) 1 (c) 2 (d) infinite

Sol. (*a*) $\because$ $3^{\log_3 (x^2 - 4x+3)} = (x-3)$ ...(i)

Eq. (i) is defined, if $x^2 - 4x + 3 > 0$

$\Rightarrow$ $(x-1)(x-3) > 0$

$\Rightarrow$ $x < 1$ or $x > 3$ ...(ii)

Eq. (i) reduces to $x^2 - 4x + 3 = x - 3 \Rightarrow x^2 - 5x + 6 = 0$

$\therefore$ $x = 2, 3$ ...(iii)

From Eqs. (ii) and (iii), use get $x \in \phi$

$\therefore$ Number of real roots = 0

● **Ex. 7** *If* $\log_6 a + \log_6 b + \log_6 c = 6$, *where* $a, b, c \in N$ *and a, b, c are in GP and* $b - a$ *is a square of an integer, then the value of* $a + b - c$ *is*

(a) 21 (b) 15 (c) 9 (d) 3

Sol. (*b*) $\because$ $\log_6 a + \log_6 b + \log_6 c = 6$

$\Rightarrow$ $\log_6 (abc) = 6$

$\Rightarrow$ $abc = 6^6$

$\Rightarrow$ $b^3 = 6^6$ $[\because b^2 = ac]$

$\Rightarrow$ $b = 36$

Also, $b - a = 36 - a$ is a square for $a = 35, 32, 27, 20, 11$

Now, $c = \frac{b^2}{a} = \frac{36^2}{a}$ is an integer for $a = 27$

$\therefore$ $a = 27, b = 36, c = 48$

Hence, $a + b - c = 27 + 36 - 48 = 15$

● **Ex. 8** *If* $x = \log_{2a}\left(\frac{bcd}{2}\right)$, $y = \log_{3b}\left(\frac{acd}{3}\right)$, $z = \log_{4c}\left(\frac{abd}{4}\right)$ *and* $w = \log_{5d}\left(\frac{abc}{5}\right)$ *and* $\frac{1}{x+1} + \frac{1}{y+1} + \frac{1}{z+1} + \frac{1}{w+1} = \log_{abcd} N + 1$, *the value of N is*

(a) 40 (b) 80

(c) 120 (d) 160

Sol. (*c*) $\because$ $x = \log_{2a}\left(\frac{bcd}{2}\right)$

$\Rightarrow$ $x + 1 = \log_{2a}\left(\frac{2abcd}{2}\right) = \log_{2a}(abcd)$

$\therefore$ $\frac{1}{x+1} = \log_{abcd} 2a$

Similarly, $\frac{1}{y+1} = \log_{abcd} 3b$, $\frac{1}{z+1} = \log_{abcd} 4c$

and $\frac{1}{w+1} = \log_{abcd} 5d$

$\therefore$ $\frac{1}{x+1} + \frac{1}{y+1} + \frac{1}{z+1} + \frac{1}{w+1} = \log_{abcd}(2a \cdot 3b \cdot 4c \cdot 5d)$

$= \log_{abcd}(120abcd)$

$= \log_{abcd} 120 + 1$

$= \log_{abcd} N + 1$ [given]

Hence, $N = 120$

JEE Type Solved Examples :
More than One Correct Option Type Questions

- This section contains **4 multiple choice examples.** Each example has four choices (a), (b), (c) and (d) out of **which more than one** may be correct.

● **Ex. 9** *The equation*
$(\log_{10} x + 2)^3 + (\log_{10} x - 1)^3 = (2 \log_{10} x + 1)^3$ *has*

(a) no natural solution (b) two rational solutions

(c) no prime solution (d) one irrational solution

Sol. (*b, c, d*) Let $\log_{10} x + 2 = a$ and $\log_{10} x - 1 = b$

$\therefore$ $a + b = 2\log_{10} x + 1$, then given equation reduces to

$$a^3 + b^3 = (a+b)^3$$

$\Rightarrow$ $3ab(a+b) = 0 \Rightarrow a = 0$ or $b = 0$ or $a + b = 0$

$\Rightarrow$ $\log_{10} x + 2 = 0$ or $\log_{10} x - 1 = 0$

or $2\log_{10} x + 1 = 0$

$\Rightarrow$ $x = 10^{-2}$ or $x = 10$ or $x = 10^{-1/2}$

Hence, $x = \frac{1}{100}$ or $x = 10$ or $x = \frac{1}{\sqrt{10}}$

● **Ex. 10** *The value of* $\frac{\log_5 9 \cdot \log_7 5 \cdot \log_3 7}{\log_3 \sqrt{6}} + \frac{1}{\log_4 \sqrt{6}}$ *is co-prime with*

(a) 1 (b) 3 (c) 4 (d) 5

Sol. (*a, b, d*) Let $P = \dfrac{\log_5 9 \cdot \log_7 5 \cdot \log_3 7}{\log_3 \sqrt{6}} + \dfrac{1}{\log_4 \sqrt{6}}$

$$= \frac{\log_3 9}{\log_3 \sqrt{6}} + \log_{\sqrt{6}} 4 = \log_{\sqrt{6}} 9 + \log_{\sqrt{6}} 4$$

$$= \log_{\sqrt{6}}(36) = \log_{\sqrt{6}}(\sqrt{6})^4 = 4 \Rightarrow P = 4$$

which is co-prime with 1, 3, 4 and 5.

● ***Ex. 11*** *Which of the following quantities are irrational for the quadratic equation*
$(\log_{10} 8)x^2 - (\log_{10} 5)x = 2(\log_2 10)^{-1} - x$?

(a) Sum of roots (b) Product of roots
(c) Sum of coefficients (d) Discriminant

Sol. (*c, d*) $\because$ $(\log_{10} 8)x^2 - (\log_{10} 5)x = 2(\log_2 10)^{-1} - x$

$\Rightarrow$ $(3\log_{10} 2)x^2 + (1 - \log_{10} 5)x - 2\log_{10} 2 = 0$

$\Rightarrow$ $(3\log_{10} 2)x^2 + (\log_{10} 2)x - 2\log_{10} 2 = 0$

Now, Sum of roots $= -\dfrac{1}{3}$ = Rational

Product of roots $= -\dfrac{2}{3}$ = Rational

Sum of coefficients $= 3\log_{10} 2 + \log_{10} 2 - 2\log_{10} 2$
$= 2\log_{10} 2$ = Irrational

Discriminant $= (\log_{10} 2)^2 + 24(\log_{10} 2)^2$
$= 25\,(\log_{10} 2)^2$ = Irrational

● ***Ex. 12*** *The system of equations*
$\log_{10}(2000xy) - \log_{10} x \cdot \log_{10} y = 4$
$\log_{10}(2yz) - \log_{10} y \cdot \log_{10} z = 1$
and $\log_{10}(zx) - \log_{10} z \cdot \log_{10} x = 0$
has two solutions (x_1, y_1, z_1) *and* (x_2, y_2, z_2), *then*

(a) $x_1 + x_2 = 101$ (b) $y_1 + y_2 = 25$
(c) $x_1 x_2 = 100$ (d) $z_1 z_2 = 100$

Sol. (*a, b, c, d*) Let $\log_{10} x = a$, $\log_{10} y = b$ and $\log_{10} z = c$

Then, given equations reduces to

$a + b - ab = 4 - \log_{10} 2000 = \log_{10} 5$ ...(i)

$b + c - bc = 1 - \log_{10} 2 = \log_{10} 5$ ...(ii)

and $c + a - ca = 0$ ...(iii)

From Eqs. (i) and (ii), we get

$a + b - ab = b + c - bc$

$\Rightarrow$ $(c - a) - b(c - a) = 0$

$\Rightarrow$ $(c - a)(1 - b) = 0$

$1 - b \neq 0, c - a = 0 \Rightarrow c = a$

From Eq. (iii), we get

$2a - a^2 = 0 \Rightarrow a = 0, 2$

Then, $c = a \Rightarrow c = 0, 2$

and $b = \log_{10} 5, 2 - \log_{10} 5$

$\therefore$ $\log_{10} x = 0, 2 \Rightarrow x = 10^0, 10^2$

$\Rightarrow$ $x = 1, 100$

$\Rightarrow$ $x_1 = 1, x_2 = 100$

and $\log_{10} y = \log_{10} 5, 2 - \log_{10} 5$
$= \log_{10} 5, \log_{10} 20$

$\Rightarrow$ $y = 5, 20$

$\Rightarrow$ $y_1 = 5, y_2 = 20$

and $\log_{10} z = 0, 2 \Rightarrow z = 10^0, 10^2$

$\Rightarrow$ $z = 1, 100$

$\Rightarrow$ $z_1 = 1, z_2 = 100$

Finally, $x_1 + x_2 = 1 + 100 = 101$, $y_1 + y_2 = 5 + 20 = 25$,
$x_1 x_2 = 1 \times 100 = 100$ and $z_1 z_2 = 1 \times 100 = 100$

JEE Type Solved Examples : Passage Based Questions

- This section contains **2 solved passages** based upon each of the passage **3 multiple choice** examples have to be answered. Each of these examples has four choices (a), (b), (c) and (d) out of which **ONLY ONE** is correct.

Passage I
(Ex. Nos. 13 to 15)

Suppose that $\log_{10}(x - 2) + \log_{10} y = 0$ and
$\sqrt{x} + \sqrt{(y - 2)} = \sqrt{(x + y)}$.

13. The value of x is

(a) $2 + \sqrt{2}$ (b) $1 + \sqrt{2}$ (c) $2\sqrt{2}$ (d) $4 - \sqrt{2}$

14. The value of y is

(a) 2 (b) $2\sqrt{2}$ (c) $1 + \sqrt{2}$ (d) $2 + 2\sqrt{2}$

15. If $x^{2t^2 - 6} + y^{6 - 2t^2} = 6$, the value of $t_1\, t_2\, t_3\, t_4$ is

(a) 1 (b) 2 (c) 4 (d) 8

Sol. (Ex. Nos. 13-15)

$\because$ $\log_{10}(x - 2) + \log_{10} y = 0$

$\therefore$ $x - 2 > 0, y > 0$

$\Rightarrow$ $x > 2, y > 0$ ...(i)

and $\log_{10}\{(x - 2)y\} = 0$

$\Rightarrow$ $(x - 2)y = 10^0 = 1$

$\therefore$ $(x - 2)y = 1$ ...(ii)

Also, given that $\sqrt{x} + \sqrt{(y - 2)} = \sqrt{(x + y)}$

$\therefore$ $x \geq 0, y - 2 \geq 0, x + y \geq 0$

$\Rightarrow \quad x \geq 0, y \geq 2$ ...(iii)

On squaring both sides, we get

$$x + y - 2 + 2\sqrt{x}\sqrt{(y-2)} = x + y$$

$\Rightarrow \quad \sqrt{x}\sqrt{y-2} = 1$

$\Rightarrow \quad x(y-2) = 1$ ...(iv)

From Eqs. (i) and (iii),we get

$$x > 2, y \geq 2$$

and from Eqs. (ii) and (iv), we get $y = x$

From Eq. (ii), $(x-2)x = 1$

$\Rightarrow \quad x^2 - 2x - 1 = 0$

$\therefore \quad x = \dfrac{2 \pm \sqrt{4+4}}{2}$ [neglect –ve sign, since $x > 2$]

13. (*b*) $x = (\sqrt{2} + 1)$.

14. (*c*) $y = x = \sqrt{1+1}$

15. (*d*) $\because \quad x^{2t^2-6} + y^{6-2t^2} = 6$

$\Rightarrow \quad x^{2t^2-6} + (x^{-1})^{2t^2-6} = 6$

$\Rightarrow \quad (x^2)^{t^2-3} + (x^{-2})^{t^2-3} = 6$

$\Rightarrow \quad (3+2\sqrt{2})^{t^2-3} + (3-2\sqrt{2})^{t^2-3} = 6$

Now, we get $\quad t^2 - 3 = \pm 1$

$\Rightarrow \quad t^2 = 4, 2$

$\therefore \quad t = \pm 2, \pm\sqrt{2}$

$\therefore \quad t_1 t_2 t_3 t_4 = (2)(-2)(\sqrt{2})(-\sqrt{2}) = 8$

Passage II

(Ex. Nos. 16 to 18)

If $10^{\log_p \{\log_q (\log_r x)\}} = 1$ and $\log_q \{\log_r (\log_p x)\} = 0$.

16. The value of x is

(a) q^r (b) r^q (c) r^p (d) rq

Sol. (*b*) $\because \quad 10^{\log_p [\log_q (\log_r x)]} = 1 = 10^0$

$\Rightarrow \quad \log_p \{\log_q (\log_r x)\} = 0$

$\Rightarrow \quad \log_q (\log_r x) = 1 \Rightarrow \log_r x = q$

$\Rightarrow \quad x = r^q$ ...(i)

and $\quad \log_q \{\log_r (\log_p x)\} = 0$

$\Rightarrow \quad \log_r (\log_p x) = 1 \Rightarrow \log_p x = r$

$\therefore \quad x = p^r$ ...(ii)

From Eqs. (i) and (ii), we get $\quad x = r^q = p^r$

17. The value of p is

(a) $r^{q/r}$ (b) rq (c) 1 (d) $r^{r/q}$

Sol. (*a*) $\because \quad r^q = p^r$ ...(iii)

$\Rightarrow \quad p = r^{q/r}$

18. The value of q is

(a) $r^{p/r}$ (b) $p\log_p r$ (c) $r\log_r p$ (d) $r^{r/p}$

Sol. (*c*) From Eq. (iii),

$$q \log r = r \log p \Rightarrow q = r\left(\frac{\log p}{\log r}\right) = r \log_r p$$

JEE Type Solved Examples : Single Integer Answer Type Questions

- This section contains **2 examples.** The answer to each example is a **single digit integer** ranging from **0** to **9** (both inclusive).

● ***Ex. 19*** *If x_1 and x_2 are the solutions of the equation $x^{\log_{10} x} = 100x$ such that $x_1 > 1$ and $x_2 < 1$, the value of $\dfrac{x_1 x_2}{2}$ is*

Sol. (5) $\because \quad x^{\log_{10} x} = 100x$

Taking logarithm on both sides on base 10, then we get

$$\log_{10} x \cdot \log_{10} x = \log_{10} 100 + \log_{10} x$$

$\Rightarrow \quad (\log_{10} x)^2 - \log_{10} x - 2 = 0$

$\Rightarrow \quad (\log_{10} x - 2)(\log_{10} x + 1) = 0$

$\therefore \quad \log_{10} x = 2, -1 \Rightarrow x = 10^2, 10^{-1}$

$\therefore \quad x_1 = 100, x_2 = \dfrac{1}{10}$

$\therefore \quad \dfrac{x_1 x_2}{2} = 5$

● ***Ex. 20*** *If $(31.6)^a = (0.0000316)^b = 100$, the value of $\dfrac{1}{a} - \dfrac{1}{b}$ is*

Sol. (3) $\because \quad (31.6)^a = (0.0000316)^b = 100$

$\Rightarrow \quad a \log_{10}(31.6) = b \log_{10}(0.0000316) = \log_{10} 100$

$\Rightarrow \quad a \log_{10}(31.6) = b \log_{10}(31.6 \times 10^{-6}) = 2$

$\Rightarrow \quad a \log_{10}(31.6) = b \log_{10}(31.6) - 6b = 2$

$\Rightarrow \quad \dfrac{2}{a} = \log_{10}(31.6)$

and $\quad \dfrac{2}{b} = \log_{10}(31.6) - 6$

$\therefore \quad \dfrac{2}{a} - \dfrac{2}{b} = 6$

$\Rightarrow \quad \dfrac{1}{a} - \dfrac{1}{b} = 3$

JEE Type Solved Examples :
Matching Type Questions

- This section contains **2 examples.** Examples 24 and 25 have three statements (A, B and C) given in Column I and four statements (p, q, r and s) in **Column II.** Any given statement in **Column I** can have correct matching with one or more statement(s) given in **Column II.**

Ex. 21

	Column I		Column II
(A)	If x_1 and x_2 satisfy the equation $(x+1)^{\log_{10}(x+1)} = 100(x+1)$, then the value of $(x_1+1)(x_2+1)+5$ is	(p)	irrational
(B)	The product of all values of x which make the following statement true $(\log_3 x)(\log_5 9) - \log_x 25 + \log_3 2 = \log_3 54$, is	(q) (r)	rational prime
(C)	If $\log_b a = -3$, $\log_b c = 4$ and if the value of x satisfying the equation $a^{3x} = c^{x-1}$ is expressed in the form p/q, where p and q are relatively prime, then q is	(s)	composite
		(t)	twin prime

Sol. A $\rightarrow$ (q, s, t), B $\rightarrow$ (p), C $\rightarrow$ (q, r)

(A) $(x+1)^{\log_{10}(x+1)} = 100(x+1)$

Taking logarithm on both sides on base 10, then we get

$$\log_{10}(x+1)\cdot\log_{10}(x+1) = \log_{10}100 + \log_{10}(x+1)$$

$$\Rightarrow \quad \{\log_{10}(x+1)\}^2 = 2 + \log_{10}(x+1)$$

$$\Rightarrow \quad \{\log_{10}(x+1)\}^2 - \log_{10}(x+1) - 2 = 0$$

$$\Rightarrow \{\log_{10}(x+1) - 2\}\{\log_{10}(x+1) + 1\} = 0$$

$$\therefore \quad \log_{10}(x+1) = 2, -1$$

$$\Rightarrow \quad (x+1) = 10^2, 10^{-1}$$

$$\therefore \quad (x_1+1)(x_2+1) = 10^2 \times 10^{-1} = 10$$

$$\Rightarrow \quad (x_1+1)(x_2+1) + 5 = 10 + 5 = 15 = 3 \times 5$$

(B) $\because (\log_3 x)(\log_5 9) - \log_x 25 + \log_3 2 = \log_3 54$

$$\Rightarrow \quad 2\log_5 x - 2\log_x 5 = \log_3 54 - \log_3 2 = \log_3(27) = 3$$

Let $\log_5 x = \lambda$, then

$$2\lambda - \frac{2}{\lambda} = 3$$

$$\Rightarrow \quad 2\lambda^2 - 3\lambda - 2 = 0$$

$$\Rightarrow \quad 2\lambda^2 - 4\lambda + \lambda - 2 = 0$$

$$2\lambda(\lambda - 2) + 1(\lambda - 2) = 0 \Rightarrow \lambda = -\frac{1}{2}, 2$$

$$\therefore \quad \log_5 x = -\frac{1}{2}, 2$$

$$\Rightarrow \quad x = 5^{-1/2}, 5^2 \text{ or } x = \frac{1}{\sqrt{5}}, 25$$

$$\therefore \text{ Product of the values of } x = \frac{1}{\sqrt{5}} \times 25 = 5\sqrt{5}$$

(C) $\because \quad \log_b a = -3$ and $\log_b c = 4$

$$\therefore \quad \log_c a = -\frac{3}{4} \qquad \text{...(i)}$$

and $\quad a^{3x} = c^{x-1}$

$$\Rightarrow \quad 3x\log a = (x-1)\log c$$

$$\Rightarrow \quad 3x\log_c a = x - 1$$

$$\Rightarrow \quad 3x \times -\frac{3}{4} = x - 1 \qquad \text{[from Eq. (i)]}$$

$$\Rightarrow \quad -9x = 4x - 4 \text{ or } x = \frac{4}{13}$$

$$\therefore \quad q = 13 \qquad \text{[prime and rational]}$$

Ex. 22

	Column I		Column II
(A)	If α and β are the roots of $ax^2 + bx + c = 0$, where $a = 2^{\log_2 3} - 3^{\log_3 2}$, $b = 1 + 2^{\sqrt{\log_2 3}} - 3^{\sqrt{\log_3 2}}$ and $c = \log_2 \log_2 \sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{2}}}}}}$, then HM of α and β is	(p) (q)	divisible by 2 divisible by 4
(B)	The sum of the solutions of the equation	(r)	divisible by 6
	$\lvert x-1\rvert^{\log_2 x^2 - 2\log_x 4} = (x-1)^7$ is	(s)	divisible by 8
(C)	If $5(\log_y x + \log_x y) = 26$, $xy = 64$, then the value of $\lvert x - y\rvert$ is	(t)	divisible by 10

Sol. A $\rightarrow$ (p, q, r), B $\rightarrow$ (p, r), C $\rightarrow$ (p, r, t)

(A) $\because a = 3 - 2 = 1, b = 1, c = \log_2 \log_2 2^{2^{-6}}$

$$= \log_2(2^{-6}) = -6$$

The equation reduces to $x^2 + x - 6 = 0$

$$\therefore \quad \alpha + \beta = -1, \ \alpha\beta = -6$$

$$\therefore \quad \text{HM} = \frac{2\alpha\beta}{\alpha + \beta} = \frac{2(-6)}{(-1)} = 12$$

(B) Obviously, $x = 2$ is a solution. Since, LHS is positive, $x - 1 > 0$. The equation reduces to

$$\log_2 x^2 - 2\log_x 4 = 7$$

$$\Rightarrow \quad 2\lambda - \frac{4}{\lambda} = 7, \text{ where } \lambda = \log_2 x$$

$\Rightarrow \quad 2\lambda^2 - 7\lambda - 4 = 0 \Rightarrow \lambda = 4, -\frac{1}{2}$

$\therefore \quad \log_2 x = 4, -\frac{1}{2} \Rightarrow x = 2^4, 2^{-1/2}$

$\Rightarrow \quad x = 16, \frac{1}{\sqrt{2}}$

$\Rightarrow \quad x = 16, x \neq \frac{1}{\sqrt{2}} \quad [\because x > 1]$

$\therefore$ Solutions are $x = 2, 16$

$\therefore$ Sum of solutions $= 2 + 16 = 18$

(C) If $\alpha = \log x, \beta = \log y$

$\therefore \quad \log_y x + \log_x y = \frac{\alpha}{\beta} + \frac{\beta}{\alpha}$

$\therefore \quad 5(\log_y x + \log_x y) = 26$

$\Rightarrow \quad \frac{\alpha}{\beta} + \frac{\beta}{\alpha} = \frac{26}{5}$

Let $\frac{\alpha}{\beta} = \lambda$, then $\lambda + \frac{1}{\lambda} = \frac{26}{5}$

$\Rightarrow \quad 5\lambda^2 - 26\lambda + 5 = 0$

$\Rightarrow \quad 5\lambda^2 - 25\lambda - \lambda + 5 = 0$

$\Rightarrow \quad (\lambda - 5)(5\lambda - 1) = 0$

$\Rightarrow \quad \lambda = 5, \frac{1}{5}$

$\therefore \quad \frac{\alpha}{\beta} = 5, \frac{1}{5} \Rightarrow \frac{\alpha}{\beta} = 5$

$\Rightarrow \quad \alpha = 5\beta \quad$ ...(i)

and $\alpha + \beta = \log x + \log y = \log(xy) = \log(64)$

$\therefore \quad \alpha + \beta = 6\log 2 \quad$ ...(ii)

From Eqs. (i) and (ii), we get

$\beta = \log 2$ and $\alpha = 5\log 2$

$\Rightarrow \quad y = 2, x = 32$ or $y = 32, x = 2$

$\therefore \quad |x - y| = 30$

JEE Type Solved Examples :
Statement I and II Type Questions

- **Directions** Example numbers 23 to 24 are Assertion-Reason type examples. Each of these examples contains two statements:
  Statement-1 (Assertion) and **Statement-2** (Reason)
  Each of these examples also has four alternative choices, only one of which is the correct answer. You have to select the correct choice as given below.
  (a) Statement-1 is true, Statement-2 is true; Statement-2 is a correct explanation for Statement-1
  (b) Statement-1 is true, Statement-2 is true; Statement-2 is not a correct explanation for Statement-1
  (c) Statement-1 is true, Statement-2 is false
  (d) Statement-1 is false, Statement-2 is true

● ***Ex. 23*** **Statement-1** *If* $N = \left(\frac{1}{0.4}\right)^{20}$, *then N contains 7 digits before decimal.*

Statement-2 *Characteristic of the logarithm of N to the base* 10 *is* 7.

Sol. (*d*) $\because \quad N = \left(\frac{1}{0.4}\right)^{20} = \left(\frac{10}{2^2}\right)^{20}$

$\Rightarrow \log_{10} N = 20(1 - 2\log_{10} 2) = 20(1 - 2 \times 0.3010)$

$= 20 \times 0.3980 = 7.9660$

Since, characteristic of $\log_{10} N$ is 7, therefore the number of digits in N will be 7 + 1, i.e. 8.

Hence, Statement-1 is false and Statement-2 is true.

● ***Ex. 24*** **Statement-1** *If* $p, q \in N$ *satisfy the equation* $x^{\sqrt{x}} = (\sqrt{x})^x$ *and* $q > p$, *then q is a perfect number.*

Statement-2 *If a number is equal to the sum of its factor, then number is known as perfect number.*

Sol. (*d*) $\because \quad x^{\sqrt{x}} = (\sqrt{x})^x$

Taking logarithm on both sides on base e, then

$\ln(x)^{\sqrt{x}} = \ln(\sqrt{x})^x$

$\Rightarrow \quad \sqrt{x}\ln x = x\ln\sqrt{x} \Rightarrow \sqrt{x}\ln x = \frac{x}{2}\ln x$

$\Rightarrow \quad \ln x\left(\sqrt{x} - \frac{x}{2}\right) = 0$

$\Rightarrow \quad \ln x \cdot \sqrt{x} \cdot \left(1 - \frac{\sqrt{x}}{2}\right) = 0$

$\Rightarrow \quad \ln x = 0, \sqrt{x} = 0, 1 - \frac{\sqrt{x}}{2} = 0$

$\therefore \quad x = 1, 0, 4$

$\because \quad x \in N$

$\therefore \quad x = 1, 4 \Rightarrow p = 1$ and $q = 4$

$\because \quad 4 = 1 \times 2 \times 2 \Rightarrow 4 \neq 1 + 2 + 2$

$\therefore$ q is not a perfect number.

Hence, Statement-1 is false and Statement-2 is true.

Subjective Type Questions

- In this section, there are **21 subjective** solved examples.

Ex. 25 *Prove that* $\log_3 5$ *is an irrational.*

Sol. Let $\log_3 5$ is rational.

$\therefore \quad \log_3 5 = \frac{p}{q}$, where p and q are co-prime numbers.

$\Rightarrow \quad 5 = 3^{p/q} \Rightarrow 3^p = 5^q$

which is not possible, hence our assumption is wrong.
Hence, $\log_3 5$ is an irrational.

Ex. 26 *Find the value of the expression*

$$(\log 2)^3 + \log 8 \cdot \log 5 + (\log 5)^3.$$

Sol. $\because \quad \log 2 + \log 5 = \log(2 \cdot 5) = \log 10 = 1 \quad$ ...(i)

$\Rightarrow \quad (\log 2 + \log 5)^3 = 1$

$\Rightarrow \quad (\log 2)^3 + (\log 5)^3 + 3 \log 2 \log 5 (\log 2 + \log 5) = 1^3$

$\Rightarrow \quad (\log 2)^3 + (\log 5)^3 + \log 2^3 \log 5 (1) = 1 \quad$ [from Eq. (i)]

$\Rightarrow \quad (\log 2)^3 + \log 8 \log 5 + (\log 5)^3 = 1$

Ex. 27 *If* $\lambda^{\log_3 5} = 81$, *find the value of* $\lambda^{(\log_3 5)^2}$.

Sol. $\because \quad \lambda^{\log_3 5} = 81$

$\therefore (\lambda^{\log_3 5})^{\log_3 5} = (81)^{\log_3 5}$

$\Rightarrow \quad \lambda^{(\log_3 5)^2} = 3^{4 \log_3 5} = 3^{\log_3 5^4} = 5^4 = 625$

Ex. 28 *Find the product of the positive roots of the equation* $\sqrt{(2009)}(x)^{\log_{2009} x} = x^2$.

Sol. Given, $\sqrt{(2009)}(x)^{\log_{2009} x} = x^2$

Taking logarithm both sides on base 2009, then

$$\log_{2009} \sqrt{(2009)} + \log_{2009} x \cdot \log_{2009} x = \log_{2009} x^2$$

$\Rightarrow \quad \frac{1}{2} + (\log_{2009} x)^2 = 2 \log_{2009} x \quad$ [for $x > 0$]

$\Rightarrow \quad (\log_{2009} x)^2 - 2 \log_{2009} x + \frac{1}{2} = 0$

If roots are x_1 and x_2, then $\log_{2009} x_1 + \log_{2009} x_2 = 2$

$\Rightarrow \quad \log_{2009}(x_1 x_2) = 2 \text{ or } x_1 x_2 = (2009)^2$

Ex. 29 *Prove that* $\log_7 11$ *is greater than* $\log_8 5$.

Sol. $\because \quad 11 > 5$

$\Rightarrow \quad \log 11 > \log 5 \quad$ ...(i)

and $\quad 8 > 7$

$\Rightarrow \quad \log 8 > \log 7 \quad$ ...(ii)

From Eqs. (i) and (ii), we get

$$\log 11 \cdot \log 8 > \log 7 \cdot \log 5$$

$\Rightarrow \quad \frac{\log 11}{\log 7} > \frac{\log 5}{\log 8} \Rightarrow \log_7 11 > \log_8 5$

Ex. 30 *Given,* $a^2 + b^2 = c^2$. *Prove that*

$\log_{b+c} a + \log_{c-b} a = 2 \log_{c+b} a \cdot \log_{c-b} a, \forall a > 0, a \neq 1$

$$c - b > 0, c + b > 0$$
$$c - b \neq 1, c + b \neq 1.$$

Sol. LHS $= \log_{b+c} a + \log_{c-b} a$

$$= \frac{1}{\log_a(c+b)} + \frac{1}{\log_a(c-b)}$$

$$= \frac{\log_a(c+b) + \log_a(c-b)}{\log_a(c+b) \log_a(c-b)}$$

$$= \frac{\log_a(c^2 - b^2)}{\log_a(c+b) \cdot \log_a(c-b)}$$

$$= \frac{\log_a a^2}{\log_a(c+b) \cdot \log_a(c-b)} \quad [\because c^2 - b^2 = a^2]$$

$$= \frac{2 \log_a a}{\log_a(c+b) \cdot \log_a(c-b)}$$

$$= \frac{2}{\log_a(c+b) \cdot \log_a(c-b)}$$

$$= 2 \log_{c+b} a \cdot \log_{c-b} a = \text{RHS}$$

Ex. 31 *Let* $a > 0, c > 0, b = \sqrt{ac}$, a, c *and* $ac \neq 1, N > 0$.

Prove that $\frac{\log_a N}{\log_c N} = \frac{\log_a N - \log_b N}{\log_b N - \log_c N}$.

Sol. RHS $= \frac{\log_a N - \log_b N}{\log_b N - \log_c N} = \frac{\frac{1}{\log_N a} - \frac{1}{\log_N b}}{\frac{1}{\log_N b} - \frac{1}{\log_N c}}$

$$= \frac{(\log_N b - \log_N a)}{(\log_N c - \log_N b)} \cdot \frac{\log_N c}{\log_N a}$$

$$= \frac{\log_N \left(\frac{b}{a}\right)}{\log_N \left(\frac{c}{b}\right)} \cdot \frac{\log_a N}{\log_c N} = \frac{\log_a N}{\log_c N} = \text{LHS}$$

$$\left[\because b = \sqrt{ac} \Rightarrow b^2 = ac \Rightarrow \frac{b}{a} = \frac{c}{b}\right]$$

● **Ex. 32** *If* $a^x = b, b^y = c, c^z = a, x = \log_b a^{k_1}, y = \log_c b^{k_2}$, $z = \log_a c^{k_3}$, *find the minimum value of* $3k_1 + 6k_2 + 12k_3$.

Sol. $\because \quad a = c^z = (b^y)^z \qquad [\because c = b^y]$

$= b^{yz} = (a^x)^{yz} = a^{xyz} \qquad [\because b = a^x]$

$\therefore \quad xyz = 1$

Also, $xyz = \log_b a^{k_1} \cdot \log_c b^{k_2} \cdot \log_a c^{k_3}$

$= k_1 \cdot k_2 \cdot k_3 \cdot \log_b a \cdot \log_c b \cdot \log_a c$

$1 = k_1 k_2 k_3$

$\because \quad$ AM $\geq$ GM

$$\therefore \quad \frac{3k_1 + 6k_2 + 12k_3}{3} \geq (3k_1 \cdot 6k_2 \cdot 12k_3)^{1/3}$$

$= (3 \cdot 6 \cdot 12 \cdot k_1 k_2 k_3)^{1/3}$

$= (3 \cdot 6 \cdot 12)^{1/3} \qquad [\because k_1 k_2 k_3 = 1]$

$= (2^3 \cdot 3^3)^{1/3} = 6$

or $\;3k_1 + 6k_2 + 12k_3 \geq 18$

$\therefore$ Minimum value of $3k_1 + 6k_2 + 12k_3$ is 18.

● **Ex. 33** *If* $x = 1 + \log_a bc, y = 1 + \log_b ca, z = 1 + \log_c ab$, *prove that* $xyz = xy + yz + zx$.

Sol. $\because \quad x = 1 + \log_a bc = 1 + \dfrac{\log bc}{\log a} = 1 + \dfrac{\log b + \log c}{\log a}$

$$= \frac{\log a + \log b + \log c}{\log a}$$

or $\quad \dfrac{1}{x} = \dfrac{\log a}{\log a + \log b + \log c} \qquad \text{...(i)}$

Similarly, $\dfrac{1}{y} = \dfrac{\log b}{\log a + \log b + \log c} \qquad \text{...(ii)}$

and $\quad \dfrac{1}{z} = \dfrac{\log c}{\log a + \log b + \log c} \qquad \text{...(iii)}$

On adding Eqs. (i), (ii) and (iii), we get

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1$$

or $\quad xyz = xy + yz + zx$

● **Ex. 34** *If* $\dfrac{\ln a}{(b-c)} = \dfrac{\ln b}{(c-a)} = \dfrac{\ln c}{(a-b)}$, *prove that*

$$a^{b+c} \cdot b^{c+a} \cdot c^{a+b} = 1$$

Also, prove that $\quad a^{b+c} + b^{c+a} + c^{a+b} \geq 3$.

Sol. Since, $a > 0, b > 0, c > 0$

$$\frac{\ln a}{(b-c)} = \frac{\ln b}{(c-a)} = \frac{\ln c}{(a-b)}$$

$$= \frac{(b+c)\ln a + (c+a)\ln b + (a+b)\ln c}{0}$$

[using ratio and proportion]

$\therefore \quad (b+c)\ln a + (c+a)\ln b + (a+b)\ln c = 0$

$\Rightarrow \quad \ln a^{b+c} + \ln b^{c+a} + \ln c^{a+b} = 0$

$\Rightarrow \quad \ln \{a^{b+c} \cdot b^{c+a} \cdot c^{a+b}\} = 0$

$\Rightarrow \quad a^{b+c} \cdot b^{c+a} \cdot c^{a+b} = e^0 = 1 \qquad \text{...(i)}$

Again, AM $\geq$ GM

$$\Rightarrow \quad \frac{a^{b+c} + b^{c+a} + c^{a+b}}{3} \geq (a^{b+c} \cdot b^{c+a} \cdot c^{a+b})^{1/3}$$

$= (1)^{1/3} = 1 \qquad$ [from Eq. (i)]

or $\quad a^{b+c} + b^{c+a} + c^{a+b} \geq 3$

● **Ex. 35** *Simplify* $5^{\log_{1/5}(1/2)} + \log_{\sqrt{2}}\left(\dfrac{4}{\sqrt{7} + \sqrt{3}}\right) + \log_{1/2}\left(\dfrac{1}{10 + 2\sqrt{21}}\right)$.

Sol. $\because \quad 5^{\log_{1/5}\left(\frac{1}{2}\right)} = 5^{\log_5(2)} = 2$

$$\log_{\sqrt{2}}\left(\frac{4}{\sqrt{7} + \sqrt{3}}\right) = \log_{\sqrt{2}}\left(\frac{4(\sqrt{7} - \sqrt{3})}{(\sqrt{7} + \sqrt{3})(\sqrt{7} - \sqrt{3})}\right)$$

$= \log_{\sqrt{2}}(\sqrt{7} - \sqrt{3})$

$= \log_{2^{1/2}}(\sqrt{7} - \sqrt{3})^1$

$= \dfrac{1}{1/2}\log_2(\sqrt{7} - \sqrt{3})$

$= \log_2(\sqrt{7} - \sqrt{3})^2 = \log_2(10 - 2\sqrt{21})$

and $\log_{1/2}\left(\dfrac{1}{10 + 2\sqrt{21}}\right) = \log_2(10 + 2\sqrt{21})$

Hence,

$$5^{\log_{1/5}(1/2)} + \log_{\sqrt{2}}\left(\frac{4}{\sqrt{7} + \sqrt{3})}\right) + \log_{1/2}\left(\frac{1}{10 + 2\sqrt{21}}\right)$$

$= 2 + \log_2(10 - 2\sqrt{21}) + \log_2(10 + 2\sqrt{21})$

$= 2 + \log_2\{(10 - 2\sqrt{21})(10 + 2\sqrt{21})\}$

$= 2 + \log_2(100 - 84) = 2 + \log_2(2)^4 = 2 + 4 = 6$

● **Ex. 36** *Find the square of the sum of the roots of the equation* $\log_2 x \cdot \log_3 x \cdot \log_5 x = \log_2 x \cdot \log_3 x + \log_3 x \cdot \log_5 x + \log_5 x \cdot \log_2 x$.

Sol. Let $\log_2 x = A$, $\log_3 x = B$ and $\log_5 x = C$, then the given equation can be written as

$$ABC = AB + BC + CA = ABC\left(\frac{1}{C} + \frac{1}{A} + \frac{1}{B}\right)$$

$$\Rightarrow \quad ABC\left(\frac{1}{A} + \frac{1}{B} + \frac{1}{C} - 1\right) = 0$$

or $A = 0, B = 0, C = 0, \dfrac{1}{A} + \dfrac{1}{B} + \dfrac{1}{C} - 1 = 0$

$$\underbrace{\log_2 x = 0, \log_3 x = 0, \log_5 x = 0}_{x>0},\ \underbrace{\log_x 2 + \log_x 3 + \log_x 5 = 0}_{x>0,\ x\neq 1}$$

or $\quad x = 2^0, x = 3^0, x = 5^0, \log_x(2\cdot 3\cdot 5) = 0$

or $\quad x = 1, x = 1, x = 1, x = 30$

$\therefore$ Roots are 1 and 30.

Hence, the required value

$$= (1 + 30)^2 = (31)^2 = 961$$

Ex. 37 *Given that* $\log_2 a = \lambda, \log_4 b = \lambda^2$ *and* $\log_{c^2}(8) = \dfrac{2}{\lambda^3 + 1}$, *write* $\log_2\left(\dfrac{a^2 b^5}{c^4}\right)$ *as a function of 'λ'* $(a, b, c > 0, c \neq 1)$.

Sol. $\because \log_2 a = \lambda \Rightarrow a = 2^\lambda$

$\Rightarrow \quad \log_4 b = \lambda^2$

$\Rightarrow \quad b = 4^{\lambda^2} = 2^{2\lambda^2}$

and $\quad \log_{c^2}(8) = \dfrac{2}{\lambda^3 + 1}$

$\Rightarrow \quad \dfrac{3}{2}\log_c 2 = \dfrac{2}{\lambda^3 + 1}$

$\Rightarrow \quad \log_c 2 = \dfrac{4}{3(\lambda^3 + 1)}$

or $\quad \log_2 c = \dfrac{3(\lambda^3 + 1)}{4}$ or $c = 2^{\left\{\frac{3(\lambda^3+1)}{4}\right\}}$

$\therefore \quad \log_2\left(\dfrac{a^2 b^5}{c^4}\right) = \log_2(a^2 b^5 c^{-4})$

$$= \log_2\{2^{2\lambda}\cdot 2^{10\lambda^2}\cdot 2^{-3(\lambda^3+1)}\}$$

$$= \log_2\{2^{2\lambda + 10\lambda^2 - 3(\lambda^3+1)}\}$$

$$= 2\lambda + 10\lambda^2 - 3(\lambda^3 + 1)$$

Ex. 38 *Given that* $\log_2 3 = a, \log_3 5 = b, \log_7 2 = c$, *express the logarithm of the number 63 to the base 140 in terms of a, b and c.*

Sol. $\because \log_2 3 = a$ ...(i)

$\Rightarrow \quad b = \log_3 5 = \dfrac{\log_2 5}{\log_2 3} = \dfrac{\log_2 5}{a}$ [from Eq. (i)]

$\therefore \quad \log_2 5 = ab$ ...(ii)

and $\quad \log_7 2 = c$

$\Rightarrow \quad \dfrac{1}{\log_2 7} = c$ or $\log_2 7 = \dfrac{1}{c}$ ...(iii)

Now, $\quad \log_{140} 63 = \dfrac{\log_2 63}{\log_2 140} = \dfrac{\log_2(3^2 \times 7)}{\log_2(2^2 \times 5 \times 7)}$

$$= \frac{2\log_2 3 + \log_2 7}{2 + \log_2 5 + \log_2 7} = \frac{2a + \frac{1}{c}}{2 + ab + \frac{1}{c}}$$

[from Eqs. (i), (ii) and (iii)]

$$= \left(\frac{2ac + 1}{2c + abc + 1}\right)$$

Ex. 39 *Show that the sum of the roots of the equation* $x + 1 = 2\log_2(2^x + 3) - 2\log_4(1980 - 2^{-x})$ *is* $\log_2 11$.

Sol. Given,

$$x + 1 = 2\log_2(2^x + 3) - 2\log_4(1980 - 2^{-x})$$

$$= 2\log_2(2^x + 3) - 2\log_{2^2}(1980 - 2^{-x})^1$$

$$= 2\log_2(2^x + 3) - 2\cdot\frac{1}{2}\log_2(1980 - 2^{-x})$$

$$= \log_2(2^x + 3)^2 - \log_2(1980 - 2^{-x})$$

$$= \log_2\left\{\frac{(2^x + 3)^2}{1980 - 2^{-x}}\right\}$$

or $2^{x+1} = \dfrac{(2^x + 3)^2}{1980 - 2^{-x}}$

$\Rightarrow \quad 1980(2^{x+1}) - 2 = 2^{2x} + 9 + 6\cdot 2^x$

$\Rightarrow \quad 2^{2x} - 3954\cdot 2^x + 11 = 0$ ...(i)

If x_1, x_2 are the roots of Eq. (i), then

$$2^{x_1}\cdot 2^{x_2} = 11 \text{ or } 2^{x_1 + x_2} = 11$$

$\Rightarrow \quad x_1 + x_2 = \log_2 11$

Ex. 40 *Solve the following equations for x and y* $\log_{100}|x + y| = \dfrac{1}{2}, \log_{10} y - \log_{10}|x| = \log_{100} 4$.

Sol. $\because \quad \log_{100}|x + y| = \dfrac{1}{2}$

$\Rightarrow \quad |x + y| = (100)^{1/2} = 10$

$\Rightarrow \quad |x + y| = 10$ ...(i)

and $\quad \log_{10} y - \log_{10}|x| = \log_{100} 4, y > 0$

$\Rightarrow \quad \log_{10}\left(\dfrac{y}{|x|}\right) = \log_{10^2} 2^2 = \dfrac{2}{2}\log_{10} 2$

$\Rightarrow \quad \log_{10}\left(\dfrac{y}{|x|}\right) = \log_{10} 2 \Rightarrow \dfrac{y}{|x|} = 2$

$\Rightarrow \quad y = 2|x|$ ...(ii)

From Eqs. (i) and (ii),we get

$$|x + 2|x|| = 10 \quad \text{...(iii)}$$

***Case* I** If $x > 0$, then $|x| = x$

From Eq. (iii),

$$|x + 2x| = 10$$

$\Rightarrow \quad 3|x| = 10 \Rightarrow |x| = \frac{10}{3}$

$\therefore \quad x = \frac{10}{3}, y = \frac{20}{3}$ [from Eq. (ii)]

***Case* II** If $x < 0$, then $|x| = -x$

From Eq. (iii),

$$|x - 2x| = 10$$

$\Rightarrow \quad |-x| = 10 \Rightarrow |x| = 10$

$\therefore \quad -x = 10$

$\Rightarrow \quad x = -10$

From Eq. (ii), $\quad y = 20$

Hence, solutions are $\left\{\frac{10}{3}, \frac{20}{3}\right\}$, $\{-10, 20\}$.

Ex. 41 *Solve the following equation for x*

$\frac{6}{5}a^{\log_a x \cdot \log_{10} a \cdot \log_a 5} - 3^{\log_{10}(x/10)} = 9^{\log_{100} x + \log_4 2}$.

Sol. $\because \frac{6}{5} \cdot a^{\log_a x \cdot \log_{10} a \cdot \log_a 5} - 3^{\log_{10}(x/10)} = 9^{\log_{100} x + \log_4 2}$

$\Rightarrow \quad \frac{6}{5} \cdot x^{\log_{10} 5} - 3^{(\log_{10} x - 1)} = 3^{2\left(\frac{1}{2}\log_{10} x + \frac{1}{2}\right)}$ [by property]

$\Rightarrow \quad \frac{6}{5} \cdot 5^{\log_{10} x} - \frac{3^{\log_{10} x}}{3} = 3^{\log_{10} x + 1}$ [by property]

Let $\log_{10} x = \lambda$, then

$\Rightarrow \quad \frac{6}{5} \cdot 5^{\lambda} - \frac{3^{\lambda}}{3} = 3 \cdot 3^{\lambda}$

$\Rightarrow \quad \frac{6}{5} \cdot 5^{\lambda} = 3^{\lambda}\left(\frac{1}{3} + 3\right) = \frac{10}{3} \cdot 3^{\lambda}$

$\Rightarrow \quad 5^{\lambda - 2} = 3^{\lambda - 2}$ which is possible only, where $\lambda = 2$.

$\Rightarrow \quad \log_{10} x = 2$

$\therefore \quad x = 10^2 = 100$

Ex. 42 *Find the value of x satisfying the equation*

$$|x - 1|^{\log_3 x^2 - 2\log_x 9} = (x - 1)^7.$$

Sol. The given equation is,

$$|x - 1|^{\log_3 x^2 - 2\log_x 9} = (x - 1)^7 \quad \text{...(i)}$$

This equation is defined for

$x^2 > 0, x > 0, x \neq 1$ and $x - 1 \geq 1$

$\Rightarrow \quad x \geq 2$, then Eq. (i) reduces to

$(x - 1)^{\log_3 x^2 - 2\log_x 9} = (x - 1)^7$

Taking log on both sides, then

$$(\log_3 x^2 - 2\log_x 9)\log(x - 1) = 7\log(x - 1)$$

$\Rightarrow \quad \log(x - 1)\{\log_3 x^2 - 2\log_x 9 - 7\} = 0$

$\Rightarrow \quad \log(x - 1)\left\{2\log_3 x - \frac{4}{\log_3 x} - 7\right\} = 0$

$\Rightarrow \quad \log(x - 1)\{2(\log_3 x)^2 - 7\log_3 x - 4\} = 0$

$\Rightarrow \quad \log(x - 1)(\log_3 x - 4)(2\log_3 x + 1) = 0$

$\Rightarrow \quad \log(x - 1) = 0, \log_3 x = 4, \log_3 x = -\frac{1}{2}$

$\Rightarrow \quad x - 1 = (10)^0, x = 3^4, x = 3^{-1/2}$

$\Rightarrow \quad x - 1 = 1, x = 81, x = \frac{1}{\sqrt{3}}$

$\therefore \quad x = 2, 81 \quad \left[\because x \geq 2, \therefore x \neq \frac{1}{\sqrt{3}}\right]$

Ex. 43 *Find all real numbers x which satisfy the equation* $2\log_2 \log_2 x + \log_{1/2} \log_2(2\sqrt{2}x) = 1$.

Sol. Given,

$$2\log_2 \log_2 x + \log_{1/2} \log_2(2\sqrt{2}x) = 1$$

$\Rightarrow \quad 2\log_2 \log_2 x - \log_2 \log_2(2\sqrt{2}x) = 1$

$\Rightarrow \quad 2\log_2 \log_2 x - \log_2\{\log_2(2\sqrt{2}) + \log_2 x\} = 1$

$\Rightarrow \quad 2\log_2 \log_2 x - \log_2\left\{\frac{3}{2} + \log_2 x\right\} = 1$

Let $\log_2 x = \lambda$, then $2\log_2 \lambda - \log_2\left(\frac{3}{2} + \lambda\right) = 1$

$\Rightarrow \quad \log_2 \lambda^2 - \log_2\left(\frac{3}{2} + \lambda\right) = 1$

$\Rightarrow \quad \log_2\left\{\frac{\lambda^2}{\frac{3}{2} + \lambda}\right\} = 1 \Rightarrow \frac{\lambda^2}{\frac{3}{2} + \lambda} = 2^1$

$\Rightarrow \quad \lambda^2 = 3 + 2\lambda \Rightarrow \lambda^2 - 2\lambda - 3 = 0$

$\Rightarrow \quad (\lambda - 3)(\lambda + 1) = 0$

$\therefore \quad \lambda = 3, -1$

or $\quad \log_2 x = 3, -1$

$\Rightarrow \quad x = 2^3, 2^{-1}$

$\Rightarrow \quad x = 8, \frac{1}{2}$ ...(i)

But the given equation is valid only when,

$x > 0, 2\sqrt{2}x > 0, \log_2 x > 0, \log_2(2\sqrt{2}x) > 0$

$\Rightarrow \quad x > 0, x > 0, x > 1, x > \frac{1}{2\sqrt{2}}$

Hence, $x > 1$

From Eq. (i), the solution of the given equation is $x = 8$.

Ex. 44 *Solve for x,*

$\log_{3/4} \log_8(x^2 + 7) + \log_{1/2} \log_{1/4}(x^2 + 7)^{-1} = -2$.

Sol. Given,

$$\log_{3/4} \log_8 (x^2 + 7) + \log_{1/2} \log_{1/4} (x^2 + 7)^{-1} = -2$$

$$\Rightarrow \log_{3/4} \log_{2^3} (x^2+7) + \log_{2^{-1}} \log_{2^{-2}} (x^2+7)^{-1} = -2$$

$$\Rightarrow \log_{3/4}\left\{\frac{1}{3}\log_2(x^2+7)\right\} - \log_2\left\{\frac{1}{2}\log_2(x^2+7)\right\} = -2$$

Let $\log_2(x^2+7) = 6\lambda$ ...(i)

Then, $\log_{3/4}(2\lambda) - \log_2(3\lambda) = -2$

$$\Rightarrow \frac{\log_2(2\lambda)}{\log_2(3/4)} - \log_2(3\lambda) = -2$$

$$\Rightarrow \frac{1+\log_2\lambda}{\log_2 3 - \log_2 4} - (\log_2 3 + \log_2\lambda) = -2$$

$$\Rightarrow \frac{1+\log_2\lambda}{\log_2 3 - 2} - (\log_2 3 + \log_2\lambda) = -2$$

Again, let $\log_2\lambda = A$ and $\log_2 3 = B$, then

$$\frac{1+A}{B-2} - (B+A) = -2$$

$$\Rightarrow 1 + A - B^2 - AB + 2B + 2A = -2B + 4$$

$$\Rightarrow A(3-B) = B^2 - 4B + 3 = (B-1)(B-3)$$

$$\Rightarrow A = -(B-1)$$

$[\because B - 3 \neq 0, \text{ i.e. } \log_2 3 \neq 3]$

$$\Rightarrow A + B = 1 \Rightarrow \log_2\lambda + \log_2 3 = 1$$

$$\Rightarrow \log_2(3\lambda) = 1$$

$$\Rightarrow 3\lambda = 2$$

$$\Rightarrow 3\cdot\frac{1}{6}\log_2(x^2+7) = 2 \quad \text{[from Eq. (i)]}$$

$$\Rightarrow \log_2(x^2+7) = 4$$

$$\Rightarrow x^2 + 7 = 2^4 = 16 \text{ or } x^2 = 9$$

$$\therefore x = \pm 3$$

- ***Ex. 45*** *Prove that*

$$2^{\left(\sqrt{\log_a \sqrt[4]{ab} + \log_b \sqrt[4]{ab}} - \sqrt{\log_a \sqrt[4]{b/a} + \log_b \sqrt[4]{a/b}}\right)\sqrt{\log_a b}} = \begin{cases} 2, & b \geq a > 1. \\ 2^{\log_a b}, & 1 < b < a \end{cases}.$$

Sol. Since, $\sqrt{\log_a \sqrt[4]{ab} + \log_b \sqrt[4]{ab}} = \sqrt{\frac{1}{4}\log_a(ab) + \frac{1}{4}\log_b(ab)}$

$$= \sqrt{\frac{1}{4}(1 + \log_a b + \log_b a + 1)}$$

$$= \sqrt{\left(\frac{\log_a b + \frac{1}{\log_a b} + 2}{4}\right)} = \sqrt{\left(\frac{\sqrt{|\log_a b|} + \frac{1}{\sqrt{|\log_a b|}}}{2}\right)^2}$$

$$= \frac{1}{2}\left(\sqrt{|\log_a b|} + \frac{1}{\sqrt{|\log_a b|}}\right)$$

and $\sqrt{\log_a \sqrt[4]{(b/a)} + \log_b \sqrt[4]{(a/b)}}$

$$= \sqrt{\frac{1}{4}\log_a\left(\frac{b}{a}\right) + \frac{1}{4}\log_b\left(\frac{a}{b}\right)}$$

$$= \sqrt{\frac{1}{4}(\log_a b - 1 + \log_b a - 1)}$$

$$= \sqrt{\frac{\log_a b + \frac{1}{\log_a b} - 2}{4}}$$

$$= \frac{\left|\sqrt{|\log_a b|} - \frac{1}{\sqrt{|\log_a b|}}\right|}{2}$$

$\therefore \sqrt{\log_a \sqrt[4]{ab} + \log_b \sqrt[4]{ab}} - \sqrt{\log_a \sqrt[4]{b/a} + \log_b \sqrt[4]{(a/b)}}$

P (say)

$$= \frac{1}{2}\left\{\sqrt{|\log_a b|} + \frac{1}{\sqrt{|\log_a b|}} + \left|\sqrt{|\log_a b|} - \frac{1}{\sqrt{|\log_a b|}}\right|\right\}$$

***Case* I** If $b \geq a > 1$, then

$$P = \frac{1}{2}\left\{\sqrt{|\log_a b|} + \frac{1}{\sqrt{|\log_a b|}} - \sqrt{|\log_a b|} + \frac{1}{\sqrt{|\log_a b|}}\right\}$$

$$= \frac{1}{\sqrt{\log_a b}}$$

$$\therefore 2^{P\sqrt{\log_a b}} = 2^1 = 2$$

***Case* II** If $1 < b < a$, then

$$P = \frac{1}{2}\left\{\sqrt{|\log_a b|} + \frac{1}{\sqrt{|\log_a b|}} + \sqrt{|\log_a b|} - \frac{1}{\sqrt{|\log_a b|}}\right\}$$

$$= \sqrt{|\log_a b|}$$

$$\therefore 2^{P\sqrt{\log_a b}} = 2^{\log_a b}$$