10 VECTOR ALGEBRA

Objectives

After studying the material of this chapter, you should be able to :

- Understand Representation of vectors and their types.
- Understand Addition of Vectors and its properties.
- Understand Multiplication of a vector by a scalar and its properties.
- Understand Position Vector, Distance Formula, Section Formula and Centroid & Incentre.
- Understand Product of two Vectors — Scalar and Vector and their properties.
- Understand Scalar Triple Product and its properties.

In Mathematics alone each generation builds a new story to the old structure

—*Herman Hankel*

SUB CHAPTER 10.1 Vectors

Chapter at Glance

INTRODUCTION

In Mathematics, Physics and Engineering, we usually come across with two types of quantities, namely scalar quantities and vector quantities.

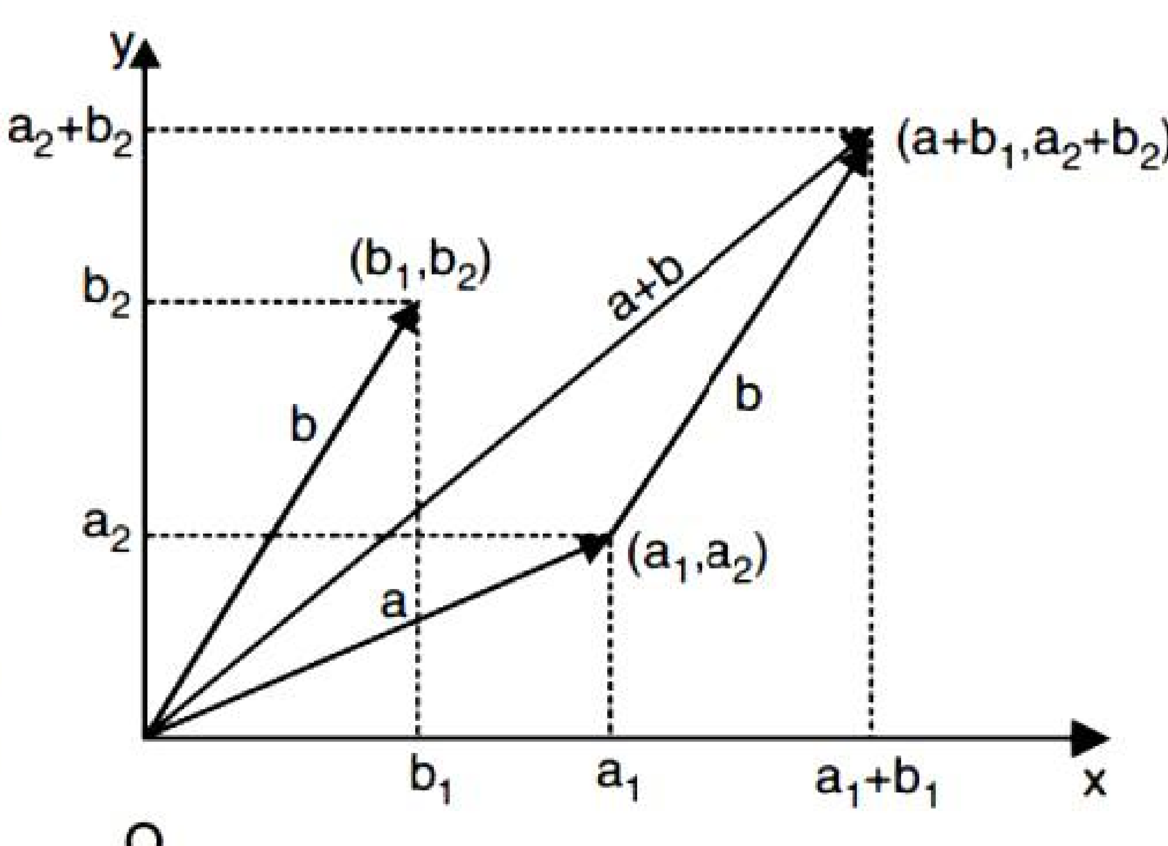
Scalar. *A quantity, which has only magnitude but is not related to any fixed direction in space, is called scalar.*

For examples : Mass, length, volume, density, time, temperature ; etc. are all scalar quantities.

If the unit of mesaurement is fixed, then a real number is sufficient to represent a scalar.

Vector. *A quantity, which has magnitude as well as direction, is called vector.*

For examplea : Force, velocity, acceleration, displacement, weight, momentum; etc. are all vector quantities.



In this chapter, we will study the following concepts :

- Vectors – Types and Properties
- Operations on Vectors
- Algebraic and Geometrical properties
- Product of two vectors : Scalar and Vector product
- Salar Triple Product.

10.1. REPRESENTATION OF VECTORS

(*i*) Since a vector has both magnitude and direction, therefore, it can be conveniently represented by a *directed line segment.*

Let P be an arbitrary point in space and Q any other point. Then the st. line PQ has both magnitude and direction.

Then the directed line $\overrightarrow{PQ}$ represents a vector.

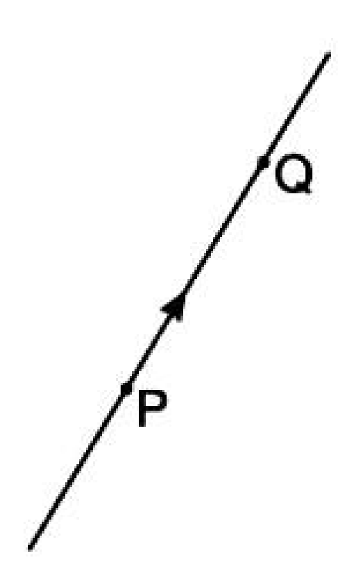


Fig.

Here P is called the **origin** or **initial point** while Q, the **terminal point** or **end point.**

Note. *The two end points P and Q are not interchangeable.*

To every directed line $\overrightarrow{PQ}$, the following are associated :

Its (I) *length* (II) *support* and (III) *sense.*

(I) Length. *The length of the directed line segment* $\overrightarrow{PQ}$ *is the distance between P and Q.*

(II) Support. *The line of an unlimited length of which a directed line segment is a part is called the support.*

(III) Sense. *The sense of the directed line segment is from its initial point to its terminal point.*

Notations. We denote the vector by claredone (bold) letters or by single letter with an arrow or bar over its head.

Thus $\mathbf{a}$, $\vec{a}$, $\bar{a}$ may denote the vector $\overrightarrow{PQ}$.

(*ii*) *The non-negative number, which measures the magnitude of a vector, is called its* **module** *or* **modulus.**

The module of vector $\vec{a}$ is denoted by a or $|\vec{a}|$.

10.2. TYPES OF VECTORS

(i) Zero Vector. *A vector whose length or magnitude or modulus is zero but direction indeterminate is called a zero vector.*

This is also known as **null vector.**

This is denoted by $\vec{0}$.

In this case, the initial and terminal points coincide.

Thus $\overrightarrow{AA}$, $\overrightarrow{OO}$; etc. are zero vectors.

(ii) Proper Vector. *Any non-zero vector is called a proper vector.*

(iii) Unit Vector. *A unit vector is that whose length or magnitude or modulus is unity.*

The sign $\wedge$ stands for **unit vector.**

Thus $\hat{a}$ denotes a unit vector and is read as '*a cap.*'

$$\therefore \qquad \vec{a} = |\vec{a}|\hat{a} \qquad \Rightarrow \qquad \hat{a} = \frac{\vec{a}}{|\vec{a}|}.$$

Thus, if we divide a vector by its magnitude, we get a unit vector in the original direction.

(iv) Co-initial Vectors. *All vectors, having the same initial point, are called co-initial vectors.*

Thus $\overrightarrow{AA}, \overrightarrow{AB}, \overrightarrow{AD}$; etc. are all co-initial vectors.

(v) Like and Unlike Vectors. *Vectors are said to be **like** if they have the same direction and **unlike** if they have opposite directions irrespective of their magnitudes.*

(vi) Collinear or Parallel Vectors. *Vectors are said to be **collinear** or **parallel** if they have the same line of action or have the lines of action parallel to one another irrespective of their magnitudes.* **(Karnataka B. 2017)**

(vii) Coplanar Vectors. *Vectors are said to be coplanar if they are parallel to the same plane or they lie in the same plane.*

(viii) Free and Localised Vectors. *When there is no restriction to choose the origin of a vector, then it is called a **free vector.** When there is restriction to choose a certain specified point, then it is called a **localised vector** or **sliding vector.***

(ix) Negative of a Vector. *The vector, which has the same magnitude as that of the vector* $\vec{a}$ *but has the direction opposite to that of* $\vec{a}$, *is called the negative of* $\vec{a}$ *and is denoted as* $-\vec{a}$.

Thus, if $\vec{a} = \overrightarrow{AB}$, then $-\vec{a} = \overrightarrow{BA}$.

(x) Reciprocal Vector. *The vector, which has the same direction as that of* $\vec{a}$ *but has magnitude reciprocal to that of* $\vec{a}$, *is called the reciprocal of* $\vec{a}$ *and is denoted as* $\vec{a}^{-1}$.

(xi) Equal Vectors.

Two vectors are said to be equal if they have :

(*I*) *the same length,*

(*II*) *the same or parallel supports* and

(*III*) *the same sense.*

Thus $\overrightarrow{LM} = \overrightarrow{PQ} = \overrightarrow{RS}$, and as such these are equal vectors.

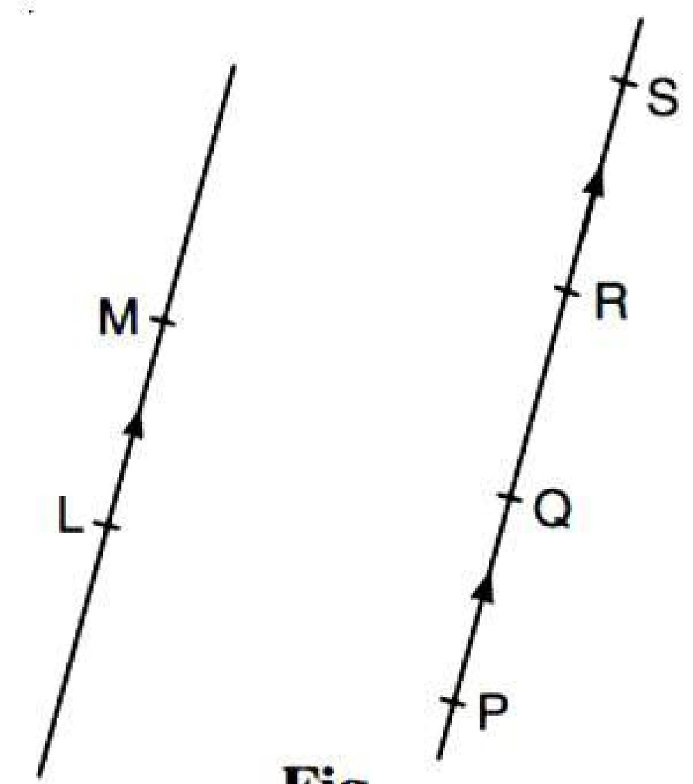


Fig.

KEY POINT

1. Two vectors are not equal if they (I) have different magnitudes (II) do not have the same or parallel supports (III) have the same magnitudes and parallel supports but have different senses.

2. Vectors, defined above, are such that each of them is subject to its parallel displacement without changing its magnitude and direction.

Such vectors are called **free vectors.**

ILLUSTRATIVE EXAMPLES

Example 1. Classify the following measures as scalar and vector quantities :

(i) 40°

(ii) 50 watt

(iii) 10 gm/cm^3

(iv) 20 m/sec towards north

(v) 5 seconds. *(N.C.E.R.T.)*

Solution. (*i*) Angle-scalar

(*ii*) Power-scalar

(*iii*) Density-scalar

(*iv*) Velocity-vector

(*v*) Time-scalar.

Example 2. In the figure, which of the vectors are :

(i) Collinear

(ii) Equal

(iii) Co-initial. *(N.C.E.R.T.)*

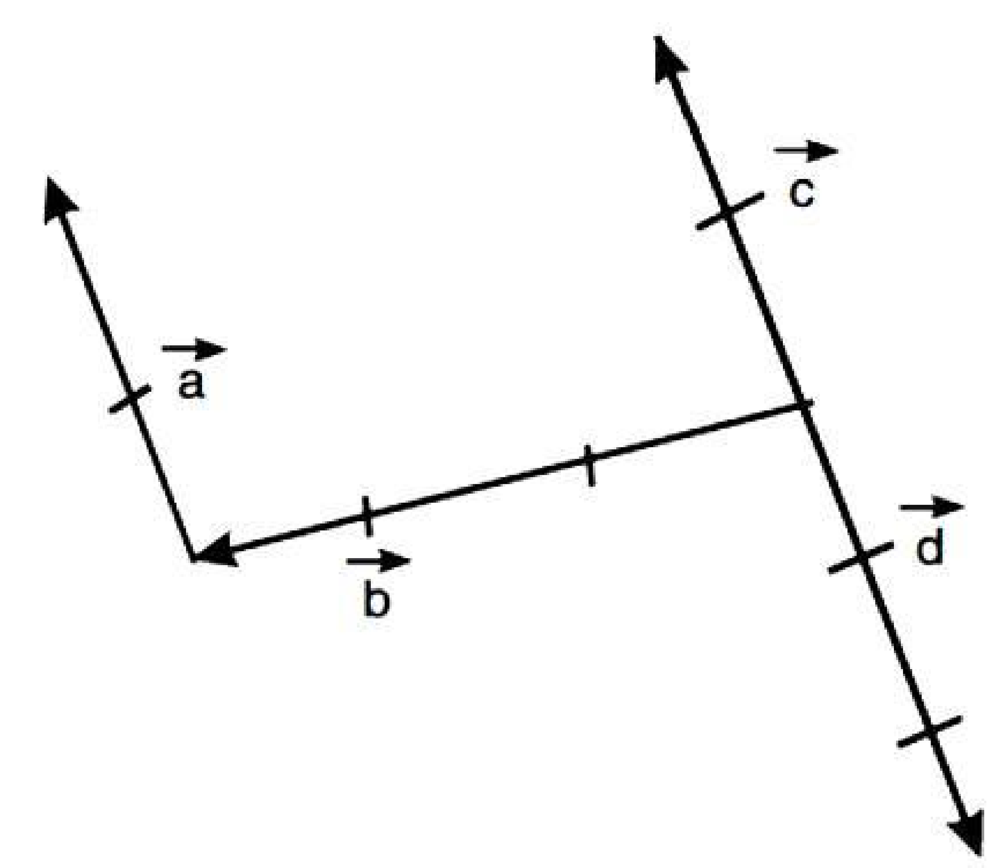


Fig.

Solution. (*i*) $\vec{a}$, $\vec{c}$ and $\vec{d}$ are collinear vectors.

(*ii*) $\vec{a}$ and $\vec{c}$ are equal vectors.

(*iii*) $\vec{b}$, $\vec{c}$ and $\vec{d}$ are co-initial vectors.

EXERCISE 10 (a)

Fast Track Answer Type Questions

FTATQ

1. Represent the following graphically a displacement of :

(*i*) 40 km, 30° west of south

(*ii*) 40 km, 30° east of south

(*iii*) 40 km, 30° west of north. *(N.C.E.R.T.)*

2. Classify the following measures as scalars and vectors :

(*i*) 10 kg

(*ii*) 40°

(*iii*) 2 metres north-west

(*iv*) 40 watt

(*v*) 10^{-19} coulomb

(*vi*) 20 m/s^2

(*vii*) 1000 cm^3

(*viii*) 10 Newton

(*ix*) 30 km/h. (N.C.E.R.T.)

3. Classify the following as scalar and vector quantities :

(*i*) time-period

(*ii*) distance

(*iii*) force

(*iv*) velocity

(*v*) work done. (N.C.E.R.T.)

4. In the figure, identify the following vectors :

(*i*) Co-initial (*ii*) Equal. (N.C.E.R.T.)

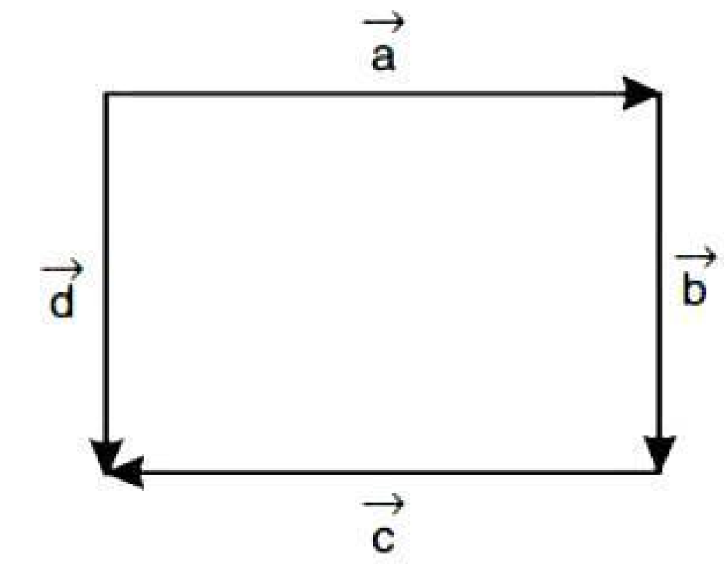


Fig.

5. A girl walks 4 km westward, then she walks 3 km in a direction 30° east of north and stops. Determine the girl's displacement from her initial point of departure. (N.C.E.R.T.)

6. Answer the following as true or false :

(*i*) $\vec{a}$ and $-\vec{a}$ are collinear

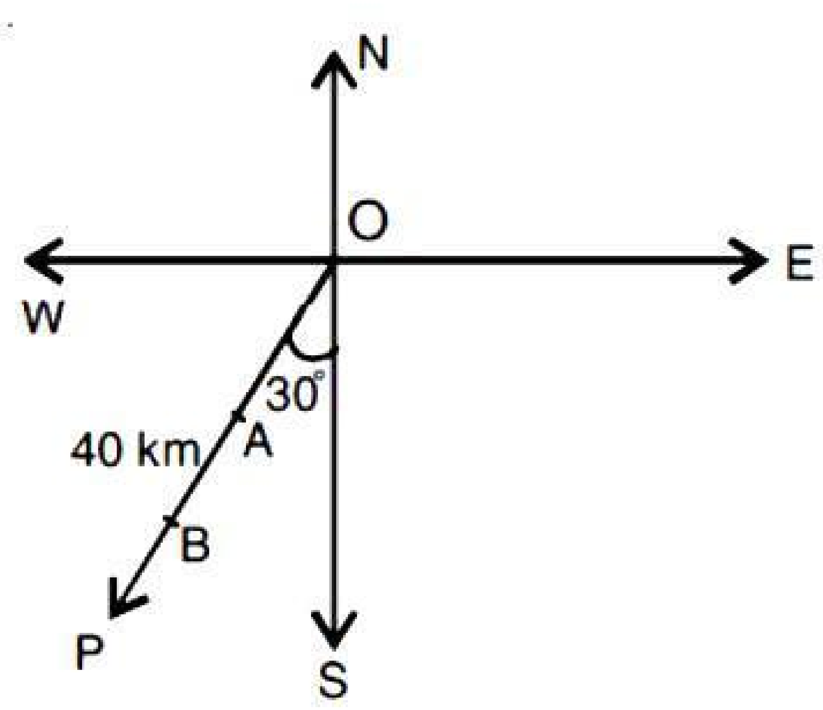
(*ii*) Two collinear vectors are always equal in magnitude

(*iii*) Two vectors having same magnitude are collinear

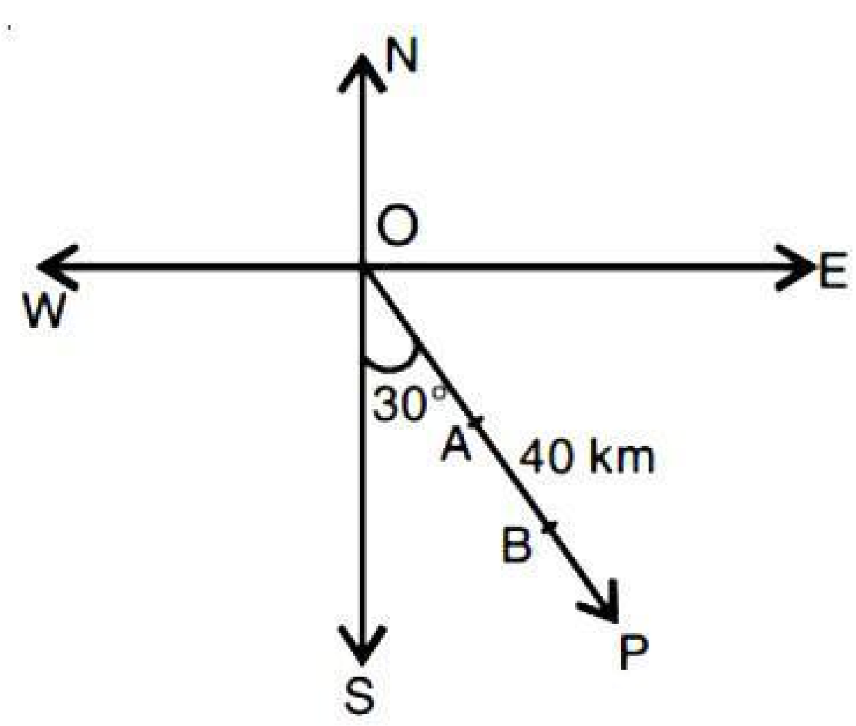
(*iv*) Two collinear vectors having the same magnitude are equal. (N.C.E.R.T.)

Answers

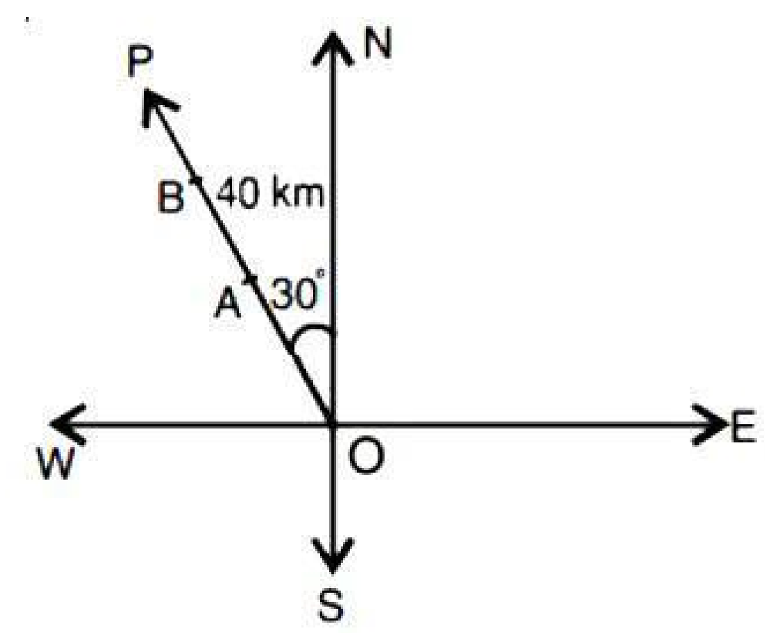
1. (*i*)



(*ii*)



(*iii*)



2. (*i*) Mass-scalar (*ii*) Angle-scalar

(*iii*) Distance-vector (*iv*) Power-scalar

(*v*) Charge-scalar (*vi*) Acceleration-vector

(*vii*) Volume-scalar (*viii*) Force-vector

(*ix*) Speed-scalar.

3. (*i*) Scalar (*ii*) Scalar (*iii*) Vector (*iv*) Vector (*v*) Scalar.

4. (*i*) $\vec{a}, \vec{b}; \vec{b}, \vec{c}; \vec{c}, \vec{d}; \vec{d}, \vec{a}$ (*ii*) $\vec{a}, \vec{c}; \vec{b}, \vec{d}$.

5. $-\frac{5}{2}\hat{i}+\frac{3\sqrt{3}}{2}\hat{j}$.

6. (*i*) True (*ii*) False (*iii*) False (*iv*) False.

10.3. ADDITION OF VECTORS

Let $\vec{a}$ and $\vec{b}$ be two given vectors.

Take any point O.

Let $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{AB} = \vec{b}$, so that the terminal point of $\vec{a}$ is the initial point of $\vec{b}$.

Then the vector $\overrightarrow{OB}$ is defined as the *vector sum* (*resultant*) of $\vec{a}$ and $\vec{b}$.

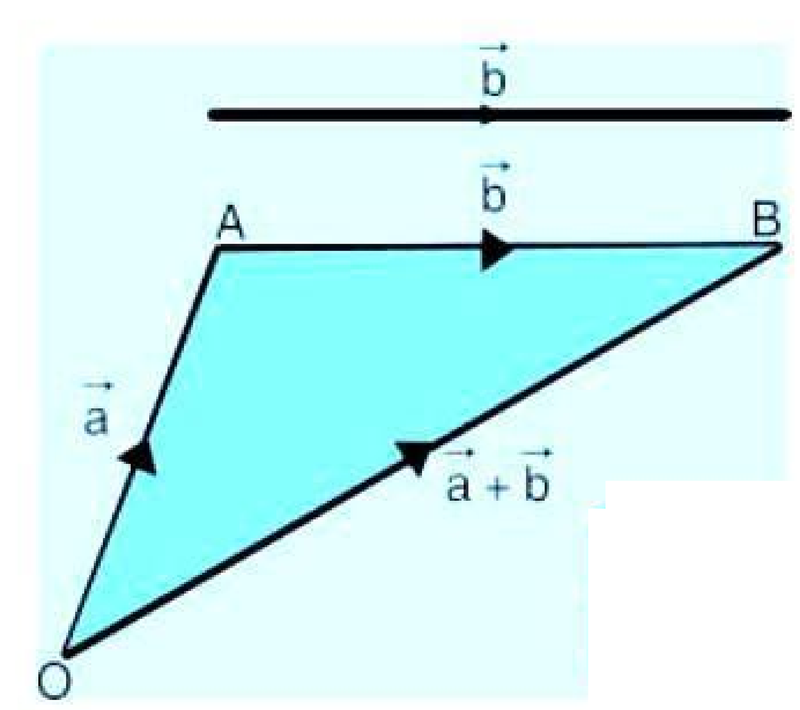


Fig.

Thus $\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB} = \vec{a} + \vec{b}$.

This is known as the **Triangle Law of vector addition,** which is stated as :

If two vectors are represented by the two sides of a triangle, taken in order, their sum is represented by the third side of the triangle, taken in the opposite order.

Theorem. *Vector addition is independent of the choice of the point O.*

Proof. Let O and O′ be any two points.

Further, let $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{AB} = \vec{b}$ and $\overrightarrow{O'A'} = \vec{a}$, $\overrightarrow{A'B'} = \vec{b}$.

Since OA = O′A′ and OA || O′A′ in the same sense,

∴ OO′A′A is a || gm.

∴ $\quad OO' = AA'$ and $OO' \parallel AA'$ ...(1)

Similarly, $\quad AA' = BB'$ and $AA' \parallel BB'$ ...(2)

From (1) and (2),

$$OO' = BB' \text{ and } OO' \parallel BB'.$$

∴ OO′B′B is a || gm.

∴ $\quad OB = O'B'$ and $OB \parallel O'B'$.

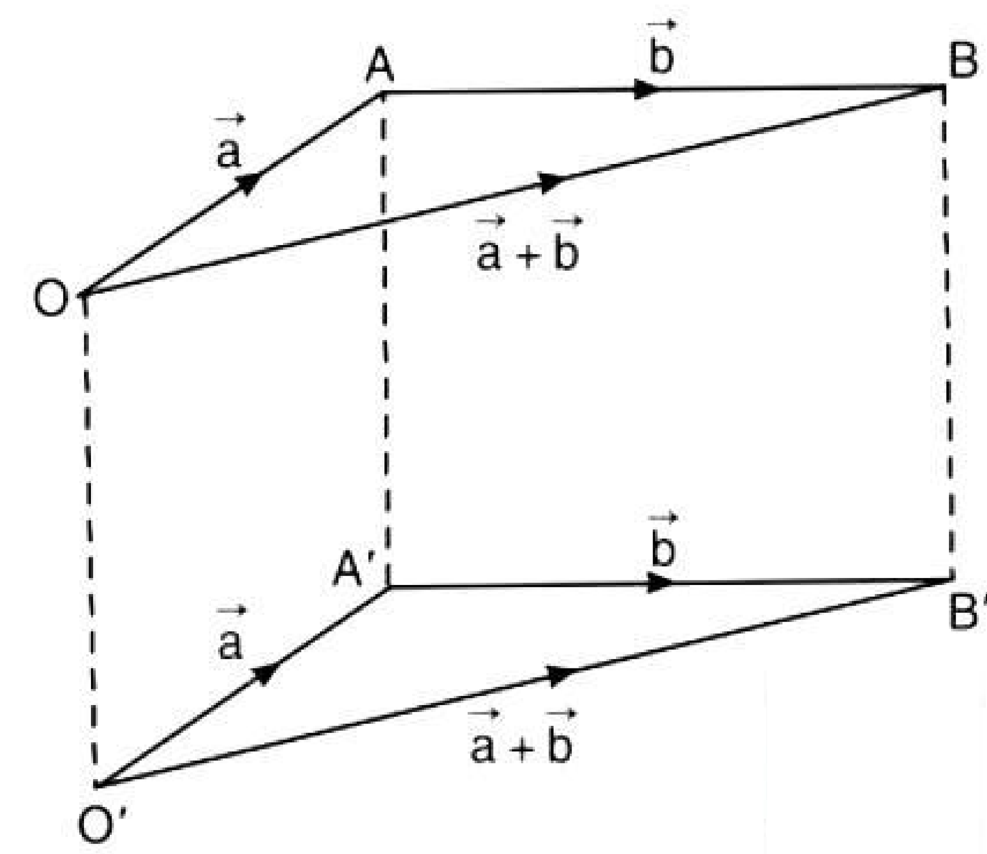


Fig.

Thus $\quad \overrightarrow{OB} = \overrightarrow{O'B'} = \vec{a} + \vec{b}$.

Hence, vector addition is independent of the choice of the point O.

10.3.1. POLYGON LAW OF VECTOR ADDITION

Let $\vec{a_1}, \vec{a_2}, ..., \vec{a_n}$ be n given vectors.

Take any point O.

Let $\quad \overrightarrow{OA_1} = \vec{a_1}$, $\overrightarrow{A_1A_2} = \vec{a_2}$, ..., $\overrightarrow{A_{n-1}A_n} = \vec{a_n}$.

Then $\vec{a_1} + \vec{a_2} + ... + \vec{a_n}$

$$= \overrightarrow{OA_1} + \overrightarrow{A_1A_2} + ... + \overrightarrow{A_{n-1}A_n}$$

$$= \overrightarrow{OA_2} + \overrightarrow{A_2A_3} + ... + \overrightarrow{A_{n-1}A_n}$$

$$= \overrightarrow{OA_3} + \overrightarrow{A_3A_4} + ... + \overrightarrow{A_{n-1}A_n}$$

$$= ....$$

$$= \overrightarrow{OA_n}.$$

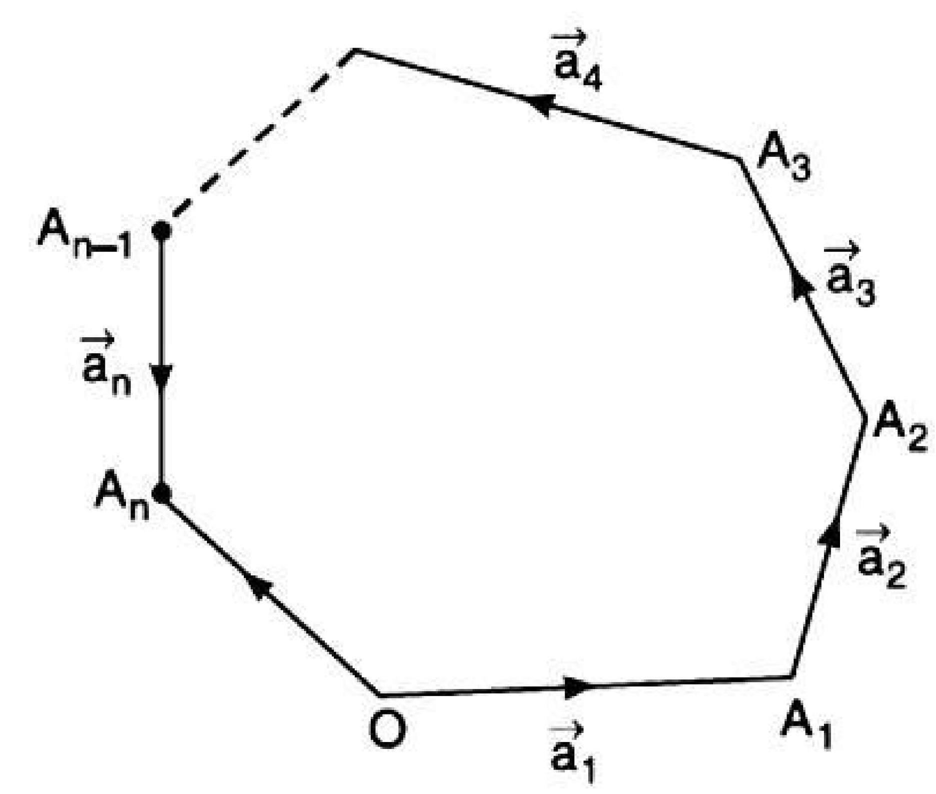


Fig.

Hence, the required sum is represented by $\overrightarrow{OA_n}$.

This is known as **Polygon Law of vector addition.**

If n vectors are represented by n sides of a polygon, taken in order, their sum is represented by the closing side of the polygon, taken in the opposite order.

10.4. PROPERTIES OF VECTOR ADDITION

Property I. Commutative Law. *If $\vec{a}$ and $\vec{b}$ be any two vectors,*

then $\vec{a}+\vec{b} = \vec{b}+\vec{a}$.

Proof. Let $\vec{a}$ and $\vec{b}$ be two vectors represented by $\overrightarrow{OA}$ and $\overrightarrow{AB}$ respectively, so that

$$\overrightarrow{OB} = \overrightarrow{OA}+\overrightarrow{AB} = \vec{a}+\vec{b} \quad ...(1)$$

Complete the || gm OABC.

Then $\overrightarrow{OC} = \overrightarrow{AB} = \vec{b}$ and $\overrightarrow{CB} = \overrightarrow{OA} = \vec{a}$.

$$\therefore \quad \overrightarrow{OB} = \overrightarrow{OC}+\overrightarrow{CB} = \vec{b}+\vec{a} \quad ...(2)$$

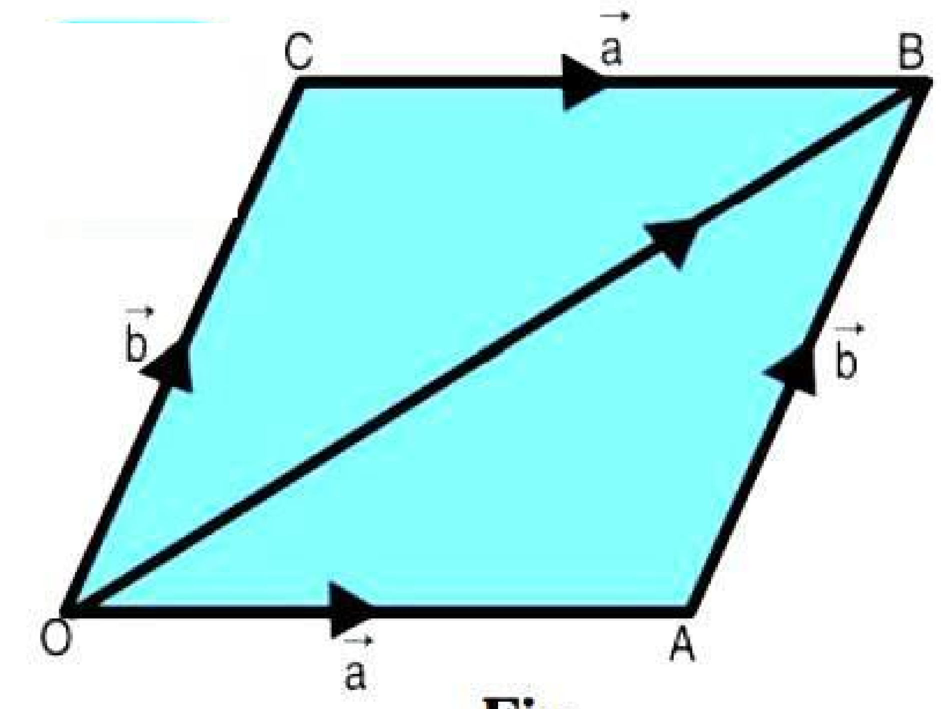


Fig.

Hence, $\vec{a}+\vec{b} = \vec{b}+\vec{a}$. [$\because$ *each* $= \overrightarrow{OB}$]

Property II. Associative Law. *If $\vec{a}, \vec{b}$ and $\vec{c}$ be any three vectors, then* $\vec{a}+(\vec{b}+\vec{c}) = (\vec{a}+\vec{b})+\vec{c}$.

Proof. With any point O as **origin,** let

$$\overrightarrow{OA} = \vec{a},\ \overrightarrow{AB} = \vec{b} \text{ and } \overrightarrow{BC} = \vec{c}.$$

Then $\vec{b}+\vec{c} = \overrightarrow{AB}+\overrightarrow{BC} = \overrightarrow{AC}$

$$\therefore \quad \vec{a}+(\vec{b}+\vec{c}) = \overrightarrow{OA}+\overrightarrow{AC} = \overrightarrow{OC} \quad ...(1)$$

Again, $\vec{a}+\vec{b} = \overrightarrow{OA}+\overrightarrow{AB} = \overrightarrow{OB}$

$$\therefore \quad (\vec{a}+\vec{b})+\vec{c} = \overrightarrow{OB}+\overrightarrow{BC} = \overrightarrow{OC} \quad ...(2)$$

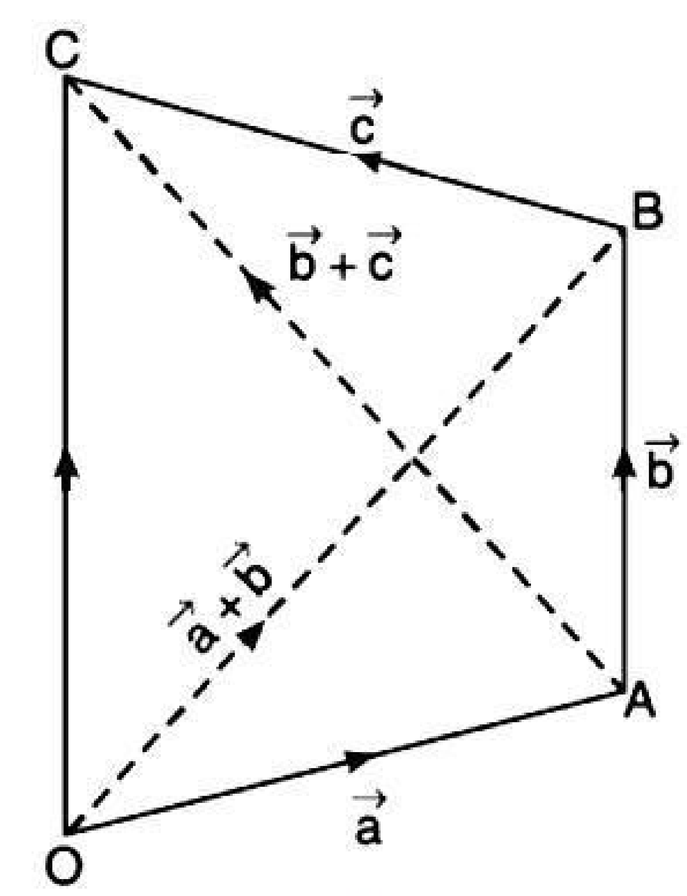


Fig.

Hence, $\vec{a}+(\vec{b}+\vec{c}) = (\vec{a}+\vec{b})+\vec{c}$. [$\because$ *each* $= \overrightarrow{OC}$]

KEY POINT

The sum of three vectors $\vec{a}, \vec{b}, \vec{c}$ is **independent** of the order in which they are added and is written as $\vec{a}+\vec{b}+\vec{c}$.

Property III. Zero vector is an additive identity. *If $\vec{a}$ be any vector and $\vec{0}$ is a zero vector, then* $\vec{a}+\vec{0} = \vec{a} = \vec{0}+\vec{a}$.

Proof. Let $\overrightarrow{OA} = \vec{a}$ and $\overrightarrow{AA} = \vec{0}$ $\therefore$ $\vec{a}+\vec{0} = \overrightarrow{OA}+\overrightarrow{AA} = \overrightarrow{OA} = \vec{a}$.

Similarly, $\vec{0}+\vec{a} = \vec{a}$.

Hence, $\vec{a}+\vec{0} = \vec{a} = \vec{0}+\vec{a}$ for every vector $\vec{a}$.

Property IV. Additive Inverse. *If $\vec{a}$ be any vector, then there exists a vector $-\vec{a}$ s.t. $\vec{a}+(-\vec{a})=\vec{0}=(-\vec{a})+\vec{a}$.*

Proof. Let $\overrightarrow{OA}=\vec{a}$. Then $\overrightarrow{AO}=-\vec{a}$. $\quad\therefore\ \vec{a}+(-\vec{a})=\overrightarrow{OA}+\overrightarrow{AO}=\overrightarrow{OO}=\vec{0}$.

Similarly, $(-\vec{a})+\vec{a}=\vec{0}$.

Hence, $\vec{a}+(-\vec{a})=\vec{0}=(-\vec{a})+\vec{a}$.

10.5. SUBTRACTION OF VECTORS

Definition

If $\vec{a}$ and $\vec{b}$ be any two vectors, then the subtraction of $\vec{b}$ from $\vec{a}$ is defined as the addition of $-\vec{b}$ to $\vec{a}$ and is written as $\vec{a}+(-\vec{b})=\vec{a}-\vec{b}$.

10.6. MULTIPLICATION OF A VECTOR BY A SCALAR (REAL NUMBER)

Definition

Let m be any scalar (a + ve real number) and $\vec{a}$ be any vector, then the product $m\vec{a}$ or $\vec{a}m$ of the vector $\vec{a}$ and the scalar m is a vector whose :

(i) magnitude is $|m|$ times that of $\vec{a}$,

(ii) support is the same or parallel to that of $\vec{a}$ and

(iii) sense is the same or opposite to that of $\vec{a}$ according as m is + ve or – ve.

AN IMPORTANT RESULT

If $\vec{a}$ and $\vec{b}$ are collinear or parallel vectors, then $\vec{b}=k\vec{a}$, where 'k' is some scalar. **| Remember**

10.7. PROPERTIES OF MULTIPLICATION OF A VECTOR BY A SCALAR (REAL NUMBER)

Property I. Associative Law.

If $\vec{a}$ be any vector and m, n are any scalars, then $m(n\vec{a})=mn(\vec{a})$.

Proof. Let $n\vec{a}=\vec{b}$. Thus $m(n\vec{a})=m\vec{b}$.

$\therefore\ \vec{b}$ represents a vector having modulus $|n|$ times that of $\vec{a}$

$\Rightarrow m\vec{b}$ represents a vector having modulus $|m|$ times that of $\vec{b}$

$\Rightarrow m\vec{b}$ represents a vector having modulus $|m|\,|n|$ times that of $\vec{a}$

$\Rightarrow m\vec{b}$ represents a vector having modulus $|mn|$ times that of $\vec{a}$ ... (1)

Also, $mn\ (\vec{a})$ represents a vector having modulus $|\,mn\,|$ times that of $\vec{a}$ ... (2)

$\therefore$ (1) and (2) $\Rightarrow m\ (n\ \vec{a})$ and $mn\ (\vec{a})$ represent the same vector.

Hence, $m\ (n\vec{a}) = mn\ (\vec{a})$.

Property II. Distributive Law.

If $\vec{a}$ is any vector and m, n are any scalars, then :

$$(m + n)\ \vec{a} = m\ \vec{a} + n\ \vec{a}.$$

Or

To prove that the scalar multiplication of vectors distributes the addition of vectors.

Proof. *When m and n are both + ve.*

Let $\overrightarrow{OA} = \vec{a}$.

Take two points A′, A″ on OA (produced) such that $\overrightarrow{OA'} = m\vec{a}$ and $\overrightarrow{A'A''} = n\vec{a}$.

Then $\overrightarrow{OA''} = \overrightarrow{OA'} + \overrightarrow{A'A''}$

$\therefore \quad (m + n)\ \vec{a} = m\vec{a} + n\vec{a}$.

Similarly, we can prove other cases.

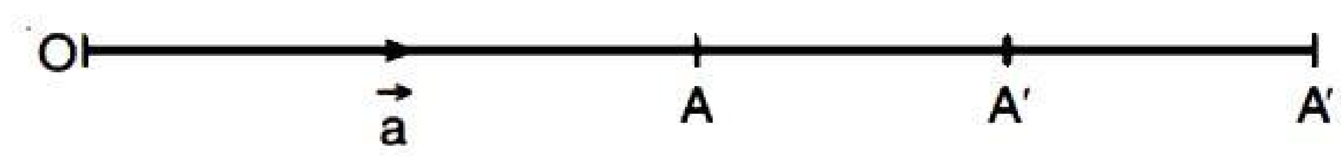


Fig.

Property III. Distributive Law. *If $\vec{a}$ and $\vec{b}$ be any two vectors and m is any scalar, then* $m(\vec{a} + \vec{b}) = m\vec{a} + m\vec{b}$.

Proof. Let $\overrightarrow{OA} = \vec{a}$ and $\overrightarrow{AB} = \vec{b}$ so that

$\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB} = \vec{a} + \vec{b}$ ... (1)

Case I. *When m > 0.*

Produce OA to A′ and OB to B′ so that $\overrightarrow{OA'} = m\vec{a}$.

Through A′, draw a st. line parallel to AB so as to meet OB (produced) in B′.

Δs OAB and OA′B′ are similar.

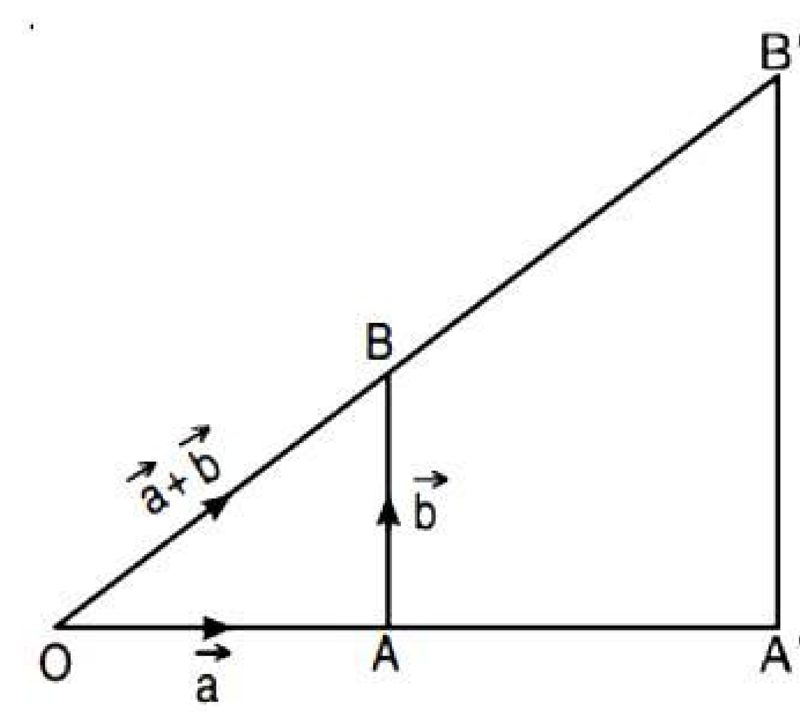


Fig.

$\therefore \quad \overrightarrow{A'B'} = m\ \overrightarrow{AB} = m\ \vec{b}$

and $\quad \overrightarrow{OB'} = m\ \overrightarrow{OB} = m(\vec{a} + \vec{b})$ ... (2) [*Using (1)*]

Also $\quad \overrightarrow{OB'} = \overrightarrow{OA'} + \overrightarrow{A'B'} = m\ \vec{a} + m\vec{b}$ ...(3)

From (2) and (3), $m(\vec{a} + \vec{b}) = m\ \vec{a} + m\vec{b}$.

Case II. *When m < 0.*

Produce AO to A′ and BO to B′ so that $\overrightarrow{OA'} = m\vec{a}$ and $\overrightarrow{OB'} = m\vec{b}$.

Through A′, draw a st. line parallel to AB so as to meet BO (produced) in B′.

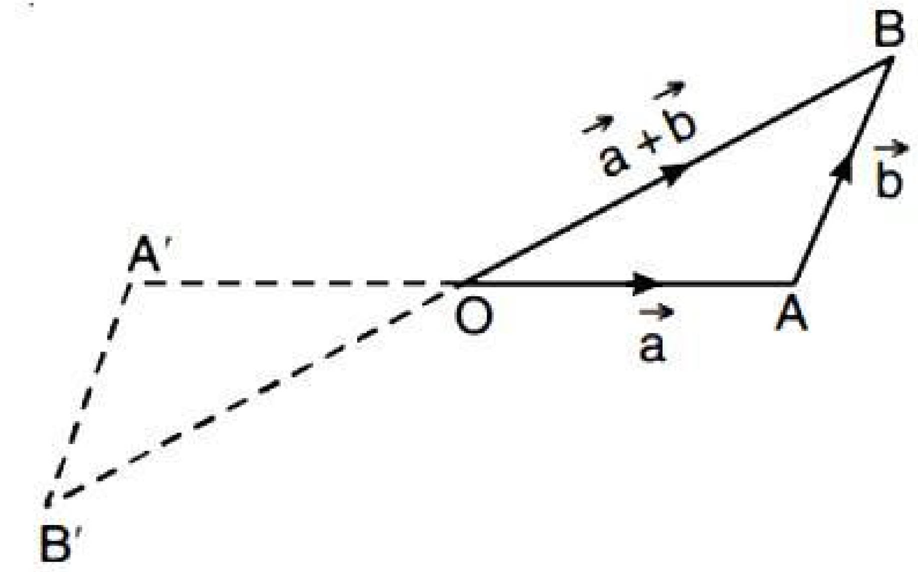


Fig.

$\therefore \Delta$s OAB and OA′B′ are similar.

$$\therefore \quad \overrightarrow{A'B'} = m\,\overrightarrow{AB} = m\,\vec{b}$$

and $$\overrightarrow{OB'} = m\,\overrightarrow{OB} = m\,(\vec{a}+\vec{b}) \qquad \ldots (4)$$

Also, $$\overrightarrow{OB'} = \overrightarrow{OA'} + \overrightarrow{A'B'} = m\,\overrightarrow{OA} + m\,\overrightarrow{AB} = m\,\vec{a} + m\,\vec{b} \qquad \ldots (5)$$

From (4) and (5), $m\,(\vec{a}+\vec{b}) = m\,\vec{a} + m\,\vec{b}$.

Property IV. Existence of Identity.

If $\vec{a}$ is any vector, then $1.\ \vec{a} = \vec{a}$, where 1 is the identity scalar.

Proof. $1.\ \vec{a}$ represents a vector having modulus 1 time that of $\vec{a}$ and which has the same support and the same sense. Hence, $1.\ \vec{a} = \vec{a}$.

ILLUSTRATIVE EXAMPLES

Example 1. Find the sum of the vectors :

$\vec{a} = \hat{i} - 2\hat{j} + \hat{k}, \vec{b} = -2\hat{i} + 4\hat{j} + 5\hat{k}$ and $\vec{c} = \hat{i} - 6\hat{j} - 7\hat{k}$.

(C.B.S.E. 2012)

Solution. Sum of the vectors $= \hat{a} + \hat{b} + \hat{c}$

$= (\hat{i} - 2\hat{j} + \hat{k}) + (-2\hat{i} + 4\hat{j} + 5\hat{k}) + (\hat{i} - 6\hat{j} - 7\hat{k})$

$= (\hat{i} - 2\hat{i} + \hat{i}) + (-2\hat{j} + 4\hat{j} - 6\hat{j}) + (\hat{k} + 5\hat{k} - 7\hat{k})$

$= -4\hat{j} - \hat{k}$.

Example 2. Prove that the resultant of the vectors represented by the sides $\overrightarrow{AB}$ and $\overrightarrow{AC}$ of a triangle ABC is $2\,\overrightarrow{AD}$, where D is the mid-point of [BC].

Solution. Complete the || gm. ABEC.

Since BE is equal and parallel to AC,

$$\therefore \quad \overrightarrow{AB} + \overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BE} = \overrightarrow{AE} \qquad \ldots (1)$$

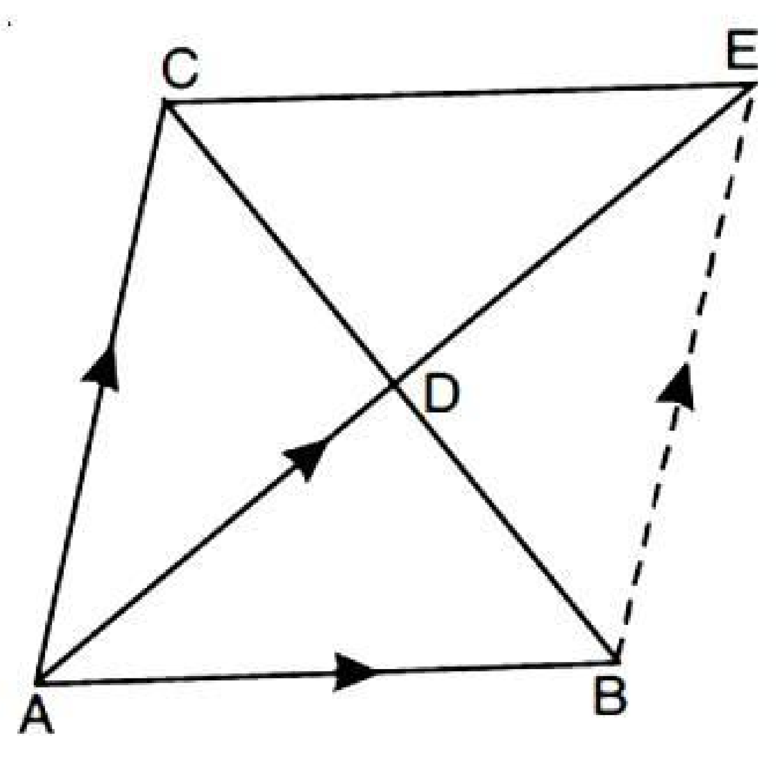


Fig.

Again, since the diagonals of a || gm. bisect each other,

$$\therefore \quad AE = 2AD.$$

Since $\overrightarrow{AE}$ and $\overrightarrow{AD}$ have the same direction

and $$|\overrightarrow{AE}| = 2\,|\overrightarrow{AD}|, \ \therefore\ \overrightarrow{AE} = 2\,\overrightarrow{AD}.$$

From (1), $\overrightarrow{AB} + \overrightarrow{AC} = 2\,\overrightarrow{AD}$, which proves the result.

Example 3. Show that the sum of three vectors determined by the medians of a triangle directed from the vertices is zero.

Solution. Let ABC be the given triangle and AD, BE, CF be the medians.

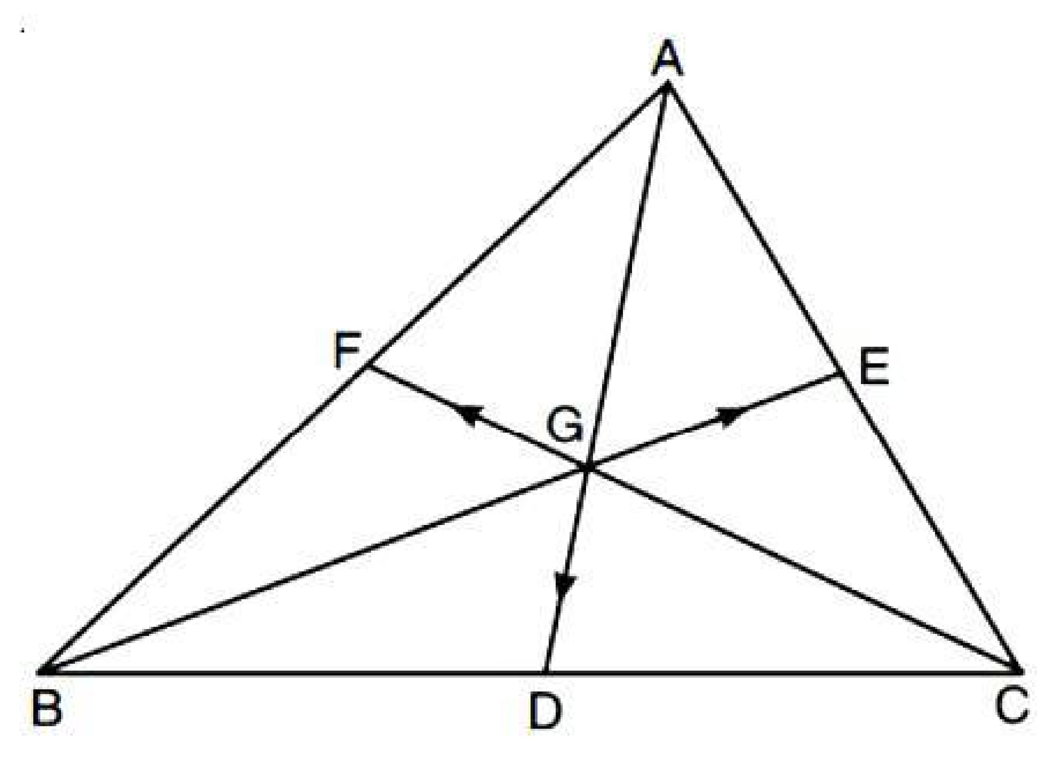


Fig.

Now $$\overrightarrow{AB} + \overrightarrow{AC} = 2\,\overrightarrow{AD} \qquad [Ex.\ 2]$$

$$\Rightarrow \quad \overrightarrow{AD} = \frac{1}{2}\overrightarrow{AB} + \frac{1}{2}\overrightarrow{AC} \qquad \ldots (1)$$

Similarly, $\overrightarrow{BE} = \frac{1}{2}\overrightarrow{BC} + \frac{1}{2}\overrightarrow{BA}$ ... (2)

and $\overrightarrow{CF} = \frac{1}{2}\overrightarrow{CA} + \frac{1}{2}\overrightarrow{CB}$ ... (3)

Adding (1), (2) and (3), $\overrightarrow{AD} + \overrightarrow{BE} + \overrightarrow{CF}$

$$= \frac{1}{2}\left(\overrightarrow{AB} + \overrightarrow{BA}\right) + \frac{1}{2}\left(\overrightarrow{BC} + \overrightarrow{CB}\right) + \frac{1}{2}\left(\overrightarrow{CA} + \overrightarrow{AC}\right)$$

$$= \frac{1}{2}(\vec{0}) + \frac{1}{2}(\vec{0}) + \frac{1}{2}(\vec{0}) = \vec{0}.$$

Hence, the result.

Example 4. In the figure, M is the mid-point of [AB] and N is the mid-point of [CD] and O is the mid-point of [MN]. Prove that :

(i) $\overrightarrow{\mathbf{OA}} + \overrightarrow{\mathbf{OB}} + \overrightarrow{\mathbf{OC}} + \overrightarrow{\mathbf{OD}} = \vec{\mathbf{0}}$

(ii) $\overrightarrow{\mathbf{BC}} + \overrightarrow{\mathbf{AD}} = 2\overrightarrow{\mathbf{MN}}$.

Solution.

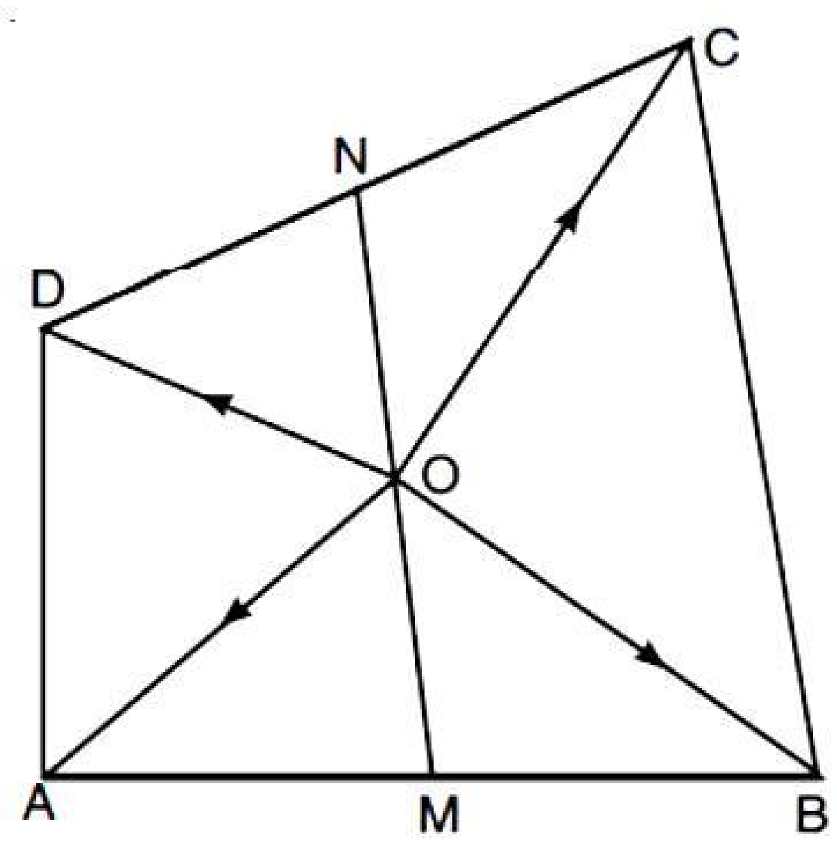


Fig.

(*i*) Since M is the mid-point of [AB],

$\therefore$ $\overrightarrow{OA} + \overrightarrow{OB} = 2\,\overrightarrow{OM}$ ...(1) [*Ex. 2*]

Similarly, $\overrightarrow{OC} + \overrightarrow{OD} = 2\,\overrightarrow{ON}$ ...(2),

where N is the mid-point of [CD].

Adding (1) and (2), we get :

$$\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} + \overrightarrow{OD} = 2\,(\overrightarrow{OM} + \overrightarrow{ON})$$

$$= 2\,(\vec{0})$$

[$\because$ *O is the mid-point of [MN]*]

$$= \vec{0}.$$

(*ii*) In quad. MNCB, $\overrightarrow{BM} + \overrightarrow{MN} + \overrightarrow{NC} = \overrightarrow{BC}$ ...(3)

In quad. ADNM, $\overrightarrow{AM} + \overrightarrow{MN} + \overrightarrow{ND} = \overrightarrow{AD}$ ...(4)

Adding (3) and (4), we get :

$$\overrightarrow{BC} + \overrightarrow{AD} = (\overrightarrow{BM} + \overrightarrow{AM}) + 2\overrightarrow{MN} + (\overrightarrow{NC} + \overrightarrow{ND})$$

$$= \vec{0} + 2\overrightarrow{MN} + \vec{0}.$$

Hence, $\overrightarrow{BC} + \overrightarrow{AD} = 2\overrightarrow{MN}$.

Example 5. ABCD is a parallelogram and P the intersection of the diagonals; O is any point. Show that :

$$\overrightarrow{\mathbf{OA}} + \overrightarrow{\mathbf{OB}} + \overrightarrow{\mathbf{OC}} + \overrightarrow{\mathbf{OD}} = 4\overrightarrow{\mathbf{OP}}.$$

Solution. Since the diagonals of a || gm bisect each other,

$\therefore$ P is the mid-point of [AC] and [BD] both.

$\therefore$ $\overrightarrow{OA} + \overrightarrow{OC} = 2\overrightarrow{OP}$ ...(1) [*Ex. 2*]

and $\overrightarrow{OB} + \overrightarrow{OD} = 2\overrightarrow{OP}$ ... (2) [*Ex. 2*]

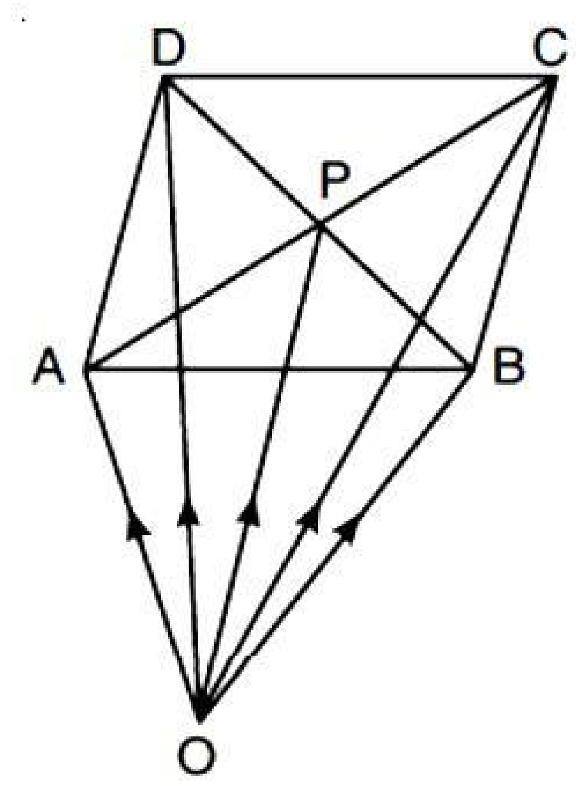


Fig.

Adding, $\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} + \overrightarrow{OD} = 4\overrightarrow{OP}$,

which proves the result.

Example 6. (a) What is the geometric significance of the relation $|\vec{\mathbf{a}} + \vec{\mathbf{b}}| = |\vec{\mathbf{a}} - \vec{\mathbf{b}}|$ **?**

(b) Prove geometrically that $|\vec{\mathbf{a}} + \vec{\mathbf{b}}| \le |\vec{\mathbf{a}}| + |\vec{\mathbf{b}}|$.

Solution. (*a*) Let $\overrightarrow{AB} = \vec{a}$ and $\overrightarrow{BC} = \vec{b}$.

Complete the || gm ABCD.

Join AC and BD.

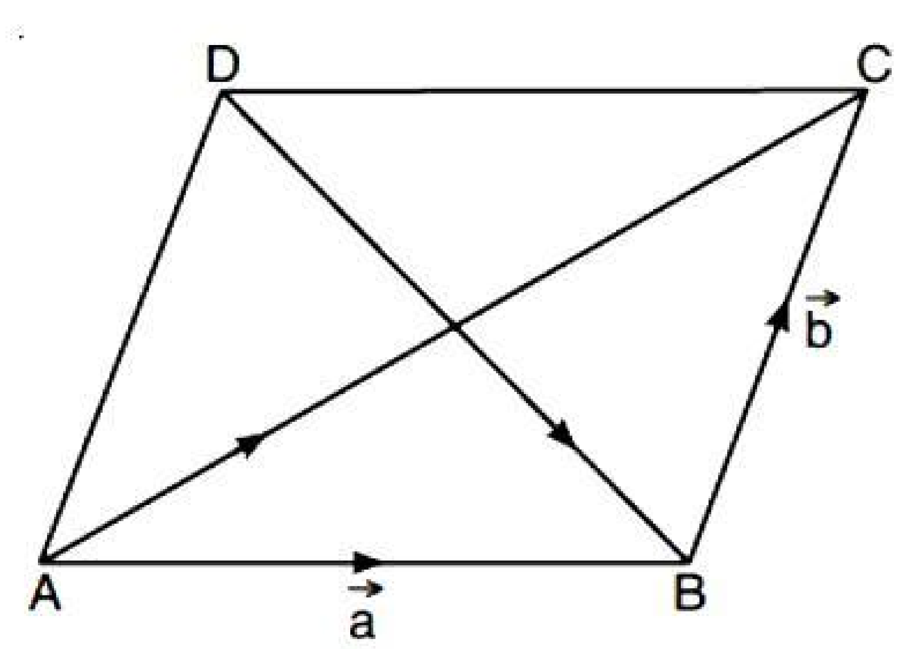


Fig.

Now $\qquad \vec{AC} = \vec{AB} + \vec{BC} = \vec{a} + \vec{b}$.

$\therefore \qquad |\vec{AC}| = |\vec{a} + \vec{b}|$.

Since $\qquad \vec{AD} = \vec{BC} = \vec{b}$,

$\therefore \qquad \vec{AD} + \vec{DB} = \vec{AB}$

$\Rightarrow \qquad \vec{DB} = \vec{AB} - \vec{AD} = \vec{a} - \vec{b}$.

$\therefore \qquad |\vec{DB}| = |\vec{a} - \vec{b}|$.

From $|\vec{a} + \vec{b}| = |\vec{a} - \vec{b}|$, we see that the two diagonals AC and BD of the || gm ABCD are equal and || gm reduces to rectangle, by geometry.

Hence, ABCD is rectangle and $\vec{a} \perp \vec{b}$.

(*b*) In ΔABC, AC < AB + BC

$\Rightarrow \qquad |\vec{a} + \vec{b}| < |\vec{a}| + |\vec{b}| \qquad \ldots (1)$

This result always holds if A, B, C are not collinear.

They will form a triangle and the sum of any two sides of a triangle is greater than the third side.

When A, B, C are collinear, then

$$AC = AB + BC$$

$\Rightarrow \qquad |\vec{AC}| = |\vec{AB}| + |\vec{BC}|$

$\Rightarrow \qquad |\vec{a} + \vec{b}| = |\vec{a}| + |\vec{b}| \qquad \ldots (2)$

Combining (1) and (2), we get :

$$|\vec{a} + \vec{b}| \le |\vec{a}| + |\vec{b}|.$$

Example 7. If the sum of two unit vectors is a unit vector, show that the magnitude of their difference is $\sqrt{3}$.

(Mizoram B. 2018; Karnataka B. 2013)

Solution. We have :

$$|\vec{a}| = |\vec{b}| = 1, |\vec{a} + \vec{b}| = 1.$$

Let $\qquad \vec{AB} = \vec{a}, \vec{BC} = \vec{b}$.

Then $\qquad \vec{AC} = \vec{AB} + \vec{BC} = \vec{a} + \vec{b}$

and $\qquad \vec{DB} = \vec{DA} + \vec{AB} = -\vec{AD} + \vec{AB}$

$$= \vec{AB} - \vec{AD} = \vec{a} - \vec{b}.$$

By the question, $|\vec{AB}| = |\vec{BC}| = |\vec{AC}| = 1$

$\Rightarrow$ ΔABC is equilateral, each of its angles being 60°

$\Rightarrow$ $\angle$DAB = 2 × 60° = 120° and $\angle$ADB = 30°.

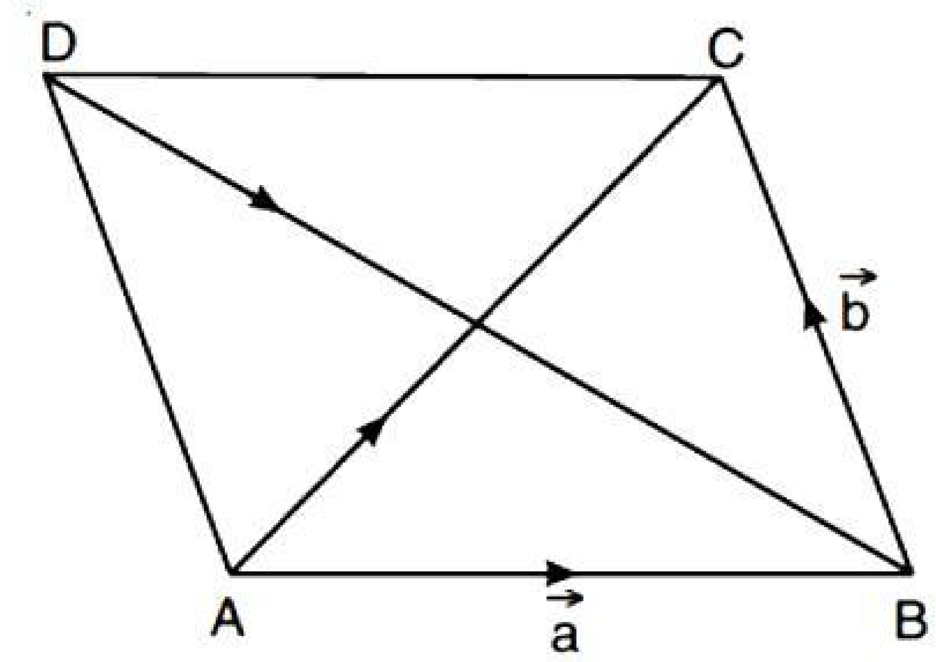


Fig.

In ΔADB, by *Sine Formula*, we have :

$$\frac{DB}{\sin \angle DAB} = \frac{AB}{\sin \angle ADB}$$

$$\Rightarrow \qquad \frac{|\vec{DB}|}{\sin 120^\circ} = \frac{|\vec{AB}|}{\sin 30^\circ}$$

$$\Rightarrow \qquad |\vec{DB}| = \frac{\sin 120^\circ}{\sin 30^\circ} |\vec{AB}|$$

$$= \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} \times 1 = \sqrt{3}.$$

Hence, $\qquad |\vec{a} - \vec{b}| = \sqrt{3}$.

EXERCISE 10 (b)

Fast Track Answer Type Questions

FTATQ

1. Find the sum of the following vectors :

(*i*) $\vec{a} = \hat{i} - 3\hat{k}, \ \vec{b} = 2\hat{j} - \hat{k}, \ \vec{c} = 2\hat{i} - 3\hat{j} + 2\hat{k}$

(C.B.S.E. 2012)

(*ii*) $\vec{a} = \hat{i} - 2\hat{j}, \ \vec{b} = -2\hat{i} - 3\hat{j}, \ \vec{c} = 2\hat{i} + 3\hat{k}$.

(C.B.S.E. 2012)

2. (*a*) Give an example of vectors $\vec{a}$ and $\vec{b}$ such that $|\vec{a}| = |\vec{b}|$ but $\vec{a} \ne \vec{b}$. ***(C.B.S.E. Sample Paper 2018)***

(*b*) For what value of '*a*' the vectors :

$2\hat{i} - 3\hat{j} + 4\hat{k}$ and $a\hat{i} + 6\hat{j} - 8\hat{k}$ are collinear ?

(C.B.S.E. 2011)

(*c*) (*i*) If $\vec{a} = -\vec{b}$, is it true that $|\vec{a}| = |\vec{b}|$?

(*ii*) If $|\vec{a}| = |\vec{b}|$, is it true that $\vec{a} = \pm\vec{b}$?

(*iii*) If $|\vec{a}| = |\vec{b}|$, is it true that $\vec{a} = \vec{b}$?

(*iv*) $k\vec{a} = \vec{0}$ gives rise to what alternatives for k and $\vec{a}$?

(*d*) If $\vec{a} = 2\hat{i} + 3\hat{j}$ and $\vec{b} = 3\hat{i} + 4\hat{j}$, find the magnitude of $\vec{a} + \vec{b}$.

3. For two non-zero vectors $\vec{a}$ and $\vec{b}$, write when $|\vec{a} + \vec{b}| = |\vec{a}| + |\vec{b}|$ holds.

4. Vectors drawn from the origin to the points A, B and C are respectively $\vec{a}, \vec{b}$ and $4\vec{a} - 3\vec{b}$. Find $\overrightarrow{AC}$ and $\overrightarrow{BC}$.

Very Short Answer Type Questions

VSATQ

5. Give a condition that the three vectors $\vec{a}, \vec{b}$ and $\vec{c}$ form the three sides of a triangle. What are other possibilities ?

6. D, E, F are mid-points of the sides of the triangle ABC. Show that for any point O, the system of concurrent forces represented by $\overrightarrow{OA}, \overrightarrow{OB}, \overrightarrow{OC}$ is equivalent to the system represented by $\overrightarrow{OD}, \overrightarrow{OE}, \overrightarrow{OF}$.

7. In pentagon ABCDE, prove that :

$$\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DE} + \overrightarrow{EA} = \vec{0}.$$

Short Answer Type Questions

SATQ

8. ABCD is a parallelogram and AC, BD are its diagonals. Show that :

$$\overrightarrow{AC} + \overrightarrow{BD} = 2\overrightarrow{BC}\ ;\ \overrightarrow{AC} - \overrightarrow{BD} = 2\overrightarrow{AB}.$$

9. ABCDEF is a regular hexagon. Show that :

(*i*) $\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} + \overrightarrow{OD} + \overrightarrow{OE} + \overrightarrow{OF} = \vec{0}$

(*ii*) $\overrightarrow{AB} + \overrightarrow{AC} + \overrightarrow{AD} + \overrightarrow{AE} + \overrightarrow{AF} = 3\overrightarrow{AD}$

(*iii*) $\overrightarrow{AB} + \overrightarrow{AC} + \overrightarrow{AD} + \overrightarrow{AE} + \overrightarrow{AF} = 6\overrightarrow{AO}$,

where O is the centre of the hexagon.

10. Prove that $|\vec{a}| - |\vec{b}| \le |\vec{a} - \vec{b}|$.

11. If $\vec{a} + 5\vec{b} = \vec{c}$ and $\vec{a} - 7\vec{b} = 2\vec{c}$, then show that $\vec{a}$ has the same direction as that of $\vec{c}$ and opposite direction to that of $\vec{b}$.

Answers

1. (*i*) $3\hat{i} - \hat{j} - 2\hat{k}$ (*ii*) $\hat{i} - 5\hat{j} + 3\hat{k}$.

2. $\vec{a} = 2\hat{i} + \hat{j} + \hat{k}$ and $\vec{b} = \hat{i} + 2\hat{j} + \hat{k}$.

(*b*) $a = -4$

(*c*) (*i*) Yes (*ii*) Yes (*iii*) No (*iv*) Either $k = 0$ or $\vec{a} = \vec{0}$

(*d*) $\sqrt{74}$.

3. When $\vec{a}, \vec{b}$ and $\vec{a} + \vec{b}$ are collinear.

4. $\overrightarrow{AC} = 3(\vec{a} - \vec{b})$; $\overrightarrow{BC} = 4(\vec{a} - \vec{b})$.

5. $\vec{a} + \vec{b} + \vec{c} = \vec{0}$. Other possibilities are ;

$\vec{a} = \vec{b} + \vec{c}$, $\vec{b} = \vec{a} + \vec{c}$ and $\vec{c} = \vec{a} + \vec{b}$.

Hints to Selected Questions

11. (*i*) Eliminate $\vec{b}$ (*ii*) Eliminate $\vec{c}$.

10.8. (A) DEFINITIONS

(*i*) **Linear Combination (L.C.).** A vector $\vec{r}$ is said to be a linear combination of the vectors $\vec{a_1}, \vec{a_2}, \vec{a_3}, \ldots$ if there exist scalars $x_1, x_2, x_3, \ldots\ldots$, such that $\vec{r} = x_1\vec{a_1} + x_2\vec{a_2} + x_3\vec{a_3} + \ldots$

(*ii*) **Linearly Dependent (L.D.).** A system of vectors $\vec{a_1}, \vec{a_2}, ..., \vec{a_n}$ is said to be linearly dependent if there exist scalars $x_1, x_2, ..., x_n$ (not all zero) such that $x_1\vec{a_1} + x_2\vec{a_2} + ... + x_n\vec{a_n} = \vec{0}$.

(*iii*) **Linearly Independent (L.I.).** If the system of vectors is not linearly dependent, then it is said to be linearly independent and in that case $x_1 = 0, x_2 = 0, ...$ *i.e.* all the scalars are zero. Thus :

A system of vectors $\vec{a_1}, \vec{a_2}, ..., \vec{a_n}$ is said to be linearly independent if there exist scalars $x_1, x_2, ..., x_n$ (all zero) such that $x_1\vec{a_1} + x_2\vec{a_2} + ... + x_n\vec{a_n} = \vec{0}$.

KEY POINT

1. $\vec{r}$ and $x\vec{r}$ are collinear vectors, where x is a scalar.
2. $x\vec{a} + y\vec{b}$ represents a vector coplanar with vectors $\vec{a}$ and $\vec{b}$, where x, y are scalars.

(B) THEOREMS

Theorem I. *If $\vec{a}, \vec{b}$ be two non-zero, non-collinear vectors and x, y are two scalars such that $x\vec{a} + y\vec{b} = \vec{0}$, then $x = 0, y = 0$.*

Proof. Given : $x\vec{a} + y\vec{b} = \vec{0}$ ... (1)

Suppose that $x \neq 0$.

Then (1) can be written as $x\vec{a} = -y\vec{b}$

$$\Rightarrow \quad \vec{a} = -\frac{y}{x}\vec{b} \qquad ... (2)$$

Since $\frac{y}{x}$ is a scalar, [$\because$ *x and y are scalars*]

$\therefore$ (2) expresses $\vec{a}$ as a product of $\vec{b}$ by a scalar $\Rightarrow$ $\vec{a}$ and $\vec{b}$ are collinear,

which is a contradiction because $\vec{a}$ and $\vec{b}$ are given to be non-collinear.

Thus our supposition *i.e.* $x \neq 0$ is wrong.

Hence, $x = 0$. Similarly, $y = 0$.

Theorem II. *If $\vec{a}, \vec{b}, \vec{c}$ be three non-zero, non-coplanar vectors and x, y, z are three scalars such that $x\vec{a} + y\vec{b} + z\vec{c} = \vec{0}$, then $x = 0, y = 0, z = 0$.*

Proof. Given : $x\vec{a} + y\vec{b} + z\vec{c} = \vec{0}$ ... (1)

Suppose that $x \neq 0$. Then (1) can be written as $x\vec{a} = -y\vec{b} - z\vec{c}$

$$\Rightarrow \quad \vec{a} = -\frac{y}{x}\vec{b} - \frac{z}{x}\vec{c} \qquad ... (2)$$

Since $\frac{y}{x}$ and $\frac{z}{x}$ are scalars, [$\because$ *x, y and z are scalars*]

$\therefore$ (2) expresses $\vec{a}$ as a linear combination of $\vec{b}$ and $\vec{c}$ $\Rightarrow$ $\vec{a}$ is coplanar with $\vec{b}$ and $\vec{c}$, which is a contradiction because $\vec{a}, \vec{b}$ and $\vec{c}$ are given to be non-coplanar.

Thus our supposition *i.e.* $x \neq 0$ is wrong.

Hence, $x = 0$.

Similarly, $y = 0$ and $z = 0$.

Theorem III. Resolution of a vector in terms of coplanar vectors.

If $\vec{a}, \vec{b}$ *be two given non-collinear vectors, then every vector* $\vec{r}$ *can be expressed uniquely as a linear combination* $x\vec{a} + y\vec{b}$ *; x, y being scalars.*

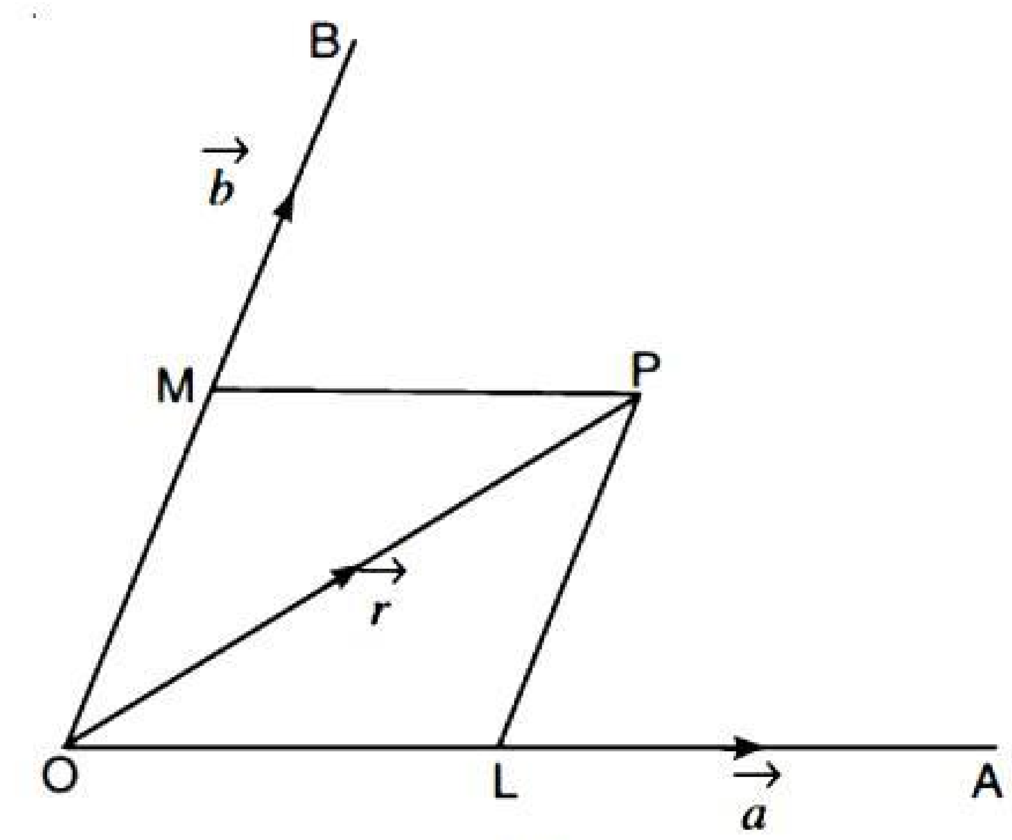


Fig.

Proof. Let $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OB} = \vec{b}$, where O is any point.

Also, let $\overrightarrow{OP} = \vec{r}$.

Clearly, OA, OB and OP, being intersecting at O, are coplanar.

Through P, draw lines parallel to OB and OA meeting OA and OB in L, M respectively.

We have : $\overrightarrow{OP} = \overrightarrow{OL} + \overrightarrow{LP}$

$= \overrightarrow{OL} + \overrightarrow{OM}$ ...(1) [$\because$ *LP = OM and LP || OM*]

Now $\overrightarrow{OL}$ and $\overrightarrow{OA}$ are collinear vectors, $\therefore$ $\overrightarrow{OL} = x\overrightarrow{OA} = x\vec{a}$, where x is a scalar.

Similarly, $\overrightarrow{OM} = y\vec{b}$, where y is a scalar.

$\therefore$ From (1), $\overrightarrow{OP} = x\vec{a} + y\vec{b}$. Hence, $\vec{r} = x\vec{a} + y\vec{b}$.

Uniqueness. If possible, let $r = x\vec{a} + y\vec{b}$ and $\vec{r} = x'\vec{a} + y'\vec{b}$ be two different ways of representing $\vec{r}$.

Then $x\vec{a} + y\vec{b} = x'\vec{a} + y'\vec{b}$

$\Rightarrow$ $(x - x')\vec{a} + (y - y')\vec{b} = \vec{0}$.

Now $\vec{a}$ and $\vec{b}$ are non-collinear vectors.

$\therefore$ $x - x' = 0$ and $y - y' = 0$ [*Th. I*]

$\Rightarrow$ $x' = x$ and $y' = y$.

Hence, the uniqueness is established.

Theorem IV. Non-coplanar Vectors.

If $\vec{a}, \vec{b}, \vec{c}$ *be three given non-coplanar vectors, then any vector* $\vec{r}$ *can be expressed uniquely as a linear combination* $x\vec{a} + y\vec{b} + z\vec{c}$ *; x, y, z being scalars.*

Proof. Let $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OB} = \vec{b}$ and $\overrightarrow{OC} = \vec{c}$, where O is any point.

Also, let $\overrightarrow{OP} = \vec{r}$.

Here the lines OA, OB, OC are not coplanar.

On OP as diagonal, construct a parallelopiped whose three coterminous edges OA′, OB′, OC′ are along OA, OB, OC respectively.

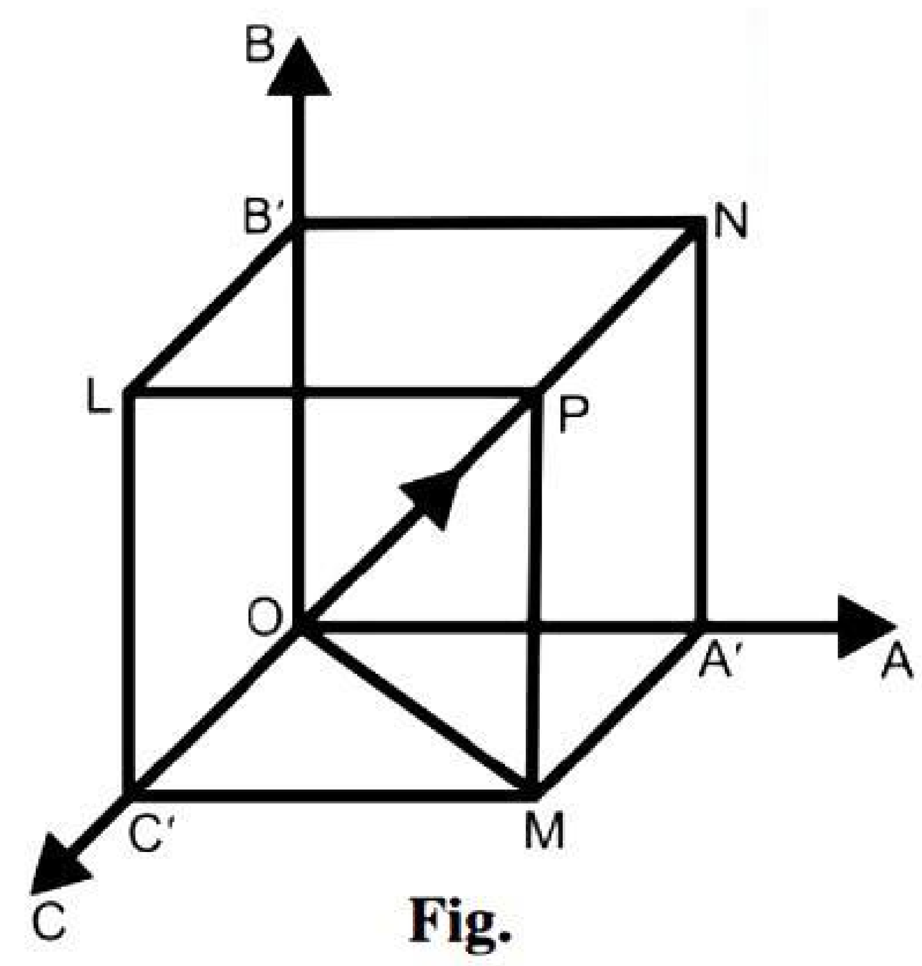


Fig.

Since $\overrightarrow{OA'}$ is collinear with $\overrightarrow{OA}$,

$\therefore\ \overrightarrow{OA'} = x\ \overrightarrow{OA} = x\ \vec{a}$, where x is a scalar.

Similarly, $\overrightarrow{OB'} = y\,\vec{b}$ and $\overrightarrow{OC'} = z\,\vec{c}$, where y, z are scalars.

Now $$\overrightarrow{OP} = \overrightarrow{OM} + \overrightarrow{MP} = \left(\overrightarrow{OC'} + \overrightarrow{C'M}\right) + \overrightarrow{MP}$$

$$= \overrightarrow{OC'} + \overrightarrow{OA'} + \overrightarrow{OB'}$$

$$[\because \overrightarrow{C'M} = \overrightarrow{OA'} \text{ and } \overrightarrow{MP} = \overrightarrow{A'N} = \overrightarrow{OB'}]$$

$$= \overrightarrow{OA'} + \overrightarrow{OB'} + \overrightarrow{OC'}.$$

$$\therefore\quad \vec{r} = x\,\vec{a} + y\,\vec{b} + z\,\vec{c}.$$

Hence, $\vec{r}$ can be expressed as a linear combination of $\vec{a}, \vec{b}$ and $\vec{c}$.

Uniqueness. If possible, let $\vec{r} = x\vec{a} + y\vec{b} + z\vec{c}$ and $\vec{r} = x'\vec{a} + y'\vec{b} + z'\vec{c}$

be two different ways of representing $\vec{r}$.

Then $x\vec{a} + y\vec{b} + z\vec{c} = x'\vec{a} + y'\vec{b} + z'\vec{c} \Rightarrow (x - x')\ \vec{a} + (y - y')\ \vec{b} + (z - z')\ \vec{c} = \vec{0}$.

Now $\vec{a}$, $\vec{b}$ and $\vec{c}$ are non-coplanar vectors, $\therefore\ x - x' = 0,\ y - y' = 0$ and $z - z' = 0$ [*Th. II*]
$\Rightarrow x' = x,\ y' = y$ and $z' = z$.
Hence, the uniqueness is established.

Note. In the relation $\vec{r} = x\vec{a} + y\vec{b} + z\vec{c}$,

$\vec{r}$ is called the **resultant** of the three vectors $x\vec{a}, y\vec{b}, z\vec{c}$ and $x\vec{a}, y\vec{b}, z\vec{c}$ are called the **components** of the vector $\vec{r}$.

10.9. UNIT VECTORS

(a) (i) The unit vectors $\hat{i}, \hat{j}$.

We know that if the sum of two vectors $\vec{a}$ and $\vec{b}$ is a third vector $\vec{c}$, then $\vec{a}$ and $\vec{b}$ are called **components** of $\vec{c}$.

A given vector can be resolved into components in various ways. In general, we resolve a given vector along the co-ordinate axes.

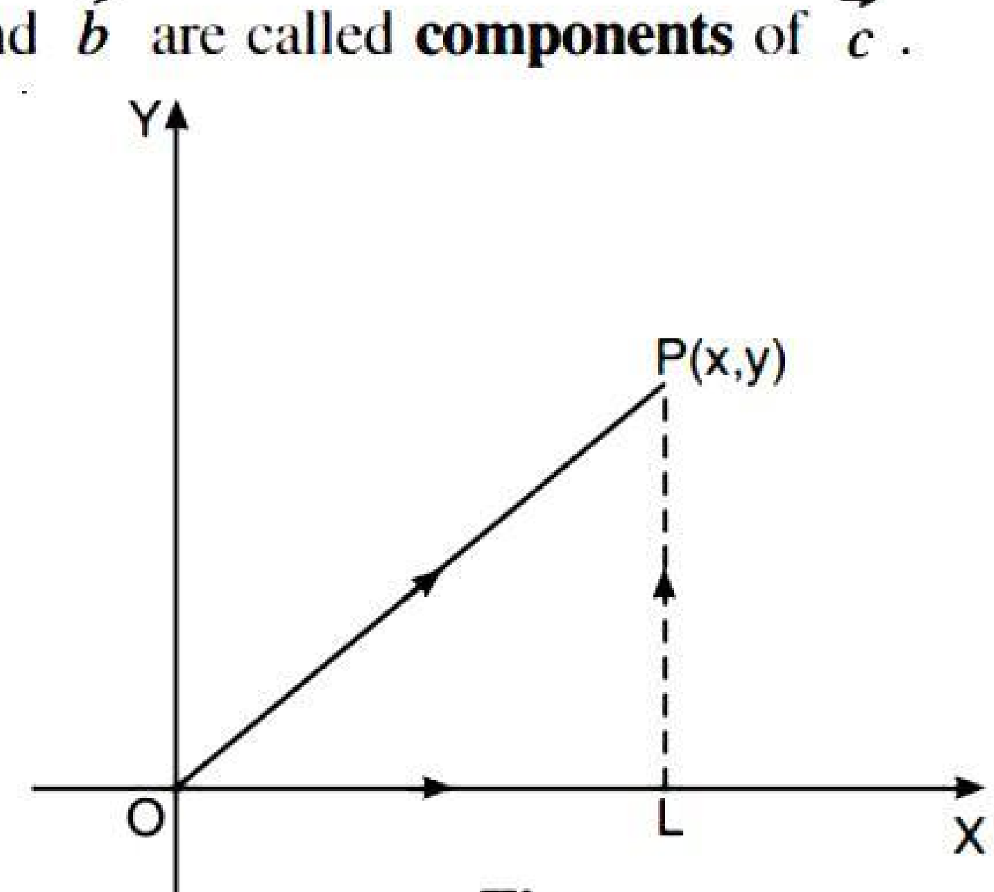


Fig.

Let $\hat{i}$ and $\hat{j}$ be the unit vectors along OX and OY respectively.

Let P (x, y) be any point in the XOY plane.

Then $\overrightarrow{OP} = \overrightarrow{OL} + \overrightarrow{LP}$ ... (1),

where PL $\perp$ OX.

Since the magnitude of $\overrightarrow{OL}$ is x and direction is along x-axis, which is given by $\hat{i}$, similarly, magnitude of $\overrightarrow{LP}$ is y and direction is along y-axis, which is given by $\hat{j}$.

From (1), $\vec{r} = x\hat{i} + y\hat{j}$, where $\overrightarrow{OP} = \vec{r}$.

(ii) Distance Formula.

If P (x_1, y_1) and Q (x_2, y_2) be two points, we have :

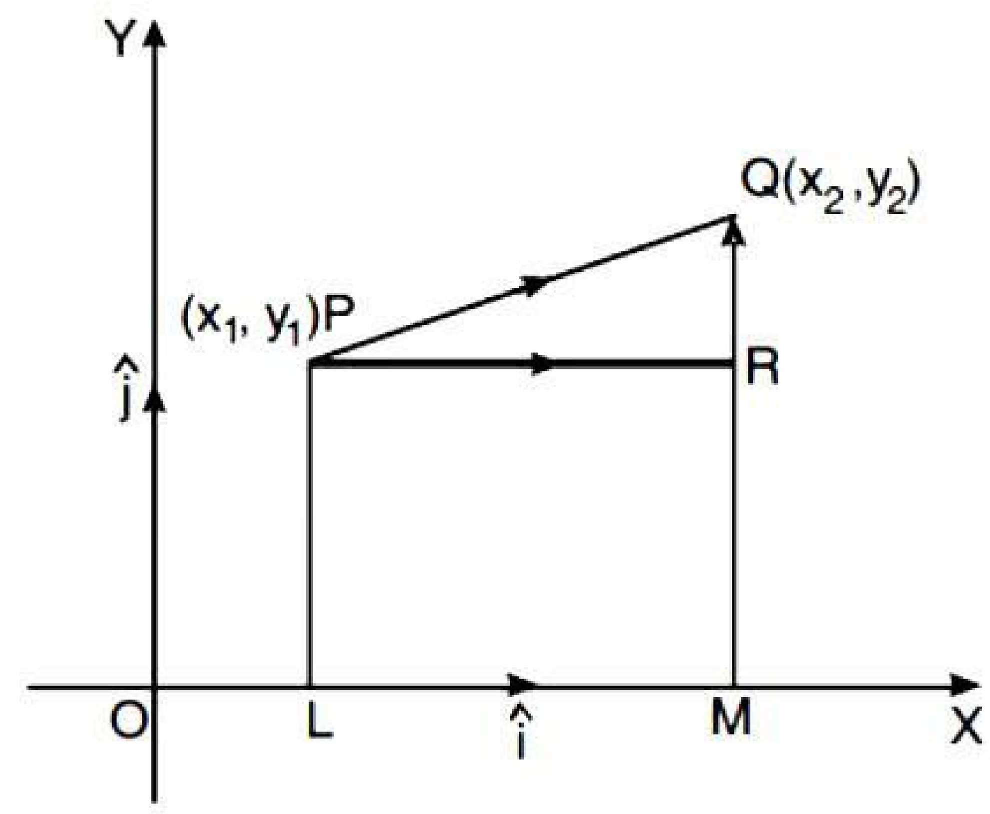


Fig.

$$\overrightarrow{PQ} = \overrightarrow{PR} + \overrightarrow{RQ}$$

$$= (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j}$$

$$[\because |\overrightarrow{PR}| = x_2 - x_1 \text{ and } |\overrightarrow{RQ}| = y_2 - y_1]$$

$$\therefore \quad |\overrightarrow{PQ}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2},$$

which is the required distance formula.

(b) The unit vectors $\hat{i}, \hat{j}, \hat{k}$.

(*i*) Consider three mutually perpendicular st. lines OX, OY, OZ with the right-handed rotation. Take these lines as co-ordinate axes with O as the origin.

Through O, draw a vector $\overrightarrow{OP} = \vec{r}$.

On OP as diagonal, construct a rectangular parallelopiped whose three coterminous edges OA, OB, OC are along OX, OY, OZ respectively.

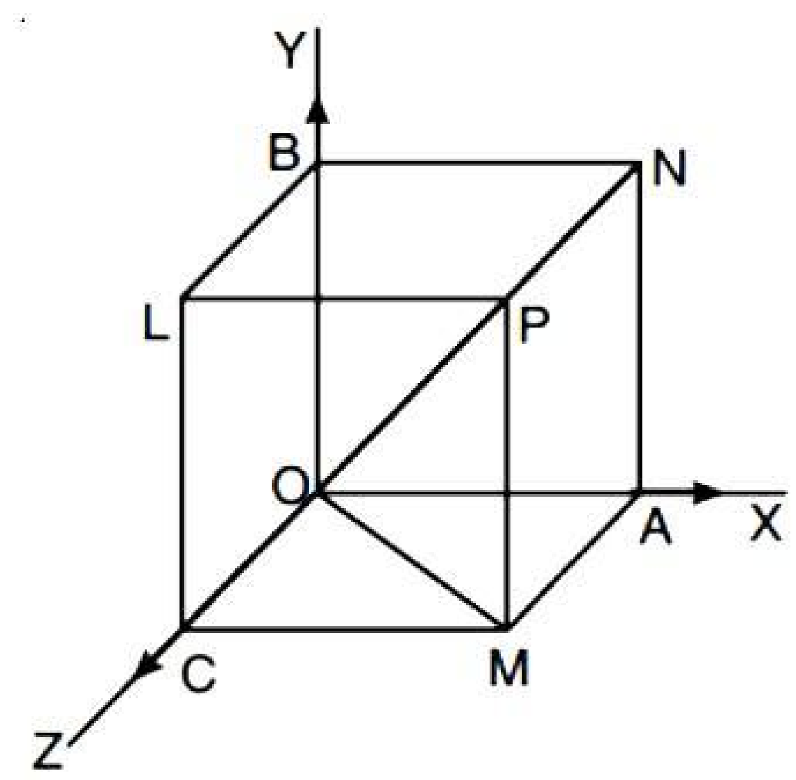


Fig.

Let $\hat{i}, \hat{j}, \hat{k}$ be the unit vectors along OX, OY, OZ respectively.

If $\quad OA = x, OB = y, OC = z,$

then $\quad \overrightarrow{OA} = x\hat{i}, \overrightarrow{OB} = y\hat{j}$ and $\overrightarrow{OC} = z\hat{k}$ ...(1)

Now
$$\overrightarrow{OP} = \overrightarrow{OM} + \overrightarrow{MP} = (\overrightarrow{OC} + \overrightarrow{CM}) + \overrightarrow{MP}$$

$$= \overrightarrow{OC} + \overrightarrow{OA} + \overrightarrow{OB} \qquad [\because \overrightarrow{CM} = \overrightarrow{OA} \text{ and } \overrightarrow{MP} = \overrightarrow{AN} = \overrightarrow{OB}]$$

$$= \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}.$$

$$\therefore \quad \vec{r} = x\hat{i} + y\hat{j} + z\hat{k} \qquad \text{... (2) [Using (1)]}$$

This is called rectangular resolution of $\vec{r}$, (x, y, z) are called **co-ordinates** of P referred to OX, OY, OZ and $x\hat{i}, y\hat{j}, z\hat{k}$ are called **resolved parts** of $\vec{r}$ in the directions of $\hat{i}, \hat{j}, \hat{k}$ respectively.

(*ii*) Further, if α, β, γ be the angles which OP makes with OX, OY, OZ respectively,

then $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are called the **direction-cosines** of OP, which are generally denoted by l, m, n respectively.

If OP = r, then $x = lr$, $y = mr$, $z = nr$ ...(3)

From (2), $\vec{r} = lr\hat{i} + mr\hat{j} + nr\hat{k}$

$$\Rightarrow \quad \frac{\vec{r}}{r} = l\hat{i} + m\hat{j} + n\hat{k} \qquad \text{i.e. } \hat{r} = l\hat{i} + m\hat{j} + n\hat{k} \qquad ...(4)$$

Hence, the direction-cosines of $\vec{r}$ are the co-efficients of $\hat{i}$, $\hat{j}$, $\hat{k}$ in the rectangular resolution of unit vector $\hat{r}$.

(*iii*) Also,

$$\begin{aligned} OP^2 &= OM^2 + MP^2 = (OC^2 + CM^2) + MP^2 \\ &= OC^2 + OA^2 + OB^2 \qquad [\because CM = OA \text{ and } MP = OB] \\ &= OA^2 + OB^2 + OC^2 \end{aligned}$$

i.e. $r^2 = x^2 + y^2 + z^2 \quad \Rightarrow \quad |\vec{r}|^2 = x^2 + y^2 + z^2$ ...(5)

Hence, the square of the modulus of a vector is equal to the sum of the squares of its rectangular components.

(*iv*) Also, from $r^2 = x^2 + y^2 + z^2$, on dividing both sides by r^2, we get :

$$1 = \frac{x^2}{r^2} + \frac{y^2}{r^2} + \frac{z^2}{r^2} = l^2 + m^2 + n^2. \qquad [\text{Using } (3)]$$

Hence, the sum of the squares of the direction-cosines is unity.

(*v*) Distance between two points.

If P_1, P_2 be two points such that :

$$\overrightarrow{OP_1} = \vec{r_1} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k} \text{ and } \overrightarrow{OP_2} = \vec{r_2} = x_2\hat{i} + y_2\hat{j} + z_2\hat{k},$$

where O is the origin,

then

$$\begin{aligned} \overrightarrow{P_1P_2} &= \overrightarrow{P_1O} + \overrightarrow{OP_2} = \overrightarrow{OP_2} - \overrightarrow{OP_1} \\ &= (x_2\hat{i} + y_2\hat{j} + z_2\hat{k}) - (x_1\hat{i} + y_1\hat{j} + z_1\hat{k}) \\ &= (x_2 - x_1)\,\hat{i} + (y_2 - y_1)\,\hat{j} + (z_2 - z_1)\,\hat{k}. \end{aligned}$$

$$\therefore \quad |\overrightarrow{P_1P_2}|^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2.$$

Hence,

$$|\overrightarrow{\mathbf{P_1P_2}}| = \boldsymbol{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}}.$$

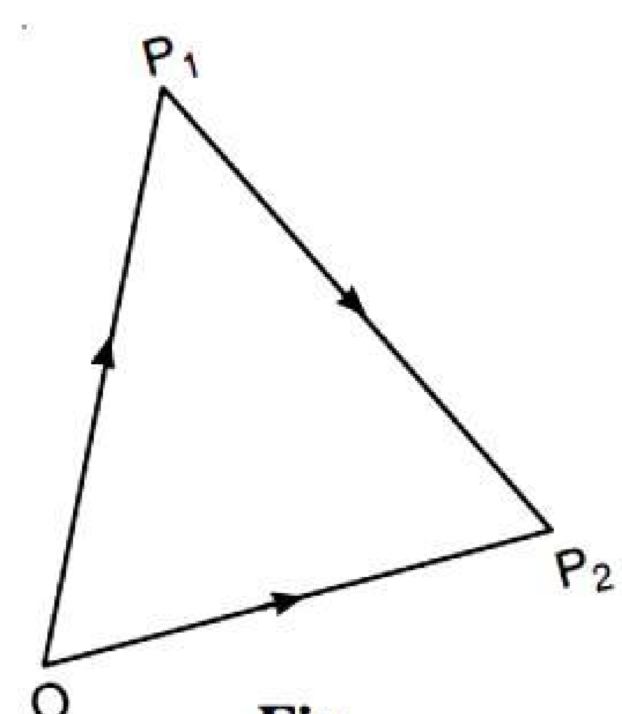


Fig.

Notation. *The vector of the form* $x\hat{i} + y\hat{j} + z\hat{k}$ *is denoted by* (x, y, z).

10.10. POSITION VECTOR

(*a*) **Def.** *Assume any point O, called the origin of reference.*

Mark an arbitrary point P in space. Then $\overrightarrow{OP}$ is called the position vector of the point P referred to the point O.

(*b*) **To represent a vector in terms of the position vectors of its end points.**

Let A and B be two points whose position vectors w.r.t. O as origin are $\vec{a}$ and $\vec{b}$ respectively.

Fig.

Then $\overrightarrow{OA} = \vec{a}$ and $\overrightarrow{OB} = \vec{b}$.

From ΔOAB, $\overrightarrow{OA} + \overrightarrow{AB} = \overrightarrow{OB}$

$\Rightarrow$ $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \vec{b} - \vec{a}$,

which is the reqd. representation.

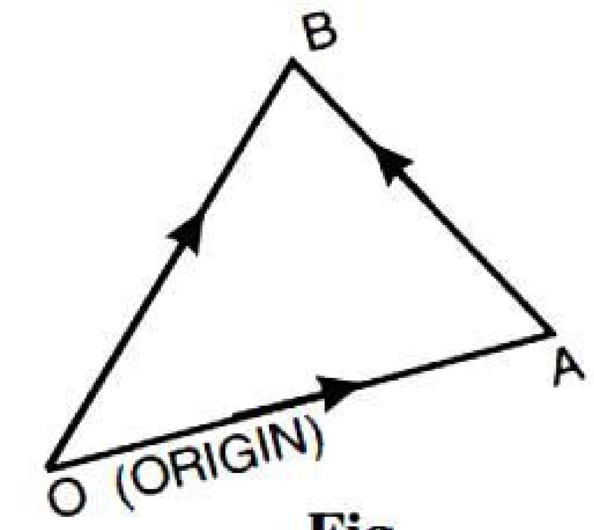


Fig.

Aid to Memory

$\overrightarrow{AB}$ = (*Position vector of B*) – (*Position vector of A*).

10.11. SECTION FORMULAE

To find the position vector of a point, which divides the line joining two points in a given ratio $m : n$.

Case I. *Internal Division.*

Let the position vectors of A and B referred to O as origin be $\vec{a}$ and $\vec{b}$ respectively.

$\therefore$ $\overrightarrow{OA} = \vec{a}$ and $\overrightarrow{OB} = \vec{b}$.

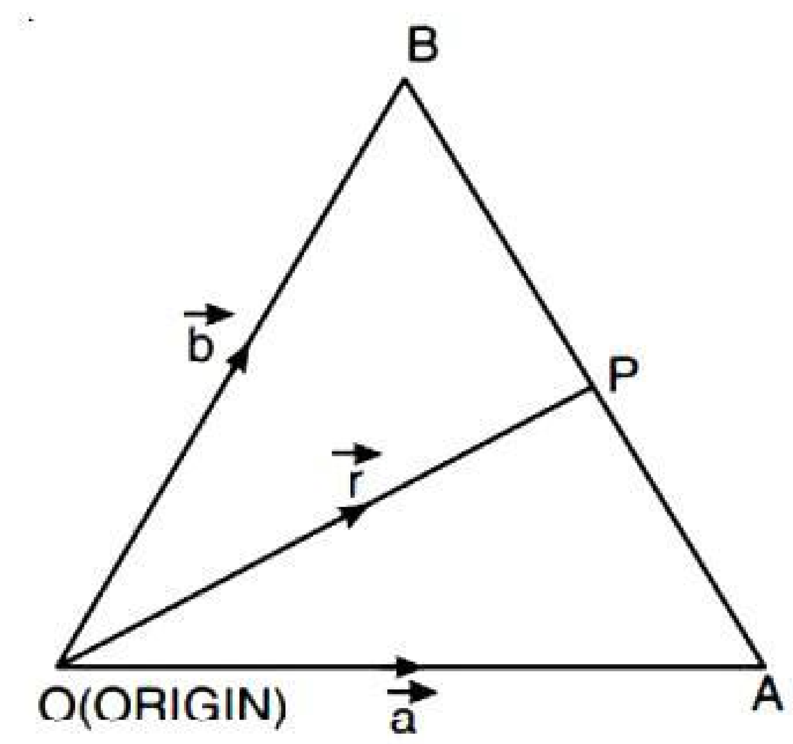


Fig.

Let $\overrightarrow{OP} = \vec{r}$. Let P divide [AB] internally in the ratio $m : n$.

Then $\dfrac{AP}{PB} = \dfrac{m}{n}$ $\Rightarrow$ $n\,\overrightarrow{AP} = m\,\overrightarrow{PB}$

$\Rightarrow$ $n\,(\overrightarrow{OP} - \overrightarrow{OA}) = m\,(\overrightarrow{OB} - \overrightarrow{OP})$ $\Rightarrow$ $n(\vec{r} - \vec{a}) = m\,(\vec{b} - \vec{r})$

$\Rightarrow$ $(m+n)\,\vec{r} = m\vec{b} + n\vec{a}$

$\Rightarrow$ $\vec{r} = \dfrac{m\vec{b} + n\vec{a}}{m+n}$, where $m + n \neq 0$.

Case II. *External Division.*

Here $\dfrac{AP}{PB} = -\dfrac{m}{n}$ $\Rightarrow$ $n\,\overrightarrow{AP} = -m\,\overrightarrow{PB}$

$\Rightarrow$ $n\,(\overrightarrow{OP} - \overrightarrow{OA}) = -m\,(\overrightarrow{OB} - \overrightarrow{OP})$

$\Rightarrow$ $n\,(\vec{r} - \vec{a}) = -m\,(\vec{b} - \vec{r})$

$\Rightarrow$ $(m-n)\,\vec{r} = m\vec{b} - n\vec{a}$

$\Rightarrow$ $\vec{r} = \dfrac{m\vec{b} - n\vec{a}}{m-n}$, where $m - n \neq 0$.

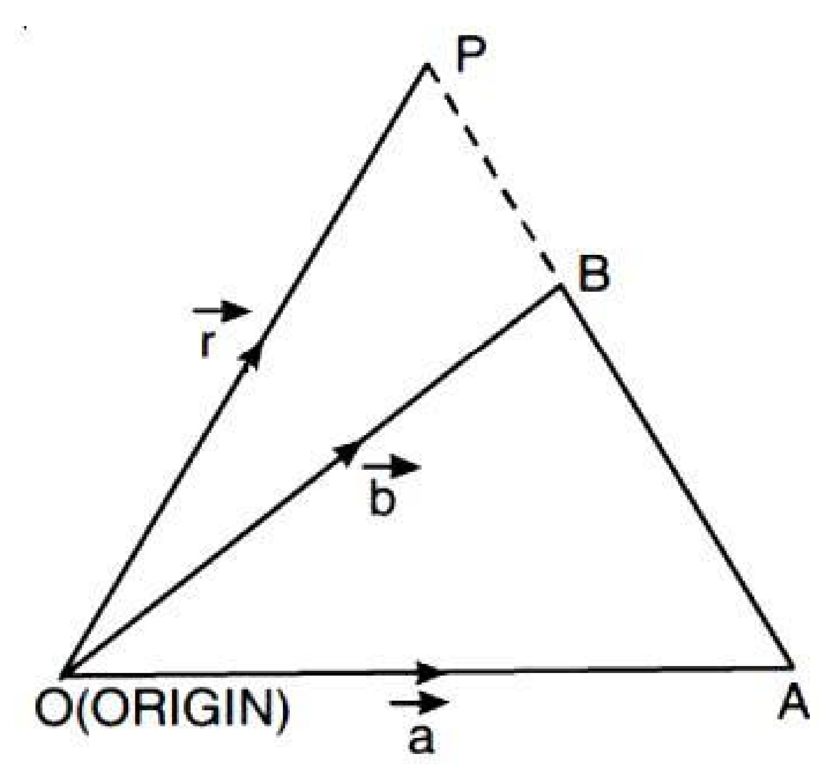


Fig.

Cor. Mid-point Formula. If P is the mid-point of [AB], then $m : n = 1 : 1$.

$\therefore$ The position vector of P is given by :

$\therefore$ $\vec{r} = \dfrac{1\,(\vec{b}) + 1\,(\vec{a})}{1+1}$ $\Rightarrow$ $\vec{\mathbf{r}} = \dfrac{\vec{\mathbf{a}} + \vec{\mathbf{b}}}{\mathbf{2}}$.

10.12. COLLINEARITY OF POINTS

The necessary and sufficient condition for three points having position vectors $\vec{a}, \vec{b}, \vec{c}$ *to be collinear is that there exist three scalars x, y, z (not all zero) such that* $x\vec{a} + y\vec{b} + z\vec{c} = \vec{0}$, *where* $x + y + z = 0$.

The condition is necessary.

Let A, B, C represent the position vectors $\vec{a}, \vec{b}, \vec{c}$ respectively referred to the origin O.

Since A, B, C are distinct collinear points,
∴ C divides [AB] in a certain ratio ; say $m : l$.

∴ The position vector $\vec{c}$ of the point C is given by :

$$\vec{c} = \frac{l\vec{a} + m\vec{b}}{l + m} \quad \Rightarrow \quad (l + m)\,\vec{c} = l\vec{a} + m\vec{b}$$

$$\Rightarrow l\vec{a} + m\vec{b} - (l + m)\,\vec{c} = \vec{0}.$$

Take $l = x,\ m = y,\ -(l + m) = z. \quad \therefore\ x\vec{a} + y\vec{b} + z\vec{c} = \vec{0},$

where $x + y + z = l + m - (l + m) = 0$, which is the reqd. necessary condition.

The condition is sufficient.

Here $\quad x\vec{a} + y\vec{b} + z\vec{c} = \vec{0}$ ...(1),

where $\quad x + y + z = 0$ ...(2) [*Given*]

Since x, y, z are not given to be all zero; let $x \neq 0$.

From (1), $\quad y\vec{b} + z\vec{c} = -x\vec{a}$

$$\Rightarrow \quad \frac{y\vec{b} + z\vec{c}}{y + z} = -\frac{x}{y + z}\,\vec{a} = -\frac{x}{-x}\vec{a} = \vec{a}.$$ [*Using (2)*]

Thus the point A having position vector $\vec{a}$ divides the join of B and C in the ratio $z : y$ internally.
Thus A, B, C are collinear.
Hence, the condition is sufficient.

10.13. COPLANARITY OF POINTS

The necessary and sufficient condition for four points having position vectors $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ *to be coplanar is that there exist four scalars l, m, n and p (not all zero) such that :*

$$l\vec{a} + m\vec{b} + n\vec{c} + p\vec{d} = \vec{0}, \text{ where } l + m + n + p = 0.$$

The condition is necessary.

Let A, B, C, D be four given points with position vectors $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ respectively referred to the origin O.

Any plane through A $(\vec{a})$, B $(\vec{b})$ and C $(\vec{c})$ is :

$$\vec{\mathbf{r}} = (\mathbf{1 - s - t})\ \vec{\mathbf{a}} + \mathbf{s}\vec{\mathbf{b}} + \mathbf{t}\vec{\mathbf{c}}$$ ...(1) **(*Note this step*)**

Since A, B, C, D are coplanar,

∴ D $(\vec{d})$ lies on (1).

∴ $\quad \vec{d} = (1 - s - t)\ \vec{a} + s\vec{b} + t\vec{c}$

$\Rightarrow \quad (1 - s - t)\ \vec{a} + s\vec{b} + t\vec{c} - \vec{d} = \vec{0}$ ...(2)

$\therefore$ (2) becomes : $l\vec{a} + m\vec{b} + n\vec{c} + p\vec{d} = \vec{0}$,
where $l + m + n + p = 1 - s - t + s + t - 1 = 0$.
Hence, the condition is necessary.
The condition is sufficient.
Let us assume that there exists a relation

$$l\vec{a} + m\vec{b} + n\vec{c} + p\vec{d} = \vec{0} \qquad \text{...(3)},$$

where $$l + m + n + p = 0 \qquad \text{...(4) [Given]}$$
and the scalars l, m, n, p are not all zero.

Suppose $p \neq 0$. Then dividing (3) and (4) by p, we get :

$$\frac{l}{p}\vec{a} + \frac{m}{p}\vec{b} + \frac{n}{p}\vec{c} + \vec{d} = \vec{0}$$

i.e. $$\vec{d} = -\frac{l}{p}\vec{a} - \frac{m}{p}\vec{b} - \frac{n}{p}\vec{c} \qquad \text{...(5)}$$

and $$\frac{l}{p} + \frac{m}{p} + \frac{n}{p} + 1 = 0$$

i.e. $$-\frac{l}{p} = \left(1 + \frac{m}{p} + \frac{n}{p}\right) \qquad \text{....(6)}$$

Putting the value of $-\frac{l}{p}$ from (6) in (5), we get :

$$\vec{d} = \left(1 + \frac{m}{p} + \frac{n}{p}\right)\vec{a} - \frac{m}{p}\vec{b} - \frac{n}{p}\vec{c}$$

$\Rightarrow$ $$= (1 - s - t)\,\vec{a} + s\vec{b} + t\vec{c}, \text{ where } s = -\frac{m}{p},\ t = -\frac{n}{p}.$$

This shows that $D(\vec{d})$ lies on the plane through A, B and C.

Thus the four points A, B, C and D are coplanar.
Hence, the condition is sufficient.

10.14. CENTROID

Definition

If there be n points whose position vectors relative to any origin O be given by $\vec{a}_1, \vec{a}_2, \vec{a}_3, \ldots\ldots\ldots, \vec{a}_n$, *then the point G whose position vector is :*

$$\overrightarrow{OG} = \frac{\vec{a}_1 + \vec{a}_2 + \vec{a}_3 + \ldots\ldots + \vec{a}_n}{n}$$

is called the **centroid** *or* **centre** *or* **mean position** *of the given points.*

Theorem. *The centroid of the given points is independent of the origin of the vectors.*

Proof. Let $A_1, A_2, \ldots, A_n$ be n given points whose position vectors are $\vec{a}_1, \vec{a}_2, \ldots\ldots, \vec{a}_n$ respectively relative to the origin O.

Then the centroid is given by :

$$\vec{OG} = \frac{m_1 \vec{a_1} + m_2 \vec{a_2} + ...+ m_n \vec{a_n}}{m_1 + m_2 + ...+ m_n} \quad ...(1)$$

[*Def.*]

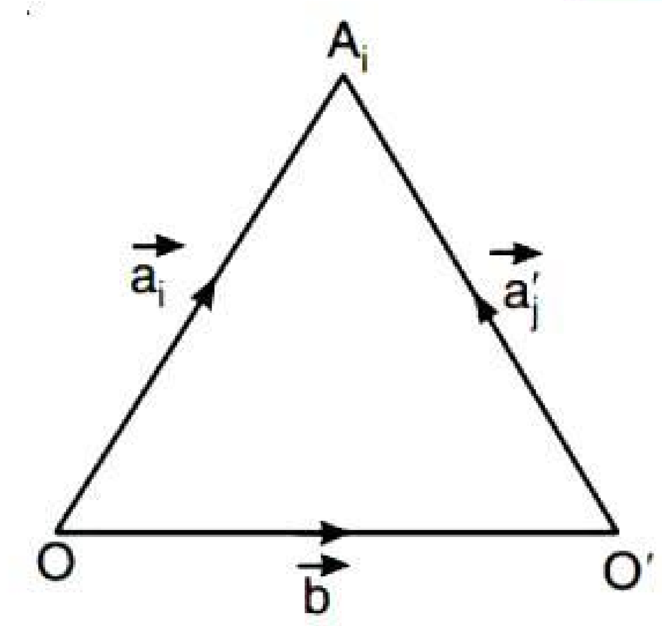


Fig.

Let O′ be the new origin whose position vector is $\vec{b}$ relative to O.

Let $\vec{a_i'}$ be the position vector of A_i relative to O′ so that :

$$\vec{a_i'} = \vec{a_i} - \vec{b} \quad ....(2)$$

The centroid G′ relative to the new origin O′ is given by :

$$\vec{O'G'} = \frac{m_1 \vec{a_{1'}} + m_2 \vec{a_{2'}} + ...+ m_n \vec{a_{n'}}}{m_1 + m_2 + ...+ m_n}.$$

But from (2), $\vec{a_1'} = \vec{a_1} - \vec{b}, \vec{a_2'} = \vec{a_2} - \vec{b}, ..., \vec{a_n'} = \vec{a_n} - \vec{b}$.

$$\therefore \quad \vec{O'G'} = \frac{m_1(\vec{a_1} - \vec{b}) + m_2(\vec{a_2} - \vec{b}) + ...+ m_n(\vec{a_n} - \vec{b})}{m_1 + m_2 + ...+ m_n}$$

$$= \frac{m_1 \vec{a_1} + m_2 \vec{a_2} + ...+ m_n \vec{a_n}}{m_1 + m_2 + ...+ m_n} - \vec{b}$$

$$= \vec{OG} - \vec{OO'} \quad [\textit{Using (1)}]$$

$$= \vec{O'O} + \vec{OG} = \vec{O'G}.$$

Thus G′ and G coincide.

Hence, the centroid is independent of the origin of vectors.

10.15. INCENTRE

If A $(\vec{\alpha})$, B $(\vec{\beta})$ and C $(\vec{\gamma})$ are the vertices of a triangle ABC, then the position vector of its incentre is $\dfrac{a\vec{\alpha} + b\vec{\beta} + c\vec{\gamma}}{a+b+c}$, where $|\vec{BC}| = a$, $|\vec{CA}| = b$ and $|\vec{AB}| = c$.

Proof. Let the internal bisector of ∠A meet [BC] in D. Then D divides [BC] in the ratio [AB] : [AC] *i.e.*. $c : b$

$$\Rightarrow \quad \frac{|BD|}{|DC|} = \frac{c}{b} \quad ...(1)$$

∴ P. V. of D is :

$$\frac{c\vec{\gamma} + b\vec{\beta}}{c+b} = \frac{b\vec{\beta} + c\vec{\gamma}}{b+c}.$$

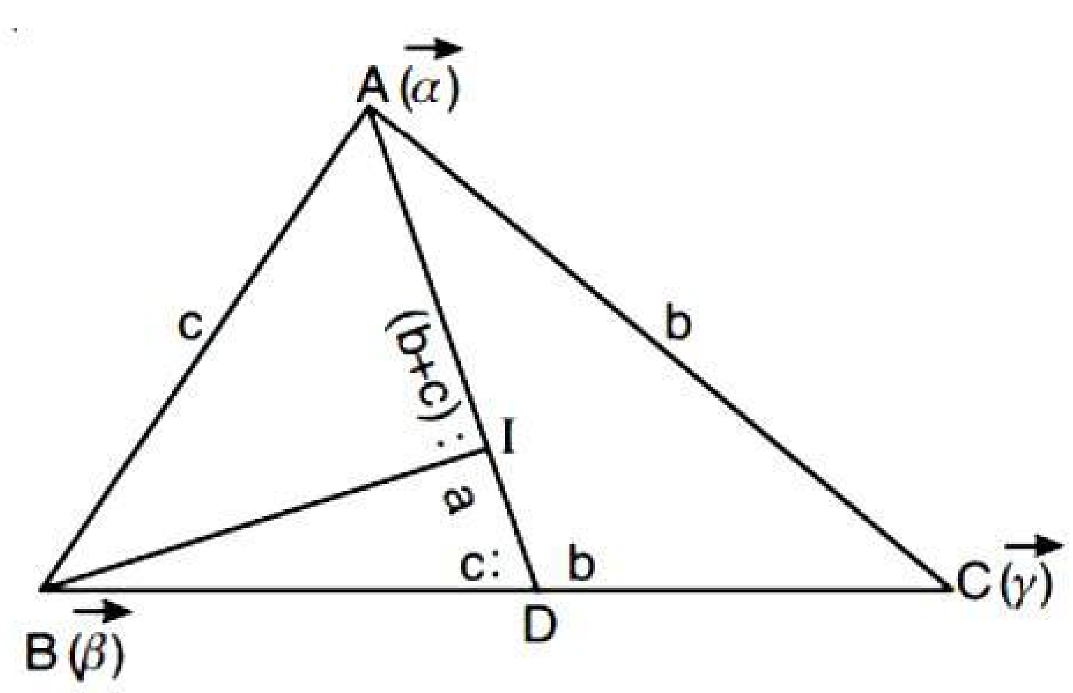


Fig.

Let the internal bisector of ∠B meet [AD] in I.
Then I divides [AD] in the ratio
| BA | : | BD | *i.e.* c : | BD |

$$\Rightarrow \quad \frac{|AI|}{|ID|} = \frac{c}{|BD|} \quad ....(2)$$

From (1), $\dfrac{|\text{BD}|}{|\text{DC}|} = \dfrac{c}{b}$ $\Rightarrow$ $\dfrac{|\text{BD}|}{|\text{BC}|-|\text{BD}|} = \dfrac{c}{b}$

$\Rightarrow$ $\dfrac{|\text{BD}|}{a-|\text{BD}|} = \dfrac{c}{b}$ $\Rightarrow$ $b\,|\text{BD}| = ca - c\,|\text{BD}|$

$\Rightarrow$ $(b+c)\,|\text{BD}| = ca$ $\Rightarrow$ $|\text{BD}| = \dfrac{ca}{b+c}$

$\Rightarrow$ $\dfrac{c}{|\text{BD}|} = \dfrac{b+c}{a}$ ...(3)

From (2) and (3), $\dfrac{|\text{AI}|}{|\text{ID}|} = \dfrac{b+c}{a}$

$\Rightarrow$ I divides [AD] in the ratio $(b+c) : c$.
Hence,

$$\text{P. V. of I} = \frac{(b+c)(\text{P.V.of D})+a(\text{P.V. of A})}{b+c+a}$$

$$= \frac{(b+c)\left(\dfrac{b\vec{\beta}+c\vec{\gamma}}{b+c}\right)+a\vec{\alpha}}{(b+c)+a} = \frac{a\vec{\alpha}+b\vec{\beta}+c\vec{\gamma}}{a+b+c}.$$

The symmetry of the P.V. of I indicates that the internal bisectors of $\angle$B and $\angle$C also meet at I, which is called the **incentre**.

Hence, the internal bisectors of the angles of a triangle are concurrent at the point I, the incentre.

Frequently Asked Questions

FAQs

Example 1. Find the position vector of a point, which divides the join of the points with position vectors $\vec{a}-2\vec{b}$ and $2\vec{a}+\vec{b}$ externally in the ratio 2 : 1. *(C.B.S.E. 2016)*

Solution. Here $\vec{a}-2\vec{b}$ and $2\vec{a}+\vec{b}$ are position vectors of two points.

$$\therefore \text{ Required position vector} = \frac{2(2\vec{a}+\vec{b})-1.(\vec{a}-2\vec{b})}{2-1}$$

$$= \frac{3\vec{a}+4\vec{b}}{1} = 3\vec{a}+4\vec{b}.$$

Example 2. The two vectors $\hat{j}+\hat{k}$ and $3\hat{i}-\hat{j}+4\hat{k}$ represent the two sides AB and AC respectively of a ΔABC. Find the length of the median through A. *(C.B.S.E. 2016)*

Solution. Take A as the origin (0, 0, 0).

$\therefore$ The co-ordinates of B and C are (0, 1, 1) and (3, –1, 4) respectively.

Now D, the mid-point of [BC],

is $\dfrac{0+3}{2}, \dfrac{1-1}{2}, \dfrac{1+4}{2}$ *i.e.* $\dfrac{3}{2}, 0, \dfrac{5}{2}$.

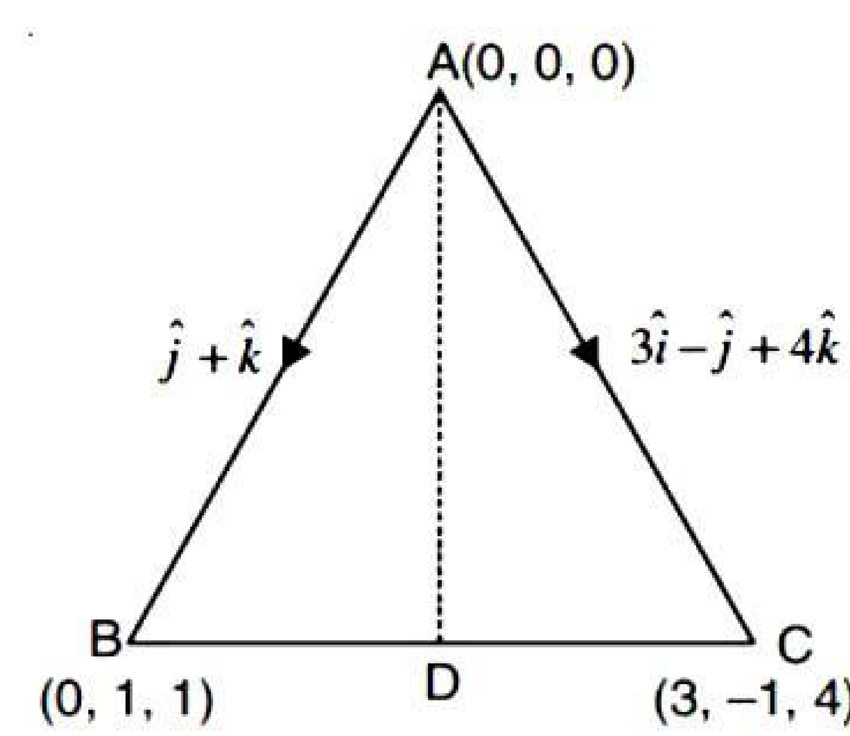


Fig.

$\therefore$ Length of median AD = |AD|

$$= \sqrt{\left(\frac{3}{2}-0\right)^2+(0-0)^2+\left(\frac{5}{2}-0\right)^2}$$

$$= \sqrt{\frac{9}{4}+\frac{25}{4}} = \sqrt{\frac{34}{4}} = \frac{1}{2}\sqrt{34} \text{ units.}$$

Example 3. If $\vec{a}=4\hat{i}-\hat{j}+\hat{k}$ and $\vec{b}=2\hat{i}-2\hat{j}+\hat{k}$, then find a unit vector parallel to the vector $\vec{a}+\vec{b}$. *(A.I.C.B.S.E. 2016)*

Solution. $\vec{a}+\vec{b} = (4\hat{i}-\hat{j}+\hat{k})+(2\hat{i}-2\hat{j}+\hat{k})$

$$= 6\hat{i}-3\hat{j}+2\hat{k}.$$

Hence, unit vector parallel to the vector $\vec{a}+\vec{b}$

$$= \frac{6\hat{i}-3\hat{j}+2\hat{k}}{\sqrt{36+9+4}} = \frac{1}{7}(6\hat{i}-3\hat{j}+2\hat{k}).$$

Example 4. If $\vec{a} = 2\hat{i}+\hat{j}-2\hat{k}$, then evaluate $|\vec{a}|$.

(Uttarakhand. B. 2015)

Solution. We have : $\vec{a} = 2\hat{i}+\hat{j}-2\hat{k}$.

$$\therefore \quad |\vec{a}| = \sqrt{(2)^2+(1)^2+(-2)^2}$$

$$= \sqrt{4+1+4} = \sqrt{9} = 3.$$

Example 5. Find the vector joining the points P (2, 3, 0) and Q (– 1, – 2, – 4) directed from P to Q. *(N.C.E.R.T.)*

Solution. Since the vector is directed from P to Q,

$\therefore$ P is the initial point and Q is the terminal point.

$\therefore$ Reqd. vector $= \overrightarrow{PQ}$

$$= (-\hat{i}-2\hat{j}-4\hat{k})-(2\hat{i}+3\hat{j}+0\hat{k})$$

$$= (-1-2)\,\hat{i}+(-2-3)\,\hat{j}+(-4-0)\,\hat{k}$$

$$= -3\hat{i}-5\hat{j}-4\hat{k}.$$

Example 6. If $\vec{a} = x\hat{i}+2\hat{j}-z\hat{k}$ and $\vec{b} = 3\hat{i}-y\hat{j}+\hat{k}$ are two equal vectors, then write the value of x + y + z.

(C.B.S.E. 2013)

Solution. Here $\vec{a}=\vec{b} \Rightarrow x\hat{i}+2\hat{j}-z\hat{k} = 3\hat{i}-y\hat{j}+\hat{k}$.

Comparing, $x = 3$, $2 = -y$ *i.e.* $y = -2$, $-z = 1$ *i.e.* $z = -1$.

Hence, $x + y + z = 3 - 2 - 1 = 0$.

Example 7. Write the direction-ratios of the vector $\vec{r} = \hat{i}+\hat{j}-2\hat{k}$, and hence calculate its direction-cosines.

(N.C.E.R.T.)

Solution. We have : $\vec{r} = \hat{i}+\hat{j}-2\hat{k}$.

Direction ratios of $\vec{r}$ are $< 1, 1, -2 >$.

Direction-cosines of $\vec{r}$ are given by :

$$l = \frac{a}{|\vec{r}|} = \frac{1}{\sqrt{1+1+4}} = \frac{1}{\sqrt{6}},$$

$$m = \frac{b}{|\vec{r}|} = \frac{1}{\sqrt{1+1+4}} = \frac{1}{\sqrt{6}}$$

and $$n = \frac{c}{|\vec{r}|} = \frac{-2}{\sqrt{1+1+4}} = -\frac{2}{\sqrt{6}}.$$

Hence, the direction-cosines of $\vec{r}$ are $< \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}} >$.

Example 8. Write a unit vector in the direction of the sum of the vectors :

$\vec{a} = 2\hat{i}+2\hat{j}-5\hat{k}$ and $\vec{b} = 2\hat{i}+\hat{j}-7\hat{k}$.

(C.B.S.E. 2014)

Solution. $\vec{a}+\vec{b} = (2\hat{i}+2\hat{j}-5\hat{k})+(2\hat{i}+\hat{j}-7\hat{k})$

$$= 4\hat{i}+3\hat{j}-12\hat{k}.$$

Hence, the unit vector in the direction of $\vec{a}+\vec{b}$

$$= \frac{4\hat{i}+3\hat{j}-12\hat{k}}{\sqrt{16+9+144}} = \frac{1}{13}(4\hat{i}+3\hat{j}-12\hat{k}).$$

Example 9. Find the unit vector in the direction of the sum of the vectors :

$\vec{a} = 2\hat{i}-\hat{j}+2\hat{k}$ and $\vec{b} = -\hat{i}+\hat{j}+3\hat{k}$. *(N.C.E.R.T.)*

Solution. We have : $\vec{a} = 2\hat{i}-\hat{j}+2\hat{k}$

and $\vec{b} = -\hat{i}+\hat{j}+3\hat{k}$.

$\therefore \quad \vec{c} = \vec{a}+\vec{b}$

$$= (2\hat{i} - \hat{j} + 2\hat{k}) + (-\hat{i} + \hat{j} + 3\hat{k})$$

$$= \hat{i} + 0.\hat{j} + 5\hat{k}.$$

$$\therefore \quad |\vec{c}| = \sqrt{1^2 + 0^2 + 5^2} = \sqrt{1+0+25} = \sqrt{26}.$$

$$\therefore \quad \text{Reqd. unit vector} = \hat{c} = \frac{\vec{c}}{|\vec{c}|}$$

$$= \frac{\hat{i} + 0\hat{j} + 5\hat{k}}{\sqrt{26}} = \frac{\hat{i} + 5\hat{k}}{\sqrt{26}}.$$

Example 10. Find a vector of magnitude 5 units, and parallel to the resultant of the vectors :

$\vec{a} = 2\hat{i} + 3\hat{j} - \hat{k}$ and $\vec{b} = \hat{i} - 2\hat{j} + \hat{k}$. *(C.B.S.E. 2011)*

Solution. We have : $\vec{a} = 2\hat{i} + 3\hat{j} - \hat{k}$

and $\vec{b} = \hat{i} - 2\hat{j} + \hat{k}$.

Resultant of vectors $\vec{a}$ and $\vec{b}$ = $\vec{a} + \vec{b}$

$$= (2\hat{i} + 3\hat{j} - \hat{k}) + (\hat{i} - 2\hat{j} + \hat{k})$$

$$= 3\hat{i} + \hat{j}.$$

$\therefore$ Reqd. vector of magnitude 5 units and parallel to $(\vec{a} + \vec{b})$

$$= 5\frac{(\vec{a} + \vec{b})}{|\vec{a} + \vec{b}|}$$

$$= 5\frac{3\hat{i} + \hat{j}}{\sqrt{9+1+0}}$$

$$= \frac{5}{\sqrt{10}}(3\hat{i} + \hat{j})$$

$$= \frac{15}{\sqrt{10}}\hat{i} + \frac{5}{\sqrt{10}}\hat{j}.$$

Example 11. Prove that if $\vec{u} = u_1\hat{i} + u_2\hat{j}$ and $\vec{v} = v_1\hat{i} + v_2\hat{j}$ are non-zero vectors, then they are parallel if and only if $u_1v_2 - u_2v_1 = 0$.

Solution. We know that $\vec{u}$ is parallel to $\vec{v}$ if and only if there exists a non-zero scalar k such that $\vec{u} = k\vec{v}$

i.e. if and only if $u_1\hat{i} + u_2\hat{j} = k(v_1\hat{i} + v_2\hat{j})$

i.e. if and only if $(u_1 - kv_1)\hat{i} + (u_2 - kv_2)\hat{j} = \vec{0}$

i.e. if and only if $u_1 - kv_1 = 0$ and $u_2 - kv_2 = 0$

i.e. if and only if $k = \frac{u_1}{v_1}$ and $k = \frac{u_2}{v_2}$

i.e. if and only if $\frac{u_1}{v_1} = \frac{u_2}{v_2}$

i.e. if and only if $u_1v_2 - u_2v_1 = 0$,

which is true.

Example 12. Find the value of 'p' for which the vectors :

$3\hat{i} + 2\hat{j} + 9\hat{k}$ and $\hat{i} - 2p\hat{j} + 3\hat{k}$ are parallel.

(A.I.C.B.S.E. 2014)

Solution. The given vectors $3\hat{i} + 2\hat{j} + 9\hat{k}$ and $\hat{i} - 2p\hat{j} + 3\hat{k}$ are parallel

if $\quad \frac{3}{1} = \frac{2}{-2p} = \frac{9}{3}$ if $3 = \frac{1}{-p} = 3 \quad$ [*Ex. 11*]

if $\quad p = -\frac{1}{3}$.

Example 13. Show that the points :

A $(2\hat{i} - \hat{j} + \hat{k})$, B $(\hat{i} - 3\hat{j} - 5\hat{k})$, C $(3\hat{i} - 4\hat{j} - 4\hat{k})$ are the vertices of a right-angled triangle. *(N.C.E.R.T)*

Solution. Let O be the origin of reference.

$$\therefore \quad \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$$

$$\Rightarrow \quad \overrightarrow{AB} = (\hat{i} - 3\hat{j} - 5\hat{k}) - (2\hat{i} - \hat{j} + \hat{k})$$

$$= -\hat{i} - 2\hat{j} - 6\hat{k},$$

$$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB}$$

$\Rightarrow \quad \overrightarrow{BC} = (3\hat{i} - 4\hat{j} - 4\hat{k}) - (\hat{i} - 3\hat{j} - 5\hat{k})$

$= 2\hat{i} - \hat{j} + \hat{k}$

and $\quad \overrightarrow{CA} = \overrightarrow{OA} - \overrightarrow{OC}$

$\Rightarrow \quad \overrightarrow{CA} = (2\hat{i} - \hat{j} + \hat{k}) - (3\hat{i} - 4\hat{j} - 4\hat{k})$

$= -\hat{i} + 3\hat{j} + 5\hat{k}.$

Now $\quad |\overrightarrow{AB}| = \sqrt{(-1)^2 + (-2)^2 + (-6)^2}$

$= \sqrt{1 + 4 + 36} = \sqrt{41}$,

$|\overrightarrow{BC}| = \sqrt{(2)^2 + (-1)^2 + (1)^2}$

$= \sqrt{4 + 1 + 1} = \sqrt{6}$

and $\quad |\overrightarrow{CA}| = \sqrt{(-1)^2 + (3)^2 + (5)^2}$

$= \sqrt{1 + 9 + 25} = \sqrt{35}.$

Thus $\quad AB^2 = BC^2 + CA^2. \qquad [\because\ 41 = 6 + 35]$

Hence, the triangle is a right-angled triangle.

Example 14. The position vectors of A, B, C are $2\hat{i} + \hat{j} - \hat{k}$, $3\hat{i} - 2\hat{j} + \hat{k}$ and $\hat{i} + 4\hat{j} - 3\hat{k}$ respectively. Show that A, B and C are collinear.

Solution. $\overrightarrow{AB}$ = P.V. of B – P.V. of A

$= (3\hat{i} - 2\hat{j} + \hat{k}) - (2\hat{i} + \hat{j} - \hat{k})$

$= \hat{i} - 3\hat{j} + 2\hat{k} \qquad ....(1)$

and $\quad \overrightarrow{BC}$ = P.V. of C – P.V. of B

$= (\hat{i} + 4\hat{j} - 3\hat{k}) - (3\hat{i} - 2\hat{j} + \hat{k})$

$= -2\hat{i} + 6\hat{j} - 4\hat{k} = -2(\hat{i} - 3\hat{j} + 2\hat{k})$

$= -2\overrightarrow{AB}. \qquad$ [*Using (1)*]

Thus, either AB || BC or A, B, C are collinear.

But B is the common point.

Hence, A,B, C are collinear.

Example 15. Show that the following vectors are coplanar :

$$\mathbf{2\hat{i} - \hat{j} + \hat{k},\ \hat{i} - 3\hat{j} - 5\hat{k},\ 3\hat{i} - 4\hat{j} - 4\hat{k}.}$$

Solution. Let $\vec{a} = 2\hat{i} - \hat{j} + \hat{k}$, $\vec{b} = \hat{i} - 3\hat{j} - 5\hat{k}$

and $\quad \vec{c} = 3\hat{i} - 4\hat{j} - 4\hat{k}.$

The three vectors are coplanar if we can express any one vector in terms of the other two vectors.

Let $\vec{b} = x\vec{a} + y\vec{c}$, where x, y are scalars.

Then

$\hat{i} - 3\hat{j} - 5\hat{k} = x(2\hat{i} - \hat{j} + \hat{k}) + y(3\hat{i} - 4\hat{j} - 4\hat{k})$

$= (2x + 3y)\hat{i} + (-x - 4y)\hat{j} + (x - 4y)\hat{k}.$

Comparing coefficients of like vectors, we have :

$2x + 3y = 1 \qquad ...(1)$

$-x - 4y = -3 \qquad ...(2)$

and $\quad x - 4y = -5 \qquad ...(3)$

Solving (2) and (3), we get :

$x = -1,\ y = 1.$

Putting in (1),

$2(-1) + 3(1) = 1 \quad \Rightarrow \quad 1 = 1$, which is true.

Hence, the given vectors are coplanar.

Example 16. Show that the four points A, B, C, D with position vectors $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ respectively, such that $3\vec{a} - 2\vec{b} + 5\vec{c} - 6\vec{d} = \vec{0}$, are coplanar. Also find the position vector of the point of intersection of the lines AC and BD.

Solution. We have : $3\vec{a} - 2\vec{b} + 5\vec{c} - 6\vec{d} = \vec{0}$

$\Rightarrow \quad \mathbf{3\vec{a} - 3\vec{b} + \vec{b} - \vec{c} + 6\vec{c} - 6\vec{d} = \vec{0}}$

(*Note this step*)

$\Rightarrow \quad 3(\vec{a} - \vec{b}) + (\vec{b} - \vec{c}) + 6(\vec{c} - \vec{d}) = \vec{0}$

$$\Rightarrow \quad 3\overrightarrow{BA} + \overrightarrow{CB} + 6\overrightarrow{DC} = \vec{0}$$

$$\Rightarrow \quad 3\overrightarrow{AB} + 1.\overrightarrow{BC} + 6\overrightarrow{CD} = \vec{0},$$

which shows that the vectors $\overrightarrow{AB}, \overrightarrow{BC}, \overrightarrow{CD}$ are linearly dependent and hence, the four points A, B, C, D are coplanar.

Now $3\vec{a} + 5\vec{c} = 2\vec{b} + 6\vec{d}$

$$\Rightarrow \quad \frac{3\vec{a} + 5\vec{c}}{8} = \frac{2\vec{b} + 6\vec{d}}{8} \quad [\textit{Dividing by } 8]$$

$$\Rightarrow \quad \frac{3\vec{a} + 5\vec{c}}{3+5} = \frac{2\vec{b} + 6\vec{d}}{2+6}$$

$\Rightarrow$ The point P, which divides [AC] in the ratio 5 : 3, also divides [BD] in the ratio 6 : 2

$\Rightarrow$ The lines AC and BD intersect in P whose position vector is :

$$\frac{3\vec{a} + 5\vec{c}}{8} \text{ or } \frac{2\vec{b} + 6\vec{d}}{8}.$$

EXERCISE 10 (c)

Fast Track Answer Type Questions

FTATQ

1. The direction-ratios of the vector $\vec{a} = 6\hat{i} - 3\hat{j} + 2\hat{k}$ are .... **(Fill in the Blank)** *(Jammu B. 2017)*

2. Write two different vectors having same :

(*i*) magnitude (*ii*) direction. *(N.C.E.R.T.)*

3. Find the sum of the vectors :

$$\vec{a} = \hat{i} - 2\hat{j} + \hat{k}, \ \vec{b} = -2\hat{i} + 4\hat{j} + 5\hat{k}$$

and $\vec{c} = \hat{i} - 6\hat{j} - 7\hat{k}$. *(N.C.E.R.T.)*

4. Find a unit vector in the direction from :

(*i*) P(3, 2) towards Q(5, 6)

(*ii*) P(1, 2) towards Q(4, 5).

5. (*a*) Find the magnitude of the vector

(*i*) $\frac{1}{\sqrt{3}}\hat{i} + \frac{1}{\sqrt{3}}\hat{j} - \frac{1}{\sqrt{3}}\hat{k}$ *(N.C.E.R.T.)*

(*ii*) $\hat{i} - 3\hat{j} + 4\hat{k}$. *(Bihar B. 2014)*

(*b*) Find the values of x for which $x(\hat{i} + \hat{j} + \hat{k})$ is a unit vector. *(N.C.E.R.T.; Kashmir B. 2017 ; Jammu B. 2012)*

6. Find the unit vector in the direction of the vector :

(*a*) $\vec{a} = \hat{i} + \hat{j} + 2\hat{k}$ *(N.C.E.R.T.; Kashmir B. 2016)*

(*b*) (*i*) $\vec{a} = 2\hat{i} + 3\hat{j} + \hat{k}$ *(N.C.E.R.T.; H.B. 2016)*

(*ii*) $\vec{a} = 3\hat{i} + 2\hat{j} + 6\hat{k}$ *(Jammu B. 2013)*

(*iii*) $\vec{b} = 2\hat{i} + \hat{j} + 2\hat{k}$

(A.I.C.B.S.E. 2011; C.B.S.E. 2009)

(*iv*) $\vec{a} = 2\hat{i} - 3\hat{j} + 6\hat{k}$. *(Assam B. 2016)*

7. Find the unit vector in the direction of vector $\overrightarrow{PQ}$, where P and Q are the points (1, 2, 3) and (4, 5, 6) respectively. *(N.C.E.R.T. ; Jammu B. 2013)*

8. (*i*) Find the values of x and y so that the vectors $2\hat{i} + 3\hat{j}$ and $x\hat{i} + y\hat{j}$ are equal. *(N.C.E.R.T.)*

(*ii*) Find the values of x, y and z so that the vectors :

$\vec{a} = x\hat{i} + 2\hat{j} + z\hat{k}$ and $\vec{y} = 2\hat{i} + y\hat{j} + \hat{k}$ are equal.

(N.C.E.R.T.)

9. (*a*) Let $\vec{a} = \hat{i} + 2\hat{j}$ and $\vec{b} = 2\hat{i} + \hat{j}$. Is $|\vec{a}| = |\vec{b}|$? Are the vectors $\vec{a}$ and $\vec{b}$ equal ? *(N.C.E.R.T.)*

(*b*) If $\vec{a} = \vec{b} + \vec{c}$, then is it true that $|\vec{a}| = |\vec{b}| + |\vec{c}|$? Justify your answer. *(N.C.E.R.T.)*

10. Find the direction-cosines of the vector $\hat{i} + 2\hat{j} + 3\hat{k}$. *(N.C.E.R.T.; Jammu B. 2015)*

11. (*i*) Show that the direction-cosines of a vector equally inclined to the axes OX, OY and OZ are $< \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} >$. *(N.C.E.R.T.)*

(*ii*) Show that the vector $\hat{i}+\hat{j}+\hat{k}$ is equally inclined to the axes OX, OY, OZ. *(N.C.E.R.T.)*

12. For given vectors, $\vec{a}=2\hat{i}-\hat{j}+2\hat{k}$ and $\vec{b}=-\hat{i}+\hat{j}-\hat{k}$, find the unit vector in the direction of the vector $\vec{a}+\vec{b}$. *(N.C.E.R.T.; Assam B. 2018)*

13. (*i*) A and B are two points with position vectors $2\vec{a}-3\vec{b}$ and $6\vec{b}-\vec{a}$ respectively. Write the position vector of a point, which divides the line segment AB internally in the ratio 1 : 2. *(A.I.C.B.S.E. 2013)*

(*ii*) P and Q are two points with position vectors $3\vec{a}-2\vec{b}$ and $\vec{a}+\vec{b}$ respectively. Write the position vector of a point R, which divides the line segment PQ in the ratio 2 : 1 externally. *(A.I.C.B.S.E. 2013)*

(*iii*) L and M are two points with position vectors $2\vec{a}-\vec{b}$ and $\vec{a}+2\vec{b}$ respectively. Write the position vector of a point N, which divides the line segment LM in the ratio 2 : 1 externally. *(A.I.C.B.S.E. 2013)*

14. (*i*) Find the position vector of the mid-point of the vector joining points :

P (2, 3, 4) and Q (4, 1, – 2). *(N.C.E.R.T.)*

(*ii*) Find the position vector of the mid-point of the line-segment AB, where A is the point (3, 4, – 2) and B is the point (1, 2, 4). *(C.B.S.E. 2010 C)*

15. Find a vector in the direction of :

(*i*) $\vec{a} = 5\hat{i}-\hat{j}+2\hat{k}$, which has magnitude 8 units *(Kashmir B. 2017)*

(*ii*) $\vec{a} = 2\hat{i}-\hat{j}+2\hat{k}$, which has magnitude 6 units *(C.B.S.E. 2010 C)*

(*iii*) $\vec{a} = \hat{i}-2\hat{j}+2\hat{k}$, which has magnitude 15 units. *(C.B.S.E. 2010)*

(*iv*) $\vec{a} = -2\hat{i}+\hat{j}+2\hat{k}$, which has magnitude 9 units. *(A.I.C.B.S.E. 2010)*

16. Find the scalar components and magnitude of the vector joining the points P (x_1, y_1, z_1) and Q (x_2, y_2, z_2). *(N.C.E.R.T.)*

17. (*a*) If $|\vec{a}| = 3$, what is :

(*i*) $|5\vec{a}|$ (*ii*) $|-2\vec{a}|$ (*iii*) $|0\vec{a}|$?

(*b*) If $\vec{a}=3\hat{i}-2\hat{j}+\hat{k}$, $\vec{b}=2\hat{i}-4\hat{j}-3\hat{k}$, find $|\vec{a}-2\vec{b}|$.

18. If two vectors are $\vec{a}=\hat{i}+2\hat{j}$ and $\vec{b}=2\hat{i}+\hat{j}$; Is $|\vec{a}|=|\vec{b}|$? Are the vectors $\vec{a}$ and $\vec{b}$ equal ? *(Uttarakhand B. 2013)*

Very Short Answer Type Questions

VSATQ

19. Let $\vec{a}$ be a given vector whose initial point is $P(x_1, y_1)$ and terminal point is $Q(x_2, y_2)$. Find the magnitude and components of the vector along x and y directions : P (2, 3) ; Q (4, 6).

20. In the following, find the components of the vector $\overrightarrow{PQ}$ along x and y directions whose magnitude is M, and makes an angle θ with the x-axis :

M = 15, θ = 30°.

21. If the position vectors of the points A and B are : $7\hat{i}+3\hat{j}-\hat{k}$ and $2\hat{i}-5\hat{j}+4\hat{k}$ respectively, find the magnitude and direction-cosines of the vector $\overrightarrow{AB}$.

22. Find the position vector of the centroid of the ΔABC when the position vectors of its vertices are A (1, 3, 0), B (2, 1, 1) and C (0, – 1, 0).

23. Show that the vectors $\vec{a}=2\hat{i}+3\hat{j}$ and $\vec{b}=4\hat{i}+6\hat{j}$ are parallel.

24. (*i*) Find a unit vector in the direction of $(\vec{a}+\vec{b})$, where : $\vec{a}=2\hat{i}+2\hat{j}-5\hat{k}$ and $\vec{b}=2\hat{i}+\hat{j}+3\hat{k}$. *(N.C.E.R.T.)*

(*ii*) If $\vec{a} = 2\hat{i}-\hat{j}+2\hat{k}$ and $\vec{b} = 6\hat{i}+2\hat{j}+3\hat{k}$, find a unit vector parallel to $\vec{a}+\vec{b}$. *(Kerala B. 2014)*

(*iii*) Find the unit vector in the direction of $\vec{a}-\vec{b}$, where :

$$\vec{a}=\hat{i}+3\hat{j}-\hat{k},\ \vec{b}=3\hat{i}+2\hat{j}+\hat{k}.$$

(Kerala B. 2016)

25. If $\vec{a} = \hat{i}+\hat{j}+\hat{k}$, $\vec{b} = 2\hat{i}-\hat{j}+3\hat{k}$, $\vec{c} = \hat{i}-2\hat{j}+\hat{k}$, then find a unit vector parallel to the vector $2\vec{a}-\vec{b}+3\vec{c}$.

(P.B. 2014)

26. (*a*) Find the condition that the vectors $\vec{a} = k\hat{i} + l\hat{j}$ and $\vec{b} = l\hat{i} + k\hat{j}$ $(k, l \neq 0)$ are parallel.

(*b*) Show that the vectors $2\hat{i}-3\hat{j}+4\hat{k}$ and $-4\hat{i}+6\hat{j}-8\hat{k}$ are collinear. *(N.C.E.R.T.)*

Short Answer Type Questions

SATQ

27. (*a*) Find the position vector of R, which divides the line joining $P(3\vec{a} - 2\vec{b})$ and $Q(\vec{a} + \vec{b})$ in the ratio 2 : 1

(*i*) internally and (*ii*) externally. *(N.C.E.R.T)*

(*b*) Find the position vector of R, which divides the line joining two points $P(2\vec{a} + \vec{b})$ and $Q(\vec{a} - 3\vec{b})$ externally in the ratio 1 : 2. Also, show that P is the middle point of the segment RQ. *(N.C.E.R.T.; C.B.S.E. 2010)*

28. Show that the following points are collinear :

(*i*) A(– 2, 1), B(– 5, – 1), C (1, 3)

(*ii*) A (1, 2, 7), B (2, 6, 3), C (3, 10, – 1)

(N.C.E.R.T.; Kerala B. 2018; H.P.B. 2016)

(*iii*) A (2, 0, 3), B (3, 2, –1), C (1, – 2, – 5)

(Meghalaya B. 2018)

29. If $\vec{a} = -2\hat{i} + 3\hat{j} + 5\hat{k}$, $\vec{b} = \hat{i} + 2\hat{j} + 3\hat{k}$ and $\vec{c} = 7\hat{i} - \hat{k}$ are position vectors of three points A, B, C respectively, prove that A, B, C are collinear.

(N.C.E.R.T.; C.B.S.E. 2009 C)

30. Show that the following vectors are coplanar :

(*i*) $\hat{i} - \hat{j} + \hat{k}$, $6\hat{i} - \hat{k}$ and $4\hat{i} + 2\hat{j} - 3\hat{k}$

(*ii*) $3\hat{i} - 2\hat{j} + 4\hat{k}$, $6\hat{i} + 3\hat{j} + 2\hat{k}$, $5\hat{i} + 7\hat{j} + 3\hat{k}$ and $2\hat{i} + 2\hat{j} + 5\hat{k}$.

31. Show that the points A(3, – 2, 1), B (1, – 3, 5), C (2, 1, – 4) do not form a right-angled triangle.

32. If the position vectors of the vertices of a triangle are :

$\vec{A} = \hat{i} + 2\hat{j} + 3\hat{k}$, $\vec{B} = 2\hat{i} + 3\hat{j} + \hat{k}$, $\vec{C} = 3\hat{i} + \hat{j} + 2\hat{k}$,

show that the triangle is an equilateral one.

33. If $\vec{a} = \hat{i}+\hat{j}+\hat{k}$, $\vec{b} = 4\hat{i}-2\hat{j}+3\hat{k}$ and $\vec{c} = \hat{i}-2\hat{j}+\hat{k}$, find a vector of magnitude 6 units, which is parallel to the vector $2\vec{a} - \vec{b} + 3\vec{c}$.

(A.I.C.B.S.E. 2010)

Long Answer Type Questions

LATQ

34. Show that the four points A, B, C, D with position vectors $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ respectively, are coplanar if and only if $3\vec{a} - 2\vec{b} + \vec{c} - 2\vec{d} = \vec{0}$. HOTS

35. Show that the four points P, Q, R, S with position vectors $\vec{p}, \vec{q}, \vec{r}, \vec{s}$ respectively such that $5\vec{p} - 2\vec{q} + 6\vec{r} - 9\vec{s} = \vec{0}$, are coplanar. Also find the position vector of the point of intersection of the lines PR and QS.

36. Prove that the necessary and sufficient condition for three vectors $\vec{a}, \vec{b}$ and $\vec{c}$ to be coplanar is that there exist scalars *l*, *m*, *n* (not all zero simultaneously) such that $l\vec{a} + m\vec{b} + n\vec{c} = \vec{0}$.

Answers

1. <6, –3, 2>

2. (*i*) $\hat{i} + \hat{j} + \hat{k}, \hat{i} - \hat{j} + \hat{k}$ (*ii*) $\hat{i} + \hat{j} + \hat{k}, 2\hat{i} + 2\hat{j} + 2\hat{k}$.

3. $-4\hat{j} - \hat{k}$.

4. (*i*) $\frac{1}{\sqrt{5}}(\hat{i} + 2\hat{j})$ (*ii*) $\frac{1}{\sqrt{2}}(\hat{i} + \hat{j})$.

5. (*a*) (*i*) 1 (*ii*) $\sqrt{26}$ (*b*) $\pm\frac{1}{\sqrt{3}}$.

6. (*a*) $\frac{1}{\sqrt{6}}\hat{i} + \frac{1}{\sqrt{6}}\hat{j} + \frac{2}{\sqrt{6}}\hat{k}$

(*b*) (*i*) $\frac{2}{\sqrt{14}}\hat{i} + \frac{3}{\sqrt{14}}\hat{j} + \frac{1}{\sqrt{14}}\hat{k}$

(*ii*) $\frac{1}{7}\left(3\hat{i} + 2\hat{j} + 6\hat{k}\right)$ (*iii*) $\frac{1}{3}\left(2\hat{i} + \hat{j} + 2\hat{k}\right)$

(*iv*) $\frac{1}{7}(2\hat{i} - 3\hat{j} + 6\hat{k})$.

7. $\frac{1}{\sqrt{3}}(\hat{i}+\hat{j}+\hat{k})$.

8. (*i*) $x = 2, y = 3$ (*ii*) $x = 2, y = 2, z = 1$.

9. (*a*) Yes and No (*b*) No.

10. $<\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}>$.

12. $\frac{1}{\sqrt{2}}(\hat{i}+\hat{k})$.

13. (*i*) $\vec{a}$ (*ii*) $-\vec{a}+4\vec{b}$ (*iii*) $5\vec{b}$.

14. (*i*) $3\hat{i}+2\hat{j}+\hat{k}$ (*ii*) $2\hat{i}+3\hat{j}+\hat{k}$.

15. (*i*) $\frac{8}{\sqrt{30}}(5\hat{i}-\hat{j}+2\hat{k})$ (*ii*) $2(2\hat{i}-\hat{j}+2\hat{k})$

(*iii*) $5(\hat{i}-2\hat{j}+2\hat{k})$ (*iv*) $3(-2\hat{i}+\hat{j}+2\hat{k})$.

16. $x_2 - x_1, y_2 - y_1, z_2 - z_1$;

$$\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}.$$

17. (*a*) (*i*) 15 (*ii*) 6 (*iii*) 0 (*b*) $\sqrt{86}$.

18. Yes ; No.

19. $\sqrt{13}$; 2, 3.

20. $\frac{15\sqrt{3}}{2}, \frac{15}{2}$.

21. $\sqrt{114}$; $< -\frac{5}{\sqrt{114}}, -\frac{8}{\sqrt{114}}, \frac{5}{\sqrt{114}} >$.

22. 1, 1, $\frac{1}{3}$.

24. (*i*) $\frac{4}{\sqrt{29}}\hat{i}+\frac{3}{\sqrt{29}}\hat{j}-\frac{2}{\sqrt{29}}\hat{k}$

(*ii*) $\frac{8}{3\sqrt{10}}\hat{i}+\frac{1}{3\sqrt{10}}\hat{j}+\frac{5}{3\sqrt{10}}\hat{k}$

(*iii*) $-\frac{2}{3}\hat{i}+\frac{1}{3}\hat{j}-\frac{2}{3}\hat{k}$.

25. $\frac{1}{\sqrt{22}}(3\hat{i}-3\hat{j}+2\hat{k})$. **26.** (*a*) $k^2 = l^2$.

27. (*a*) (*i*) $\frac{5}{3}\vec{a}$ (*ii*) $4\vec{b}-\vec{a}$ (*b*) $3\vec{a}+5\vec{b}$.

33. $2\left(\hat{i}-2\hat{j}+2\hat{k}\right)$.

35. $\frac{5\vec{p}+6\vec{r}}{11}$ *or* $\frac{2\vec{q}+9\vec{s}}{11}$.

Hints to Selected Questions

23. $\vec{b} = 2\vec{a}$.

26. Given vectors are parallel $\Rightarrow \frac{k}{l} = \frac{l}{k} \Rightarrow k^2 = l^2$.

28. (*i*) $\overrightarrow{BC} = -2\overrightarrow{AB}$; etc.

29. $\overrightarrow{AB} = 2\overrightarrow{BC}$; etc.

30. (*i*) Let $\vec{b} = x\vec{a} + y\vec{c}$. Then etc.

31. Here $BC^2 \neq AB^2 + CA^2$.

32. Prove that AB = BC = CA.

APPLICATION OF VECTORS

ILLUSTRATIVE EXAMPLES

Example 1. D, E, F are the middle points of the sides [BC], [CA], [AB] respectively of a triangle ABC. Show that :

(i) FE is parallel to BC and half of its length,

(ii) the sum of the vectors $\overrightarrow{AD}$, $\overrightarrow{BE}$, $\overrightarrow{CF}$ is zero,

(iii) the medians have a common point of trisection i.e. they are concurrent. *(H.B. 2010)*

Solution. Let $\vec{a}$, $\vec{b}$ and $\vec{c}$ be the position vector of A, B and C respectively.

(*i*) Since D is the mid-point of [BC],

$\therefore$ position vector of D is $\frac{\vec{b}+\vec{c}}{2}$.

Similarly, the position vector of E is $\frac{\vec{c}+\vec{a}}{2}$

and that of F is $\frac{\vec{a}+\vec{b}}{2}$.

Let O be the origin of reference (not shown in fig.)

Now $\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = \vec{c} - \vec{b}$

and $\overrightarrow{FE} = \overrightarrow{OE} - \overrightarrow{OF} = \dfrac{\vec{c}+\vec{a}}{2} - \dfrac{\vec{a}+\vec{b}}{2} = \dfrac{\vec{c}-\vec{b}}{2}$.

$\therefore \quad \overrightarrow{FE} = \frac{1}{2}\overrightarrow{BC}$.

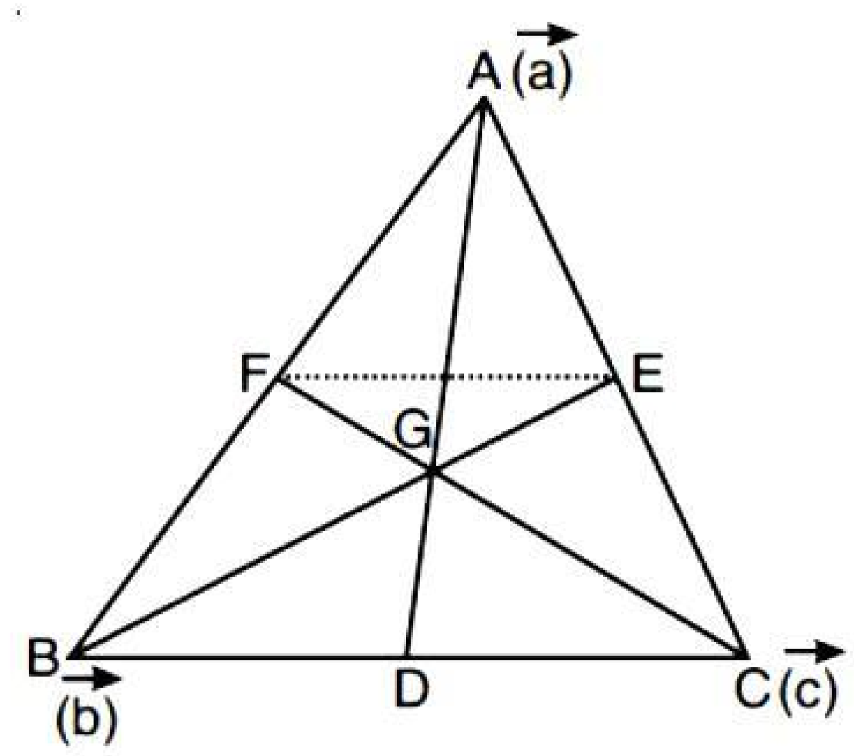


Fig.

Hence, FE || BC and FE $= \frac{1}{2}$BC.

(*ii*) $\overrightarrow{AD} = \overrightarrow{OD} - \overrightarrow{OA} = \dfrac{\vec{b}+\vec{c}}{2} - \vec{a} = \dfrac{\vec{b}+\vec{c}-2\vec{a}}{2}$.

Similarly, $\overrightarrow{BE} = \dfrac{\vec{c}+\vec{a}-2\vec{b}}{2}$ and $\overrightarrow{CF} = \dfrac{\vec{a}+\vec{b}-2\vec{c}}{2}$.

Adding, $\overrightarrow{AD} + \overrightarrow{BE} + \overrightarrow{CF}$

$$= \left(\frac{\vec{b}+\vec{c}-2\vec{a}}{2}\right) + \left(\frac{\vec{c}+\vec{a}-2\vec{b}}{2}\right) + \left(\frac{\vec{a}+\vec{b}-2\vec{c}}{2}\right)$$

$$= \frac{\vec{b}+\vec{c}-2\vec{a}+\vec{c}+\vec{a}-2\vec{b}+\vec{a}+\vec{b}-2\vec{c}}{2} = \vec{0}.$$

(*iii*) Let G be the point, which divides [AD] in the ratio 2 : 1.

$\therefore$ The position vector of G

$$= \frac{1(\vec{a}) + 2\left(\dfrac{\vec{b}+\vec{c}}{2}\right)}{1+2} \quad [\textit{Section Formula}]$$

$$= \frac{\vec{a}+\vec{b}+\vec{c}}{3}.$$

The symmetry of this result shows that G also lies on [BE] and [CF] at their point of trisection.

Hence, the medians are concurrent at G, which divides each median in the ratio 2 : 1.

Example 2. Show, by vector method, that the angular bisectors of a triangle are concurrent and find an expression for the position vector of the point of concurrency in terms of the position vectors of the vertices.

Solution. Let $\vec{a_1}, \vec{b_1}, \vec{c_1}$ be the position vectors of the vertices of ΔABC.

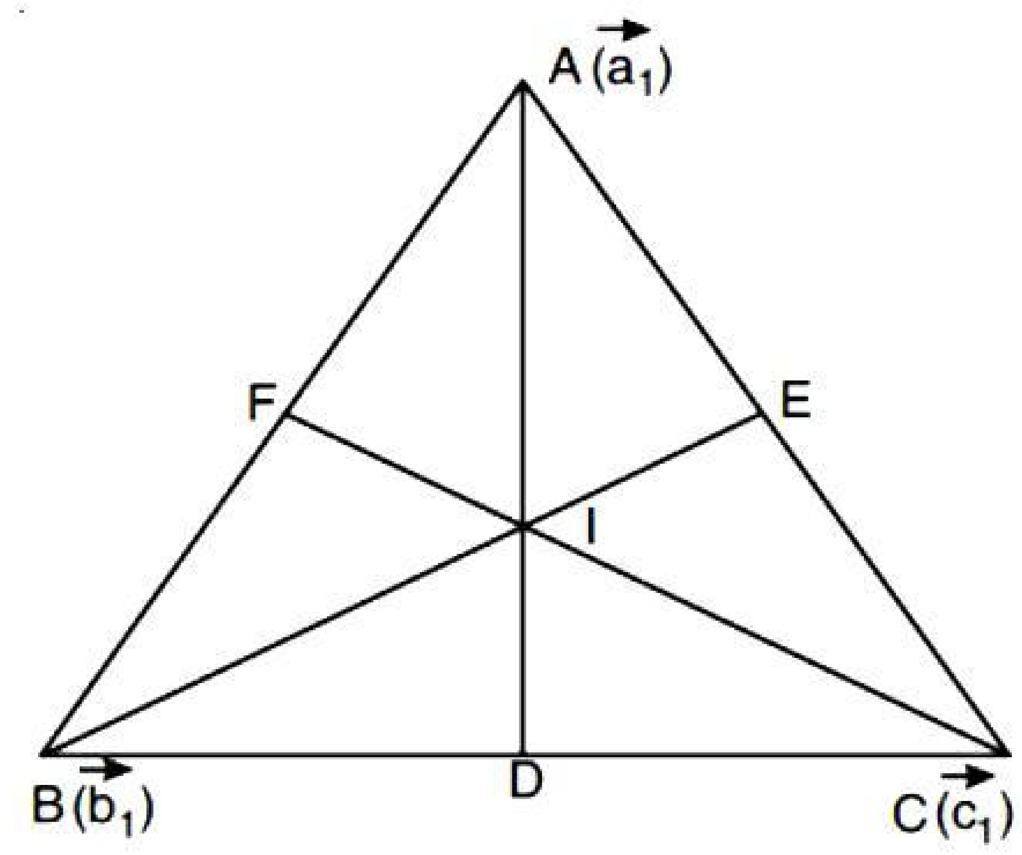


Fig.

Let the bisectors of angles A, B and C meet the opposite sides BC, CA and AB at D, E and F respectively.

Let I be the point, where AD and BE meet.

$\therefore$ Position vector of D is $\left(\dfrac{b\vec{b_1} + c\vec{c_1}}{b+c}\right)$, where

CA $= b$ and AB $= c$.

Also, $\dfrac{BA}{AI} = \dfrac{BD}{DI} \Rightarrow \dfrac{c}{AI} = \dfrac{ac}{(b+c)\,DI}$

$$\Rightarrow \frac{AI}{DI} = \frac{b+c}{a}.$$

Hence, the position vector of I is $\dfrac{a\vec{a_1} + b\vec{b_1} + c\vec{c_1}}{a+b+c}$.

The symmetry of the result shows that the point I also lies on the intersection of the bisectors of angles B and C.

Hence, the three bisectors are concurrent.

Example 3. Prove, by vector method, that the diagonals of a parallelogram bisect each other ; conversely, if the diagonals of a quadrilateral bisect each other, it is a parallelogram.

Solution. Let ABCD be a || gm.

Let $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ be the position vectors of A, B, C, D respectively referred to O as origin.

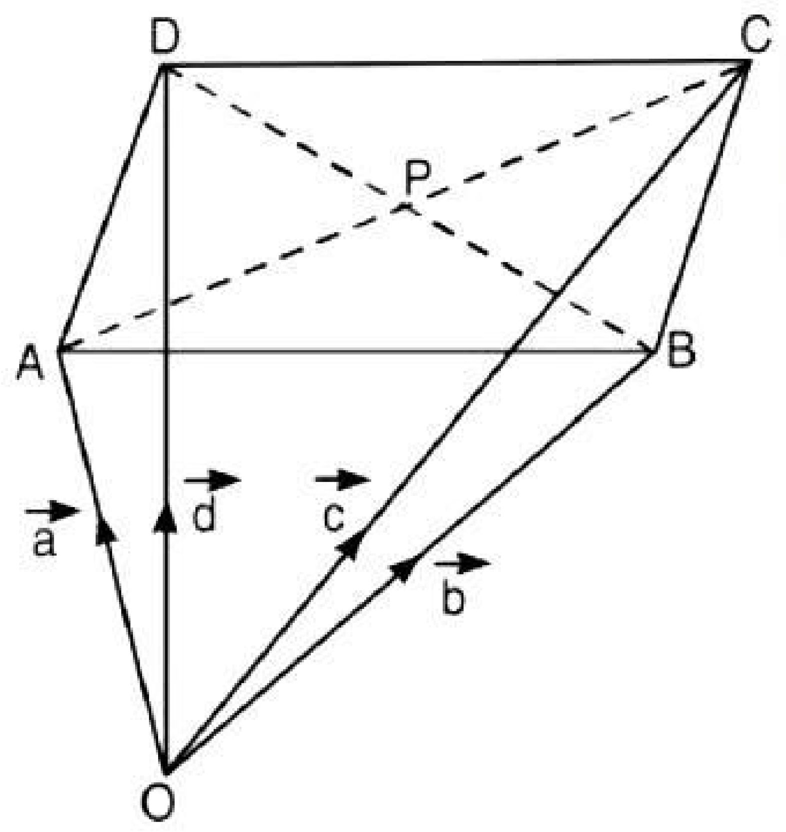


Fig.

Since ABCD is a || gm.,

$\therefore \quad AB = DC$ and AB || DC

$\therefore \quad \overrightarrow{AB} = \overrightarrow{DC}$

$\Rightarrow \quad \overrightarrow{OB} - \overrightarrow{OA} = \overrightarrow{OC} - \overrightarrow{OD}$

$\Rightarrow \quad \vec{b} - \vec{a} = \vec{c} - \vec{d}$

$\Rightarrow \quad \vec{b} + \vec{d} = \vec{a} + \vec{c}$

$\Rightarrow \quad \dfrac{\vec{a} + \vec{c}}{2} = \dfrac{\vec{b} + \vec{d}}{2}$ *[Dividing by 2]*

$\Rightarrow$ Mid-points of [AC] and [BD] coincide.

Hence, diagonals of a || gm. bisect each other.

Conversely. Since the diagonals [AC] and [BD] have the same mid-point,

$\therefore \quad \dfrac{\vec{a} + \vec{c}}{2} = \dfrac{\vec{b} + \vec{d}}{2}$

$\Rightarrow \quad \vec{a} + \vec{c} = \vec{b} + \vec{d}$

$\Rightarrow \quad \vec{b} - \vec{a} = \vec{c} - \vec{d}$

$\Rightarrow \quad \overrightarrow{OB} - \overrightarrow{OA} = \overrightarrow{OC} - \overrightarrow{OD}$

$\Rightarrow \quad \overrightarrow{AB} = \overrightarrow{DC}$

$\Rightarrow \quad AB = DC$ and AB || DC ...(1)

Similarly, $AD = BC$ and AD || BC ...(2)

Combining (1) and (2), ABCD is a || gm.

Example 4. Show that the diagonals of quadrilateral bisect each other if and only if it is a parallelogram, by using vector method. ***(H.B. 2010)***

Solution. Let ABCD be the quadrilateral whose vertices have position vectors $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ respectively.

Now [AC] and [BD] bisect each other,

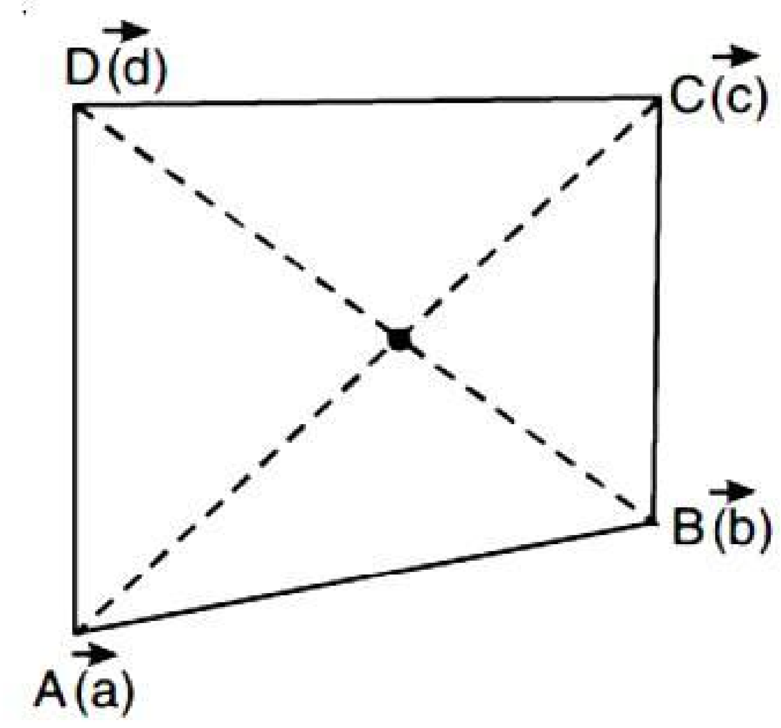


Fig.

$\therefore \quad \dfrac{\vec{a} + \vec{c}}{2} = \dfrac{\vec{b} + \vec{d}}{2}$

$\Rightarrow \quad \vec{a} + \vec{c} = \vec{b} + \vec{d}$

$\Rightarrow \quad \vec{b} - \vec{a} = \vec{c} - \vec{d}$

$\Rightarrow \quad \overrightarrow{AB} = \overrightarrow{DC}$

$\Rightarrow \quad AB = DC$, AB || DC

$\Rightarrow$ ABCD is a || gm.

Conversely. If ABCD is a parallelogram,

then $\quad \overrightarrow{AB} = \overrightarrow{DC}$

$\Rightarrow \quad \vec{b} - \vec{a} = \vec{c} - \vec{d}$

$\Rightarrow \quad \vec{a} + \vec{c} = \vec{b} + \vec{d}$

$\Rightarrow \quad \dfrac{\vec{a} + \vec{c}}{2} = \dfrac{\vec{b} + \vec{d}}{2}$

$\Rightarrow$ diagonals [AC] and [BD] bisect each other.

Example 5. Show that the st. line joining the mid-points of two non-parallel sides of a trapezium is parallel to the bases and is equal to half of the sum of their lengths.

Solution. Let ABCD be a trapezium whose sides AB and DC are parallel.

Let A be the origin of reference.

Let $\vec{b}$ and $\vec{d}$ be the position vectors of B and D respectively.

Since DC || AB, *[Given]*

$\therefore \quad \overrightarrow{DC}$ and $\overrightarrow{AB}$ are parallel

$\Rightarrow \quad \overrightarrow{DC} = m\,\overrightarrow{AB}$, where '$m$' is some scalar

$\Rightarrow \quad \overrightarrow{DC} = m\,\vec{b}$.

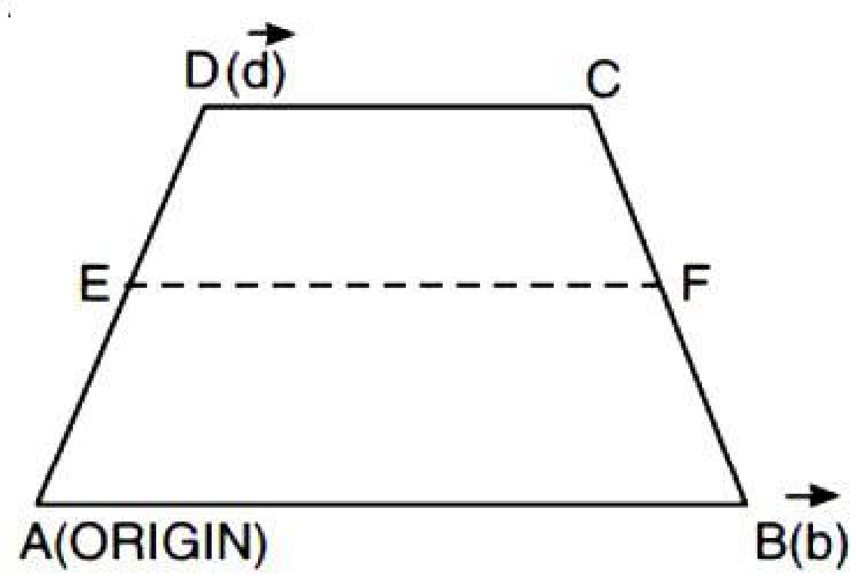


Fig.

Now $\quad \overrightarrow{AC} = \overrightarrow{AD} + \overrightarrow{DC} = \vec{d} + m\,\vec{b}$.

If E is the mid-point of [AD], then its position vector is

$\dfrac{\vec{0}+\vec{d}}{2}$ *i.e.* $\dfrac{\vec{d}}{2}$.

If F is the mid-pt. of [BC], then its position vector is

$$\frac{\vec{b}+(\vec{d}+m\,\vec{b})}{2} = \frac{\vec{d}}{2} + \frac{(m+1)\,\vec{b}}{2}.$$

$\therefore \quad \overrightarrow{EF} = \overrightarrow{AF} - \overrightarrow{AE}$

$$= \frac{\vec{d}}{2} + \frac{(m+1)\,\vec{b}}{2} - \frac{\vec{d}}{2} = \frac{(m+1)\,\vec{b}}{2}$$

$\Rightarrow \quad \overrightarrow{EF} = \dfrac{m+1}{2}\overrightarrow{AB}$.

Hence, EF || AB and hence || DC.

Also, $\quad \overrightarrow{EF} = \dfrac{m+1}{2}\overrightarrow{AB}$.

$\therefore \quad \dfrac{EF}{AB} = \dfrac{m+1}{2}$

$\Rightarrow \quad EF = \dfrac{m+1}{2} AB = \dfrac{1}{2}(AB + m\,AB)$.

But $\quad \overrightarrow{DC} = m\,\overrightarrow{AB}$.

$\therefore \quad DC = m\,AB$.

Hence, $\quad EF = \dfrac{1}{2}(AB + DC)$.

EXERCISE 10 (d)

Short Answer Type Questions

SATQ

1. If $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ respectively, are position vectors representing the vertices A, B, C, D of a parallelogram, then write $\vec{d}$ in terms of $\vec{a}, \vec{b}$ and $\vec{c}$. *(Kerala B. 2015)*

2. If Q is the point of intersection of the medians of a triangle ABC, prove that $\overrightarrow{QA} + \overrightarrow{QB} + \overrightarrow{QC} = \vec{0}$.

3. G is the centroid of a triangle ABC, show that :

$$\overrightarrow{GA} + \overrightarrow{GB} + \overrightarrow{GC} = \vec{0}.$$

4. Let $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ be the position vectors of the four distinct points, P, Q, R, S respectively. If $\vec{b} - \vec{a} = \vec{c} - \vec{d}$, show that PQRS is a parallelogram.

5. Show that the line joining any vertex of a parallelogram to the mid-point of an opposite side divides the opposite diagonal in the ratio 2 : 1.

6. Show that the lines joining the mid-points of the consecutive sides of a quadrilateral form a parallelogram.

Long Answer Type Questions

LATQ

8. Show that if P, A, B are any three points,

then $\lambda\,\overrightarrow{PA} + \mu\,\overrightarrow{PB} = (\lambda + \mu)\,\overrightarrow{PC}$,

where C divides [AB] in the ratio $\mu : \lambda$.

Answers

1. $\vec{d} = \vec{a} + \vec{c} - \vec{b}$.

Hints to Selected Questions

4. $\vec{b} - \vec{a} = \vec{c} - \vec{d} \Rightarrow \vec{a} - \vec{b} = \vec{d} - \vec{c} \Rightarrow \overrightarrow{QP} = \overrightarrow{RS}$; etc.

SUB CHAPTER
10.2
Product of Vectors

10.16. INTRODUCTION

In the previous sub-chapter, we have done elementary topics and definitions of vectors. We shall deal with the product of vectors in the present course of study.

10.17. PRODUCT OF TWO VECTORS

The product of two vectors can be defined in two ways : (*i*) Scalar product (*ii*) Vector product.

(*i*) Scalar product of two vectors is always a *scalar quantity.*

(*ii*) Vector product of two vectors is always a *vector quantity.*

10.18. SCALAR PRODUCT

Definition

Let $\vec{a}$ and $\vec{b}$ be any two vectors. The scalar product of $\vec{a}$ and $\vec{b}$ is defined as :

(i) $\vec{a}\,.\,\vec{b} = |\vec{a}||\vec{b}|\cos\theta$, *where 'θ' is the angle ($0 \le \theta < \pi$) between $\vec{a}$ and $\vec{b}$; $\vec{a}, \vec{b}$ being non-zero vectors*

(ii) $\vec{a}\,.\,\vec{b} = 0$ *if either $\vec{a}$ or $\vec{b}$ is a zero vector.*

Notation. The scalar product of two vectors is written as $\vec{a}\,.\,\vec{b}$ or $(\vec{a}\,.\,\vec{b})$.

($\vec{a}\,.\,\vec{b}$ *is read as* $\vec{a}$ *dot* $\vec{b}$)

Because of this, the scalar product is sometimes called the **dot product.**

Hence, **$\vec{a}\,.\,\vec{b} = |\vec{a}||\vec{b}|\cos\theta$, which is +ve, –ve or zero according as 'θ' is acute, obtuse or a right angle.**

Geometrical Interpretation. *The scalar product of two vectors is the product of the modulus of either vector and the scalar component of the other in its direction.*

Let $\overrightarrow{OA} = \vec{a}, = \overrightarrow{OB} = \vec{b}$

and $\angle BOA = \theta.$

By *def.*, $\vec{a}\,.\,\vec{b} = |\vec{a}||\vec{b}|\cos\theta,$

where $|\vec{a}| = a = OA$

and $|\vec{b}| = b = OB.$

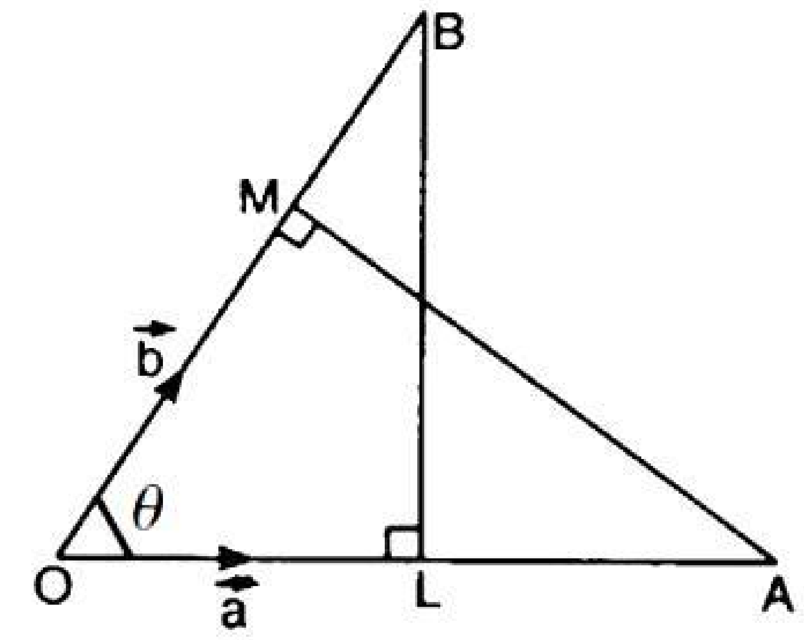


Fig.

From B, draw BL $\perp$ OA.

From A, draw AM $\perp$ OB.

$\therefore \quad OL = \text{Projection of } \overrightarrow{OB} \text{ on } \overrightarrow{OA} = OB \cos\theta = b\cos\theta$

and $\quad OM = \text{Projection of } \overrightarrow{OA} \text{ on } \overrightarrow{OB} = OA\cos\theta = a\cos\theta.$

$\therefore \quad \vec{a}\,.\,\vec{b} = ab\cos\theta = a\,(b\cos\theta) = OA\,(\text{Projection of } \overrightarrow{OB} \text{ on } \overrightarrow{OA})$

$= |\vec{a}|\,(\text{Projection of } \vec{b} \text{ in the direction of } \vec{a}).$

Again, $\vec{a}\,.\,\vec{b} = ab\cos\theta = b\,(a\cos\theta) = OB\,(\text{Projection of } \overrightarrow{OA} \text{ on } \overrightarrow{OB})$

$= |\vec{b}|\,(\text{Projection of } \vec{a} \text{ in the direction of } \vec{b}).$

KEY POINT

If $\vec{a}$ and $\vec{b}$ are two vectors, then the projection of $\vec{b}$ in the direction of $\vec{a} = \dfrac{\vec{a}\,.\,\vec{b}}{|\vec{a}|}$.

10.19. CONDITION OF PERPENDICULARITY

Theorem. *If $\vec{a}$ and $\vec{b}$ are perpendicular vectors, then $\vec{a}\,.\,\vec{b} = 0$.*

Proof. We have : $\vec{a}\,.\,\vec{b} = |\vec{a}||\vec{b}|\cos\theta$ ...(1),

where 'θ' is the angle between the vectors $\vec{a}$ and $\vec{b}$.

Since the vectors $\vec{a}$ and $\vec{b}$ are perpendicular, [*Given*]

$\therefore \quad \theta = \dfrac{\pi}{2} \quad \Rightarrow \quad \cos\theta = \cos\left(\dfrac{\pi}{2}\right) = 0.$

$\therefore$ From (1), $\vec{a}\,.\,\vec{b} = 0.$

Conversely. *If $\vec{a}\,.\,\vec{b} = 0$, then either at least one of the two vectors is a zero vector or two vectors are perpendicular.*

Proof. Here $\vec{a}\,.\,\vec{b} = 0$ *i.e.* $|\vec{a}||\vec{b}|\cos\theta = 0$, where 'θ' is the angle between the vectors $\vec{a}$ and $\vec{b}$.

$\therefore \quad$ Either $\vec{a} = 0$ or $\vec{b} = 0$ or $\cos\theta = 0$ *i.e.* Either $|\vec{a}| = 0$ or $|\vec{b}| = 0$ or $\theta = \dfrac{\pi}{2}$

i.e. Either $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$ or $\vec{a}, \vec{b}$ are perpendicular vectors.
Hence, the result.

10.20. IMPORTANT RESULTS

(I) When the two vectors are like parallel.

Here $\quad \theta = 0° \quad \therefore \cos\theta = \cos 0° = 1. \quad \therefore \quad \vec{a}\,.\,\vec{b} = |\vec{a}||\vec{b}|\cos\theta = ab\,(1) = ab.$

(II) When the two vectors are unlike parallel.

Here $\quad \theta = 180° \quad \therefore \cos\theta = \cos 180° = -1. \quad \therefore \quad \vec{a}\,.\,\vec{b} = |\vec{a}||\vec{b}|\cos\theta = ab\,(-1) = -ab.$

(III) $\vec{a} \cdot \vec{a} = a^2$.

Putting $\vec{b} = \vec{a}$, we have : $\vec{a} \cdot \vec{a} = |\vec{a}||\vec{a}|$ cos (angle between $\vec{a}$ and $\vec{a}$) $= a^2 \cos 0° = a^2 (1) = a^2$.

(IV) Since $\hat{i}, \hat{j}, \hat{k}$ are unit vectors perpendicular to each other,

$\therefore \quad \hat{i} \cdot \hat{j} = \hat{j} \cdot \hat{k} = \hat{k} \cdot \hat{i} = 0 \quad$ and $\quad \hat{i}^2 = \hat{j}^2 = \hat{k}^2 = 1$.

These results can be remembered with the help of the following **table :**

TABLE

$\odot$	$\hat{i}$	$\hat{j}$	$\hat{k}$
$\hat{i}$	1	0	0
$\hat{j}$	0	1	0
$\hat{k}$	0	0	1

10.21. PROPERTIES OF SCALAR OR DOT PRODUCT

Property I. Commutative Law. *If* $\vec{a}, \vec{b}$ *be any two vectors, then* $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$

Proof. Let 'θ' be the angle between the two vectors $\vec{a}$ and $\vec{b}$.

$\therefore$ By *def.*, $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta = ab\cos\theta = ba\cos\theta$ [$\because$ *a, b, being numbers, are commutative*]

$= |\vec{b}||\vec{a}| \cos\theta = \vec{b} \cdot \vec{a}$ [*Def.*]

Hence, $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$

Property II. Associative law does not hold.

If $\vec{a}, \vec{b}, \vec{c}$ *be any three vectors, then* $(\vec{a} \cdot \vec{b}) \cdot \vec{c} \neq \vec{a} \cdot (\vec{b} \cdot \vec{c})$

Proof. Since $\vec{a} \cdot \vec{b}$ is a scalar, $\therefore$ $(\vec{a} \cdot \vec{b}) \cdot \vec{c}$ cannot be defined.

Similarly, $\vec{a} \cdot (\vec{b} \cdot \vec{c})$ cannot be defined. Hence, $(\vec{a} \cdot \vec{b}) \cdot \vec{c} \neq \vec{a} \cdot (\vec{b} \cdot \vec{c})$.

Property III. *If m is any non-zero scalar, then* $(m\vec{a}) \cdot \vec{b} = m(\vec{a} \cdot \vec{b}) = \vec{a} \cdot (m\vec{b})$.

Proof. Case I. *When m > 0.*

Here $m\vec{a}$ has the same direction as that of $\vec{a}$.

$\therefore$ The angle between $m\vec{a}$ and $\vec{b}$ = the angle between $\vec{a}$ and $\vec{b} = \theta$ (say) ...(1)

$\therefore \; (m\vec{a}) \cdot \vec{b} = |m\vec{a}||\vec{b}| \cos\angle(m\vec{a}, \vec{b}) = m\,ab\cos\angle(\vec{a}, \vec{b})$

$= m(ab\cos\theta) = m[|\vec{a}||\vec{b}|\cos\theta]$ [*Using (1)*]

$= m(\vec{a} \cdot \vec{b})$

Case II. *When $m < 0$.*

Here $m\vec{a}$ has the opposite direction to that of $\vec{a}$.

If the angle between $\vec{a}$ and $\vec{b} = \theta$, then the angle between $m\vec{a}$ and $\vec{b} = \pi - \theta$.

$$\therefore \quad (m\vec{a}) \cdot \vec{b} = |m\vec{a}||\vec{b}| \cos \angle (m\vec{a}, \vec{b})$$

$$= -\,m ab \cos (\pi - \theta) = m ab \cos \theta = m(|\vec{a}||\vec{b}| \cos \theta) \qquad [\because m < 0]$$

$$= m(\vec{a} \cdot \vec{b})$$

Thus in both cases, $(m\vec{a}) \cdot \vec{b} = m(\vec{a} \cdot \vec{b})$ ...(2)

Similarly, $\vec{a} \cdot (m\vec{b}) = m(\vec{a} \cdot \vec{b})$ ...(3)

Hence, (2) and (3) $\Rightarrow$ $(m\vec{a}) \cdot \vec{b} = m(\vec{a} \cdot \vec{b}) = \vec{a} \cdot (m\vec{b})$, which proves the result.

Property IV. Distributive Laws.

If $\vec{a}, \vec{b}, \vec{c}$ are three vectors, then :

(*i*) $\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$ **(Left Law)**

(*ii*) $(\vec{b} + \vec{c}) \cdot \vec{a} = \vec{b} \cdot \vec{a} + \vec{c} \cdot \vec{a}$ **(Right law)**

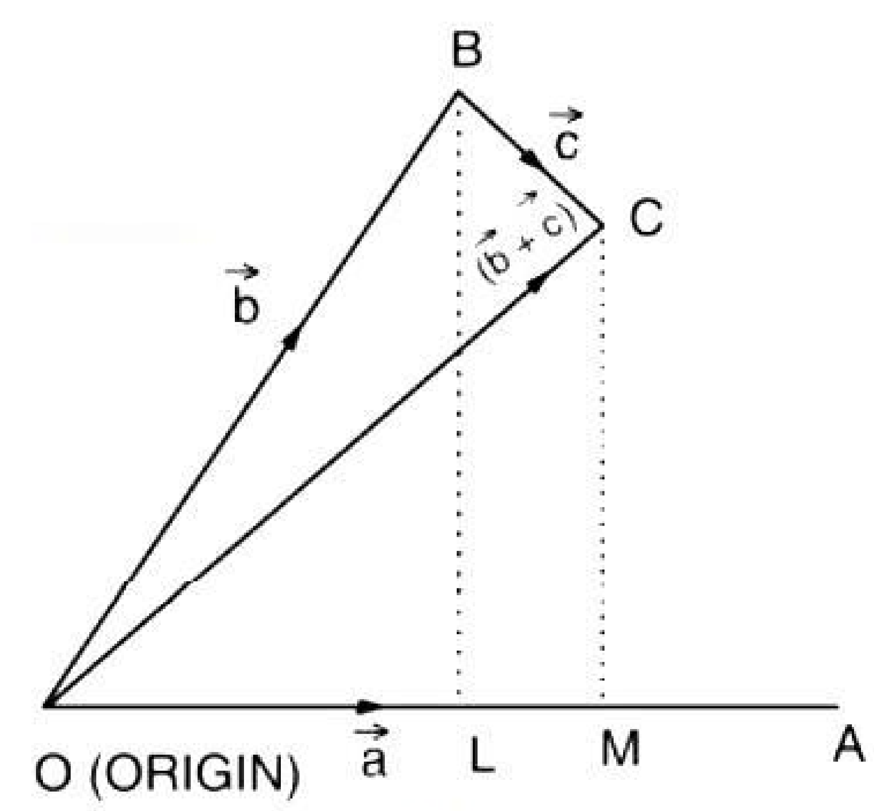


Fig.

Proof. Let O be the origin.

Let $\overrightarrow{OA} = \vec{a}, \overrightarrow{OB} = \vec{b}, \overrightarrow{BC} = \vec{c}$ so that $\overrightarrow{OC} = \overrightarrow{OB} + \overrightarrow{BC} = \vec{b} + \vec{c}$.

Draw BL $\perp$ OA and CM $\perp$ OA.

$\therefore$ OL = Projection of $\overrightarrow{OB}$ on $\overrightarrow{OA}$, OM = Projection of $\overrightarrow{OC}$ on $\overrightarrow{OA}$

$\therefore$ LM = Projection of $\overrightarrow{BC}$ on $\overrightarrow{OA}$.

(*i*) $\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \overrightarrow{OC}$

= (Modulus of $\vec{a}$) (Scalar component of $\overrightarrow{OC}$ along $\vec{a}$)

= a (OM) = a (OL + LM) = a (OL) + a (LM)

= (Modulus of $\vec{a}$) (Scalar component of $\overrightarrow{OB}$ along $\vec{a}$) + (Modulus of $\vec{a}$) (Scalar component of $\overrightarrow{BC}$ along $\vec{a}$)

= $\vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$, which proves the result.

(*ii*) $(\vec{b} + \vec{c}) \cdot \vec{a} = \vec{a} \cdot (\vec{b} + \vec{c})$ [*Commutative Law*]

$= \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$ [*Part* (*i*)]

Hence, $(\vec{b} + \vec{c}) \cdot \vec{a} = \vec{b} \cdot \vec{a} + \vec{c} \cdot \vec{a}$ [*Commutative Law*]

Cor. $\vec{a} \cdot (\vec{b} - \vec{c}) = \vec{a} \cdot [\vec{b} + (-\vec{c})] = \vec{a} \cdot \vec{b} + \vec{a} \cdot (-\vec{c}) = \vec{a} \cdot \vec{b} - \vec{a} \cdot \vec{c}$

Property V. $\vec{a} \cdot \vec{a} \geq 0$, *where* $\vec{a}$ *is any vector.*

Proof. $\vec{a} \cdot \vec{a} = |\vec{a}||\vec{a}| \cos 0° = a^2 (1) = a^2$.

This, being a perfect square, is never negative. Hence, $\vec{a} \cdot \vec{a} \geq 0$.

Property VI. *If* $\vec{a}, \vec{b}$ *are any two vectors, then* :

(*i*) $|\vec{a} + \vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 + 2\vec{a} \cdot \vec{b}$

(*ii*) $|\vec{a} - \vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 - 2\vec{a} \cdot \vec{b}$

(*iii*) $(\vec{a} + \vec{b}) \cdot (\vec{a} - \vec{b}) = |\vec{a}|^2 - |\vec{b}|^2$

(*iv*) $\vec{a} \cdot \vec{b} = \frac{1}{4}[|\vec{a} + \vec{b}|^2 - |\vec{a} - \vec{b}|^2]$.

Proof. (*i*)
$$|\vec{a} + \vec{b}|^2 = (\vec{a} + \vec{b}) \cdot (\vec{a} + \vec{b}) \qquad [\because \vec{a} \cdot \vec{a} = |\vec{a}|^2]$$
$$= (\vec{a} + \vec{b}) \cdot \vec{a} + (\vec{a} + \vec{b}) \cdot \vec{b} \qquad [\textit{Left Distributive Law}]$$
$$= \vec{a} \cdot \vec{a} + \vec{b} \cdot \vec{a} + \vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{b} \qquad [\textit{Right Distributive Law}]$$
$$= |\vec{a}|^2 + 2\vec{a} \cdot \vec{b} + |\vec{b}|^2 \qquad [\because \vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}]$$
$$= |\vec{a}|^2 + |\vec{b}|^2 + 2\vec{a} \cdot \vec{b}$$

(*ii*)
$$|\vec{a} - \vec{b}|^2 = (\vec{a} - \vec{b}) \cdot (\vec{a} - \vec{b}) \qquad [\because \vec{a} \cdot \vec{a} = |\vec{a}|^2]$$
$$= (\vec{a} - \vec{b}) \cdot \vec{a} - (\vec{a} - \vec{b}) \cdot \vec{b} \qquad [\textit{Left Distributive Law}]$$
$$= \vec{a} \cdot \vec{a} - \vec{b} \cdot \vec{a} - \vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{b} \qquad [\textit{Right Distributive Law}]$$
$$= |\vec{a}|^2 - 2\vec{a} \cdot \vec{b} + |\vec{b}|^2 \qquad [\because \vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}]$$
$$= |\vec{a}|^2 + |\vec{b}|^2 - 2\vec{a} \cdot \vec{b}$$

(*iii*)
$$(\vec{a} + \vec{b}) \cdot (\vec{a} - \vec{b}) = (\vec{a} + \vec{b}) \cdot \vec{a} - (\vec{a} + \vec{b}) \cdot \vec{b} \qquad [\textit{Left Distributive Law}]$$
$$= \vec{a} \cdot \vec{a} + \vec{b} \cdot \vec{a} - \vec{a} \cdot \vec{b} - \vec{b} \cdot \vec{b} \qquad [\textit{Right Distributive Law}]$$
$$= |\vec{a}|^2 - |\vec{b}|^2. \qquad [\because \vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}]$$

(*iv*) Using (*i*) and (*ii*),
$$|\vec{a} + \vec{b}|^2 - |\vec{a} - \vec{b}|^2 = (|\vec{a}|^2 + |\vec{b}|^2 + 2\vec{a} \cdot \vec{b}) - (|\vec{a}|^2 + |\vec{b}|^2 - 2\vec{a} \cdot \vec{b}) = 4\vec{a} \cdot \vec{b}$$

Hence,
$$\vec{a} \cdot \vec{b} = \frac{1}{4}[|\vec{a} + \vec{b}|^2 - |\vec{a} - \vec{b}|^2].$$

10.22. THEOREM

If $\vec{a} = (a_1, a_2, a_3)$ *and* $\vec{b} = (b_1, b_2, b_3)$, *then* $(\vec{a} \cdot \vec{b}) = (a_1 b_1, a_2 b_2, a_3 b_3)$.

Proof. Here $\vec{a} = a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k}$ and $\vec{b} = b_1 \hat{i} + b_2 \hat{j} + b_3 \hat{k}$.

$\therefore \qquad \vec{a} \cdot \vec{b} = (a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k}) \cdot (b_1 \hat{i} + b_2 \hat{j} + b_3 \hat{k})$

$$= a_1b_1 \hat{i} \cdot \hat{i} + a_1b_2 \hat{i} \cdot \hat{j} + a_1b_3 \hat{i} \cdot \hat{k} + a_2b_1 \hat{j} \cdot \hat{i} + a_2b_2 \hat{j} \cdot \hat{j} + a_2b_3 \hat{j} \cdot \hat{k} + a_3b_1 \hat{k} \cdot \hat{i} + a_3b_2 \hat{k} \cdot \hat{j} + a_3b_3 \hat{k} \cdot \hat{k}$$

[*Applying Distributive Law*]

$$= a_1b_1 (1) + a_1b_2 (0) + a_1b_3 (0) + a_2b_1 (0) + a_2b_2 (1) + a_2b_3 (0) + a_3b_1 (0) + a_3b_2 (0) + a_3b_3 (1)$$
$$= a_1b_1 + a_2b_2 + a_3b_3.$$

Hence, $\quad \vec{a} \cdot \vec{b} = (a_1b_1, a_2b_2, a_3b_3)$.

Cor. *To prove that* $a = \sqrt{a_1^2 + a_2^2 + a_3^2}$.

Proof. If $\vec{b} = \vec{a}$, then $b_1 = a_1$, $b_2 = a_2$, $b_3 = a_3$, the above result becomes :

$$\vec{a} \cdot \vec{a} = a_1 a_1 + a_2 a_2 + a_3 a_3 \Rightarrow a^2 = a_1^2 + a_2^2 + a_3^2.$$

Hence, $\quad a = \sqrt{a_1^2 + a_2^2 + a_3^2}$.

10.23. ANGLE BETWEEN TWO VECTORS

If 'θ' be the angle between two vectors $\vec{a} = (a_1, a_2, a_3)$ *and* $\vec{b} = (b_1, b_2, b_3)$, *to prove that :*

$$\cos\theta = \frac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{a_1^2 + a_2^2 + a_3^2}\,\sqrt{b_1^2 + b_2^2 + b_3^2}}.$$

Proof. $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta = ab \cos\theta$ ...(1)

But $\vec{a} \cdot \vec{b} = a_1b_1 + a_2b_2 + a_3b_3$; $a = \sqrt{a_1^2 + a_2^2 + a_3^2}$ [*Proved above*]

and $\qquad b = \sqrt{b_1^2 + b_2^2 + b_3^2}$.

$\therefore$ (1) becomes :

$$a_1b_1 + a_2b_2 + a_3b_3 = \sqrt{a_1^2 + a_2^2 + a_3^2}\,\sqrt{b_1^2 + b_2^2 + b_3^2}\,\cos\theta.$$

Hence, $$\cos\theta = \frac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{a_1^2 + a_2^2 + a_3^2}\,\sqrt{b_1^2 + b_2^2 + b_3^2}} \qquad ...(2)$$

KEY POINT

Angle between two vectors $\vec{a}$ and $\vec{b}$ is given by :

$$\theta = \cos^{-1} \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}.$$

Cor. 1. Condition of Parallelism.

The vectors $\vec{a}$ and $\vec{b}$ are parallel if there exists a scalar '*t*' such that

$$\vec{a} = t\,\vec{b} \quad \text{if} \quad (a_1, a_2, a_3) = t\,(b_1, b_2, b_3)$$

if $\qquad (a_1, a_2, a_3) = (tb_1, tb_2, tb_3)$

if $\qquad a_1 = tb_1,\ a_2 = tb_2,\ a_3 = tb_3$

if $\qquad \dfrac{a_1}{b_1} = \dfrac{a_2}{b_2} = \dfrac{a_3}{b_3}, \qquad [\because \text{each} = t]$

which is the required condition.

Cor. 2. Condition of Perpendicularity.

Here θ = angle between $\vec{a}$ and $\vec{b} = \dfrac{\pi}{2}$ $\qquad \therefore \quad \cos\theta = \cos\left(\dfrac{\pi}{2}\right) = 0.$

$\therefore$ (2) becomes : $\dfrac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{a_1^2 + a_2^2 + a_3^2}\sqrt{b_1^2 + b_2^2 + b_3^2}} = 0$

$\Rightarrow \qquad a_1b_1 + a_2b_2 + a_3b_3 = 0$, which is the reqd. condition.

10.24. COMPONENTS OF A VECTOR

To find the components of a vector $\vec{b}$ along and perpendicular to vector $\vec{a}$.

Let $\vec{a}$ and $\vec{b}$ be two vectors, represented by $\overrightarrow{OA}$ and $\overrightarrow{OB}$ respectively. Let 'θ' be angle between them. Draw BL $\perp$ OA.

In ΔOBL, $\overrightarrow{OB} = \overrightarrow{OL} + \overrightarrow{LB}$

$\Rightarrow \qquad \vec{b} = \overrightarrow{OL} + \overrightarrow{LB} \qquad$ ...(1)

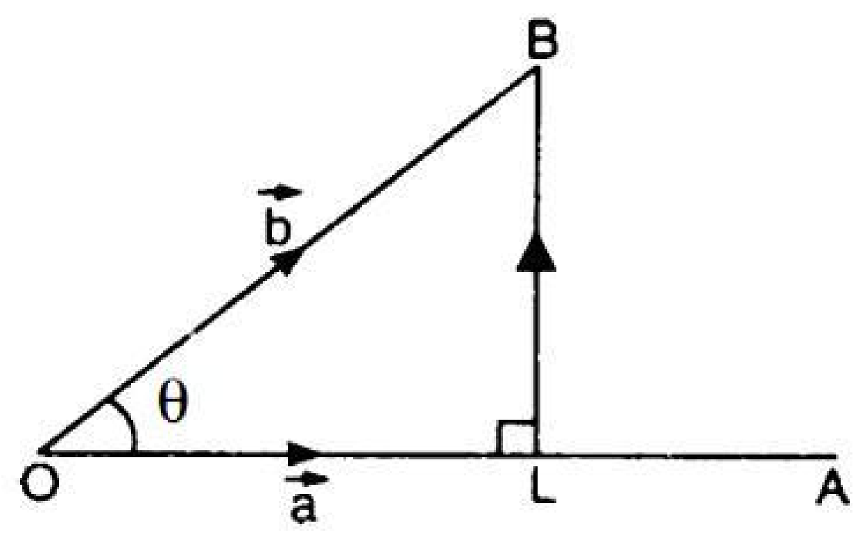


Fig.

Thus $\overrightarrow{OL}$ and $\overrightarrow{LB}$ are the components of $\vec{b}$ along $\vec{a}$ and perpendicular to $\vec{a}$ respectively.

Now $\overrightarrow{OL} = \text{OL}\,\hat{a} = (\text{OB}\cos\theta)\,\hat{a} = (|\vec{b}|\cos\theta)\,\hat{a}$

$$= \left(|\vec{b}|\,\frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}\right)\hat{a} \qquad \left[\because \cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}\right]$$

$$= \left(\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}\right)\hat{a} = \left(\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}\right)\frac{\vec{a}}{|\vec{a}|} = \left(\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2}\right)\vec{a}.$$

From (1), $\overrightarrow{LB} = \vec{b} - \overrightarrow{OL} = \vec{b} - \left(\dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2}\right)\vec{a}.$

Hence, the components of $\vec{b}$ along and perpendicular to $\vec{a}$ are $\left(\dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2}\right)\vec{a}$ and $\vec{b} - \left(\dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2}\right)\vec{a}$ respectively.

Frequently Asked Questions

FAQs

Example 1. Find the magnitude of each of two vectors $\vec{a}$ and $\vec{b}$ having the same magnitude such that the angle between them is 60° and their scalar product is $\frac{9}{2}$. *(C.B.S.E. 2018)*

Solution. By the question, $|\vec{a}| = |\vec{b}|$ ...(1)

Now, $\vec{a}.\vec{b} = |\vec{a}|\,|\vec{b}|\cos\theta$

$\Rightarrow \frac{9}{2} = |\vec{a}|\,|\vec{b}|\cos 60°$ [*Using (1)*]

$\Rightarrow \frac{9}{2} = |\vec{a}|^2\left(\frac{1}{2}\right)$

$\Rightarrow |\vec{a}|^2 = 9.$

Hence, $|\vec{a}| = |\vec{b}| = 3.$

Example 2. If $\vec{a}$ and $\vec{b}$ are perpendicular vectors, $|\vec{a}+\vec{b}| = 13$ and $|\vec{a}| = 5$, find the value of $|\vec{b}|$. *(A.I.C.B.S.E. 2014)*

Solution. We have : $|\vec{a}+\vec{b}| = 13.$

Squaring, $(\vec{a}+\vec{b})^2 = 169$

$\Rightarrow |\vec{a}|^2 + |\vec{b}|^2 + 2\vec{a}.\vec{b} = 169$

$\Rightarrow (5)^2 + |\vec{b}|^2 + 2(0) = 169$

[$\because$ $\vec{a}$ *and* $\vec{b}$ *are perpendicular* $\Rightarrow \vec{a}.\vec{b} = 0$]

$\Rightarrow |\vec{b}|^2 = 169 - 25 = 144.$

Hence, $|\vec{b}| = 12.$

Example 3. Find the projection of the vector $\hat{i}+3\hat{j}+7\hat{k}$ on the vector $2\hat{i}-3\hat{j}+6\hat{k}$. *(C.B.S.E. 2014)*

Solution. Let $\vec{a} = \hat{i}+3\hat{j}+7\hat{k}$ and $\vec{b} = 2\hat{i}-3\hat{j}+6\hat{k}$.

$\therefore$ Projection of $\vec{a}$ on $\vec{b}$ = $\frac{\vec{a}.\vec{b}}{|\vec{b}|}$

$$= \frac{(\hat{i}+3\hat{j}+7\hat{k}).(2\hat{i}-3\hat{j}+6\hat{k})}{|2\hat{i}-3\hat{j}+6\hat{k}|}$$

$$= \frac{(1)(2)+(3)(-3)+(7)(6)}{\sqrt{4+9+36}}$$

$$= \frac{2-9+42}{\sqrt{49}} = \frac{35}{7} = 5.$$

Example 4. If $\vec{a}$ and $\vec{b}$ are two unit vectors such that $\vec{a}+\vec{b}$ is also a unit vector, then find the angle between $\vec{a}$ and $\vec{b}$. *(Mizoram B. 2016; Kerala B. 2014; C.B.S.E. 2014)*

Solution. We have : $|\vec{a}| = 1 = |\vec{b}|$ and $|\vec{a}+\vec{b}| = 1.$

Squaring, $|\vec{a}+\vec{b}|^2 = 1 \Rightarrow (\vec{a}+\vec{b})^2 = 1$

$\Rightarrow |\vec{a}|^2 + |\vec{b}|^2 + 2\vec{a}.\vec{b} = 1$

$\Rightarrow (1)^2 + (1)^2 + 2|\vec{a}||\vec{b}|\cos\theta = 1,$

where 'θ' is the angle between $\vec{a}$ and $\vec{b}$

$\Rightarrow 1 + 1 + 2(1)(1)\cos\theta = 1$

$\Rightarrow \cos\theta = -\frac{1}{2}.$

Hence, $\theta = 120°.$

Example 5. Find $|\vec{x}|$, if for a unit vector $\vec{a}$, $(\vec{x}-\vec{a}).(\vec{x}+\vec{a}) = 15$. *(A.I.C.B.S.E. 2013)*

Solution. We have : $(\vec{x}-\vec{a}).(\vec{x}+\vec{a}) = 15$

$\Rightarrow \vec{x}.(\vec{x}+\vec{a}) - \vec{a}.(\vec{x}+\vec{a}) = 15$

$\Rightarrow \vec{x}.\vec{x} + \vec{x}.\vec{a} - \vec{a}.\vec{x} - \vec{a}.\vec{a} = 15$

$\Rightarrow |\vec{x}|^2 - |\vec{a}|^2 = 15$ $[\because \vec{x}.\vec{a} = \vec{a}.\vec{x}]$

$\Rightarrow |\vec{x}|^2 - 1 = 15 \Rightarrow |\vec{x}|^2 = 16.$

Hence, $|\vec{x}| = 4.$

Example 6. Find 'λ' when the projection of : $\vec{a} = \lambda\hat{i}+\hat{j}+4\hat{k}$ on $\vec{b} = 2\hat{i}+6\hat{j}+3\hat{k}$ is 4 units. *(C.B.S.E. 2012)*

Solution. The projection of $\vec{a}$ on $\vec{b}$ = $\frac{\vec{a}.\vec{b}}{|\vec{b}|}$.

By the question, $\frac{(\lambda\hat{i}+\hat{j}+4\hat{k}).(2\hat{i}+6\hat{j}+3\hat{k})}{|2\hat{i}+6\hat{j}+3\hat{k}|} = 4$

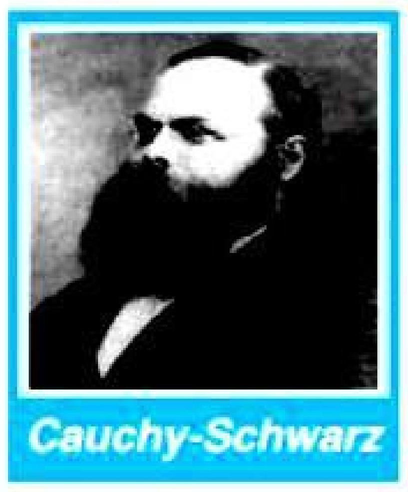
$$\frac{(\lambda)(2)+(1)(6)+(4)(3)}{\sqrt{4+36+9}}=4$$

$$\frac{2\lambda+6+12}{7}=4$$

$2\lambda+18 = 28$ $\qquad$ $2\lambda = 10.$

Hence, $\lambda = 5.$

Example 7. Cauchy-Schwarz Inequality.



For any two vectors $\vec{a}$ and $\vec{b}$, we always have :

$$|\vec{a}\,.\,\vec{b}| \le |\vec{a}||\vec{b}|.$$

(N.C.E.R.T.)

Solution. The equality holds trivially when either $\vec{a}=\vec{0}$ or $\vec{b}=\vec{0}$.

In this case, $|\vec{a}\,.\,\vec{b}|=0=|\vec{a}|\,|\vec{b}|$.

Let us assume that $|\vec{a}|\ne 0$ and $|\vec{b}|\ne 0$.

Then $\dfrac{|\vec{a}\,.\,\vec{b}|}{|\vec{a}||\vec{b}|}=|\cos\theta| \le 1.$

$\Rightarrow$ $|\vec{a}\,.\,\vec{b}| \le |\vec{a}||\vec{b}|$, which is true.

Example 8. Triangular Inequality.

For any two vectors $\vec{a}$ and $\vec{b}$, we always have :

$$|\vec{a}+\vec{b}| \le |\vec{a}|+|\vec{b}|.$$

(N.C.E.R.T.)

Solution. The equality holds trivially when either $\vec{a}=\vec{0}$ or $\vec{b}=\vec{0}$.

In this case, $|\vec{a}+\vec{b}|=0=|\vec{a}|+|\vec{b}|$.

Let us assume that $|\vec{a}|\ne 0$ and $|\vec{b}|\ne 0$.

Then $|\vec{a}+\vec{b}|^2=(\vec{a}+\vec{b})^2=(\vec{a}+\vec{b}).(\vec{a}+\vec{b})$

$= \vec{a}.\vec{a}+\vec{a}.\vec{b}+\vec{b}.\vec{a}+\vec{b}.\vec{b}$

$= |\vec{a}|^2+2\,\vec{a}.\vec{b}+|\vec{b}|^2$ $\qquad$ *[Commutative Law]*

$\le |\vec{a}|^2+2|\vec{a}.\vec{b}|+|\vec{b}|^2$ $\qquad$ $[\because x \le |x|\ \forall\ x \in \boldsymbol{R}]$

$\le |\vec{a}|^2+2|\vec{a}||\vec{b}|+|\vec{b}|^2$ $\qquad$ *[Ex. 6]*

$= (|\vec{a}|+|\vec{b}|)^2$.

Hence, $|\vec{a}+\vec{b}| \le |\vec{a}|+|\vec{b}|$.

Geometrically. *The sum of any two sides of a triangle is always greater than or equal to the third side.*

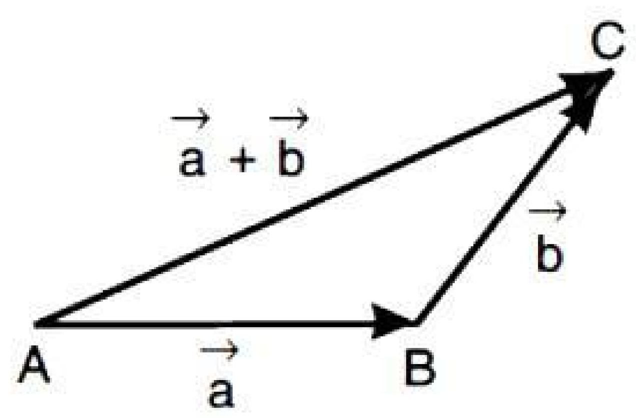


Fig.

Example 9. If two vectors $\vec{a}$ and $\vec{b}$ are such that : $|\vec{a}|=2, |\vec{b}|=1$ and $\vec{a}.\vec{b}=1$, then find the value of $(3\vec{a}-5\vec{b}).(2\vec{a}+7\vec{b})$.

(C.B.S.E. 2011)

Solution. We have : $|\vec{a}|=2, |\vec{b}|=1$

and $\qquad \vec{a}.\vec{b}=1$ $\qquad$ ...(1)

Now $(3\vec{a}-5\vec{b}).(2\vec{a}+7\vec{b})$

$= 3\vec{a}.(2\vec{a}+7\vec{b})-5\vec{b}.(2\vec{a}+7\vec{b})$

$= 6\vec{a}.\vec{a}+21\vec{a}.\vec{b}-10\vec{b}.\vec{a}-35\vec{b}.\vec{b}$

$= 6|\vec{a}|^2+21\vec{a}.\vec{b}-10\vec{a}.\vec{b}-35|\vec{b}|^2$

$[\because \vec{a}.\vec{b}=\vec{b}.\vec{a}]$

$= 6|\vec{a}|^2+11\vec{a}.\vec{b}-35|\vec{b}|^2$

$= 6(2)^2+11(1)-35(1)^2$ $\qquad$ *[Using (1)]*

$= 24+11-35=0.$

Example 10. If $\vec{a}$ and $\vec{b}$ are two vectors such that $|\vec{a}+\vec{b}|=|\vec{a}|$, then prove that the vector $2\vec{a}+\vec{b}$ is perpendicular to vector $\vec{b}$.

(C.B.S.E. 2013)

Solution. We have : $|\vec{a}+\vec{b}|=|\vec{a}|$.

Squaring, $|\vec{a}+\vec{b}|^2=|\vec{a}|^2 \Rightarrow (\vec{a}+\vec{b})^2=(\vec{a})^2$

$\Rightarrow (\vec{a}+\vec{b}).(\vec{a}+\vec{b})=\vec{a}^{\,2}$

$\Rightarrow \vec{a}.\vec{a}+\vec{a}.\vec{b}+\vec{b}.\vec{a}+\vec{b}.\vec{b}=\vec{a}^{\,2}$

$\Rightarrow \vec{a}^{\,2}+\vec{a}.\vec{b}+\vec{a}.\vec{b}+\vec{b}^{\,2}=\vec{a}^{\,2}$

$\left[\because \vec{b}.\vec{a}=\vec{a}.\vec{b} \text{ and } \vec{a}.\vec{a}=\vec{a}^{\,2}\right]$

$\Rightarrow \qquad 2\vec{a}.\vec{b}+\vec{b}^{\,2}=0$ $\qquad$ ...(1)

Now $(2\vec{a}+\vec{b}).\vec{b}=2\vec{a}.\vec{b}+\vec{b}.\vec{b}$

$= 2\vec{a}.\vec{b}+\vec{b}^{\,2}=0.$ $\qquad$ *[Using (1)]*

Hence, vector $(2\vec{a}+\vec{b})$ is perpendicular to vector $\vec{b}$.

Example 11. Find $|\vec{a}-\vec{b}|$, if two vectors $\vec{a}$ and $\vec{b}$ are such that $|\vec{a}|=2, |\vec{b}|=3$ and $\vec{a}\,.\,\vec{b}=4$.

(N.C.E.R.T. ; H.P.B. 2012)

Solution. Here $|\vec{a}-\vec{b}| = \sqrt{(\vec{a}-\vec{b}).(\vec{a}-\vec{b})}$

$$= \sqrt{\vec{a}.\vec{a} - \vec{a}.\vec{b} - \vec{b}.\vec{a} + \vec{b}.\vec{b}}$$

$$= \sqrt{|\vec{a}|^2 - 2(\vec{a}.\vec{b}) + |\vec{b}|^2}$$

$$= \sqrt{(2)^2 - 2(4) + (3)^2}$$

$$= \sqrt{4-8+9} = \sqrt{5}.$$

Example 12. Let $\vec{a}$, $\vec{b}$ and $\vec{c}$ be three vectors of magnitude 3, 2, 5 respectively. If each one is perpendicular to the sum of other two vectors, prove that :

$$\left|\vec{a}+\vec{b}+\vec{c}\right| = \sqrt{38}.$$

Solution. We have : $\left|\vec{a}\right| = 3$, $\left|\vec{b}\right| = 2$ and $\left|\vec{c}\right| = 5$.

Also, $\vec{a}.\left(\vec{b}+\vec{c}\right) = 0$, $\vec{b}.\left(\vec{c}+\vec{a}\right) = 0$, $\vec{c}.\left(\vec{a}+\vec{b}\right) = 0$

[$\because$ *Each is perp. to the sum of the other two*]

$\Rightarrow \vec{a}.\vec{b}+\vec{a}.\vec{c} = 0$, $\vec{b}.\vec{c}+\vec{b}.\vec{a} = 0$, $\vec{c}.\vec{a}+\vec{c}.\vec{b} = 0$

$\Rightarrow \vec{a}.\vec{b}+\vec{c}.\vec{a} = 0$ [$\because \vec{a}.\vec{c} = \vec{c}.\vec{a}$] ....(1)

$\vec{b}.\vec{c}+\vec{a}.\vec{b} = 0$ [$\because \vec{b}.\vec{a} = \vec{a}.\vec{b}$] ....(2)

and $\vec{c}.\vec{a}+\vec{b}.\vec{c} = 0$ [$\because \vec{c}.\vec{b} = \vec{b}.\vec{c}$] ....(3)

Adding (1), (2) and (3), we get :

$$2\left(\vec{a}.\vec{b}+\vec{b}.\vec{c}+\vec{c}.\vec{a}\right) = 0$$

$\Rightarrow \vec{a}.\vec{b}+\vec{b}.\vec{c}+\vec{c}.\vec{a} = 0$ ....(4)

Subtracting (1) from (4), $\vec{b}.\vec{c} = 0$. ....(5)

Similarly, $\vec{c}.\vec{a} = \vec{a}.\vec{b} = 0$ ...(6)

Now $\left(\vec{a}+\vec{b}+\vec{c}\right)^2 = \vec{a}^2 + \vec{b}^2 + \vec{c}^2 + 2\left(\vec{a}.\vec{b}+\vec{b}.\vec{c}+\vec{c}.\vec{a}\right)$

$$= \left|\vec{a}\right|^2 + \left|\vec{b}\right|^2 + \left|\vec{c}\right|^2 + 2(0+0+0)$$

[*Using (5) and (6)*]

$$= 3^2 + 2^2 + 5^2$$
$$= 9 + 4 + 25 = 38.$$

Hence, $\left|\vec{a}+\vec{b}+\vec{c}\right| = \sqrt{38}$.

Example 13. If $\hat{a}$ and $\hat{b}$ are unit vectors and 'θ' is the angle between them, then prove that $\cos\frac{\theta}{2} = \frac{1}{2}\left|\hat{a}+\hat{b}\right|$.

(J. & K.B. 2011)

Solution. $\left|\hat{a}+\hat{b}\right|^2 = \left(\hat{a}+\hat{b}\right).\left(\hat{a}+\hat{b}\right)$

$$\left[\because \vec{a}.\vec{a} = \left|\vec{a}\right|^2\right]$$

$$\Rightarrow \left|\hat{a}+\hat{b}\right|^2 = \hat{a}.\hat{a}+\hat{a}.\hat{b}+\hat{b}.\hat{a}+\hat{b}.\hat{b}$$

$$\Rightarrow \left|\hat{a}+\hat{b}\right|^2 = \left|\hat{a}\right|^2 + 2\hat{a}.\hat{b} + \left|\hat{b}\right|^2$$

$$\left[\because \hat{a}.\hat{b} = \hat{b}.\hat{a}\right]$$

$$\Rightarrow \left|\hat{a}+\hat{b}\right|^2 = \left|\hat{a}\right|^2 + 2\left|\hat{a}\right|\left|\hat{b}\right|\cos\theta + \left|\hat{b}\right|^2$$

$$\Rightarrow \left|\hat{a}+\hat{b}\right|^2 = 1 + 2(1)(1)\cos\theta + 1$$

$$\left[\because \left|\hat{a}\right| = \left|\hat{b}\right| = 1\right]$$

$$\Rightarrow \left|\hat{a}+\hat{b}\right|^2 = 2(1+\cos\theta) = 2\left(2\cos^2\frac{\theta}{2}\right) = 4\cos^2\frac{\theta}{2}$$

$$\Rightarrow \cos^2\frac{\theta}{2} = \frac{1}{4}\left|\hat{a}+\hat{b}\right|^2.$$

Hence, $\cos\frac{\theta}{2} = \frac{1}{2}\left|\hat{a}+\hat{b}\right|$.

Example 14. If $\vec{a}$ is any vector in space, show that :

$$\vec{\mathbf{a}} = (\vec{\mathbf{a}}.\hat{\mathbf{i}})\hat{\mathbf{i}} + (\vec{\mathbf{a}}.\hat{\mathbf{j}})\hat{\mathbf{j}} + (\vec{\mathbf{a}}.\hat{\mathbf{k}})\hat{\mathbf{k}}.$$

Solution. Let $\vec{a} = a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k}$.

$\therefore \quad \vec{a} \cdot \hat{i} = (a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k}) \cdot \hat{i}$

$= a_1(1) + a_2(0) + a_3(0) = a_1$

$\therefore \quad (\vec{a} \cdot \hat{i})\, \hat{i} = a_1 \hat{i}$.

Similarly, $(\vec{a} \cdot \hat{j})\, \hat{j} = a_2 \hat{j}$ and $(\vec{a} \cdot \hat{k}) \cdot \hat{k} = a_3 \hat{k}$.

Adding, $(\vec{a} \cdot \hat{i})\, \hat{i} + (\vec{a} \cdot \hat{j})\, \hat{j} + (\vec{a} \cdot \hat{k})\, \hat{k}$

$= a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k}$

$= \vec{a}$, which is true.

Example 15. If $\vec{a}, \vec{b}, \vec{c}$ are three vectors such that :

$|\vec{a}| = 5, |\vec{b}| = 12$ and $|\vec{c}| = 13$ and $\vec{a} + \vec{b} + \vec{c} = \vec{0}$, find the value of $\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a}$. *(C.B.S.E. 2012)*

Solution. We have : $|\vec{a}| = 5, |\vec{b}| = 12$ and $|\vec{c}| = 13$.

Now $\vec{a} + \vec{b} + \vec{c} = \vec{0}$. *[Given]*

Squaring, $|\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2 + 2(\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a}) = 0$

$(5)^2 + (12)^2 + (13)^2 + 2(\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a}) = 0$

$2(\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a}) = -(25 + 144 + 169)$.

Hence, $(\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a}) = -\frac{1}{2}(338) = -169$.

Example 16. If $\vec{a} + \vec{b} + \vec{c} = \vec{0}$ and $\left|\vec{a}\right| = 3$, $\left|\vec{b}\right| = 5$, and $\left|\vec{c}\right| = 7$, find the angle between $\vec{a}$ and $\vec{b}$. *(C.B.S.E. 2014)*

Solution. Since $\vec{a} + \vec{b} + \vec{c} = \vec{0}$,

$\therefore \quad \vec{a} + \vec{b} = -\vec{c}$.

Squaring, $\left(\vec{a} + \vec{b}\right)^2 = \vec{c}^{\,2}$

$\Rightarrow \vec{a}^{\,2} + \vec{b}^{\,2} + 2\vec{a} \cdot \vec{b} = \vec{c}^{\,2}$

$\Rightarrow \left|\vec{a}\right|^2 + \left|\vec{b}\right|^2 + 2\left|\vec{a}\right|\left|\vec{b}\right| \cos\theta = \left|\vec{c}\right|^2$,

where 'θ' is the angle between $\vec{a}$ and $\vec{b}$

$\Rightarrow (3)^2 + (5)^2 + 2\,(3)\,(5) \cos\theta = (7)^2$

$\Rightarrow \quad 9 + 25 + 30 \cos\theta = 49$

$\Rightarrow \quad 30 \cos\theta = 49 - 34 \Rightarrow \cos\theta = \frac{1}{2}$

$\Rightarrow \quad \theta = 60°$.

Hence, the angle between $\vec{a}$ and $\vec{b}$ is 60°.

Example 17. Prove that the vectors :

$\vec{a} = 2\hat{i} - \hat{j} + \hat{k}$, $\vec{b} = \hat{i} - 3\hat{j} - 5\hat{k}$, $\vec{c} = 3\hat{i} - 4\hat{j} - 4\hat{k}$

form the sides of a right-angled triangle.

(H.B. 2017, 14; Kashmir B. 2016; Jammu B. 2013)

Also find the remaining angles of the triangle.

Solution. (*a*) **To prove that the three given vectors are coplanar.**

For this, if $\vec{a}, \vec{b}, \vec{c}$ are co-planar, then any one can be expressed in terms of the other two.

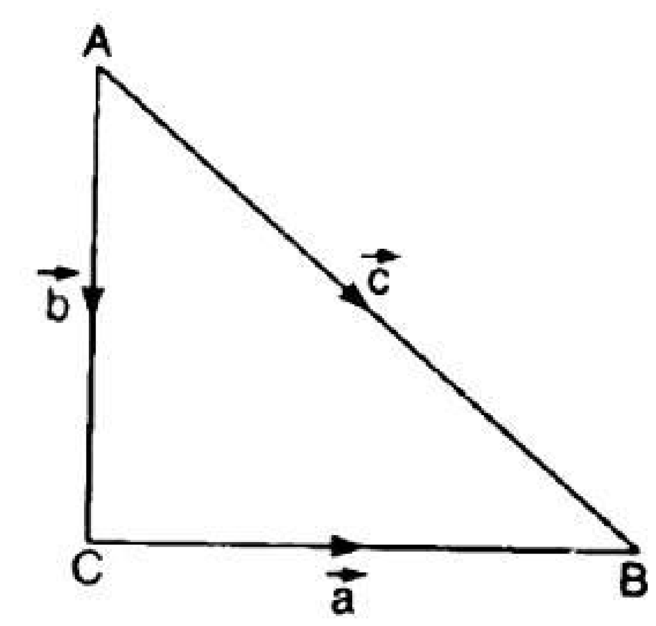


Fig.

Now, $\vec{a} + \vec{b} = (2\hat{i} - \hat{j} + \hat{k}) + (\hat{i} - 3\hat{j} - 5\hat{k})$

$= 3\hat{i} - 4\hat{j} - 4\hat{k} = \vec{c}$.

Thus, if $\overrightarrow{AC} = \vec{b}$, $\overrightarrow{CB} = \vec{a}$ and $\overrightarrow{AB} = \vec{c}$, then :

$\overrightarrow{AC} + \overrightarrow{CB} = \overrightarrow{AB}$.

Hence, the vectors $\vec{a}, \vec{b}, \vec{c}$ form the sides of a triangle.

Since $\vec{a} \cdot \vec{b} = (2\hat{i} - \hat{j} + \hat{k}) \cdot (\hat{i} - 3\hat{j} - 5\hat{k})$

$= (2)(1) + (-1)(-3) + (1)(-5)$

$= 2 + 3 - 5 = 0$,

$\therefore \quad \vec{a} \perp \vec{b}$.

Hence, the given vectors form a right-angled triangle.

(*b*) (*i*) **Angle between $\vec{b}$ and $\vec{c}$.**

If 'θ_1' is the angle between $\vec{b}$ and $\vec{c}$, then :

$$\cos\theta_1 = \frac{b_1c_1 + b_2c_2 + b_3c_3}{\sqrt{b_1^2 + b_2^2 + b_3^2}\sqrt{c_1^2 + c_2^2 + c_3^2}}$$

$$= \frac{(1)(3) + (-3)(-4) + (-5)(-4)}{\sqrt{1+9+25}\sqrt{9+16+16}}$$

$$= \frac{3+12+20}{\sqrt{35}\sqrt{41}} = \sqrt{\frac{35}{41}}.$$

Hence, $\theta_1 = \cos^{-1}\sqrt{\frac{35}{41}}$.

(*ii*) **Angle between $\vec{c}$ and $\vec{a}$.**

If 'θ_2' is the angle between $\vec{c}$ and $\vec{a}$, then :

$$\cos\theta_2 = \frac{c_1a_1 + c_2a_2 + c_3a_3}{\sqrt{c_1^2 + c_2^2 + c_3^2}\sqrt{a_1^2 + a_2^2 + a_3^2}}$$

$$= \frac{(3)(2) + (-4)(-1) + (-4)(1)}{\sqrt{9+16+16}\sqrt{4+1+1}}$$

$$= \frac{6+4-4}{\sqrt{41}\sqrt{6}} = \sqrt{\frac{6}{41}}.$$

Hence, $\theta_2 = \cos^{-1}\sqrt{\frac{6}{41}}$.

Example 18. If $\vec{a}, \vec{b}, \vec{c}$ are mutually perpendicular vectors of equal magnitude, show that $\vec{a} + \vec{b} + \vec{c}$ is equally inclined to $\vec{a}, \vec{b}$ and $\vec{c}$. Also, find the angle, which $\vec{a} + \vec{b} + \vec{c}$ makes with $\vec{a}$ or $\vec{b}$ or $\vec{c}$. ***(C.B.S.E. 2017)***

Solution. Given : $|\vec{a}| = |\vec{b}| = |\vec{c}|$ ...(1)

And $\vec{a}.\vec{b} = 0$, $\vec{b}.\vec{c} = 0$ and $\vec{c}.\vec{a} = 0$ ...(2)

Let α, β and γ be the angles which $\vec{a} + \vec{b} + \vec{c}$ makes with $\vec{a}, \vec{b}$, and $\vec{c}$ respectively.

$$\therefore \quad \cos\alpha = \frac{(\vec{a}+\vec{b}+\vec{c}).\vec{a}}{|\vec{a}+\vec{b}+\vec{c}||\vec{a}|}$$

$$= \frac{\vec{a}.\vec{a}+\vec{b}.\vec{a}+\vec{c}.\vec{a}}{|\vec{a}+\vec{b}+\vec{c}||\vec{a}|}$$

$$= \frac{|\vec{a}|^2+0+0}{|\vec{a}+\vec{b}+\vec{c}||\vec{a}|} \qquad [Using\ (2)]$$

$$= \frac{|\vec{a}|}{|\vec{a}+\vec{b}+\vec{c}|} \qquad ...(3)$$

Similarly, $\cos\beta = \dfrac{|\vec{b}|}{|\vec{a}+\vec{b}+\vec{c}|}$ ...(4)

and $\cos\gamma = \dfrac{|\vec{c}|}{|\vec{a}+\vec{b}+\vec{c}|}$ ...(5)

From (1), (3), (4) and (5),

$\cos\alpha = \cos\beta = \cos\gamma \quad \Rightarrow \quad \alpha = \beta = \gamma.$

Hence, $\vec{a}+\vec{b}+\vec{c}$ is equally inclined to $\vec{a}, \vec{b}$, and $\vec{c}$.

Also, $\alpha = \cos^{-1}\left(\dfrac{|\vec{a}|}{|\vec{a}+\vec{b}+\vec{c}|}\right)$, $\beta = \cos^{-1}\left(\dfrac{|\vec{b}|}{|\vec{a}+\vec{b}+\vec{c}|}\right)$

and $\gamma = \cos^{-1}\left(\dfrac{|\vec{c}|}{|\vec{a}+\vec{b}+\vec{c}|}\right)$.

Example 19. Find a vector $\vec{a}$ of magnitude $5\sqrt{2}$, making an angle $\frac{\pi}{4}$ with x-axis, $\frac{\pi}{2}$ with y-axis and an angle 'θ' with z-axis. ***(A.I.C.B.S.E. 2014)***

Solution. Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$.

Then $\cos\frac{\pi}{4} = \dfrac{\vec{a}.\hat{i}}{|\vec{a}||\hat{i}|} = \dfrac{(a_1\hat{i}+a_2\hat{j}+a_3\hat{k}).\hat{i}}{(5\sqrt{2})(1)}$

$\Rightarrow \quad \dfrac{1}{\sqrt{2}} = \dfrac{a_1}{5\sqrt{2}} \quad \Rightarrow \quad a_1 = 5$ ...(1)

Similarly, $a_2 = \cos\frac{\pi}{2} = 0$ ...(2)

Since $\vec{a}$ is of magnitude $5\sqrt{2}$,

$\therefore \ a_1^2 + a_2^2 + a_3^2 = 50$

$\Rightarrow \quad 25 + 0 + a_3^2 = 50$ [*Using (1) and (2)*]

$\Rightarrow a_3^2 = 50 - 25 = 25 \Rightarrow a_3 = \pm 5$ ...(3)

Since $\vec{a}$ makes an angle 'θ' with z-axis,

$$\therefore \quad \cos\theta = \frac{\vec{a}.\hat{k}}{|\vec{a}||\hat{k}|}$$

$$\Rightarrow \quad \cos\theta = \frac{(a_1\hat{i} + a_2\hat{j} + a_3\hat{k}).\hat{k}}{(5\sqrt{2})(1)}$$

$$\Rightarrow \quad \cos\theta = a_3 = \pm\frac{5}{5\sqrt{2}} \qquad [Using\ (3)]$$

i.e., $a_3 = \pm\frac{1}{\sqrt{2}}$.

Hence, $\vec{a} = 5\hat{i} \pm \frac{1}{\sqrt{2}}\hat{k}$.

Example 20. Let $\vec{a} = 4\hat{i} + 5\hat{j} - \hat{k}, \vec{b} = \hat{i} - 4\hat{j} + 5\hat{k}$ and $\vec{c} = 3\hat{i} + \hat{j} - \hat{k}$. Find a vector $\vec{d}$, which is perpendicular to both $\vec{c}$ and $\vec{b}$ and $\vec{d}.\vec{a} = 21$.

(C.B.S.E. 2018)

Solution. We have :

$$\vec{a} = 4\hat{i} + 5\hat{j} - \hat{k}$$

$$\vec{b} = \hat{i} - 4\hat{j} + 5\hat{k}$$

and $\vec{c} = 3\hat{i} + \hat{j} - \hat{k}$.

Let $\vec{d} = x\hat{i} + y\hat{j} + z\hat{k}$.

Since $\vec{d}$ is perpendicular to $\vec{c}$ and $\vec{b}$,

$\therefore$ $\vec{d}.\vec{c} = 0$

and $\vec{d}.\vec{b} = 0$

$\Rightarrow$ $(x\hat{i} + y\hat{j} + z\hat{k}).(3\hat{i} + \hat{j} - \hat{k}) = 0$

and $(x\hat{i} + y\hat{j} + z\hat{k}).(\hat{i} - 4\hat{j} + 5\hat{k}) = 0$

$\Rightarrow$ $3x + y - z = 0$ ...(1)

and $x - 4y + 5z = 0$ ...(2)

Also, $\vec{d}.\vec{a} = 21$

$\Rightarrow$ $(x\hat{i} + y\hat{j} + z\hat{k}).(4\hat{i} + 5\hat{j} - \hat{k}) = 21$

$\Rightarrow$ $4x + 5y - z = 21$ ...(3)

Multiplying (1) by 5,

$15x + 5y - 5z = 0$ ...(4)

Adding (2) and (4),

$16x + y = 0$ ...(5)

Subtracting (1) from (3),

$x + 4y = 21$ ...(6)

From (5), $y = -16x$ ...(7)

Putting in (6), $x - 64x = 21$

$\Rightarrow$ $-63x = 21$

$\Rightarrow$ $x = -\frac{1}{3}$.

Putting in (7), $y = -16\left(-\frac{1}{3}\right)$

$\Rightarrow$ $y = \frac{16}{3}$.

Putting in (1), $3\left(-\frac{1}{3}\right) + \frac{16}{3} - z = 0$

$\Rightarrow$ $z = \frac{13}{3}$.

Hence, $\vec{d} = -\frac{1}{3}\hat{i} + \frac{16}{3}\hat{j} + \frac{13}{3}\hat{k}$.

Example 21. If with reference to a right-handed system of mutually perpendicular unit vectors $\hat{i}, \hat{j}, \hat{k}$, $\vec{\alpha} = 3\hat{i} - \hat{j}$, where $\vec{\beta} = 2\hat{i} + \hat{j} - 3\hat{k}$, then express $\vec{\beta}$ in the form $\vec{\beta} = \vec{\beta}_1 + \vec{\beta}_2$, where $\vec{\beta}_1$ is parallel to $\vec{\alpha}$ and $\vec{\beta}_2$ is perpendicular to $\vec{\alpha}$.

(N.C.E.R.T. ; C.B.S.E. (F) 2013)

Solution. Since $\vec{\beta}_1$ is parallel to $\vec{\alpha}$, [*Given*]

$\therefore$ $\vec{\beta}_1 = \lambda\vec{\alpha}$ for some scalar λ ...(1)

Let $\vec{\beta}_2 = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ ...(2)

Since $\vec{\beta}_2 \perp \vec{\alpha}$, $\therefore$ $\vec{\beta}_2.\vec{\alpha} = 0$

$\Rightarrow$ $(a_1\hat{i} + a_2\hat{j} + a_3\hat{k}).(3\hat{i} - \hat{j}) = 0$

$\Rightarrow$ $3a_1 - a_2 = 0$ ...(3)

Now $\vec{\beta} = \vec{\beta}_1 + \vec{\beta}_2$

$\Rightarrow$ $\vec{\beta} = \lambda\vec{\alpha} + \vec{\beta}_2$ [*Using (1)*]

$\Rightarrow$ $2\hat{i} + \hat{j} - 3\hat{k} = \lambda(3\hat{i} - \hat{j}) + (a_1\hat{i} + a_2\hat{j} + a_3\hat{k})$

$\Rightarrow$ $2\hat{i} + \hat{j} - 3\hat{k} = (3\lambda + a_1)\hat{i} + (-\lambda + a_2)\hat{j} + a_3\hat{k}$.

Comparing co-effs., $2 = 3\lambda + a_1$,

$1 = -\lambda + a_2, -3 = a_3$

$\Rightarrow$ $a_1 = 2 - 3\lambda, a_2 = 1 + \lambda, a_3 = -3$ ...(4)

Putting in (3), $3(2 - 3\lambda) - (1 + \lambda) = 0$

$\Rightarrow$ $-10\lambda + 5 = 0$ $\Rightarrow$ $\lambda = \frac{1}{2}$.

From (4), $a_1 = 2 - 3\left(\frac{1}{2}\right) = 2 - \frac{3}{2} = \frac{1}{2}$

and $a_2 = 1 + \frac{1}{2} = \frac{3}{2}$.

Putting in (2), $\vec{\beta}_2 = \frac{1}{2}\hat{i} + \frac{3}{2}\hat{j} - 3\hat{k}$.

From (1), $\vec{\beta}_1 = \frac{1}{2}(3\hat{i} - \hat{j})$.

Hence, $\vec{\beta}_1 = \frac{1}{2}(3\hat{i} - \hat{j})$ and $\vec{\beta}_2 = \frac{1}{2}(\hat{i} + 3\hat{j} - 6\hat{k})$.

EXERCISE 10 (e)

Fast Track Answer Type Questions

FTATQ

1. Two vectors $\vec{a}$ and $\vec{b}$ are perpendicular to each other if $\vec{a}.\vec{b}=0$. (**True/False**) *(Kerala B. 2017)*

2. (*a*) Obtain the dot product of the vectors :

$$\vec{a}=\vec{i}-\hat{j}+\hat{k} \text{ and } \vec{b}=\vec{i}-\hat{k}.$$

(Meghalaya B. 2015)

(*b*) Write the magnitude of a vector $\vec{a}$ in terms of dot product. *(Kerala B. 2015)*

3. (*a*) (*i*) If $\vec{a}=7\hat{i}+\hat{j}-4\hat{k}$ and $\vec{b}=2\hat{i}+6\hat{j}+3\hat{k}$, find the projection of $\vec{a}$ on $\vec{b}$.

(*ii*) Let $\vec{a}=2\hat{i}+3\hat{j}+2\hat{k}$ and $\vec{b}=\hat{i}+2\hat{j}+\hat{k}$. Find the projection of $\vec{b}$ on $\vec{a}$. *(Meghalaya B. 2014)*

(*b*) Find the projection of the vector $\hat{i}-\hat{j}$ on the vector $\hat{i}+\hat{j}$. *(A.I.C.B.S.E. 2011)*

4. Find $\vec{a}.\vec{b}$ if :

(*i*) $\vec{a}=3\hat{i}-\hat{j}+2\hat{k}$ and $\vec{b}=2\hat{i}+3\hat{j}+3\hat{k}$ *(C.B.S.E. 2009 C)*

(*ii*) $\vec{a}=2\hat{i}-\hat{j}+\hat{k}$ and $\vec{b}=2\hat{j}-\hat{k}$. *(J. & K.B. 2011)*

5. (*a*) Evaluate the product $(3\vec{a}-5\vec{b}).(2\vec{a}+7\vec{b})$. *(N.C.E.R.T.)*

(*b*) (*i*) If $\vec{a}$ is a unit vector and

$(\vec{x}-\vec{a}).(\vec{x}+\vec{a})=8$, then find $|\vec{x}|$.

(N.C.E.R.T. ; Karnataka B. 2014; Kashmir B. 2011)

(*ii*) If $\vec{a}$ is a unit vector and $\left(\vec{x}-\vec{a}\right).\left(\vec{x}+\vec{a}\right)=80$, then find $|\vec{x}|$. *(Rajasthan B. 2012)*

6. If $\vec{p}$ is a unit vector and $\left(\vec{x}-\vec{p}\right).\left(\vec{x}+\vec{p}\right)=80$, then find $|\vec{x}|$. *(A.I.C.B.S.E. 2009)*

7. Find $|\vec{x}|$, if for unit vector $\vec{a}$,

$(\vec{x}-\vec{a}).(\vec{x}+\vec{a})=12$. *(N.C.E.R.T. ; H.P.B. 2010)*

8. (*a*) Find the angle between the vectors :

(*i*) $\hat{i}-\hat{j}$ and $\hat{j}-\hat{k}$ *(A.I.C.B.S.E. 2015)*

(*ii*) $\hat{i}+\hat{j}+\hat{k}$ and $\hat{i}+\hat{j}-\hat{k}$ *(Uttarakhand B. 2015)*

(*iii*) $\vec{a}=\hat{i}-2\hat{j}+3\hat{k}$ and $\vec{b}=3\hat{i}-2\hat{j}+\hat{k}$ *(N.C.E.R.T.; Kashmir B. 2016)*

(*iv*) $\vec{a}=\hat{i}+\hat{j}-\hat{k}$ and $\vec{b}=\hat{i}-\hat{j}+\hat{k}$ *(N.C.E.R.T.; C.B.S.E. Sample Paper 2019; Kerala B. 2016; Kashmir B. 2012)*

(*v*) $\vec{a}=3\vec{i}-2\vec{j}+\vec{k}$ and $\vec{b}=\vec{i}-2\vec{j}-3\vec{k}$ *(Meghalaya B. 2013)*

(*vi*) $\vec{a}=2\hat{i}-\hat{j}+2\hat{k}$ and $\vec{b}=6\hat{i}+2\hat{j}+3\hat{k}$. *(Kerala B. 2014)*

(*b*) Find the cosine of the acute angle which the vector :

$\sqrt{2}\hat{i}+\hat{j}+\hat{k}$ makes with *y*-axis. *(C.B.S.E. 2010)*

9. Find the angle between two vectors $\vec{a}$ and $\vec{b}$ such that :

(*i*) $\left|\vec{a}\right|=\sqrt{3}$, $\left|\vec{b}\right|=2$ and $\vec{a}.\vec{b}=\sqrt{6}$ *(A.I.C.B.S.E. 2011; H.P.B. 2010)*

(*ii*) $|\vec{a}|=\sqrt{2}$, $|\vec{b}|=2$ and $\vec{a}.\vec{b}=\sqrt{6}$. *(Meghalaya B. 2016)*

10. Find the angle between two vectors $\vec{a}$ and $\vec{b}$ with :

magnitudes 1 and 2 respectively and such that $\vec{a}.\vec{b}=1$ *(N.C.E.R.T. ; Uttarakhand B. 2013)*

11. Find the magnitude of two vectors $\vec{a}$ and $\vec{b}$ having the same magnitude and such that angle between them is $60°$ and their scalar product is $\frac{1}{2}$. *(N.C.E.R.T.)*

12. (*a*) If $\vec{a}.\vec{a}=0$ and $\vec{a}.\vec{b}=0$, what can you conclude about the vector $\vec{b}$? *(N.C.E.R.T.; C.B.S.E. (F) 2011)*

(*b*) If either $\vec{a}=\vec{0}$ or $\vec{b}=\vec{0}$, then $\vec{a}.\vec{b}=0$. But the converse need not be true. Justify your answer with an example. *(N.C.E.R.T.)*

Very Short Answer Type Questions

VSATQ

13. (*a*) Find the scalar projection of :

(*i*) $\vec{a} = 7\hat{i} + \hat{j} - 4\hat{k}$ on $\vec{b} = 2\hat{i} + 6\hat{j} + 3\hat{k}$ ***(C.B.S.E. 2015)***

(*ii*) $\vec{a} = 3\hat{i} - 2\hat{j} + \hat{k}$ on $\vec{b} = \hat{i} - 2\hat{j} - 3\hat{k}$ ***(Meghalaya B. 2016)***

(*iii*) $\vec{a} = 2\hat{i} + 3\hat{j} + 2\hat{k}$ on $\vec{b} = \hat{i} + 2\hat{j} + \hat{k}$

(N.C.E.R.T. ;P.B. 2015; Jammu B. 2013 ; Rajasthan B. 2012 ; Kashmir B. 2011 ; H.B. 2010)

(*iv*) $\vec{a} = \hat{i} - \hat{j}$ on $\vec{b} = \hat{i} + \hat{j}$

(N.C.E.R.T.; Kashmir B. 2016; Jammu B. 2013 ; H.B. 2010)

(*v*) $\vec{a} = \hat{i} + 3\hat{j} + 7\hat{k}$ on $\vec{b} = 7\hat{i} - \hat{j} + 8\hat{k}$

(N.C.E.R.T. ; Kashmir B. 2017, 11 ; Jharkhand B. 2016;)

(*b*) Find the scalar projection of $\vec{b}$ on $\vec{a}$, when :

(*i*) $\vec{a} = 2\hat{i} + 2\hat{j} - \hat{k}$ and $\vec{b} = 2\hat{i} - \hat{j} - 4\hat{k}$ ***(P.B. 2011)***

(*ii*) $\vec{a} = 2\hat{i} + \hat{j} + 2\hat{k}$ and $\vec{b} = \hat{i} + 2\hat{j} + \hat{k}$. ***(Kerala B. 2015)***

14. Find the vector projection of the vector :

(*a*) $7\hat{i} + \hat{j} - \hat{k}$ on $2\hat{i} + 6\hat{j} + 3\hat{k}$

(*b*) $2\hat{i} - \hat{j} + \hat{k}$ on $\hat{i} - 2\hat{j} + \hat{k}$. ***(C.B.S.E. Sample Paper 2018)***

15. Find 'λ' when the scalar projection of :

$\vec{a} = \lambda\hat{i} + \hat{j} + 4\hat{k}$ on $\vec{b} = 2\hat{i} + 6\hat{j} + 3\hat{k}$ is 4 units. ***(Nagaland B. 2018)***

16. Prove that :

$$\frac{1}{7}(2\hat{i} + 3\hat{j} + 6\hat{k}), \frac{1}{7}(3\hat{i} - 6\hat{j} + 2\hat{k}), \frac{1}{7}(6\hat{i} + 2\hat{j} - 3\hat{k})$$

are mutually perpendicular unit vectors. ***(Jammu B. 2015)***

17. (*i*) If $\vec{a} = 5\hat{i} - \hat{j} - 3\hat{k}$ and $\vec{b} = \hat{i} + 3\hat{j} - 5\hat{k}$, then show that the vectors $(\vec{a} + \vec{b})$ and $(\vec{a} - \vec{b})$ are perpendicular. ***(N.C.E.R.T.)***

(*ii*) If $\vec{a} = \hat{i} + 2\hat{j} - 3\hat{k}$ and

$\vec{b} = 3\hat{i} - \hat{j} + 2\hat{k}$, then

show that $(\vec{a} + \vec{b})$ is perpendicular to $(\vec{a} - \vec{b})$.

(Meghalaya B. 2017)

18. Write the value of 'p' for which :

$\vec{a} = 3\hat{i} + 2\hat{j} + 9\hat{k}$ and $\vec{b} = \hat{i} + p\hat{j} + 3\hat{k}$

are parallel. ***(A.I.C.B.S.E. 2009)***

19. Find the value of 'λ' such that the vectors $\vec{a}$ and $\vec{b}$ are perpendicular (orthogonal), where :

(*i*) $\vec{a} = 2\hat{i} + 4\hat{j} - \hat{k}$, $\vec{b} = 3\hat{i} - 2\hat{j} + \lambda\hat{k}$

(*ii*) $\vec{a} = 2\hat{i} + \lambda\hat{j} + \hat{k}$, $\vec{b} = \hat{i} - 2\hat{j} + 3\hat{k}$.

20. If $2\hat{i} + \hat{j} - 3\hat{k}$ and $m\hat{i} + 3\hat{j} - \hat{k}$ are perpendicular to each other, then find '*m*'.

Also find the area of the rectangle having these two vectors as sides. ***(Kerala B. 2015)***

21. Show that the projection of $\vec{b}$ on $\vec{a} \neq 0$ is :

$$\left(\frac{\vec{a} \cdot \vec{b}}{|\vec{a}|^2}\right)\vec{a}.$$

22. Show that $|\vec{a}|\vec{b} + |\vec{b}|\vec{a}$ is perpendicular to $|\vec{a}|\vec{b} - |\vec{b}|\vec{a}$, for any two non-zero vectors $\vec{a}$ and $\vec{b}$. ***(N.C.E.R.T.)***

Short Answer Type Questions

SATQ

23. Find a unit vector perpendicular to each of the vectors $\vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$, where :

$\vec{a} = 3\hat{i} + 2\hat{j} + 2\hat{k}$ and $\vec{b} = \hat{i} + 2\hat{j} - 2\hat{k}$.

(Kashmir B. 2015)

24. If $\vec{a} = 2\hat{i} + 2\hat{j} + 3\hat{k}$, $\vec{b} = -\hat{i} + 2\hat{j} + \hat{k}$ and $\vec{c} = 3\hat{i} + \hat{j}$ are such that $\vec{a} + \lambda\vec{b}$ is perpendicular to $\vec{c}$, find the value of 'λ'.

(Karnataka B. 2014 ; A.I.C.B.S.E. 2009 C)

25. (*i*) If $\vec{a} = \hat{i} - \hat{j} + 7\hat{k}$ and $\vec{b} = 5\hat{i} - \hat{j} + \lambda\hat{k}$, then find the value of 'λ', so that $\vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$ are perpendicular vectors. ***(A.I.C.B.S.E. 2013)***

(*ii*) If $\vec{p} = 5\hat{i} + \lambda\hat{j} - 3\hat{k}$ and $\vec{q} = \hat{i} + 3\hat{j} - 5\hat{k}$, then find the value of 'λ', so that $\vec{p} + \vec{q}$ and $\vec{p} - \vec{q}$ are perpendicular vectors. ***(A.I.C.B.S.E. 2013)***

(*iii*) If $\vec{a} = 5\hat{i} - \hat{j} + 7\hat{k}$ and $\vec{b} = \hat{i} - \hat{j} - \lambda\hat{k}$, find the value of λ for which $(\vec{a} + \vec{b})$ and $(\vec{a} - \vec{b})$ are orthogonal. ***(Mizoram B. 2016)*** (*iv*) If $\vec{a} = 3\hat{i} + \hat{j} + 9\hat{k}$ and $\vec{b} = \hat{i} + \lambda\hat{j} + 3\hat{k}$, then find the value of λ for which the vectors $(\vec{a} + \vec{b})$ and $(\vec{a} - \vec{b})$ are perpendicular to each other. *(W. Bengal B. 2017)*

26. Find the scalar product of the following pairs of vectors and the angle between them :

(*i*) $2\hat{i} - 3\hat{j} + 6\hat{k}$ and $2\hat{i} - 3\hat{j} - 5\hat{k}$

(*ii*) $\hat{i} + 3\hat{j} - 8\hat{k}$ and $-3\hat{i} - 5\hat{j} + 4\hat{k}$.

27. Show that the vectors :

$2\hat{i} - \hat{j} + \hat{k}$, $\hat{i} - 3\hat{j} - 5\hat{k}$ and $3\hat{i} - 4\hat{j} - 4\hat{k}$

form the vertices of a right angled triangle.

28. The position vectors of the vertices of ΔABC are : $3\hat{i} - 4\hat{j} - 4\hat{k}$, $2\hat{i} - \hat{j} + \hat{k}$ and $\hat{i} - 3\hat{j} - 5\hat{k}$ respectively.

(*a*) Find $\overrightarrow{AB}, \overrightarrow{BC}$ and $\overrightarrow{CA}$.

(*b*) Prove that ΔABC is a right angled triangle. *(Kerala B. 2013)*

29. If $|\vec{a} + \vec{b}| = |\vec{a} - \vec{b}|$, prove that $\vec{a}$ and $\vec{b}$ are perpendicular. *(Assam B. 2016)*

30. If $\vec{a}$ and $\vec{b}$ are perpendicular vectors, show that : $(\vec{a} + \vec{b})^2 = (\vec{a} - \vec{b})^2$.

31. Prove that $\left(\vec{a} + \vec{b}\right) \cdot \left(\vec{a} + \vec{b}\right) = \left|\vec{a}\right|^2 + \left|\vec{b}\right|^2$, if and only if $\vec{a}, \vec{b}$ are perpendicular, given $\vec{a} \neq \vec{0}$, $\vec{b} \neq \vec{0}$.

(N.C.E.R.T.; Assam B. 2018 ; H.P.B. 2010 S)

32. (*a*) If $\vec{a}, \vec{b}, \vec{c}$ are unit vectors such that $\vec{a} + \vec{b} + \vec{c} = 0$, then find the value of $\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a}$.

(Jammu B. 2017)

(*b*) Three vectors $\vec{a}, \vec{b}$ and $\vec{c}$ satisfy :

$\vec{a} + \vec{b} + \vec{c} = \vec{0}$. Evaluate $\mu = \vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a}$, if $|\vec{a}| = 1$, $|\vec{b}| = 4$ and $|\vec{c}| = 2$.

(N.C.E.R.T.; Karnataka B. 2017; H.P.B. 2012)

(*c*) If the vectors $\vec{a}, \vec{b}$ and $\vec{c}$ satisfy the condition $\vec{a} + \vec{b} + \vec{c} = \vec{0}$ and $|\vec{a}| = 3$, $|\vec{b}| = 4$ and $|\vec{c}| = 5$, then show that $\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a} = -25$. *(W. Bengal B. 2018)*

33. The scalar product of the vector $\vec{a} = \hat{i} + \hat{j} + \hat{k}$ with a unit vector along the sum of vectors $\vec{b} = 2\hat{i} + 4\hat{j} - 5\hat{k}$ and $\vec{c} = \lambda\hat{i} + 2\hat{j} + 3\hat{k}$ is equal to one. Find the value of 'λ' and hence find the unit vector along $\vec{b} + \vec{c}$.

(H.B. 2017; Jharkhand B. 2016; H.B. 2017; A.I.C.B.S.E. 2014, 09; Kashmir B. 2013)

34. If $\left|\vec{a}\right| = 3$, $\left|\vec{b}\right| = 4$ and $\left|\vec{c}\right| = 5$, such that each is perpendicular to the sum of other two, prove that $\left|\vec{a} + \vec{b} + \vec{c}\right| = 5\sqrt{2}$.

(N.C.E.R.T.; W. Bengal B. 2018; Uttarakhand B. 2013 ; H.P.B. 2012, 11)

35. If $\left|\vec{a}\right| = a$ and $\left|\vec{b}\right| = b$, prove that :

$$\left(\frac{\vec{a}}{a^2} - \frac{\vec{b}}{b^2}\right)^2 = \left(\frac{\vec{a} - \vec{b}}{ab}\right)^2.$$

36. If $\vec{a} = 3\hat{i} + \hat{j} - 4\hat{k}$, $\vec{b} = 6\hat{i} + 5\hat{j} - 2\hat{k}$ and $|\vec{c}| = 3$, find the vector $\vec{c}$, which is perpendicular to both $\vec{a}$ and $\vec{b}$.

(Nagaland B. 2015)

37. (*a*) Let $\vec{a} = \hat{i} + 4\hat{j} + 2\hat{k}$, $\vec{b} = 3\hat{i} - 2\hat{j} + 7\hat{k}$ and $\vec{c} = 2\hat{i} - \hat{j} + 4\hat{k}$. Find a vector $\vec{d}$, which is perpendicular to both $\vec{a}$ and $\vec{b}$ and (*i*) $\vec{c} \cdot \vec{d} = 15$

(N.C.E.R.T.; Jammu B. 2017, 15; H.B. 2015)

(*ii*) $\vec{c} \cdot \vec{d} = 18$

(W. Bengal B. 2018; A.I.C.B.S.E. 2010)

(*iii*) $\vec{d} \cdot \vec{c} = 18$. *(A.I.C.B.S.E. 2012)*

(*b*) Vectors $\vec{a} = \hat{i} + \hat{j} + \hat{k}$, $\vec{b} = \hat{j} + 3\hat{k}$ and $\vec{c} = \hat{i} - 2\hat{j} + \hat{k}$ are given. Find vector $\vec{d}$ if $\vec{d}$ is perpendicular to $\vec{c}$ and $\vec{d} \cdot \vec{a} = 6$, $\vec{d} \cdot \vec{b} = 11$. *(P.B. 2017)*

(*c*) Let $\vec{a} = (\hat{i} - \hat{j})$, $\vec{b} = (3\hat{j} - \hat{k})$ and $\vec{c} = (7\hat{i} - \hat{k})$. Find a vector $\vec{d}$ such that it is perpendicular to both $\vec{a}$ and $\vec{b}$, and $\vec{c} \cdot \vec{d} = 1$. *(Mizoram B. 2015)*

Long Answer Type Questions

LATQ

38. Consider A (2, 3, 4), B (4, 3, 2) and C (5, 2, – 1) be any three points.

(*a*) Find the projection of $\overrightarrow{BC}$ on $\overrightarrow{AB}$.

(*b*) Find the area of triangle ABC. *(Kerala B. 2013)*

39. Dot–products of a vector with vectors

$3\hat{i}-5\hat{k}$, $2\hat{i}+7\hat{j}$ and $\hat{i}+\hat{j}+\hat{k}$

are respectively –1, 6 and 5. Find the vector.

(Type : Mizoram B. 2017; J. & K.B. 2011)

40. If $\hat{i}+\hat{j}+\hat{k}$, $2\hat{i}+5\hat{j}$, $3\hat{i}+2\hat{j}-3\hat{k}$ and $\hat{i}-6\hat{j}-\hat{k}$ respectively are the position vectors of the points A, B, C and D, then find the angle between the straight lines AB and CD. Deduce that AB and CD are parallel.

(N.C.E.R.T.)

41. If $\vec{a} = 2\hat{i}-\hat{j}-2\hat{k}$ and $\vec{b}=7\hat{i}+2\hat{j}-3\hat{k}$, then express $\vec{b}$ in the form $\vec{b}=\vec{b_1}+\vec{b_2}$, where $\vec{b_1}$ is parallel to $\vec{a}$ and $\vec{b_2}$ is perpendicular to $\vec{a}$.

(A.I.C.B.S.E. 2017)

Answers

1. True. **2.** (*a*) 0 (*b*) $\sqrt{\vec{a}.\vec{a}}$.

3. (*a*) (*i*) $\frac{8}{7}$ (*ii*) $\frac{10}{\sqrt{17}}$ (*b*) 0.

4. (*i*) 9 (*ii*) – 3.

5. (*a*) $6|\vec{a}|^2 + 11\,\vec{a}.\vec{b} - 35|\vec{b}|^2$ (*b*) (*i*) 3 (*ii*) 9.

6. 9. **7.** $\sqrt{13}$.

8. (*a*) (*i*) 120° (*ii*) $\cos^{-1}\left(\frac{1}{3}\right)$ (*iii*) $\cos^{-1}\left(\frac{5}{7}\right)$

(*iv*) $\cos^{-1}\left(-\frac{1}{3}\right)$ (*v*) $\cos^{-1}\left(\frac{2}{7}\right)$ (*vi*) $\cos^{-1}\left(\frac{16}{21}\right)$

(*b*) $\frac{1}{2}$.

9. (*i*) 45° (*ii*) 30°. **10.** $\frac{\pi}{3}$.

11. 1, 1. **12.** (*a*) $\vec{b}=\vec{a}$

(*b*) Take any two non-zero perpendicular vectors $\vec{a}$ and $\vec{b}$.

13. (*a*) (*i*) $\frac{8}{7}$ (*ii*) $\frac{2}{7}\sqrt{14}$ (*iii*) $\frac{5}{3}\sqrt{6}$ (*iv*) 0 (*v*) $\frac{30}{57}\sqrt{114}$

(*b*) (*i*) 2 (*ii*) 2.

14. (*a*) $\frac{17}{49}(2\hat{i}+6\hat{j}+3\hat{k})$ (*b*) $\frac{5}{6}(\hat{i}-2\hat{j}+\hat{k})$.

15. $\lambda = 5$. **18.** $p=\frac{2}{3}$.

19. (*i*) – 2 (*ii*) $\frac{5}{2}$. **20.** $m=-3;\sqrt{266}$ sq. units.

23. $\frac{1}{3}(2\hat{i}-2\hat{j}-\hat{k})$. **24.** $\lambda = 8$.

25. (*i*) ± 5 (*ii*) ± 1 (*iii*) $\pm\sqrt{73}$ (*iv*) ± 9.

26. (*i*) –17 ; $\pi-\cos^{-1}\left(\frac{17}{7\sqrt{38}}\right)$

(*ii*) – 50 ; $\pi-\cos^{-1}\left(\frac{5}{\sqrt{37}}\right)$.

28. (*a*) $\overrightarrow{AB} = -\hat{i}+3\hat{j}+5\hat{k}$

$\overrightarrow{BC} = -\hat{i}-2\hat{j}-6\hat{k}$

$\overrightarrow{CA} = 2\hat{i}-\hat{j}+\hat{k}$.

32. (*a*) $-\frac{3}{2}$ (*b*) $-\frac{21}{2}$.

33. $\lambda = 1$; $\frac{3\hat{i}+6\hat{j}-2\hat{k}}{\sqrt{7}}$.

36. $\vec{c}=2\hat{i}-2\hat{j}+\hat{k}$ **OR** $\vec{c}=-2\hat{i}+2\hat{j}-\hat{k}$.

37. (*a*) (*i*) $\frac{5}{3}(32\hat{i}-\hat{j}-14\hat{k})$

(*ii*) – (*iii*) 2 $32\hat{i}-\hat{j}-14\hat{k}$

(*b*) $\hat{i}+2\hat{j}+3\hat{k}$ (*c*) $\vec{d}=\frac{1}{4}(\hat{i}+\hat{j}+3\hat{k})$.

38. (*a*) $2\sqrt{2}$ units (*b*) $\sqrt{6}$ sq. units.

39. $3\hat{i}+2\hat{k}$. **40.** π.

41. $\vec{b}=(3\hat{i}+4\hat{j}+\hat{k})+(4\hat{i}-2\hat{j}-4\hat{k})$.

Hints to Selected Questions

5. (*b*) (*i*) $(\vec{x}-\vec{a}).(\vec{x}+\vec{a})=8 \Rightarrow \vec{x}^2-\vec{a}^2=8$

$\Rightarrow \vec{x}^2-|\vec{a}|^2=8 \Rightarrow \vec{x}^2-(1)^2=8$

$\Rightarrow |\vec{x}|^2=9 \Rightarrow |\vec{x}|=3.$

8 – 10. Use $\cos\theta = \dfrac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|}$.

12.(*a*) $\vec{a}.\vec{a}=0$ and $\vec{a}.\vec{b}=0$

$\Rightarrow \vec{a}.\vec{a}-\vec{a}.\vec{b}=0 \Rightarrow \vec{a}.(\vec{a}-\vec{b})=0$

$\Rightarrow$either $\vec{a}=\vec{0}$ or $\vec{a}-\vec{b}=\vec{0}$ or $\vec{a}\perp(\vec{a}-\vec{b})$

$\Rightarrow \vec{b}=\vec{a}$.

18. Vectors are parallel if :

$\dfrac{3}{1}=\dfrac{2}{p}=\dfrac{9}{3} \Rightarrow p=\dfrac{2}{3}$.

22. $(|\vec{a}|\vec{b}+|\vec{b}|\vec{a}).(|\vec{a}|\vec{b}-|\vec{b}|\vec{a})$

$=|\vec{a}|^2\,\vec{b}.\vec{b}-|\vec{a}||\vec{b}|\vec{b}.\vec{a}+|\vec{a}||\vec{b}|\vec{a}.\vec{b}-|\vec{b}|^2\,\vec{a}.\vec{a}$

$=|\vec{a}|^2|\vec{b}|^2-|\vec{a}||\vec{b}|\vec{a}.\vec{b}+|\vec{a}||\vec{b}|\vec{a}.\vec{b}-|\vec{b}|^2|\vec{a}|^2=0$.

29. $|\vec{a}+\vec{b}|^2=|\vec{a}-\vec{b}|^2$

$\Rightarrow (\vec{a}+\vec{b})\cdot(\vec{a}+\vec{b})=(\vec{a}-\vec{b})\cdot(\vec{a}-\vec{b})$

$\Rightarrow \vec{a}\cdot\vec{b}=0.$

31. $(\vec{a}+\vec{b}).(\vec{a}+\vec{b})=|\vec{a}|^2+|\vec{b}|^2$

$\Rightarrow \vec{a}.\vec{a}+\vec{a}.\vec{b}+\vec{b}.\vec{a}+\vec{b}.\vec{b}=|\vec{a}|^2+|\vec{b}|^2$

$\Rightarrow 2\vec{a}.\vec{b}=0 \Rightarrow \vec{a}.\vec{b}=0.$

33. Unit vector along the sum $=\dfrac{(2+\lambda)\hat{i}+6\hat{j}-2\hat{k}}{\sqrt{(2+\lambda)^2+36+4}}$.

35. $\left(\dfrac{\vec{a}}{a^2}-\dfrac{\vec{b}}{b^2}\right)^2$

$=\dfrac{\vec{a}^2}{a^4}+\dfrac{\vec{b}^2}{b^4}-\dfrac{2\vec{a}.\vec{b}}{a^2b^2}=\dfrac{1}{a^2}+\dfrac{1}{b^2}-\dfrac{2\vec{a}.\vec{b}}{a^2b^2}$

$=\dfrac{a^2+b^2-2\vec{a}.\vec{b}}{a^2b^2}=\left(\dfrac{\vec{a}-\vec{b}}{ab}\right)^2$.

37. (*a*) (*i*) Let $\vec{d}=d_1\hat{i}+d_2\hat{j}+d_3\hat{k}$.

Now $\vec{d}.\vec{a}=0 \Rightarrow d_1+4d_2+2d_3=0$

$\vec{d}.\vec{b}=0 \Rightarrow 3d_1-2d_2+7d_3=0$

and $\vec{c}.\vec{d}=15 \Rightarrow 2d_1-d_2+4d_3=18$; etc.

10.25. GEOMETRICAL AND TRIGONOMETRICAL PROBLEMS

Now we shall discuss some geometrical and trigonometrical problems in which the idea of dot product is very much used.

Frequently Asked Questions

FAQs

Example 1. Prove that in a right-angled triangle, the square of the hypotenuse is equal to the sum of the squares of the other two sides.

Solution. Let AOB be a triangle, right-angled at O. Take O as the origin.

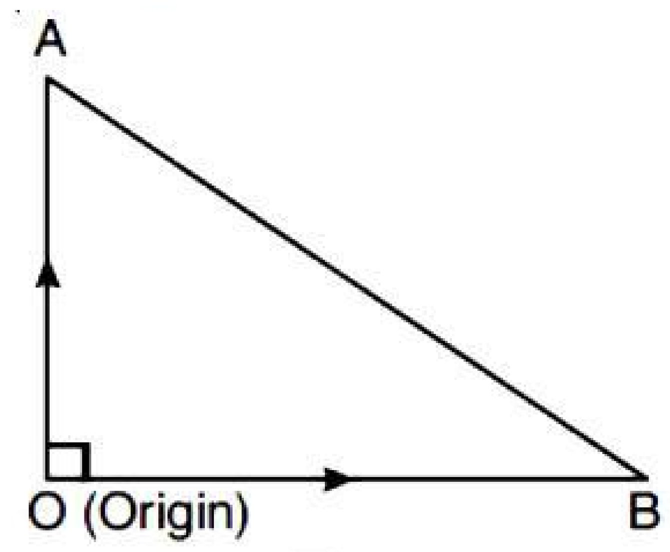


Fig.

Let the position vectors of A and B be $\vec{a}$ and $\vec{b}$ respectively.

$\therefore \quad \overrightarrow{OA}=\vec{a}$ and $\overrightarrow{OB}=\vec{b}$ so that :

$\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}=\vec{b}-\vec{a}.$

Now $\overrightarrow{OA}\perp\overrightarrow{OB} \Rightarrow \vec{a}\cdot\vec{b}=0$ ...(1)

$\therefore \; AB^2=|\overrightarrow{AB}|^2=\overrightarrow{AB}\cdot\overrightarrow{AB}=(\vec{b}-\vec{a})\cdot(\vec{b}-\vec{a})$

$=\vec{b}\cdot\vec{b}-\vec{b}\cdot\vec{a}-\vec{a}\cdot\vec{b}+\vec{a}\cdot\vec{a}$

$$= \left|\vec{b}\right|^2 - 2\,\vec{a}\cdot\vec{b} + \left|\vec{a}\right|^2 \qquad \left[\because \vec{b}\cdot\vec{a} = \vec{a}\cdot\vec{b}\right]$$

$$= \left|\vec{b}\right|^2 + \left|\vec{a}\right|^2 \qquad [\textit{Using } (1)]$$

$$= OB^2 + OA^2.$$

Hence, AOB is a triangle, right angled at O.

Example 2. Prove that in an isosceles triangle, the median to the base is perpendicular to the base.

Solution. Let ABC be an isosceles triangle in which BA = CA.

Let O be the mid-point of BC so that AO is the median.

To prove : AO $\perp$ BC.

Take O as the origin.

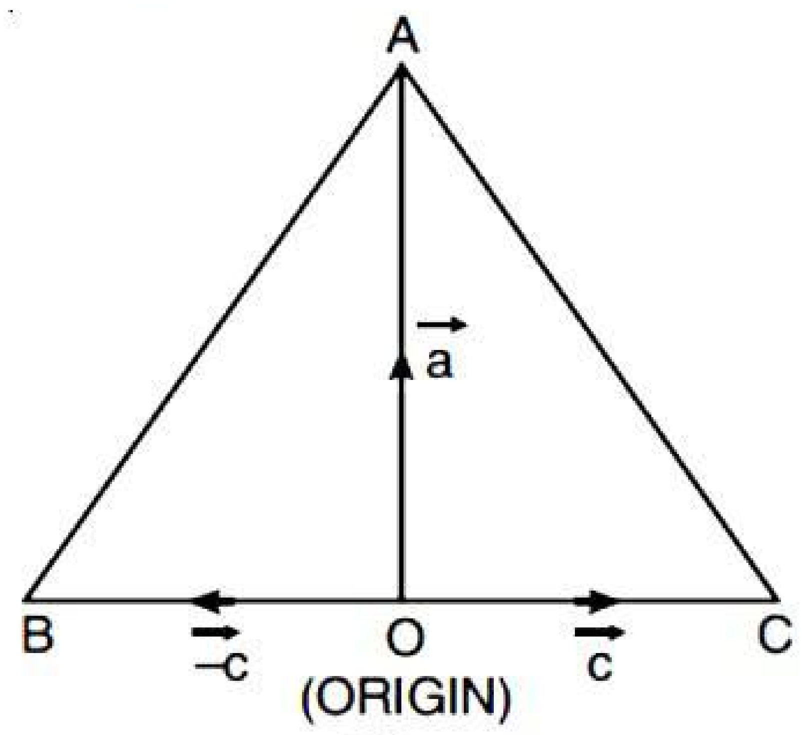


Fig.

Let the position vectors of C, B and A be $\vec{c}, -\vec{c}$ and $\vec{a}$ respectively.

Then $\overrightarrow{BA}$ = (P.V. of A) – (P.V. of B)

$$= \vec{a} - (-\vec{c}) = \vec{a} + \vec{c}$$

and $\overrightarrow{CA}$ = (P.V. of A) – (P.V. of C) = $\vec{a} - \vec{c}$.

Now BA = CA

$$\Rightarrow \quad BA^2 = CA^2 \Rightarrow \overrightarrow{BA}\cdot\overrightarrow{BA} = \overrightarrow{CA}\cdot\overrightarrow{CA}$$

$$\Rightarrow (\vec{a}+\vec{c})\cdot(\vec{a}+\vec{c}) = (\vec{a}-\vec{c})\cdot(\vec{a}-\vec{c})$$

$$\Rightarrow \vec{a}^2 + \vec{c}^2 + 2\,\vec{a}\cdot\vec{c} = \vec{a}^2 + \vec{c}^2 - 2\,\vec{a}\cdot\vec{c}$$

$$\Rightarrow \quad 4\,\vec{a}\cdot\vec{c} = 0 \quad\Rightarrow\quad \vec{a}\cdot\vec{c} = 0$$

$$\Rightarrow \quad \overrightarrow{OA}\cdot\overrightarrow{OC} = 0$$

$$\Rightarrow \quad OA \perp OC \quad\Rightarrow\quad OA \perp BC.$$

Hence, AO $\perp$ BC.

Example 3. Prove that the perpendiculars from the vertices to the opposite sides (i.e. altitudes) of a triangle are concurrent.

Solution. Let ABC be the triangle.

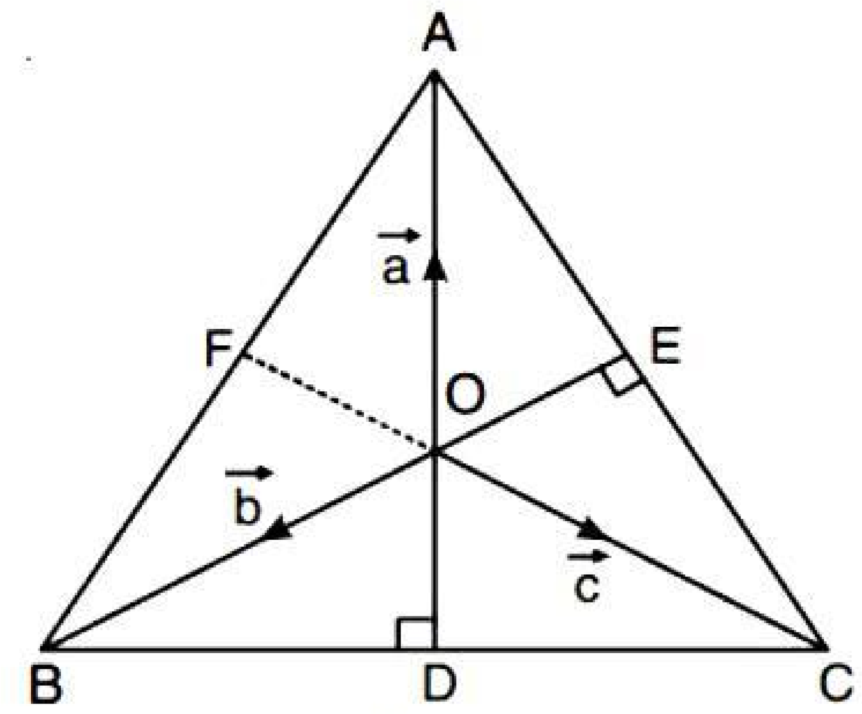


Fig.

Draw AD $\perp$ BC and BE $\perp$ CA and let them meet in O. Join CO and produce it to meet AB in F.

With O as origin, let $\vec{a}, \vec{b}, \vec{c}$ be the position vectors of A, B, C respectively.

Since $\overrightarrow{DA} \perp \overrightarrow{BC}$, $\quad\therefore\ \overrightarrow{OA} \perp \overrightarrow{BC}$

$$\Rightarrow (\vec{a} - \vec{0})\cdot(\vec{c} - \vec{b}) = 0$$

$$\Rightarrow \quad \vec{a}\cdot(\vec{c} - \vec{b}) = 0 \Rightarrow \vec{a}\cdot\vec{c} - \vec{a}\cdot\vec{b} = 0$$

$$\Rightarrow \quad \vec{a}\cdot\vec{c} = \vec{a}\cdot\vec{b} \qquad \ldots(1)$$

Since $\overrightarrow{EB} \perp \overrightarrow{CA}$, $\quad\therefore\ \overrightarrow{OB} \perp \overrightarrow{CA}$

$$\Rightarrow (\vec{b} - \vec{0})\cdot(\vec{a} - \vec{c}) = 0$$

$$\Rightarrow \quad \vec{b}\cdot(\vec{a} - \vec{c}) = 0 \Rightarrow \vec{b}\cdot\vec{a} = \vec{b}\cdot\vec{c}$$

$$\Rightarrow \quad \vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{c} \quad \ldots(2) \quad \because \vec{b}\cdot\vec{a} = \vec{a}\cdot\vec{b}$$

(1) and (2) $\Rightarrow \vec{a}\cdot\vec{c} = \vec{b}\cdot\vec{c}$

$$\Rightarrow \quad (\vec{a} - \vec{b})\cdot\vec{c}\ \text{A}= 0 \Rightarrow \vec{c}\cdot(\vec{a} - \vec{b}) = 0$$

$$\Rightarrow (\vec{c} - \vec{0})\cdot(\vec{a} - \vec{b}) = 0$$

$$\Rightarrow \overrightarrow{OC} \perp \overrightarrow{BA} \Rightarrow \overrightarrow{CF} \perp \overrightarrow{AB}.$$

Hence, the perpendiculars from the vertices to the opposite sides are concurrent.

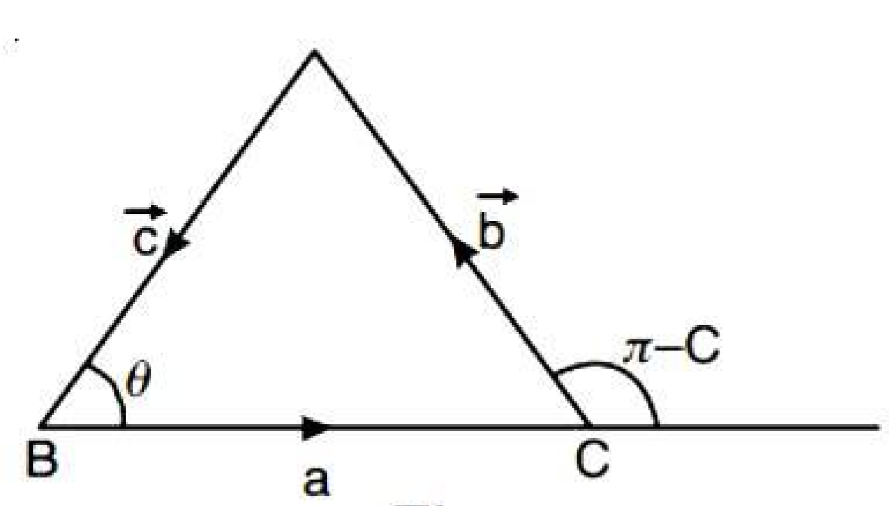
Example 4. Prove that, in any triangle ABC,

(i) cos C = $\mathbf{\dfrac{a^2 + b^2 - c^2}{2ab}}$

(ii) c = a cos B + b cos A.

(J. & K.B. 2011; P.B. 2011)

Solution. (*i*) Let BC, CA, AB represent the vectors $\vec{a}, \vec{b}, \vec{c}$ respectively.



Fig

Now $\quad \overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{CA} = \overrightarrow{0}$

$\Rightarrow \quad \overrightarrow{BC}+\overrightarrow{CA} = -\overrightarrow{AB}.$

Squaring, $\quad (\overrightarrow{BC}+\overrightarrow{CA})^2 = (\overrightarrow{AB})^2$

$\Rightarrow (\overrightarrow{BC})^2 + 2\overrightarrow{BC}\cdot\overrightarrow{CA} + (\overrightarrow{CA})^2 = (\overrightarrow{AB})^2$

$\Rightarrow \quad (\vec{a})^2 + 2\vec{a}\cdot\vec{b} + (\vec{b})^2 = (\vec{c})^2$

$\Rightarrow \; a^2 + 2ab\cos(\pi - C) + b^2 = c^2$

$\Rightarrow \; c^2 = a^2 + b^2 - 2ab\cos C.$

Hence, $\cos C = \dfrac{a^2+b^2-c^2}{2ab}$.

(*ii*) Again, $\overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{CA} = \overrightarrow{0}$

$\Rightarrow \quad \overrightarrow{AB} = -(\overrightarrow{BC}+\overrightarrow{CA})$

$\Rightarrow \quad \overrightarrow{AB}\cdot\overrightarrow{AB} = -(\overrightarrow{BC}+\overrightarrow{CA})\cdot\overrightarrow{AB}$

$\Rightarrow \quad \overrightarrow{AB}^2 = -\overrightarrow{BC}\cdot\overrightarrow{AB} - \overrightarrow{CA}\cdot\overrightarrow{AB}$

$\Rightarrow \quad (\vec{c})^2 = -\vec{a}\cdot\vec{c} - \vec{b}\cdot\vec{c}$

$\Rightarrow \quad c^2 = -ac\cos(\pi - B) - bc\cos(\pi - A)$

$\Rightarrow \quad c^2 = ac\cos B + bc\cos A.$

Hence, $\quad c = a\cos B + b\cos A.$

Example 5. Show that the diagonals of a rhombus are perpendicular to each other.

Solution. Let ABCD be the rhombus.

Take A as the origin.

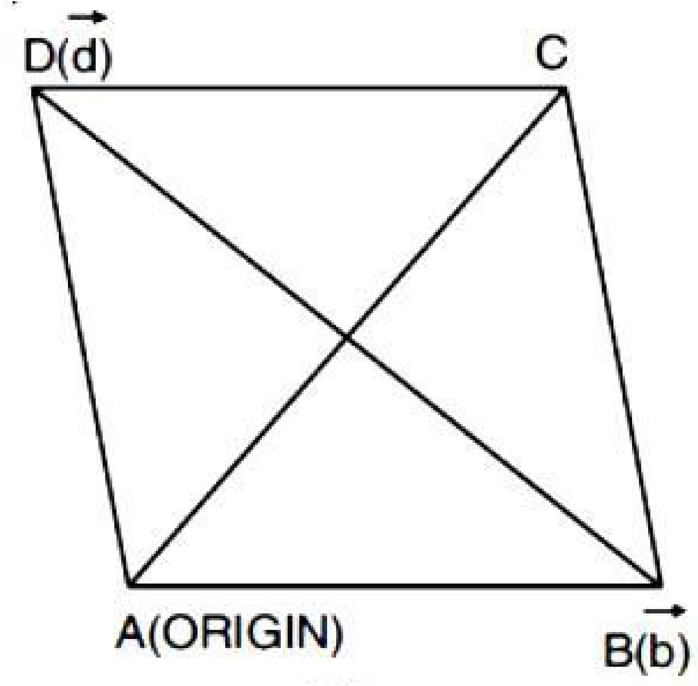


Fig.

Let $\vec{b}$ and $\vec{d}$ be the position vectors of B and D respectively.

Now $\quad \overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AB} + \overrightarrow{AD} = \vec{b} + \vec{d}$.

Thus the position vector of C is $\vec{b} + \vec{d}$.

Similarly $\overrightarrow{BD} = \vec{d} - \vec{b}$.

Since $\quad AB = AD, \quad \therefore AB^2 = AD^2$

$\Rightarrow \quad \vec{b}^2 = \vec{d}^2 \quad \Rightarrow \quad b^2 = d^2$ ...(1)

Now $\overrightarrow{AC} = \vec{b} + \vec{d}$ and $\overrightarrow{BD} = \vec{d} - \vec{b}$.

$\therefore \; \overrightarrow{AC}\,.\,\overrightarrow{BD} = (\vec{b}+\vec{d})\,.\,(\vec{d}-\vec{b}) = (\vec{d}+\vec{b})\,.\,(\vec{d}-\vec{b})$

$= \vec{d}^2 - \vec{b}^2 = d^2 - b^2 = 0$ [*Using (1)*]

Thus AC $\perp$ BD.

Hence, the result.

Tetrahedron. *A tetrahedron is a three-dimensional figure formed by four triangular faces.*

A tetrahedron with all edges equal, is called a ***regular tetrahedron***.

Example 6. In a tetrahedron, if two pairs of opposite edges are perpendicular to each other, prove that the third pair is also perpendicular and that the sum of the squares on the two opposite edges is same for each pair.

Solution. Let ABCD be the tetrahedron.

Take D as the origin.

Let the position vectors of A, B, C be $\vec{a}, \vec{b}, \vec{c}$ respectively.

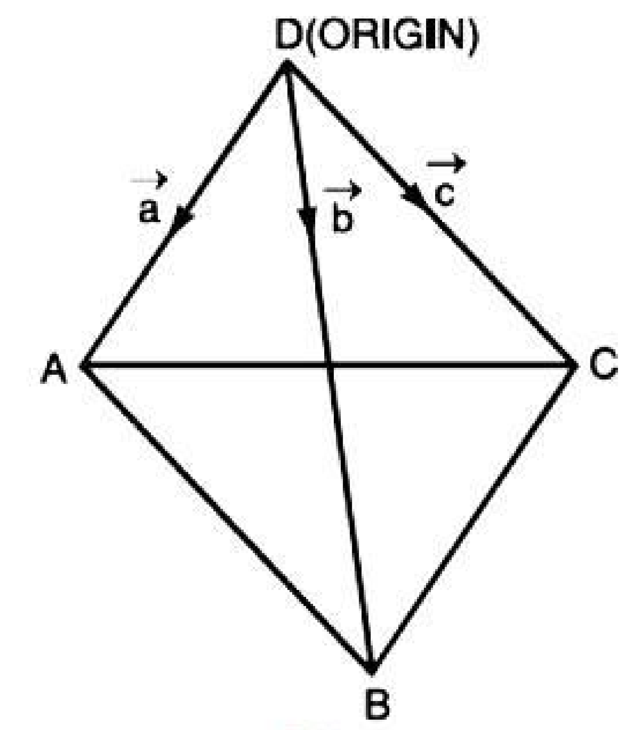


Fig.

Then $\overrightarrow{AB} = \vec{b} - \vec{a}$, $\overrightarrow{BC} = \vec{c} - \vec{b}$, $\overrightarrow{CA} = \vec{a} - \vec{c}$.

Let DA $\perp$ BC and DB $\perp$ CA.

(*i*) **To prove :** DC $\perp$ AB.

Since DA $\perp$ BC,

$\therefore \quad \overrightarrow{DA}\,.\,\overrightarrow{BC} = 0$ *i.e.* $(\vec{a} - \vec{0})\,.\,(\vec{c} - \vec{b}) = 0$

$\Rightarrow \vec{a}\,.\,\vec{c} - \vec{a}\,.\,\vec{b} = 0$ ...(1)

Since DB $\perp$ CA,

$\therefore \quad \overrightarrow{DB}\,.\,\overrightarrow{CA} = 0$ *i.e.* $(\vec{b} - \vec{0})\,.\,(\vec{a} - \vec{c}) = 0$

$\Rightarrow \vec{b}\,.\,\vec{a} - \vec{b}\,.\,\vec{c} = 0$ ...(2)

Adding (1) and (2), $\vec{a}\,.\,\vec{c} - \vec{b}\,.\,\vec{c} = 0$

$[\because \vec{a}\,.\,\vec{b} = \vec{b}\,.\,\vec{a}]$

$\Rightarrow \quad (\vec{a} - \vec{b})\,.\,\vec{c} = 0$

$\Rightarrow \quad \vec{c}\,.\,(\vec{a} - \vec{b}) = 0$

$\Rightarrow \; (\vec{c} - \vec{0})\,.\,(\vec{a} - \vec{b}) = 0$

$\Rightarrow \overrightarrow{DC}\,.\,\overrightarrow{BA} = 0 \;\Rightarrow\; \overrightarrow{DC}\,.\,\overrightarrow{AB} = 0 \;\;\therefore\;$ DC $\perp$ AB.

Hence, the first part.

(*ii*) Now $\;(DA)^2 + (BC)^2 = (\vec{a} - \vec{0})^2 + (\vec{c} - \vec{b})^2$

$= \vec{a}^2 + \vec{c}^2 + \vec{b}^2 - 2\vec{c}\,.\,\vec{b}$

$(DB)^2 + (CA)^2 = (\vec{b} - \vec{0})^2 + (\vec{a} - \vec{c})^2$

$= \vec{b}^2 + \vec{a}^2 + \vec{c}^2 - 2\vec{a}\,.\,\vec{c}$

and $\quad (DC)^2 + (AB)^2 = (\vec{c} - \vec{0})^2 + (\vec{b} - \vec{a})^2$

$= \vec{c}^2 + \vec{b}^2 + \vec{a}^2 - 2\vec{a}\,.\,\vec{c}$

But $\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{c} = \vec{c}\cdot\vec{a}$ *[From (1) and (2)]*

$\therefore (DA)^2 + (BC)^2 = (DB)^2 + (CA)^2$
$= (DC)^2 + (AB)^2$.

Hence, the second part.

Example 7. Prove, by vector method, that :
cos (A – B) = cos A cos B + sin A sin B.

(Assam B. 2015)

Solution. Let OX and OY be the co-ordinate axes and let $\hat{i}$ and $\hat{j}$ be unit vectors along OX and OY respectively.

Let $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ be unit vectors, which make angles A and B respectively with x-axis.

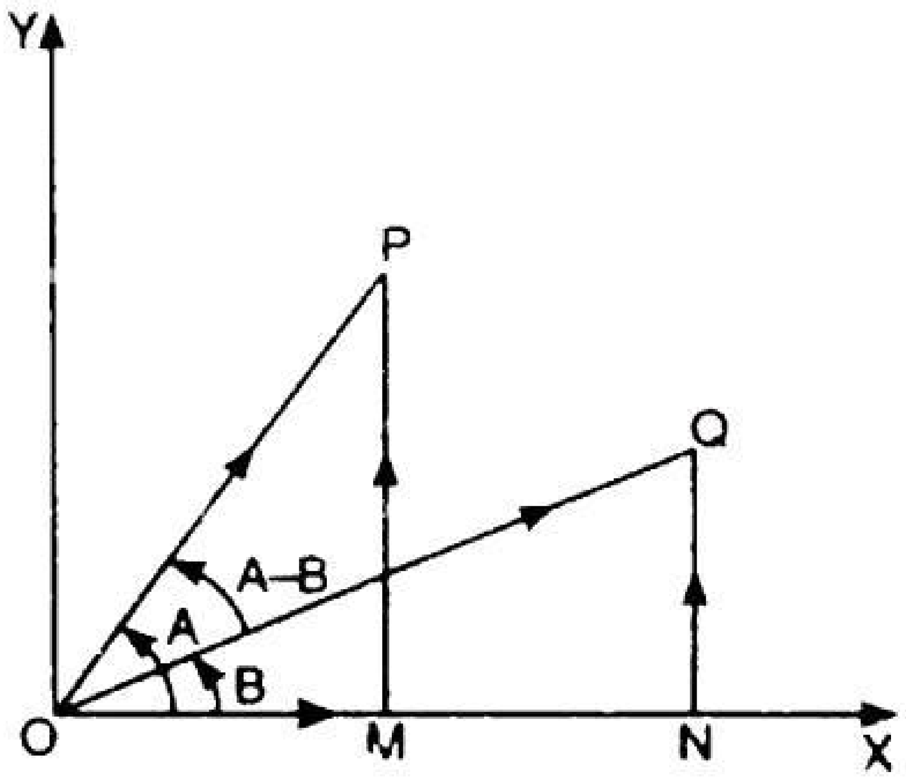


Fig.

$\therefore$ Angle between $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ is (A – B).

Now $\overrightarrow{OP} = \overrightarrow{OM} + \overrightarrow{MP} = (OM)\,\hat{i} + (MP)\,\hat{j}$

$= (OP \cos A)\,\hat{i} + (OP \sin A)\,\hat{j}$

$= \cos A\,\hat{i} + \sin A\,\hat{j}$ $[\because OP = 1]$

and $\overrightarrow{OQ} = \overrightarrow{ON} + \overrightarrow{NQ} = (ON)\,\hat{i} + (NQ)\,\hat{j}$

$= (OQ \cos B)\,\hat{i} + (OQ \sin B)\,\hat{j}$

$= \cos B\,\hat{i} + \sin B\,\hat{j}$ $[\because OQ = 1]$

$\therefore$ $\overrightarrow{OP}\cdot\overrightarrow{OQ} = (\cos A\,\hat{i} + \sin A\,\hat{j})\cdot(\cos B\,\hat{i} + \sin B\,\hat{j})$
$= \cos A \cos B + \sin A \sin B$.

But $\overrightarrow{OP}\cdot\overrightarrow{OQ} = \left|\overrightarrow{OP}\right|\left|\overrightarrow{OQ}\right| \cos(A - B)$

$= (1)(1)\cos(A - B) = \cos(A - B)$.

Hence, cos (A – B) = cos A cos B + sin A sin B.

$[\because$ *each* $= \overrightarrow{OP}\cdot\overrightarrow{OQ}]$

Example 8. Prove that an angle subtended at the circumference of a circle by any diameter is a right-angle.

Solution. Let O be the centre of the circle and AB the diameter.

Let P be any point on the circumference of the circle.

Take O as the origin.

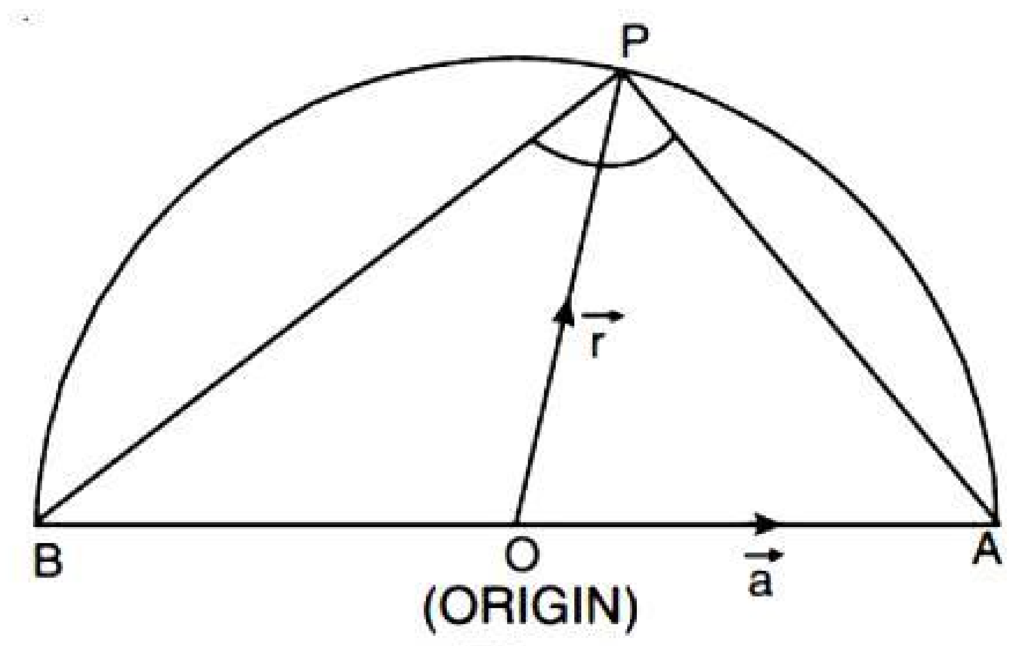


Fig.

Let $\overrightarrow{OA} = \vec{a}$. Then $\overrightarrow{OB} = -\vec{a}$.

Let $\overrightarrow{OP} = \vec{r}$.

Since OP = OA, $\therefore OP^2 = OA^2 \Rightarrow \overrightarrow{OP}^2 = \overrightarrow{OA}^2$

$\Rightarrow \vec{r}^{\,2} = \vec{a}^{\,2} \Rightarrow \vec{r}^{\,2} - \vec{a}^{\,2} = 0$

$\Rightarrow (\vec{r} - \vec{a})\cdot(\vec{r} + \vec{a}) = 0$

$\Rightarrow (\overrightarrow{OP} - \overrightarrow{OA})\cdot(\overrightarrow{OP} - \overrightarrow{OB}) = 0$

$\Rightarrow \overrightarrow{AP}\cdot\overrightarrow{BP} = 0 \Rightarrow AP \perp BP$.

Hence, $\angle APB = 90°$.

Another Form. *Prove, by vectors, that the angle in a semi-circle is a right-angle.* (Assam B. 2015; P.B. 2010 S)

EXERCISE 10 (f)

Short Answer Type Questions

SATQ

1. Prove that the right-bisectors of the sides of a triangle are concurrent.
2. Prove that in a right-angled triangle, the mid-point of the hypotenuse is equidistant from its vertices.
3. If two medians of a triangle are equal, prove that the triangle is isosceles.
4. If the diagonals of a parallelogram are equal in length, prove that it is a rectangle.
5. Prove that the sum of the squares of the diagonals of a parallelogram is equal to the sum of the squares of its sides.

In any triangle ABC, then by vectors, prove that (6 – 8) :

6. (*i*) $a^2 = b^2 + c^2 - 2bc \cos A$

(*ii*) $b^2 = c^2 + a^2 - 2ca \cos B$

(*iii*) $c^2 = a^2 + b^2 - 2ab \cos C.$ **(*Bihar B. 2013*)**

7. (*i*) $\cos B = \dfrac{c^2 + a^2 - b^2}{2ca}$ **(*P.B. 2013*)**

(*ii*) $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}.$ **(*P.B. 2013*)**

8. (*i*) $a = b \cos C + c \cos B$ **(*P.B. 2011, 10 S*)**

(*ii*) $b = c \cos A + a \cos C.$**(*P.B. 2011; J. & K.B. 2011*)**

Long Answer Type Questions

LATQ

9. Prove, by vector method, that :

$\cos (A + B) = \cos A \cos B - \sin A \sin B.$

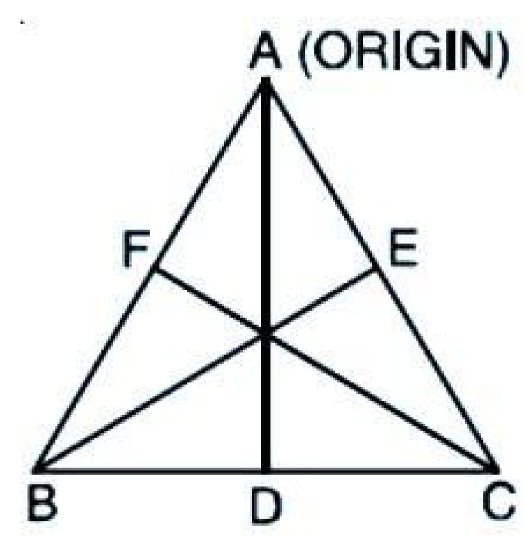
10. If D is the mid-point of the side [BC] of a triangle ABC, prove that $AB^2 + AC^2 = 2 (AD^2 + BD^2)$.

11. Prove that any two edges in a regular tetrahedron are perpendicular.

Hints to Selected Questions

3. Take A as origin. $B(\vec{b}), C(\vec{c})$ are two vertices.

P.V. of E, mid-point of [CA] is $\dfrac{\vec{c}}{2}$.



P.V. of F, mid-point of [AB] is $\dfrac{\vec{b}}{2}$.

Now $|\overrightarrow{BE}| = |\overrightarrow{CF}|$

$$\Rightarrow \quad \left|\frac{\vec{c}}{2} - \vec{b}\right| = \left|\frac{\vec{b}}{2} - \vec{c}\right|. \text{ Square ; etc.}$$

4. $\overrightarrow{AC} = \vec{c} - \vec{a}.$

$\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB} = \overrightarrow{OA} + \overrightarrow{OC} = \vec{a} + \vec{c}.$

Now $AC = OB \Rightarrow AC^2 = OB^2$

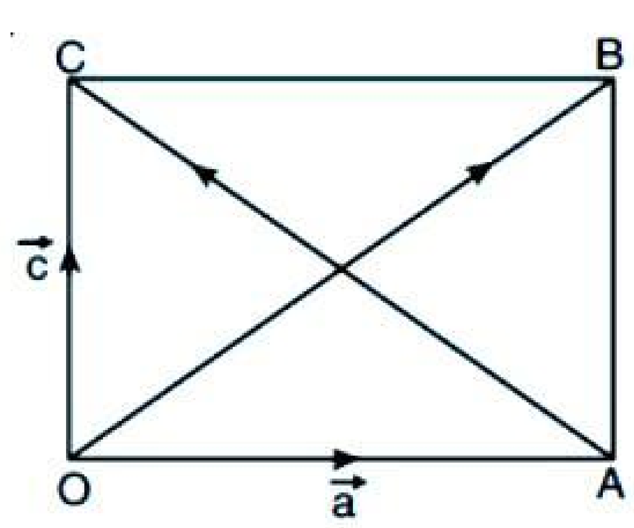


Fig.

$\Rightarrow \overrightarrow{AC} . \overrightarrow{AC} = \overrightarrow{OB} . \overrightarrow{OB}$

$\Rightarrow (\vec{c} - \vec{a}).(\vec{c} - \vec{a}) = (\vec{a} + \vec{c}).(\vec{a} + \vec{c})$

$\Rightarrow \vec{a} . \vec{c} = 0 \Rightarrow OA \perp OC$

$\Rightarrow$ OABC is a rectangle.

5. $\overrightarrow{OB} = \vec{a} + \vec{c}$, $\overrightarrow{AC} = \vec{c} - \vec{a}$

$\overrightarrow{AB} = \overrightarrow{OC} = \vec{c}$

$\overrightarrow{BC} = -\overrightarrow{OA} = -\vec{a}$, $\overrightarrow{CO} = -\vec{c}$.

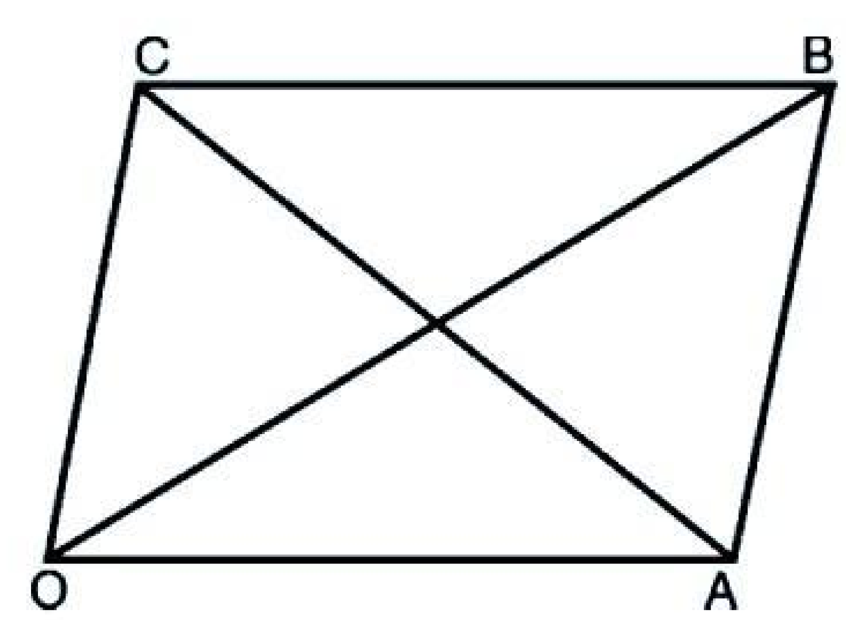


Fig.

$\therefore OB^2 + AC^2 = (\vec{a} + \vec{c})^2 + (\vec{c} - \vec{a})^2 = 2(a^2 + c^2)$

$= OA^2 + AB^2 + BC^2 + CO^2.$

7 – 8. As in Ex. 4.

9. As in Ex. 6.

10.26. WORK DONE BY A FORCE

A force acting on a particle is said to do work when the particle is displaced in a direction, which is not perpendicular to the direction of the force. The work done is a scalar quantity and its measure is defined to be the product of force and displacement. Thus if $\vec{F}, \vec{d}$ be vector representing the force and the displacement respectively, inclined at angle 'θ', the measure of the work done is $Fd \cos \theta = \vec{F}.\vec{d}$

KEY POINT

The work done is zero if $\vec{d}$ is perpendicular to $\vec{F}$ because in this case $\cos\theta = \cos\frac{\pi}{2} = 0$.

ILLUSTRATIVE EXAMPLES

Example 1. Find the work done by the force $\vec{F} = 2\hat{i} + \hat{j} + \hat{k}$ acting on a particle, if the particle is displaced from the point with position vector $2\hat{i} + 2\hat{j} + 2\hat{k}$ to the point with position vector $3\hat{i} + 4\hat{j} + 5\hat{k}$.

Solution. Here $\vec{F} = 2\hat{i} + \hat{j} + \hat{k}$

and $\vec{d} = (3\hat{i} + 4\hat{j} + 5\hat{k}) - (2\hat{i} + 2\hat{j} + 2\hat{k})$

$= \hat{i} + 2\hat{j} + 3\hat{k}$.

$\therefore$ Work done $= \vec{F}\cdot\vec{d} = (2\hat{i} + \hat{j} + \hat{k})\cdot(\hat{i} + 2\hat{j} + 3\hat{k})$

$= (2)(1) + (1)(2) + (1)(3) = 2 + 2 + 3 = 7$ units.

Example 2. Constant forces $2\hat{i} - 5\hat{j} + 6\hat{k}$, $-\hat{i} + 2\hat{j} - \hat{k}$ and $2\hat{i} + 7\hat{j}$ act on a particle, which is displaced from position $4\hat{i} - 3\hat{j} - 2\hat{k}$ to position $6\hat{i} + \hat{j} - 3\hat{k}$. Find the work done.

Solution. Let $\vec{R}$ be the resultant force and $\vec{d}$ the displacement vector.

Then $\vec{R} = (2\hat{i} - 5\hat{j} + 6\hat{k}) + (-\hat{i} + 2\hat{j} - \hat{k}) + (2\hat{i} + 7\hat{j})$

$= 3\hat{i} + 4\hat{j} + 5\hat{k}$

and $\vec{d} = (6\hat{i} + \hat{j} - 3\hat{k}) - (4\hat{i} - 3\hat{j} - 2\hat{k}) = 2\hat{i} + 4\hat{j} - \hat{k}$.

$\therefore$ Work done

$= \vec{R}\cdot\vec{d} = (3\hat{i} + 4\hat{j} + 5\hat{k})\cdot(2\hat{i} + 4\hat{j} - \hat{k})$

$= (3)(2) + (4)(4) + (5)(-1)$

$= 6 + 16 - 5 = 17$ units.

Example 3. Forces of magnitudes 5, 3, 1 units acting in the directions $6\hat{i} + 2\hat{j} + 3\hat{k}$, $3\hat{i} - 2\hat{j} + 6\hat{k}$, $2\hat{i} - 3\hat{j} - 6\hat{k}$ respectively act on a particle, which is displaced from the point (2, –1, –3) to (5, –1, 1). Find the work done by the forces.

Solution. Let $\vec{R}$ be the resultant force and $\vec{d}$ the displacement vector.

Then $\vec{R} = 5.\dfrac{6\hat{i} + 2\hat{j} + 3\hat{k}}{\sqrt{36+4+9}} + 3.\dfrac{3\hat{i} - 2\hat{j} + 6\hat{k}}{\sqrt{9+4+36}} + 1.\dfrac{2\hat{i} - 3\hat{j} - 6\hat{k}}{\sqrt{4+9+36}}$

$= \frac{5}{7}(6\hat{i} + 2\hat{j} + 3\hat{k}) + \frac{3}{7}(3\hat{i} - 2\hat{j} + 6\hat{k}) + \frac{1}{7}(2\hat{i} - 3\hat{j} - 6\hat{k})$

$= \frac{1}{7}(41\hat{i} + \hat{j} + 27\hat{k})$

and $\vec{d} = (5\hat{i} - \hat{j} + \hat{k}) - (2\hat{i} - \hat{j} - 3\hat{k}) = 3\hat{i} + 4\hat{k}$.

$\therefore$ Work done $= \vec{R}\cdot\vec{d}$

$= \frac{1}{7}(41\hat{i} + \hat{j} + 27\hat{k})\cdot(3\hat{i} + 4\hat{k})$

$= \frac{1}{7}[(41)(3) + (1)(0) + (27)(4)]$

$= \frac{1}{7}(123 + 0 + 108) = \frac{231}{7} = 33$ units.

EXERCISE 10 (g)

Very Short Answer Type Questions

VSATQ

1. In each problem, find the work done by a force $\vec{F}$ acting on a particle such that the particle is displaced from a point A to a point B :

(*i*) A : (1, 2, 0) ; B (2, – 1, 3) : $\vec{F} = 4\hat{i} + 2\hat{j} + 3\hat{k}$ ***(H.B. 2011)***

(*ii*) A : (1, 2, 0); B : (0, 2, 3) ; $\vec{F} = 4\hat{i} - 3\hat{k}$.

2. Find the work done by the force $\vec{F} = \hat{i} + 2\hat{j} + \hat{k}$ acting on a particle, if the particle is displaced from the point with position vectors $2\hat{i} + \hat{j} + \hat{k}$ to the point with position vector $3\hat{i} + 2\hat{j} + 4\hat{k}$. ***(H.B. 2011)***

3. Find the work done in moving an object along the vector $\vec{d} = 3\hat{i} + 2\hat{j} - 5\hat{k}$, if the applied force is $\vec{F} = 2\hat{i} - \hat{j} - \hat{k}$.

Short Answer Type Questions

SATQ

4. A particle acted on by constant forces $4\hat{i} + \hat{j} - 3\hat{k}$ and $3\hat{i} + \hat{j} - \hat{k}$ is displaced from the point $\hat{i} + 2\hat{j} + 3\hat{k}$ to the point $5\hat{i} + 4\hat{j} + \hat{k}$. Find the total work done by the forces.

5. Constant forces $2\hat{i} - 5\hat{j} + 6\hat{k}$ and $-\hat{i} + 2\hat{j} - \hat{k}$ act on a particle. Determine the work done when the particle is displaced from a point A with position vector $4\hat{i} - 3\hat{j} - 2\hat{k}$ to a point B with position vector $6\hat{i} + \hat{j} - 3\hat{k}$.

6. Constant forces of magnitudes 5 and 3 units acting in the directions $6\hat{i} + 2\hat{j} + 3\hat{k}$ and $3\hat{i} - 2\hat{j} + 6\hat{k}$ respectively act on a particle, which is displaced from the point (2, 2, – 1) to (4, 3, 1). Find the work done by the forces.

7. Forces $\vec{F}_1, \vec{F}_2, \vec{F}_3$ of magnitude 5, 3, 1 units respectively, act in the direction :

$6\hat{i} + 2\hat{j} + 3\hat{k}$, $3\hat{i} - 2\hat{j} + 6\hat{k}$ and $2\hat{i} - 3\hat{j} - 6\hat{k}$

respectively on a particle, which is displaced from the point (2, – 1, – 3) to the point (5, – 1, 1). Find the work done by $\vec{F}_1, \vec{F}_2, \vec{F}_3$ and $\vec{F} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3$.

Answers

1. (*i*) 7 units (*ii*) –13 units.
2. 6 units.
3. 9 units.
4. 40 units. **5.** – 15 units. **6.** $\frac{148}{7}$ units.
7. $\frac{150}{7}$ units, $\frac{99}{7}$ units, $\frac{18}{7}$ units, $\frac{231}{7}$ units.

Hints to Selected Questions

2. Here $\vec{d} = (3\hat{i} + 2\hat{j} + 4\hat{k}) - (2\hat{i} + \hat{j} + \hat{k})$

$= \hat{i} + \hat{j} + 3\hat{k}$.

$\therefore$ Work done $= \vec{F}.\vec{d} = (\hat{i} + 2\hat{j} + \hat{k}).(\hat{i} + \hat{j} + 3\hat{k})$

$= (1)(1) + (2)(1) + (1)(3)$

$= 1 + 2 + 3 = 6$ units.

4. Here $\vec{R} = (4\hat{i} + \hat{j} - 3\hat{k}) + (3\hat{i} + \hat{j} - \hat{k}) = 7\hat{i} + 2\hat{j} - 4\hat{k}$

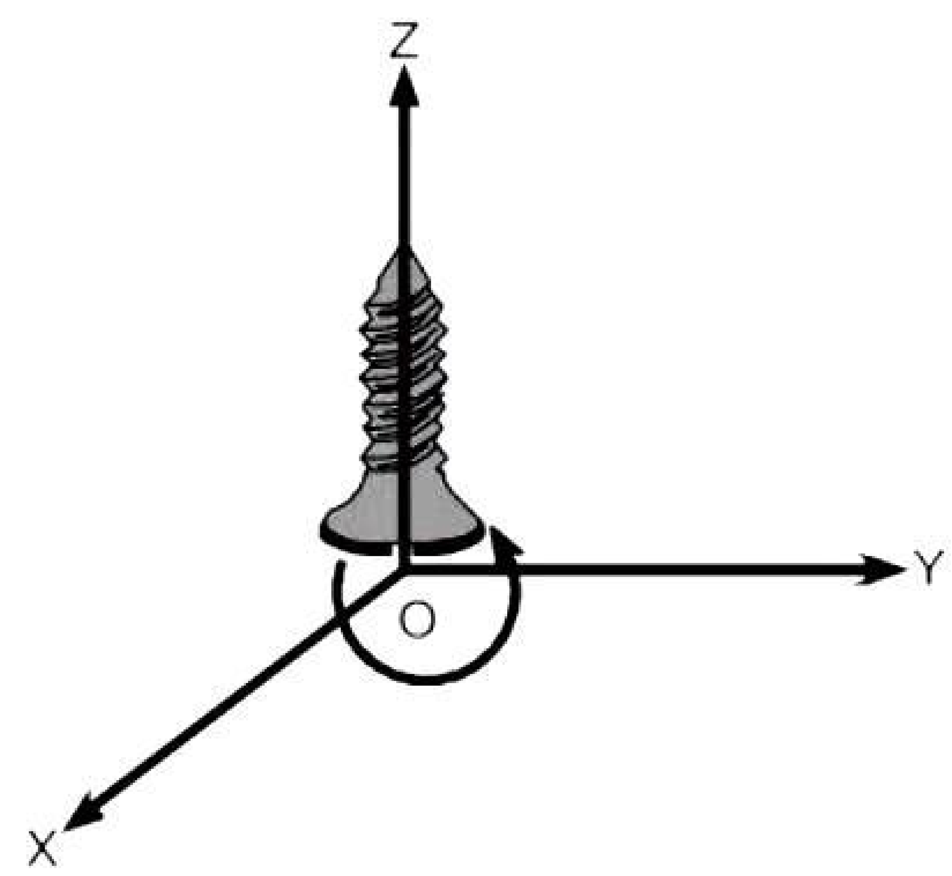
and $\vec{d} = (5\hat{i} + 4\hat{j} + \hat{k}) - (\hat{i} + 2\hat{j} + 3\hat{k})$

$= 4\hat{i} + 2\hat{j} - 2\hat{k}$.

$\therefore$ Work done $= \vec{R} \cdot \vec{d}$.

10.27 VECTOR PRODUCT

Here in this system, when the positive x-axis is rotated anti-clockwise, then the positive y-axis, a right-handed screw will go in the direction of the positive z-axis.



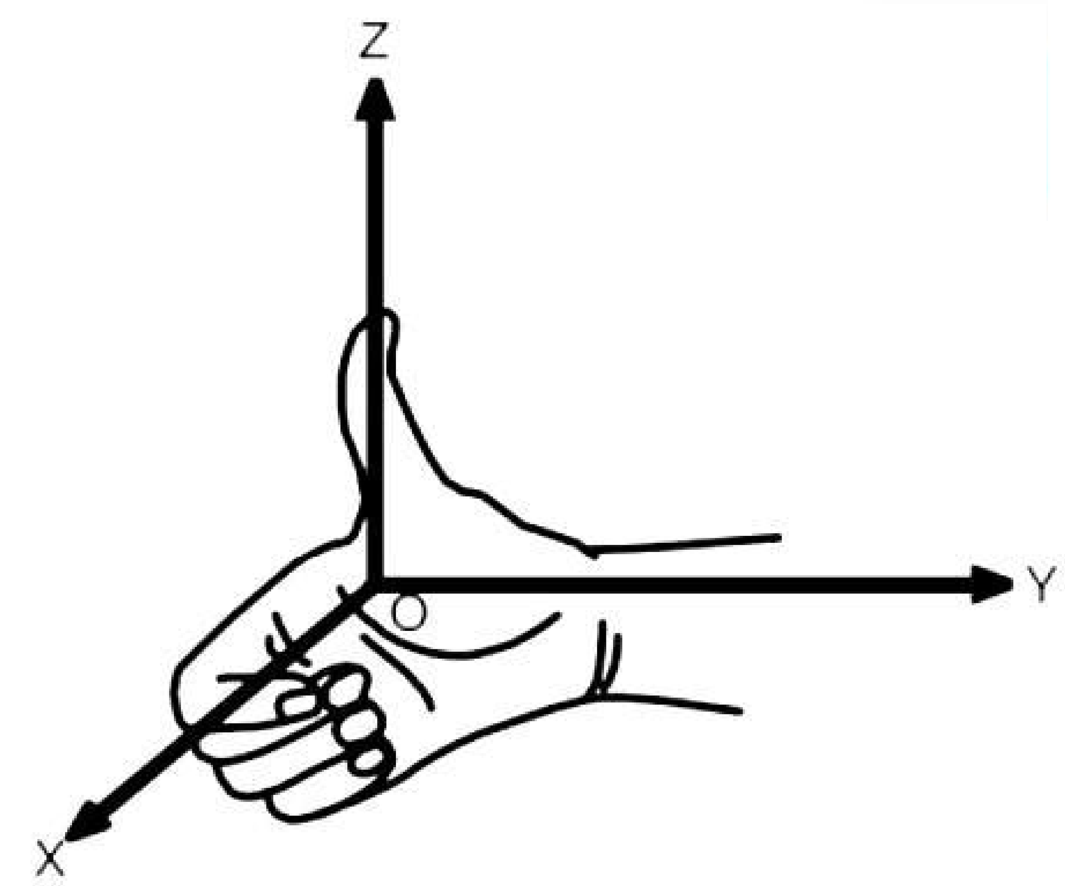


Fig.

Definition

The vector product of two vectors $\vec{a}$ and $\vec{b}$ is a vector whose magnitude is $|\vec{a}|\,|\vec{b}|\sin\theta$, where '$\theta$' is the angle between two vectors and whose direction is that of a unit vector $\hat{n}$ perpendicular to both $\vec{a}$ and $\vec{b}$.

Notation. The vector product of two vectors is written as :

$$\vec{a}\times\vec{b} \quad \text{or} \quad \left[\vec{a}\times\vec{b}\right]$$

$\vec{a}\times\vec{b}$ is read as $\vec{a}$ *cross* $\vec{b}$.

Because of this, the vector product is sometimes called the **cross product.**

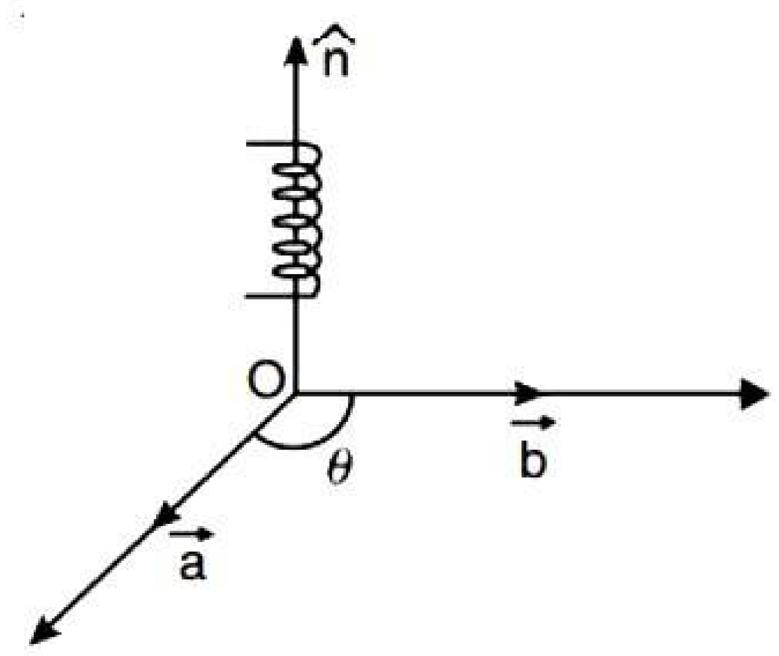


Fig.

Hence, $\boxed{\vec{\mathbf{a}}\times\vec{\mathbf{b}} = |\vec{\mathbf{a}}|\,|\vec{\mathbf{b}}|\sin\theta\,\hat{\mathbf{n}}, \text{ where } \vec{\mathbf{a}}, \vec{\mathbf{b}}, \hat{\mathbf{n}} \text{ form a right-handed system.}}$

(*ii*) **Geometrical Interpretation.**

Let OACB be a ‖ gm.

With O as origin, let $\overrightarrow{OA} = \vec{a}$ and $\overrightarrow{OB} = \vec{b}$. Let $\angle AOB = \theta$.

Area of ‖ gm. OACB $= 2\,\Delta\,OAB$

$$= 2\left(\frac{1}{2}ab\sin\theta\right) = ab\sin\theta.$$

$\therefore \quad |\vec{a}\times\vec{b}| = ab\sin\theta$

= area of the ‖gm.

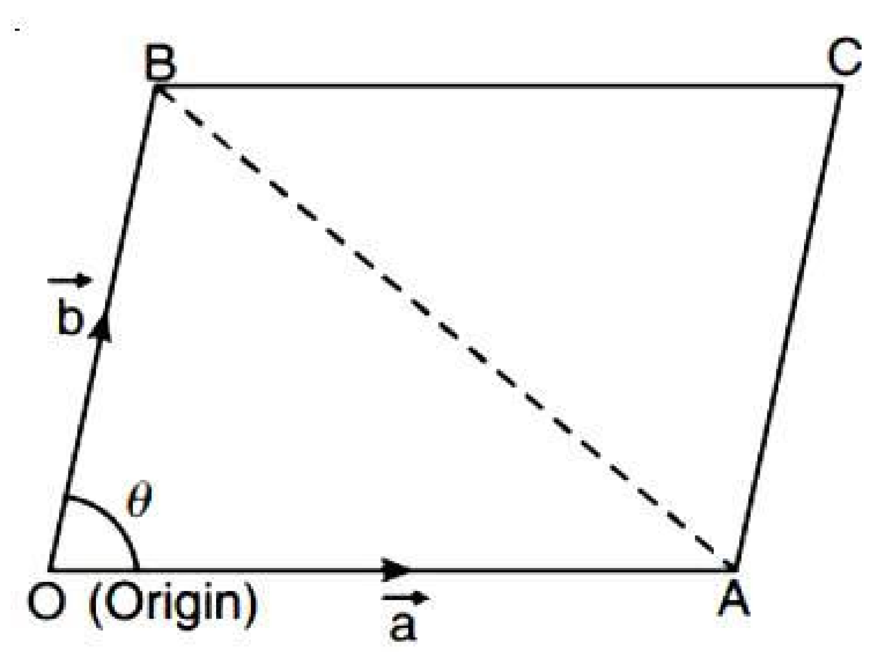


Fig.

Hence, $\vec{a} \times \vec{b}$ is a vector perpendicular to both $\vec{a}$ and $\vec{b}$ and whose magnitude is equal to the area of the parallelogram of which $\vec{a}$ and $\vec{b}$ are adjacent sides.

Note : Area of ‖ gm OBCA = $\vec{b} \times \vec{a}$.

KEY POINT

$\vec{a} \times \vec{b}$ represents a vector area.

10.28. IMPORTANT RESULTS

(I) When the two vectors are parallel.

Here $\theta = 0$ or π $\therefore$ $\sin\theta = 0$. $\therefore$ $\vec{a} \times \vec{a} = |\vec{a}||\vec{b}| \sin\theta\, \hat{n} = \vec{0}$.

Particular Cases. $\vec{a} \times \vec{a} = \vec{0}$, $\vec{b} \times \vec{b} = \vec{0}$.

Cor. When $\vec{a} \times \vec{b} = \vec{0}$, then either $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$ or $\vec{a}$ and $\vec{b}$ are parallel vectors.

(II) When the two vectors are perpendicular.

Here $\theta = \pi/2$, $\therefore$ $\sin\theta = 1$. $\therefore$ $\vec{a} \times \vec{b} = |\vec{a}||\vec{b}| \sin\theta\, \hat{n} = |\vec{a}||\vec{b}|\hat{n}$.

(III) (*a*) When $\vec{a}$ and $\vec{b}$ are unit vectors.

Here $\vec{a} \times \vec{b} = (1)(1) \sin\theta\, \hat{n}$ $\therefore$ $|\vec{a} \times \vec{b}| = \sin\theta$.

(*b*) When $\vec{a}$ and $\vec{b}$ are not unit vectors.

Here $\vec{a} \times \vec{b} = |\vec{a}||\vec{b}| \sin\theta\, \hat{n}$ $\therefore \left|\vec{a} \times \vec{b}\right| = \left|\vec{a}\right|\left|\vec{b}\right| \sin\theta$ $\Rightarrow \sin\theta = \dfrac{|\vec{a} \times \vec{b}|}{|\vec{a}||\vec{b}|}$.

(IV) Vector products of orthonormal triad of unit vectors, $\hat{i}$, $\hat{j}$ and $\hat{k}$.

Since $\hat{i}$, $\hat{j}$ and $\hat{k}$ form a right–handed system of mutually perpendicular vectors,

$\therefore$ $\hat{i} \times \hat{j}$ is a vector having modulus unity and direction parallel to $\hat{k}$.

$\therefore$ $\hat{i} \times \hat{j} = \hat{k} = -\hat{j} \times \hat{i}$. Similarly, $\hat{j} \times \hat{k} = \hat{i} = -\hat{k} \times \hat{j}$ and $\hat{k} \times \hat{i} = \hat{j} = -\hat{i} \times \hat{k}$.

Also, $\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = \vec{0}$.

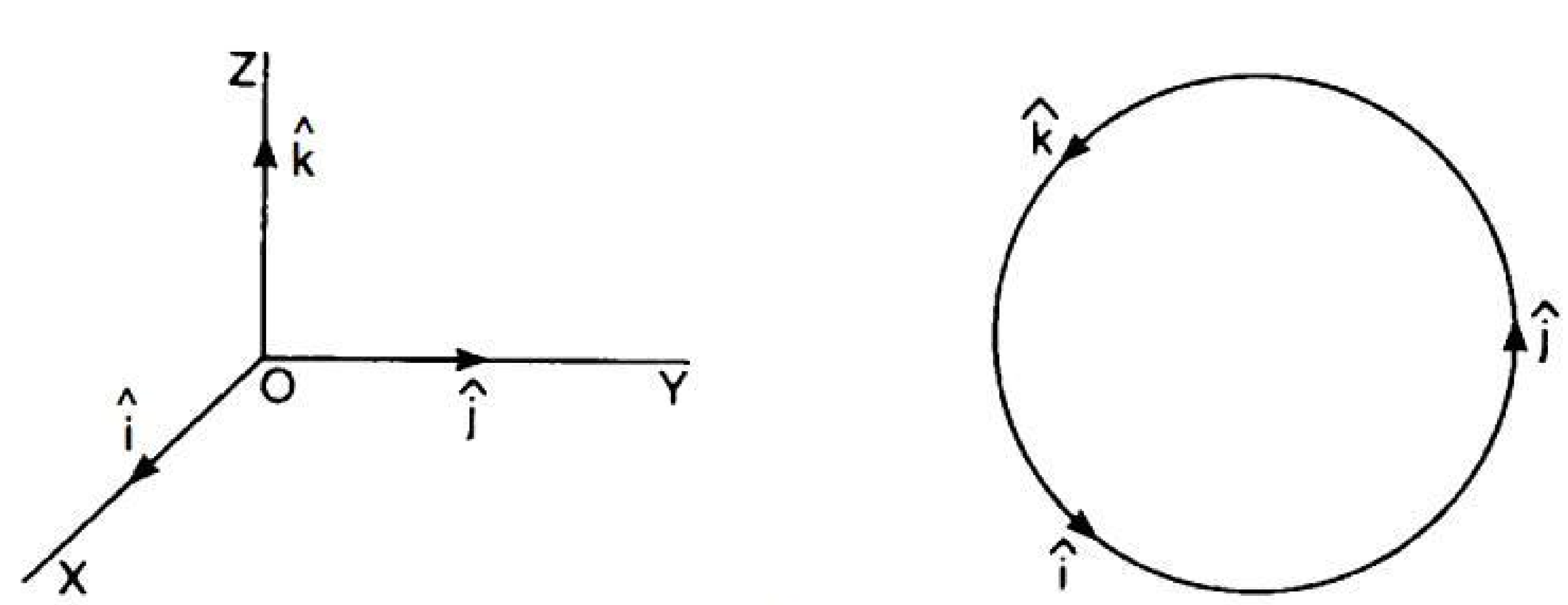


Fig.

These results can be remembered with the help of following **table** :

TABLE

$\otimes$	$\hat{i}$	$\hat{j}$	$\hat{k}$
$\hat{i}$	$\vec{0}$	$\hat{k}$	$-\hat{j}$
$\hat{j}$	$-\hat{k}$	$\vec{0}$	$\hat{i}$
$\hat{k}$	$\hat{j}$	$-\hat{i}$	$\vec{0}$

10.29. PROPERTIES OF VECTOR (OR CROSS) PRODUCT

Property I. Commutative Law. It does not hold.

If $\vec{a}, \vec{b}$ *are any two vectors, then* $\vec{a} \times \vec{b} \neq \vec{b} \times \vec{a}$.

Proof. Since the direction of translation of a right–handed screw due to rotation from $\vec{b}$ to $\vec{a}$ is opposite to that due to a rotation from $\vec{a}$ to $\vec{b}$,

$\therefore \quad \vec{b} \times \vec{a} = |\vec{b}||\vec{a}| \sin\theta\,(-\hat{n})$, where $\hat{n}$ is a unit vector, taken in the +ve direction

$$= -|\vec{a}||\vec{b}| \sin\theta\,\hat{n} = -\vec{a} \times \vec{b}.$$

Hence, $\vec{a} \times \vec{b} \neq \vec{b} \times \vec{a}$.

Property II. (*a*) $(m\vec{a}) \times \vec{b} = m(\vec{a} \times \vec{b}) = \vec{a} \times (m\vec{b})$, *where m is any non-zero scalar.*

Proof. *When* $m > 0$.

Here $(m\vec{a}) \times \vec{b} = m|\vec{a}||\vec{b}| \sin\theta\,\hat{n} = m(|\vec{a}||\vec{b}| \sin\theta\,\hat{n}) = m(\vec{a} \times \vec{b})$ ...(1)

And $(m\vec{a}) \times \vec{b} = m|\vec{a}||\vec{b}| \sin\theta\,\hat{n} = |\vec{a}|(m|\vec{b}|) \sin\theta\,\hat{n} = \vec{a} \times (m\vec{b})$ ...(2)

(1) and (2) prove the result.

Similarly, when $m < 0$.

(*b*) $\vec{a} \times (\vec{b} \times \vec{c}) \neq (\vec{a} \times \vec{b}) \times \vec{c}$, *where* $\vec{a}, \vec{b}, \vec{c}$ *are any three vectors.*

Verification. *When* $\vec{a} = \vec{b}$. **LHS** $= \vec{a} \times (\vec{a} \times \vec{c}) \neq \vec{0}$. **RHS** $= (\vec{a} \times \vec{a}) \times \vec{c} = \vec{0}$. Hence, **LHS** $\neq$ **RHS.**

Property III. Vector Product in terms of Rectangular components of vectors.

Solution. Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$.

$$\therefore \quad \vec{a} \times \vec{b} = (a_1\hat{i} + a_2\hat{j} + a_3\hat{k}) \times (b_1\hat{i} + b_2\hat{j} + b_3\hat{k})$$

$$= a_1b_1\,\hat{i} \times \hat{i} + a_1b_2\,\hat{i} \times \hat{j} + a_1b_3\,\hat{i} \times \hat{k} + a_2b_1\,\hat{j} \times \hat{i} + a_2b_2\,\hat{j} \times \hat{j} + a_2b_3\,\hat{j} \times \hat{k}$$

$$+ a_3b_1\,\hat{k} \times \hat{i} + a_3b_2\,\hat{k} \times \hat{j} + a_3b_3\,\hat{k} \times \hat{k}$$

$$= a_1b_1\,(\vec{0}) + a_1b_2\,(\hat{k}) + a_1b_3\,(-\hat{j}) + a_2b_1\,(-\hat{k}) + a_2b_2\,(\vec{0}) + a_2b_3\,(\hat{i}) + a_3b_1\,(\hat{j}) + a_3b_2\,(-\hat{i}) + a_3b_3\,(\vec{0})$$

$$= (a_2b_3 - a_3b_2)\,\hat{i} + (a_3b_1 - a_1b_3)\,\hat{j} + (a_1b_2 - a_2b_1)\,\hat{k}.$$

Hence, $\vec{a} \times \vec{b} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \mathbf{a_1} & \mathbf{a_2} & \mathbf{a_3} \\ \mathbf{b_1} & \mathbf{b_2} & \mathbf{b_3} \end{vmatrix}$.

Property IV. Distributive Law.

If $\vec{a}, \vec{b}, \vec{c}$ are any three vectors, then $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$. (N.C.E.R.T.)

Proof. Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ and $\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$.

$\therefore \quad \vec{b} + \vec{c} = (b_1 + c_1)\,\hat{i} + (b_2 + c_2)\,\hat{j} + (b_3 + c_3)\,\hat{k}$.

$$\textbf{LHS} = \vec{a} \times (\vec{b} + \vec{c}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 + c_1 & b_2 + c_2 & b_3 + c_3 \end{vmatrix} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} + \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$= \vec{a} \times \vec{b} + \vec{a} \times \vec{c} =$ **RHS**, which is true.

Frequently Asked Questions

FAQs

Example 1. If vectors $\vec{a}$ and $\vec{b}$ are such that $|\vec{a}| = 3$ and $|\vec{b}| = \frac{2}{3}$ and $\vec{a} \times \vec{b}$ is a unit vector, write the angle between $\vec{a}$ and $\vec{b}$.

(C.B.S.E. 2014; A.I.C.B.S.E. 2010)

Solution. We know that $|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}|\sin\theta$

$\Rightarrow \quad 1 = 3 \times \frac{2}{3}\sin\theta$

$\Rightarrow \quad \sin\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6}$.

Hence, the angle between $\vec{a}$ and $\vec{b}$ is $\frac{\pi}{6} = 30°$.

Example 2. Find the value of :

(i) $(\hat{\mathbf{i}} \times \hat{\mathbf{j}}).\hat{\mathbf{k}} + \hat{\mathbf{i}}.\hat{\mathbf{j}}$ (A.I.C.B.S.E. 2012)

(ii) $(\hat{\mathbf{k}} \times \hat{\mathbf{j}}).\hat{\mathbf{i}} + \hat{\mathbf{j}}.\hat{\mathbf{k}}$ (A.I.C.B.S.E. 2012)

(iii) $(\hat{\mathbf{k}} \times \hat{\mathbf{i}}).\hat{\mathbf{j}} + \hat{\mathbf{i}}.\hat{\mathbf{k}}$. (A.I.C.B.S.E. 2012)

Solution. (*i*) $(\hat{i} \times \hat{j}).\hat{k} + \hat{i}.\hat{j} = \hat{k}.\hat{k} + \hat{i}.\hat{j}$

$= 1 + 0 = 1.$

(*ii*) $(\hat{k} \times \hat{j}).\hat{i} + \hat{j}.\hat{k} = (-\hat{i}).\hat{i} + \hat{j}.\hat{k}$

$= -\hat{i}.\hat{i} + \hat{j}.\hat{k}$

$= -1 + 0 = -1.$

(*iii*) $(\hat{k} \times \hat{i}).\hat{j} + \hat{i}.\hat{k} = \hat{j}.\hat{j} + \hat{i}.\hat{k}$

$= 1 + 0 = 1.$

Example 3. If 'θ' is the angle between the vectors : $\vec{a} = \hat{i} - 2\hat{j} + 3\hat{k}$ and $\vec{b} = 3\hat{i} - 2\hat{j} + \hat{k}$, find $\sin\theta$.

(C.B.S.E. 2018)

Solution. We know that $\sin\theta = \dfrac{|\vec{a} \times \vec{b}|}{|\vec{a}||\vec{b}|}$

$$\Rightarrow \quad \sin\theta = \frac{|(\hat{i} - 2\hat{j} + 3\hat{k}) \times (3\hat{i} - 2\hat{j} + \hat{k})|}{|\hat{i} - 2\hat{j} + 3\hat{k}||3\hat{i} - 2\hat{j} + \hat{k}|} \quad ...(1)$$

Now, $(\hat{i} - 2\hat{j} + 3\hat{k}) \times (3\hat{i} - 2\hat{j} + \hat{k})$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -2 & 3 \\ 3 & -2 & 1 \end{vmatrix}$$

$$= \hat{i}(-2+6) - \hat{j}(1-9) + \hat{k}(-2+6)$$

$$= 4\hat{i} + 8\hat{j} + 4\hat{k}.$$

$\therefore |4\hat{i} + 8\hat{j} + 4\hat{k}| = \sqrt{16+64+16} = \sqrt{96} = 4\sqrt{6}$.

And, $|\hat{i} - 2\hat{j} + 3\hat{k}| = \sqrt{1+4+9} = \sqrt{14}$

$|3\hat{i} - 2\hat{j} + \hat{k}| = \sqrt{9+4+1} = \sqrt{14}$.

$\therefore$ From (1), $\sin\theta = \dfrac{4\sqrt{6}}{\sqrt{14}\sqrt{14}} = \dfrac{4\sqrt{6}}{14}$.

Hence, $\sin\theta = \dfrac{2\sqrt{6}}{7}$.

Example 4. Find 'λ' if $\left(2\hat{i}+6\hat{j}+14\hat{k}\right)\times\left(\hat{i}-\lambda\hat{j}+7\hat{k}\right) = \vec{0}$.

(A.I.C.B.S.E. 2010)

Solution. We have : $\left(2\hat{i}+6\hat{j}+14\hat{k}\right)\times\left(\hat{i}-\lambda\hat{j}+7\hat{k}\right) = \vec{0}$

$$\Rightarrow \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 6 & 14 \\ 1 & -\lambda & 7 \end{vmatrix} = \vec{0}$$

$\Rightarrow \hat{i}(42+14\lambda) - \hat{j}(14-14) + \hat{k}(-2\lambda-6) = \vec{0}$

$\Rightarrow 14\hat{i}(\lambda+3) - 2(\lambda+3)\hat{k} = \vec{0}$

$\Rightarrow \quad \lambda + 3 = 0.$

Hence, $\quad \lambda = -3.$

Example 5. Find λ and μ if :

$$(\hat{i}+3\hat{j}+9\hat{k})\times(3\hat{i}-\lambda\hat{j}+\mu\hat{k}) = \vec{0}.$$

(A.I.C.B.S.E. 2016)

Solution. We have : $(\hat{i}+3\hat{j}+9\hat{k})\times(3\hat{i}-\lambda\hat{j}+\mu\hat{k}) = \vec{0}$

$$\Rightarrow \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 3 & 9 \\ 3 & -\lambda & \mu \end{vmatrix} = \vec{0}$$

$\Rightarrow \quad \hat{i}(3\mu+9\lambda) - \hat{j}(\mu-27) + \hat{k}(-\lambda-9) = \vec{0}$

$\Rightarrow \quad 3\mu+9\lambda = 0,\ \mu-27=0,\ -\lambda-9=0$

$\Rightarrow \quad \mu+3\lambda=0,\ \mu=27,\ \lambda=-9.$

Hence, $\lambda = -9$ and $\mu = 27$.

Example 6. If $\vec{a}$ and $\vec{b}$ are two vectors such that $|\vec{a}\cdot\vec{b}| = |\vec{a}\times\vec{b}|$, then what is the angle between $\vec{a}$ and $\vec{b}$?

(A.I.C.B.S.E. 2010)

Solution. We have : $|\vec{a}\cdot\vec{b}| = |\vec{a}\times\vec{b}|$

$\Rightarrow |\vec{a}||\vec{b}|\cos\theta = |\vec{a}||\vec{b}|\sin\theta \Rightarrow \cos\theta = \sin\theta$

$\Rightarrow \quad \tan\theta = 1 \Rightarrow \theta = 45°.$

Hence, the angle between $\vec{a}$ and $\vec{b}$ = 45°.

Example 7. If $\vec{a} = \hat{i}+\hat{j}+\hat{k}$ and $\vec{b} = \hat{j}-\hat{k}$, find a vector $\vec{c}$ such that $\vec{a}\times\vec{c} = \vec{b}$ and $\vec{a}\cdot\vec{c} = 3$.

Solution. Let $\quad \vec{c} = c_1\hat{i}+c_2\hat{j}+c_3\hat{k}$ ...(1)

Now
$$\vec{a}\times\vec{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ c_1 & c_2 & c_3 \end{vmatrix}$$
$$= \hat{i}(c_3-c_2) - \hat{j}(c_3-c_1) + \hat{k}(c_2-c_1)$$

$\therefore \quad \vec{a}\times\vec{c} = \vec{b}$

$\Rightarrow (c_3-c_2)\hat{i} + (c_1-c_3)\hat{j} + (c_2-c_1)\hat{k} = \hat{j}-\hat{k}$.

Comparing, $c_3 - c_2 = 0$ ...(2)

$c_1 - c_3 = 1$ ...(3)

and $c_2 - c_1 = -1$ ...(4)

Also, $\vec{a}\cdot\vec{c} = 3$ *[Given]*

$\Rightarrow (\hat{i}+\hat{j}+\hat{k})\cdot(c_1\hat{i}+c_2\hat{j}+c_3\hat{k}) = 3$

$\Rightarrow (1)\,c_1 + (1)\,c_2 + (1)\,c_3 = 3$

$\Rightarrow \quad c_1 + c_2 + c_3 = 3$ ...(5)

From (2), $\quad c_2 = c_3$.

From (3), $\quad c_1 = 1 + c_3$.

Putting in (5), $1 + c_3 + c_3 + c_3 = 3$

$\Rightarrow \quad 3c_3 = 2 \Rightarrow c_3 = \dfrac{2}{3}$.

Also $\quad c_2 = \dfrac{2}{3}$ and $c_1 = 1 + \dfrac{2}{3} = \dfrac{5}{3}$.

These also satisfy (4). $\quad \left[\because c_2 - c_1 = \dfrac{2}{3} - \dfrac{5}{3} = -1\right]$

Putting in (1), $\quad \vec{c} = \dfrac{5}{3}\hat{i} + \dfrac{2}{3}\hat{j} + \dfrac{2}{3}\hat{k}$

$$= \frac{1}{3}\left(5\hat{i}+2\hat{j}+2\hat{k}\right).$$

Example 8. If $\vec{r} = x\hat{i}+y\hat{j}+z\hat{k}$, find :

$(\vec{r}\times\hat{i})\cdot(\vec{r}\times\hat{j}) + xy$. *(C.B.S.E. 2015)*

Solution. Here $\vec{r}\times\hat{i} = (x\hat{i}+y\hat{j}+z\hat{k})\times\hat{i}$

$= x(\hat{i}\times\hat{i}) + y(\hat{j}\times\hat{i}) + z(\hat{k}\times\hat{i})$

$= x(\vec{0}) + y(-\hat{k}) + z(\hat{j})$

$= -y\hat{k} + z\hat{j}$ ...(1)

and $\quad \vec{r}\times\hat{j} = (x\hat{i}+y\hat{j}+z\hat{k})\times\hat{j}$

$= x(\hat{i}\times\hat{j}) + y(\hat{j}\times\hat{j}) + z(\hat{k}\times\hat{j})$

$= x(\hat{k}) + y(\vec{0}) + z(-\hat{i})$

$= x\hat{k} - z\hat{i}$ ...(2)

$\therefore\ (\vec{r} \times \hat{i}) \cdot (\vec{r} \times \hat{j}) + xy$

$= (-y\hat{k} + z\hat{j}) \cdot (x\hat{k} - z\hat{i}) + xy$ *[Using (1) and (2)]*

$= -yx(\hat{k} \cdot \hat{k}) + yz(\hat{k} \cdot \hat{i}) + zx(\hat{j} \cdot \hat{k}) - z^2(\hat{j} \cdot \hat{i}) + xy$

$= -yx(1) + yz(0) + zx(0) - z^2(0) + xy$

$= -xy + xy$

$= 0.$

Example 9. If $\vec{a} \times \vec{b} = \vec{c} \times \vec{d}$, $\vec{a} \times \vec{c} = \vec{b} \times \vec{d}$, show that $(\vec{a} - \vec{d})$ is parallel to $(\vec{b} - \vec{c})$, provided $\vec{a} \neq \vec{d}$ and $\vec{b} \neq \vec{c}$. *(C.B.S.E. 2009)*

Solution. $(\vec{a} - \vec{d}) \times (\vec{b} - \vec{c})$

$= \vec{a} \times \vec{b} - \vec{a} \times \vec{c} - \vec{d} \times \vec{b} + \vec{d} \times \vec{c}$

$= \vec{c} \times \vec{d} - \vec{b} \times \vec{d} - \vec{d} \times \vec{b} + \vec{d} \times \vec{c}$

$[\because \vec{a} \times \vec{b} = \vec{c} \times \vec{d}$ *and* $\vec{a} \times \vec{c} = \vec{b} \times \vec{d}]$

$= \vec{c} \times \vec{d} - \vec{b} \times \vec{d} + \vec{b} \times \vec{d} - \vec{c} \times \vec{d} = \vec{0}.$

Hence, $(\vec{a} - \vec{d})$ is parallel to $(\vec{b} - \vec{c})$.

Example 10. Find a vector of magnitude 7 units, which is perpendicular to two vectors :

$2\hat{i} - \hat{j} + \hat{k}$ and $\hat{i} + \hat{j} - \hat{k}$. *(P.B. 2012)*

Solution. Let $\vec{a} = 2\hat{i} - \hat{j} + \hat{k}$ and $\vec{b} = \hat{i} + \hat{j} - \hat{k}$.

$\therefore$ Unit vector perp. to $\vec{a}$ and $\vec{b} = \dfrac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|}$.

But $\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$

$= \hat{i}(1 - 1) - \hat{j}(-2 - 1) + \hat{k}(2 + 1)$

$= 3\hat{j} + 3\hat{k}$.

$\therefore\ |\vec{a} \times \vec{b}| = \sqrt{0 + 9 + 9} = \sqrt{18} = 3\sqrt{2}$.

$\therefore$ Unit vector perp. to $\vec{a}$ to $\vec{b} = \dfrac{3\hat{j} + 3\hat{k}}{3\sqrt{2}} = \dfrac{\hat{j} + \hat{k}}{\sqrt{2}}$.

Hence, a unit vector of magnitude 7 units perp. to $\vec{a}$ and $\vec{b}$

$= \dfrac{7}{\sqrt{2}}(\hat{j} + \hat{k})$.

Example 11. Determine the area of a parallelogram whose adjacent sides are represented by the vectors : $\vec{a} = \hat{i} - \hat{j} + 3\hat{k}$ and $\vec{b} = 2\hat{i} - 7\hat{j} + \hat{k}$.

(Kashmir B. 2017, 12; H.P.B. 2015; Jammu B. 2015, 14)

Solution. If $\vec{a}$ and $\vec{b}$ are the vectors representing the adjacent sides of the parallelogram, then the magnitude of the area $= |\vec{a} \times \vec{b}|$.

Here $\vec{a} = \hat{i} - \hat{j} + 3\hat{k}$ and $\vec{b} = 2\hat{i} - 7\hat{j} + \hat{k}$.

$\therefore\ \vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 3 \\ 2 & -7 & 1 \end{vmatrix}$

$= \hat{i}(-1 + 21) - \hat{j}(1 - 6) + \hat{k}(-7 + 2)$

$= 20\hat{i} + 5\hat{j} - 5\hat{k}$.

$\therefore$ Magnitude of the area of the || gm.

$= |20\hat{i} + 5\hat{j} - 5\hat{k}|$

$= \sqrt{400 + 25 + 25} = \sqrt{450} = 15\sqrt{2}$ sq. units.

Example 12. Find the area of the triangle whose adjacent sides are made by the vectors :

$\vec{a} = 3\hat{i} + \hat{j} + 4\hat{k}$ and $\vec{b} = \hat{i} - \hat{j} + \hat{k}$.

(P.B. 2014; Karnataka B. 2014)

Solution. Reqd. area $= \dfrac{1}{2}|\vec{a} \times \vec{b}|$.

Now $\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 1 & 4 \\ 1 & -1 & 1 \end{vmatrix}$

$= \hat{i}(1 + 4) - \hat{j}(3 - 4) + \hat{k}(-3 - 1)$

$= 5\hat{i} + \hat{j} - 4\hat{k}$.

$\therefore\ |\vec{a} \times \vec{b}| = \sqrt{(5)^2 + (1)^2 + (-4)^2}$

$= \sqrt{25 + 1 + 16} = \sqrt{42}$.

Hence, the required area of the triangle $= \dfrac{\sqrt{42}}{2}$ sq. units.

Example 13. Find the area of the triangle having the points :

A (1, 1, 1), B (1, 2, 3) and C (2, 3, 1) as its vertices.

(N.C.E.R.T. ; P.B. 2015; H.P.B. 2015, 13; Kerala B. 2014)

Solution. Here :

$$\overrightarrow{AB} = (\hat{i} + 2\hat{j} + 3\hat{k}) - (\hat{i} + \hat{j} + \hat{k}) = \hat{j} + 2\hat{k}$$

and $\overrightarrow{AC} = (2\hat{i} + 3\hat{j} + \hat{k}) - (\hat{i} + \hat{j} + \hat{k}) = \hat{i} + 2\hat{j}$.

$\therefore$ Area of $\Delta ABC = \dfrac{1}{2} |\overrightarrow{AB} \times \overrightarrow{AC}|$.

But $\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{vmatrix}$

$$= \hat{i}\,(0-4) - \hat{j}\,(0-2) + \hat{k}\,(0-1)$$

$$= -4\hat{i} + 2\hat{j} - \hat{k}.$$

$\therefore$ Reqd. area of $\Delta ABC = \dfrac{1}{2} |-4\hat{i} + 2\hat{j} - \hat{k}|$

$$= \frac{1}{2}\sqrt{16+4+1} = \frac{\sqrt{21}}{2} \text{ sq. units.}$$

Example 14. If $\vec{a} = 2\hat{i} - 3\hat{j} + 4\hat{k}$ and $\vec{b} = 5\hat{i} + \hat{j} - \hat{k}$ represent sides of a parallelogram, then find both diagonals and a unit vector perpendicular to both diagonals of parallelogram. *(P.B. 2018)*

Solution. Let $\vec{a}$ and $\vec{b}$ represent $\overrightarrow{AB}$ and $\overrightarrow{BC}$ respectively.

Then, $\overrightarrow{AC} = \vec{a} + \vec{b}$

$$= (2\hat{i} - 3\hat{j} + 4\hat{k}) + (5\hat{i} + \hat{j} - \hat{k})$$

$$= 7\hat{i} - 2\hat{j} + 3\hat{k} = \vec{c} \quad \text{(say)}$$

and $\overrightarrow{BD} = -\vec{a} + \vec{b}$

$$= -(2\hat{i} - 3\hat{j} + 4\hat{k}) + (5\hat{i} + \hat{j} - \hat{k})$$

$$= 3\hat{i} + 4\hat{j} - 5\hat{k} = \vec{d} \text{ (say)}$$

$\vec{c} \times \vec{d}$ is perpendicular to $\vec{c}$ and $\vec{d}$

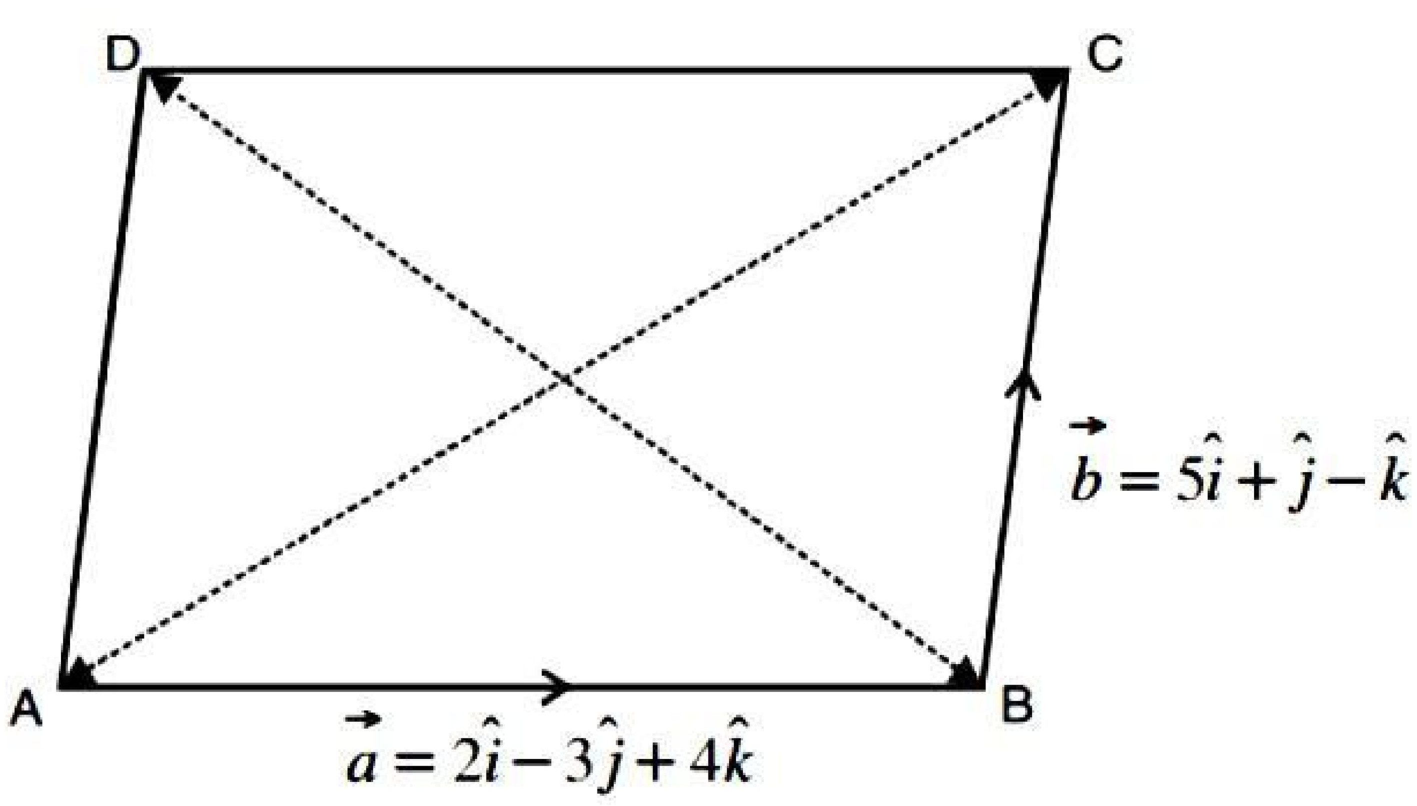


Fig.

Thus, reqd. unit vector $\hat{n} = \dfrac{\vec{c} \times \vec{d}}{|\vec{c} \times \vec{d}|}$ ....(1)

But $\vec{c} \times \vec{d} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 7 & -2 & 3 \\ 3 & 4 & -5 \end{vmatrix}$

$$= \hat{i}\,(10-12) - \hat{j}\,(-35-9) + \hat{k}\,(28+6)$$

$$= -2\hat{i} + 44\hat{j} + 34\hat{k}.$$

Also, $|\vec{c} \times \vec{d}| = \sqrt{4 + 1936 + 1156} = \sqrt{3096}$.

Hence, from (1), $\hat{n} = \dfrac{-2\hat{i} + 44\hat{j} + 34\hat{k}}{\sqrt{3096}}$.

Example 15. If $\vec{a}$, $\vec{b}$, $\vec{c}$ are the position vectors of the vertices A, B, C of a ΔABC respectively, find an expression for the area of ΔABC and hence deduce the condition for the points A, B, C to be collinear. *(H.B. 2016)*

Solution. (*i*) Let the position vectors of A, B, C be $\vec{a}$, $\vec{b}$, $\vec{c}$ respectively.

Complete the || gm ABCD.

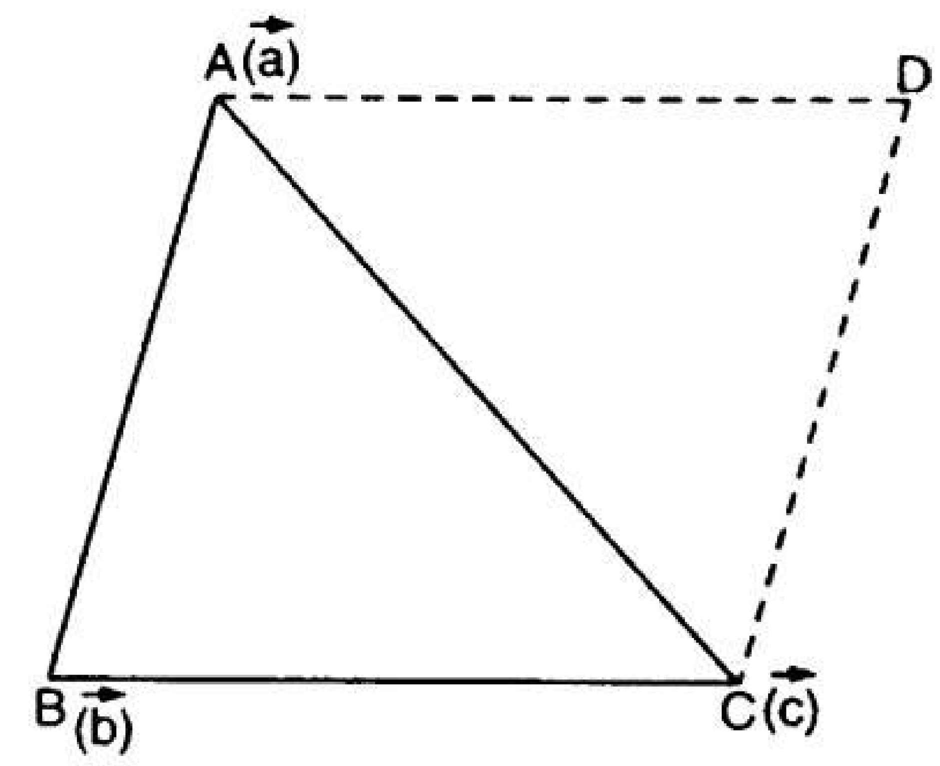


Fig.

$\therefore$ Area of ΔABC

$$= \frac{1}{2} \text{ area of ||gm. ABCD}$$

$$= \frac{1}{2} |\overrightarrow{AB} \times \overrightarrow{BC}|$$

$$= \frac{1}{2} |(\vec{b} - \vec{a}) \times (\vec{c} - \vec{b})|$$

$$= \frac{1}{2} |\vec{b} \times (\vec{c} - \vec{b}) - \vec{a} \times (\vec{c} - \vec{b})|$$

$$= \frac{1}{2} | \vec{b} \times \vec{c} - \vec{b} \times \vec{b} - \vec{a} \times \vec{c} + \vec{a} \times \vec{b} |$$

$$= \frac{1}{2} | \vec{b} \times \vec{c} - \vec{0} | - \vec{a} \times \vec{c} + \vec{a} \times \vec{b} |$$

$$= \frac{1}{2} | \vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a} |.$$

(*ii*) The three points A, B, C are collinear if $\Delta ABC = \vec{0}$

if $\frac{1}{2} | \vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a} | = 0$

if $\quad \vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a} = \vec{0}.$

Example 16. Lagrange's Identity.

Prove that

$$(\vec{\mathbf{a}} \times \vec{\mathbf{b}})^2 = |\vec{\mathbf{a}}|^2 |\vec{\mathbf{b}}|^2 - (\vec{\mathbf{a}} . \vec{\mathbf{b}})^2.$$


Joseph-Louis-Lagrange

(*Rajasthan B. 2013*)

Solution. $(\vec{a} \times \vec{b})^2 = (|\vec{a}||\vec{b}| \sin\theta\, \hat{n})^2$

$$= |\vec{a}|^2 |\vec{b}|^2 \sin^2\theta\, \hat{n}^2 = |\vec{a}|^2 |\vec{b}|^2 \sin^2\theta$$

[$\because$ $\hat{n}^2 = \hat{n} . \hat{n} = (1)(1) \cos 0° = 1$]

$$= |\vec{a}|^2 |\vec{b}|^2 (1 - \cos^2\theta)$$

$$= |\vec{a}|^2 |\vec{b}|^2 - |\vec{a}|^2 |\vec{b}|^2 \cos^2\theta$$

$$= |\vec{a}^2|^2 |\vec{b}^2|^2 - (\vec{a} . \vec{b})^2.$$

Other Forms : (I) $(\vec{\mathbf{a}} . \vec{\mathbf{b}})^2 = |\mathbf{a}|^2 |\vec{\mathbf{b}}|^2 - (\vec{\mathbf{a}} \times \vec{\mathbf{b}})^2$

(II) $(\vec{\mathbf{a}} \times \vec{\mathbf{b}})^2 + (\mathbf{a} . \vec{\mathbf{b}})^2 = |\vec{\mathbf{a}}||\vec{\mathbf{b}}|^2.$

Example 17. If $\vec{\mathbf{a}} + \vec{\mathbf{b}} + \vec{\mathbf{c}} = \vec{\mathbf{0}}$**,**

show that $\vec{\mathbf{a}} \times \vec{\mathbf{b}} = \vec{\mathbf{b}} \times \vec{\mathbf{c}} = \vec{\mathbf{c}} \times \vec{\mathbf{a}}$**.**

Solution. Here $\vec{a} + \vec{b} + \vec{c} = \vec{0} \Rightarrow \vec{a} + \vec{b} = -\vec{c}$ ...(1)

$\therefore \qquad \vec{a} \times (\vec{a} + \vec{b}) = -\vec{a} \times \vec{c}$

[*Multiplying* (*1*) *vectorially by* $\vec{a}$ *on the left*]

$\Rightarrow \quad \vec{a} \times \vec{a} + \vec{a} \times \vec{b} = -\vec{a} \times \vec{c}$

$\Rightarrow \qquad \vec{0} + \vec{a} \times \vec{b} = \vec{c} \times \vec{a}$

$\Rightarrow \qquad \vec{a} \times \vec{b} = \vec{c} \times \vec{a}$ ...(2)

Again, $\vec{b} \times (\vec{a} + \vec{b}) = -\vec{b} \times \vec{c}$

[*Multiplying* (*1*) *vectorially by* $\vec{b}$ *on the left*]

$\Rightarrow \vec{b} \times \vec{a} + \vec{b} \times \vec{b} = -\vec{b} \times \vec{c}$

$\Rightarrow \quad -\vec{a} \times \vec{b} + \vec{0} = -\vec{b} \times \vec{c}$

$\Rightarrow \qquad \vec{a} \times \vec{b} = \vec{b} \times \vec{c}$ ...(3)

Combining (2) and (3), $\vec{a} \times \vec{b} = \vec{b} \times \vec{c} = \vec{c} \times \vec{a}.$

Example 18. Show that $\vec{\mathbf{a}} \times \vec{\mathbf{b}} = \vec{\mathbf{a}} \times \vec{\mathbf{c}}$ **does not imply** $\vec{\mathbf{b}} = \vec{\mathbf{c}}$**. Illustrate geometrically.**

Solution. $\vec{a} \times \vec{b} = \vec{a} \times \vec{c} \Rightarrow \vec{a} \times \vec{b} - \vec{a} \times \vec{c} = \vec{0}$

$\Rightarrow \quad \vec{a} \times (\vec{b} - \vec{c}) = \vec{0}.$

This is possible if :

(*i*) $\vec{a} = \vec{0}$ (*ii*) $\vec{b} - \vec{c} = \vec{0}$ *i.e.* $\vec{b} = \vec{c}$

(*iii*) $\vec{a}$ is parallel to $(\vec{b} - \vec{c})$.

Hence, $\vec{a} \times \vec{b} = \vec{a} \times \vec{c}$ does not imply $\vec{b} = \vec{c}$.

[$\because$ *Any one of above three conditions can give the result*]

Geometrically :

Let $\overrightarrow{OA} = \vec{a}, \overrightarrow{OB} = \vec{b}, \overrightarrow{OC} = \vec{c}.$

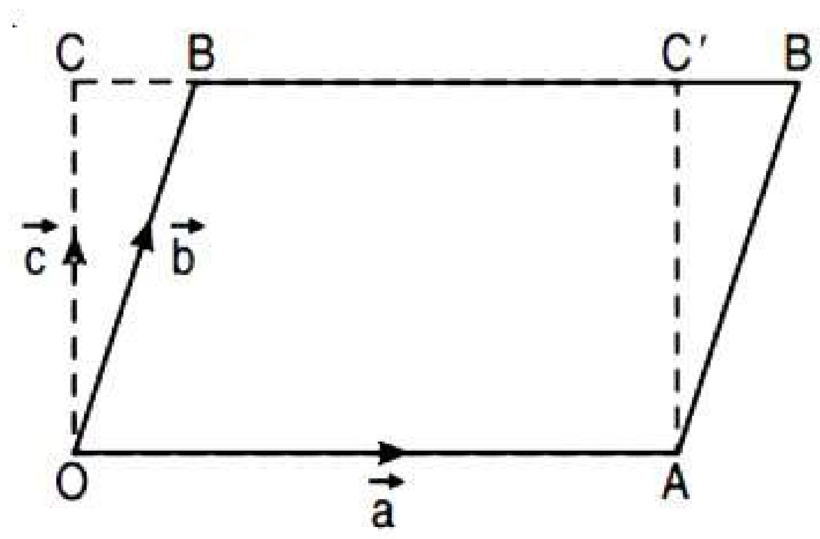


Fig.

Then $\vec{a} \times \vec{b}$ represents the area of the || gm. OAB′B

and $\vec{a} \times \vec{c}$ represents the area of the || gm. OAC′C .

Now ar (OAB′B) = ar (OAC′C)

[$\because$ *Both || gms. are on the same base OA and have same height*]

But $\overrightarrow{OB} \neq \overrightarrow{OC}.$

[$\because$ *They have different magnitudes and directions*]

Hence, $\vec{b} \neq \vec{c}.$

Example 19. Prove, by vector method, that :

sin (A + B) = sin A cos B + cos A sin B.

Solution. Let $\hat{i}$ and $\hat{j}$ be unit vectors along two mutually perpendicular lines OX and OY in the plane of the paper. Let OR and OS be any two lines in the same plane making angles A and B respectively with OX, so that $\angle ROS = A + B$.

Let $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ represent unit vectors along OR and OS respectively.

Let $\hat{k}$ be the unit vector perpendicular to $(\hat{i}, \hat{j})$ plane such that $\hat{i}, \hat{j}, \hat{k}$ form a right-handed system.

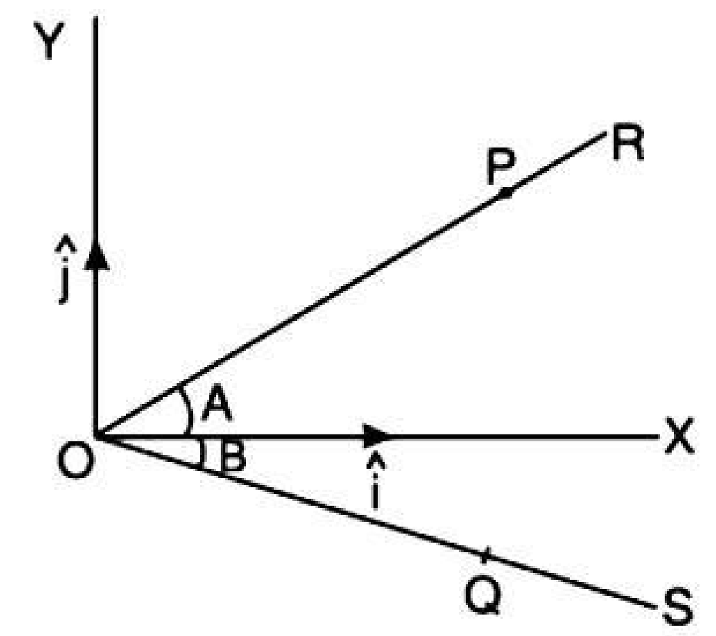


Fig.

Now $\overrightarrow{OP} = \cos A\,\hat{i} + \sin A\,\hat{j}$

and $\overrightarrow{OQ} = \cos B\,\hat{i} - \sin B\,\hat{j}$.

$\therefore\ \overrightarrow{OQ} \times \overrightarrow{OP} = (\cos B\,\hat{i} - \sin B\,\hat{j}) \times (\cos A\,\hat{i} + \sin A\,\hat{j})$

$= \cos B \cos A\ \hat{i} \times \hat{i} + \cos B \sin A\ \hat{i} \times \hat{j}$

$- \sin B \cos A\ \hat{j} \times \hat{i} - \sin B \sin A\ \hat{j} \times \hat{j}$

$= \cos B \cos A\ \vec{0} + \cos B \sin A\ \hat{k}$

$- \sin B \cos A\ (-\hat{k}) - \sin B \sin A\ \vec{0}$

$= (\sin A \cos B + \cos A \sin B)\ \hat{k}$ ...(1)

Since $\overrightarrow{OQ}$ and $\overrightarrow{OP}$ are unit vectors,

$\therefore\ \overrightarrow{OQ} \times \overrightarrow{OP} = \sin(A + B)\ \hat{k}$ ...(2)

From (1) and (2),

$\sin(A + B)\ \hat{k} = (\sin A \cos B + \cos A \sin B)\ \hat{k}$.

Hence, $\sin(A + B) = \sin A \cos B + \cos A \sin B$.

Example 20. If D, E, F are the mid-points of the sides of triangle ABC, prove that :

$$\textbf{ar}(\Delta\,\textbf{DEF}) = \frac{1}{4}\ \textbf{ar}(\Delta\,\textbf{ABC}).$$

Solution. Take A as origin.

Let position vectors of B and C be $\vec{b}$ and $\vec{c}$ respectively.

$\therefore$ P.V. of D, E, F are $\frac{1}{2}(\vec{b} + \vec{c}), \frac{1}{2}\vec{c}$ and $\frac{1}{2}\vec{b}$ respectively.

$\therefore\ \overrightarrow{DE}$ = (P.V. of E) – (P.V. of D)

$$= \frac{1}{2}\vec{c} - \frac{1}{2}(\vec{b} + \vec{c}) = -\frac{1}{2}\vec{b}$$

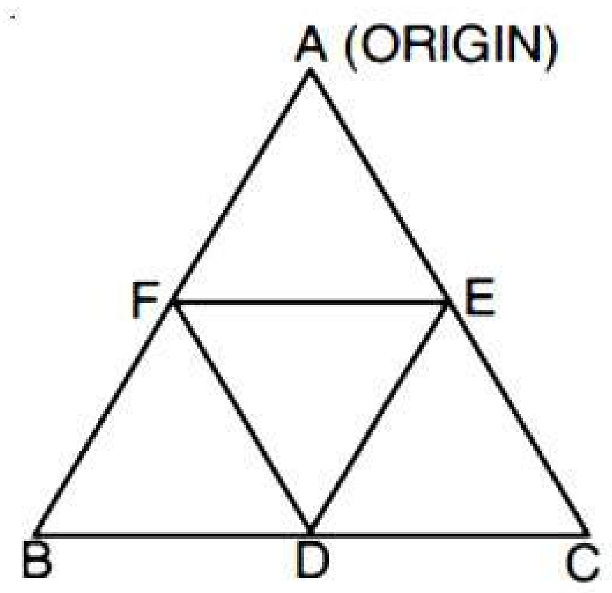


Fig.

and $\overrightarrow{DF}$ = (P.V. of F) – (P.V. of D)

$$= \frac{1}{2}\vec{b} - \frac{1}{2}(\vec{b} + \vec{c}) = -\frac{1}{2}\vec{c}.$$

$\therefore$ Vector area of $\Delta DEF = \frac{1}{2}\overrightarrow{DE} \times \overrightarrow{DF}$

$$= \frac{1}{2}\left(-\frac{1}{2}\vec{b}\right) \times \left(-\frac{1}{2}\vec{c}\right).$$

$$= \frac{1}{4}\left(\frac{1}{2}(\vec{b} \times \vec{c})\right)$$

$$= \frac{1}{4}(\text{vector area of } \Delta ABC).$$

Hence, the result.

Example 21. Show that the points A, B, C with position vectors $2\hat{i} - \hat{j} + \hat{k}$, $\hat{i} - 3\hat{j} - 5\hat{k}$ and $3\hat{i} - 4\hat{j} - 4\hat{k}$ respectively, are the vertices of a right angled triangle. Hence, find the area of the triangle. *(A.I.C.B.S.E. 2017)*

Solution. Let $(2\hat{i} - \hat{j} + \hat{k})$, $(\hat{i} - 3\hat{j} - 5\hat{k})$ and $(3\hat{i} - 4\hat{j} - 4\hat{k})$ be the position vectors of the vertices A, B and C respectively.

$\therefore\ \overrightarrow{AB} = (\hat{i} - 3\hat{j} - 5\hat{k}) - (2\hat{i} - \hat{j} + \hat{k}) = -\hat{i} - 2\hat{j} - 6\hat{k}$

$\overrightarrow{BC} = (3\hat{i} - 4\hat{j} - 4\hat{k}) - (\hat{i} - 3\hat{j} - 5\hat{k}) = 2\hat{i} - \hat{j} + \hat{k}$

and $\overrightarrow{CA} = (2\hat{i} - \hat{j} + \hat{k}) - (3\hat{i} - 4\hat{j} - 4\hat{k}) = -\hat{i} + 3\hat{j} + 5\hat{k}$.

Now, $\overrightarrow{BC}.\overrightarrow{CA} = (2\hat{i} - \hat{j} + \hat{k}).(-\hat{i} + 3\hat{j} + 5\hat{k})$

$$= (2)(-1) + (-1)(3) + (1)(5)$$

$$= -2 - 3 + 5 = 0.$$

Thus $\overrightarrow{BC} \perp \overrightarrow{CA} \Rightarrow \angle C = 90°$.

Hence, ΔABC is rt. angled.

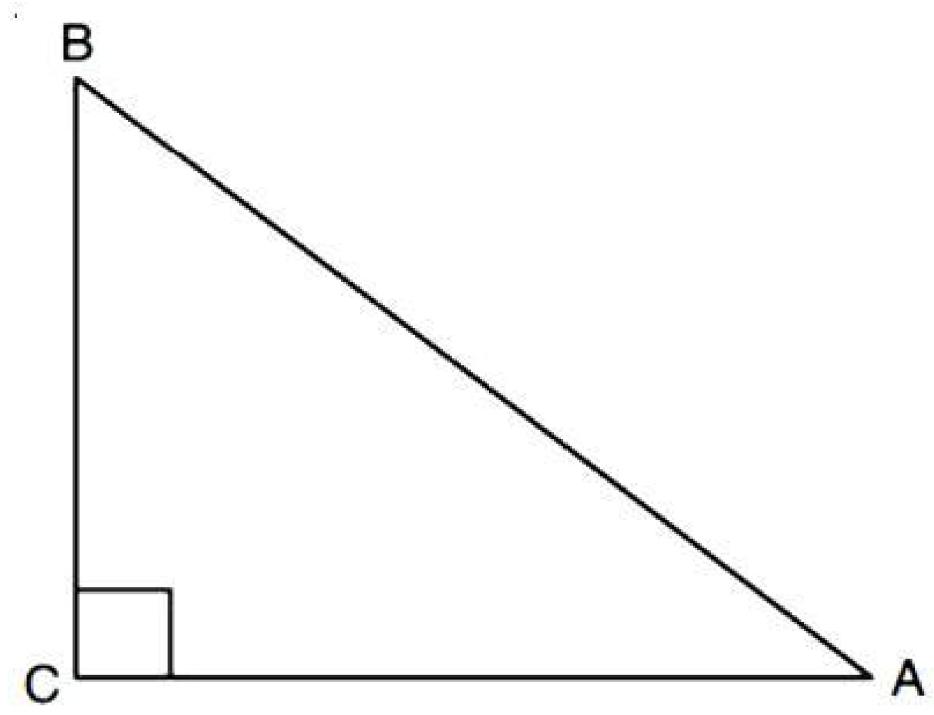


Fig.

And area of triangle $= \frac{1}{2} | \overrightarrow{CA} \times \overrightarrow{CB} |$ ...(1)

But $\overrightarrow{CA} \times \overrightarrow{CB} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 3 & 5 \\ -2 & 1 & -1 \end{vmatrix}$

$$= \hat{i}(-3-5) - \hat{j}(1+10) + \hat{k}(-1+6)$$

$$= -8\hat{i} - 11\hat{j} + 5\hat{k}.$$

$\therefore \quad \left|\overrightarrow{CA} \times \overrightarrow{CB}\right| = \sqrt{64+121+25} = \sqrt{210}.$

Hence, area of triangle $= \frac{1}{2}\sqrt{210}$ sq. units

Example 22. (a) Let $\vec{a}, \vec{b}, \vec{c}$ represent the vectors $\overrightarrow{BC}, \overrightarrow{CA}$ and $\overrightarrow{AB}$ respectively. Show that :

$$\vec{a} \times \vec{b} = \vec{b} \times \vec{c} = \vec{c} \times \vec{a}$$ *(P.B. 2010)*

and deduce the rule of sines of the triangle.

(b) If $\vec{a} + \vec{b} + \vec{c} = \vec{0}$,

show that $\vec{a} \times \vec{b} = \vec{b} \times \vec{c} = \vec{c} \times \vec{a}$.

(C.B.S.E. Sample Paper 2018; P.B. 2010)

Solution. (*a*) (*i*) Here $\overrightarrow{BC} = \vec{a}, \overrightarrow{CA} = \vec{b}$ and $\overrightarrow{AB} = \vec{c}$.

Then $\vec{a} + \vec{b} + \vec{c} = \vec{0}$. $\quad [\because \overrightarrow{BC} + \overrightarrow{CA} + \overrightarrow{AB} = \vec{0}]$

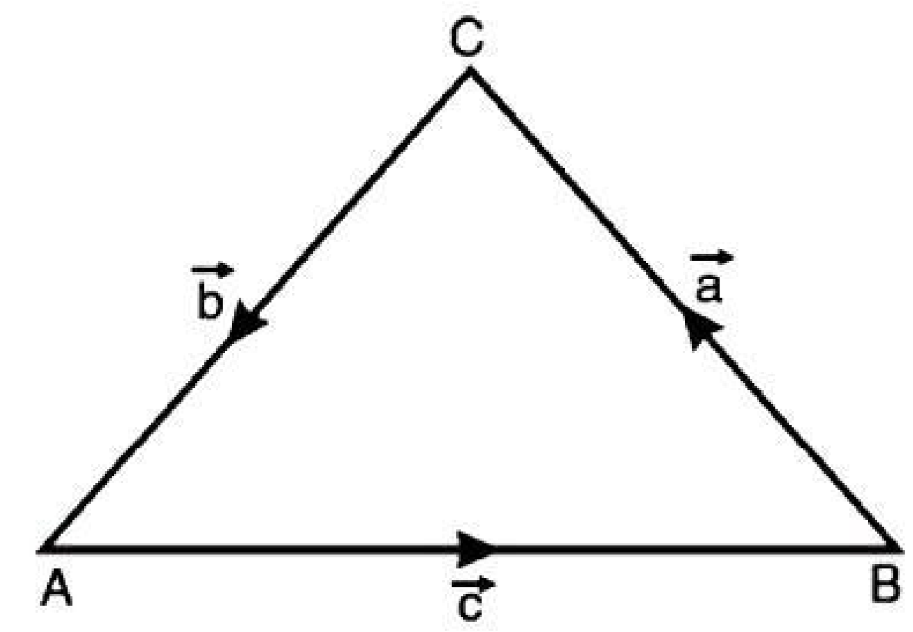


Fig.

Consider $\quad \vec{a} \times (\vec{a} + \vec{b} + \vec{c}) = \vec{0}$

$\Rightarrow \quad \vec{a} \times \vec{a} + \vec{a} \times \vec{b} + \vec{a} \times \vec{c} = \vec{0}$

$\Rightarrow \quad \vec{0} + \vec{a} \times \vec{b} - \vec{c} \times \vec{a} = \vec{0}$

$\Rightarrow \quad \vec{a} \times \vec{b} = \vec{c} \times \vec{a}$ ...(1)

Similarly, $\quad \vec{b} \times \vec{c} = \vec{a} \times \vec{b}$ ...(2)

Combining (1) and (2),

$$\vec{a} \times \vec{b} = \vec{b} \times \vec{c} = \vec{c} \times \vec{a}.$$

(*ii*) From (1), $|\vec{a} \times \vec{b}| = |\vec{c} \times \vec{a}|$

$\Rightarrow ab \sin C = ca \sin B \Rightarrow b \sin C = c \sin B$

$\Rightarrow \quad \frac{b}{\sin B} = \frac{c}{\sin C}.$

Similarly, $\frac{c}{\sin C} = \frac{a}{\sin A}.$

Combining, $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$,

which is the *Sine Rule*.

Another Form. *Using the vector method, prove that :*

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

(*b*) This is a part of part (*a*).

EXERCISE 10 (h)

Very Short Answer Type Questions

VSATQ

1. Find $\vec{a}\times\vec{b}$, if :

(*i*) $\vec{a} = -\hat{i}+3\hat{k}$ and $\vec{b} = \hat{i}+3\hat{j}-2\hat{k}$ **(J. & K. B. 2010)**

(*ii*) $\vec{a}=\hat{i}-3\hat{j}+4\hat{k}$ and $\vec{b}=2\hat{i}+\hat{j}+\hat{k}$. **(Bihar B. 2014)**

2. (*a*) Write the value of :

(*i*) $\hat{i}\cdot\left(\hat{j}\times\hat{k}\right)+\hat{j}\cdot\left(\hat{k}\times\hat{i}\right)+\hat{k}\cdot\left(\hat{j}\times\hat{i}\right)$

(*ii*) $\hat{i}\cdot\left(\hat{j}\times\hat{k}\right)+\hat{j}\cdot\left(\hat{i}\times\hat{k}\right)+\hat{k}\cdot\left(\hat{i}\times\hat{j}\right)$.

(*b*) Find the value of '*p*' if :

$$\left(2\hat{i}+6\hat{j}+27\hat{k}\right)\times\left(\hat{i}+3\hat{j}+p\hat{k}\right) = \vec{0}.$$

(A.I.C.B.S.E. 2009)

3. If $\vec{a}=\hat{i}-\hat{j}+2\hat{k}$ and $\vec{b}=2\hat{i}+\hat{j}-\hat{k}$, find :

$(2\vec{a}-\vec{b})\times(\vec{a}+2\vec{b})$.

4. Find $|\vec{a}\times\vec{b}|$, if $\vec{a}=2\hat{i}+\hat{j}+3\hat{k}$ and $\vec{b}=3\hat{i}+5\hat{j}-2\hat{k}$.

(N.C.E.R.T.; A.I.C.B.S.E. 2015; H.B. 2015)

5. If $\vec{a}=4\hat{i}+3\hat{j}+\hat{k}$ and $\vec{b}=\hat{i}-2\hat{k}$, then find $|2\vec{b}\times\vec{a}|$. **(Meghalaya B. 2017)**

6. Find the magnitude of the vector $\vec{a}\times\vec{b}$ if :

(*i*) $\vec{a}=2\hat{i}+\hat{j}+\hat{k}$ and $\vec{b}=\hat{i}-2\hat{j}+\hat{k}$

(*ii*) $\vec{a}=2\hat{i}+\hat{k}$ and $\vec{b}=\hat{i}+\hat{j}+\hat{k}$.

7. Let the vectors $\vec{a},\vec{b},\vec{c}$ be given as :

$$a_1\hat{i}+a_2\hat{j}+a_3\hat{k},\ b_1\hat{i}+b_2\hat{j}+b_3\hat{k},\ c_1\hat{i}+c_2\hat{j}+c_3\hat{k}.$$

Then show that $\vec{a}\times\left(\vec{b}+\vec{c}\right) = \vec{a}\times\vec{b}+\vec{a}\times\vec{c}$. **(N.C.E.R.T.)**

8. Prove that $\vec{a}\times\vec{b}\neq\vec{b}\times\vec{a}$ if :

$\vec{a}=2\hat{i}-3\hat{j}-\hat{k}$ and $\vec{b}=\hat{i}+4\hat{j}-2\hat{k}$.

9. Find a vector perpendicular to both :

(*i*) $\vec{a} = 3\hat{i}-2\hat{j}+\hat{k}$ and $\vec{b} = \hat{i}+4\hat{j}-\hat{k}$ **(Bihar B.2014)**

(*ii*) $\vec{a}= 4\hat{i}+2\hat{j}-\hat{k}$ and $\vec{b} = \hat{i}+4\hat{j}-\hat{k}$.

(Mizoram B. 2017)

10. If $\vec{a} = \hat{i}-2\hat{j}+3\hat{k}$ and $\vec{b} = 2\hat{i}+3\hat{j}-5\hat{k}$, find $\vec{a}\times\vec{b}$ and verify that $\vec{a}$ and $\vec{a}\times\vec{b}$ are perpendicular to each other. **(P.B. 2010)**

11. Find the unit vector perpendicular to the vectors :

(*i*) $\hat{i}-2\hat{j}+3\hat{k},\ \hat{i}+2\hat{j}-\hat{k}$ **(Meghalaya B. 2015)**

(*ii*) $3\hat{i}-2\hat{j}+\hat{k},\ \hat{i}-2\hat{j}-3\hat{k}$. **(Meghalaya B. 2013)**

12. Find a unit vector perpendicular to the plane containing the vectors :

$\vec{a}=2\hat{i}+\hat{j}+\hat{k}$ and $\vec{b}=\hat{i}+2\hat{j}+\hat{k}$.

13. Find a unit vector in the plane of vectors :

$\vec{a}=\hat{i}+2\hat{j}$ and $\vec{b}=\hat{j}+2\hat{k}$,

perpendicular to the vector $\vec{c}=2\hat{i}+\hat{j}+2\hat{k}$.

14. If $\vec{a}=3\hat{i}-\hat{j}+2\hat{k}$ and $\vec{b}=-\hat{i}-2\hat{j}+4\hat{k}$, find a unit vector along the vector $(\vec{a}\times\vec{b})$.

15. (*a*) Find a unit vector perpendicular to each of the vectors $(\vec{a}+\vec{b})$ and $(\vec{a}-\vec{b})$, where :

$\vec{a}=\hat{i}+\hat{j}+\hat{k}$ and $\vec{b}=\hat{i}+2\hat{j}+3\hat{k}$

(N.C.E.R.T. ; Kashmir B. 2011)

(*b*) Find a vector of magnitude 5 units, perpendicular to each of the vectors $\left(\vec{a}+\vec{b}\right)$ and $\left(\vec{a}-\vec{b}\right)$, where :

(*i*) $\vec{a} = \hat{i}+\hat{j}+\hat{k}$ and $\vec{b} = \hat{i}+2\hat{j}+3\hat{k}$

(Nagaland B. 2018)

(*ii*) $\vec{a} = 3\hat{i}+2\hat{j}+2\hat{k}$ and $\vec{b} = \hat{i}+2\hat{j}-2\hat{k}$.

(Kerala B. 2018 ; Kashmir B. 2017)

16. Find a vector of magnitude 9, which is perpendicular to both the vectors :

$4\hat{i}-\hat{j}+3\hat{k}$ and $-2\hat{i}+\hat{j}-2\hat{k}$.

(P.B. 2012 ; J. & K.B. 2010)

17. If $\vec{a}$ and $\vec{b}$ are unit vectors and 'θ' is the angle between them, show that $\sin\frac{\theta}{2} = \frac{1}{2}|\vec{a} - \vec{b}|$.
(Rajasthan B. 2013 ; H.P.B. 2013)

18. Determine the area of the parallelogram whose adjacent sides are :

(*i*) $2\hat{i}$ and $3\hat{j}$ *(C.B.S.E. (F) 2013)*

(*ii*) $2\hat{i}+\hat{j}$ and $3\hat{i}+\hat{j}+4\hat{k}$ *(Kerala B. 2015)*

(*iii*) $3\hat{i}+\hat{j}+4\hat{k}$ and $\hat{i}-\hat{j}+\hat{k}$
(N.C.E.R.T; Kashmir B. 2017; Jammu B. 2016; H.P.B. 2015)

(*iv*) $\hat{i}-\hat{j}+3\hat{k}$ and $2\hat{i}-7\hat{j}+\hat{k}$
(N.C.E.R.T.; Karnataka B. 2017; Kashmir B. 2016; H.P.B. 2013, 11 S; H.B. 2013)

(*v*) $\hat{i}+\hat{j}+3\hat{k}$ and $3\hat{i}+2\hat{j}+\hat{k}$.
(Uttarakhand B. 2015)

19. Find the area of the parallelogram whose diagonals are :

(*i*) $\vec{a} = \hat{i}-2\hat{j}+3\hat{k}$ and $\vec{b} = 3\hat{i}+2\hat{j}+\hat{k}$
(P.B. 2011)

(*ii*) $\vec{a} = 3\hat{i}+\hat{j}-2\hat{k}$ and $\vec{b} = \hat{i}-3\hat{j}+4\hat{k}$
(Assam B. 2016; Kashmir B. 2011)

(*iii*) $\vec{a} = 2\hat{i}-3\hat{j}+4\hat{k}$ and $\vec{b} = 2\hat{i}-\hat{j}+2\hat{k}$.
(C.B.S.E. (F) 2013)

20. Show that the area of the parallelogram with diagonals $\vec{a}$ and $\vec{b}$, is $\frac{1}{2}|\vec{a}\times\vec{b}|$. *(H.B. 2016)*

21. (*i*) Show that $|\vec{a}\times\vec{b}| = \sqrt{\vec{a}^2\,\vec{b}^2 - (\vec{a}\cdot\vec{b})^2}$.

(*ii*) Prove that $|\vec{a}\times\vec{b}| = (\vec{a}\cdot\vec{b})\tan\theta$, where '$\theta$' is the angle between $\vec{a}$ and $\vec{b}$.

(*iii*) Show that $(\vec{a}-\vec{b})\times(\vec{a}+\vec{b}) = 2(\vec{a}\times\vec{b})$.
(N.C.E.R.T.; H.B. 2016)

(*iv*) Given that $\vec{a}\cdot\vec{b} = 0$ and $\vec{a}\times\vec{b} = \vec{0}$. What can you conclude about the vectors $\vec{a}$ and $\vec{b}$? *(N.C.E.R.T.)*

22. (*i*) Find $|\vec{a}\cdot\vec{b}|$, if $|\vec{a}| = 5, |\vec{b}| = 13$ and $|\vec{a}\times\vec{b}| = 25$.

(*ii*) If $|\vec{a}| = 13, |\vec{b}| = 5$ and $\vec{a}\cdot\vec{b} = 60$, then find $|\vec{a}\times\vec{b}|$.

23. If either $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$, then $\vec{a}\times\vec{b} = \vec{0}$.

Is the converse true ? Justify your answer with an example. *(N.C.E.R.T.)*

Short Answer Type Questions

SATQ

24. Find scalars λ and μ if :

$(2\hat{i}+6\hat{j}+27\hat{k})\times(\hat{i}+\lambda\hat{j}+\mu\hat{k}) = \vec{0}$. *(Jammu B. 2016)*

25. If $\vec{a} = 2\hat{i}+\sqrt{3}\hat{j}+3\hat{k}$ and $\vec{b} = 3\hat{i}+2\hat{j}-2\sqrt{3}\hat{k}$, then :

(*i*) find the direction cosines of $\vec{b}$

(*ii*) find a vector in the direction of $\vec{a}$ that has magnitude 7 units

(*iii*) find the angle between $\vec{a}$ and $\vec{b}$

(*iv*) evaluate $(3\vec{a}-5\vec{b})\times(2\vec{a}-\vec{b})$. *(Jharkhand B. 2016)*

26. If a unit vector $\vec{a}$ makes angles $\frac{\pi}{3}$ with $\hat{i}$, $\frac{\pi}{4}$ with $\hat{j}$ and an acute angle 'θ' with $\hat{k}$, then find 'θ' and hence components of $\vec{a}$. *(N.C.E.R.T.)*

27. Prove that the unit vector perpendicular to each of the vectors $2\hat{i}-\hat{j}+\hat{k}$ and $3\hat{i}+4\hat{j}-\hat{k}$ is :

$\frac{-3\hat{i}+5\hat{j}+11\hat{k}}{\sqrt{155}}$ and that the sine of the angle between them is $\sqrt{\frac{155}{156}}$.

28. Show that the three points having position vectors $(\vec{a}-2\vec{b}+3\vec{c}), (-2\vec{a}+3\vec{b}+2\vec{c}), (-8\vec{a}+13\vec{b})$ are collinear, whatever be $\vec{a}, \vec{b}, \vec{c}$. *(Mizoram B. 2018)*

29. Find, with the help of vectors, the area of the triangle with vertices :

(*i*) A (2, 3, 5), B (3, 5, 8), C (2, 7, 8) *(C.B.S.E. 2010 C)*

(*ii*) A (1, 2, 3), B (2, –1, 4), C (4, 5, – 1)
(A.I.C.B.S.E. 2013)

(*iii*) A (1, 2, 4), B (3, 1, – 2), C (4, 3, 1)
(H.B. 2018, 13 ; P.B. 2012)

(*iv*) A (1, – 1, 2), B (2, 1, 1), C (3, – 1, 2)
(Nagaland B. 2018)

(*v*) (1, 1, 2), (2, 3, 5) and (1, 5, 5). *(Assam B. 2018)*
with reference to a rectangular system of axes.

30. Find the value of 'x' if the area of triangle is 35 square cm. with vertices $(x, 4)$, $(2, -6)$ and $(5, 4)$.

31. Find the area of the triangle OAB, where :

$$\overrightarrow{OA} = 3\hat{i} - \hat{j} + \hat{k} \text{ and } \overrightarrow{OB} = 2\hat{i} + \hat{j} - 3\hat{k}.$$

32. Prove that :

$$\vec{a} \times (\vec{b} + \vec{c}) + \vec{b} \times (\vec{c} + \vec{a}) + \vec{c} \times (\vec{a} + \vec{b}) = \vec{0}.$$

33. If $\vec{a}, \vec{b}, \vec{c}$ are three vectors such that $\vec{a} \times \vec{b} = \vec{c}, \vec{b} \times \vec{c} = \vec{a}$, prove that $\vec{a}, \vec{b}, \vec{c}$ are mutually at right angles and $|\vec{b}| = 1, |\vec{c}| = |\vec{a}|$.

34. If $\vec{a}, \vec{b}$ and $\vec{c}$ are mutually perpendicular unit vectors and $\vec{a} \times \vec{b} = \vec{c}$, show that $\vec{b} = \vec{c} \times \vec{a}$ and $\vec{a} = \vec{b} \times \vec{c}$.

35. Let $\vec{a} = \hat{i} + 4\hat{j} + 2\hat{k}, \vec{b} = 3\hat{i} - 2\hat{j} + 7\hat{k}$ and $\vec{c} = 2\hat{i} - \hat{j} + 4\hat{k}$. Find a vector $\vec{d}$, which is perpendicular to both $\vec{a}$ and $\vec{b}$ and $\vec{c} \cdot \vec{d} = 15$.

36. If $\vec{a} = \hat{i} + \hat{j} + \hat{k}$ and $\vec{b} = \hat{j} - \hat{k}$, find a vector $\vec{c}$ such that $\vec{a} \times \vec{c} = \vec{b}$ and $\vec{a} \cdot \vec{c} = 3$. **(H.B. 2018)**

37. If $\vec{a} \times \vec{b} = \vec{b} \times \vec{c} \neq \vec{0}$, show that $\vec{a} + \vec{c} = m\vec{b}$, m being a scalar.

38. Prove that $|\vec{a} \times \vec{b}|^2 = |\vec{a}|^2 |\vec{b}|^2 - (\vec{a} \cdot \vec{b})^2$

$$= \begin{vmatrix} \vec{a} \cdot \vec{a} & \vec{a} \cdot \vec{b} \\ \vec{a} \cdot \vec{b} & \vec{b} \cdot \vec{b} \end{vmatrix}.$$

39. Adjacent sides of a parallelogram are given by vectors $2\hat{i} + \hat{j} + \hat{k}$ and $\hat{i} + 5\hat{j} + \hat{k}$. Find a unit vector in the direction of its diagonal. Also, find the area of parallelogram. **(P.B. 2017)**

40. If $\vec{a}, \vec{b}, \vec{c}$ are position vectors of non-collinear points A, B and C respectively, show that :

$$\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a}$$

is perpendicular to the plane ABC. **(Mizoram B. 2018)**

41. (*a*) Prove that the normal to the plane containing three points whose position vectors are $\vec{a}, \vec{b}, \vec{c}$ lie in the direction of $\vec{b} \times \vec{c} + \vec{c} \times \vec{a} + \vec{a} \times \vec{b}$.

(*b*) Find the unit vector perpendicular to the plane ABC, where the position vectors of A, B and C are :

$2\hat{i} - \hat{j} + \hat{k}$, $\hat{i} + \hat{j} + 2\hat{k}$ and $2\hat{i} + \hat{k}$ respectively.

42. Establish, using vectors :

sin (A – B) = sin A cos B – cos A sin B.

Answers

1. (*i*) $-9\hat{i} + \hat{j} - 3\hat{k}$ (*ii*) $-7\hat{i} + 7\hat{j} + 7\hat{k}$.

2. (*a*) (*i*) – (*ii*) 1 (*b*) $p = \dfrac{27}{2}$.

3. $5(-\hat{i} + 5\hat{j} + 3\hat{k})$.

4. $\sqrt{507}$.

5. $6\sqrt{14}$.

6. (*i*) $\sqrt{35}$ (*ii*) $\sqrt{6}$.

9. (*i*) $-2\hat{i} + 4\hat{j} + 14\hat{k}$ (*ii*) $2\hat{i} + 5\hat{j} + 18\hat{k}$.

10. $\hat{i} + 11\hat{j} + 7\hat{k}$.

11. (*i*) $\dfrac{1}{\sqrt{3}}(-\hat{i} + \hat{j} + \hat{k})$ (*ii*) $\dfrac{1}{3\sqrt{5}}(4\hat{i} + 5\hat{j} - 2\hat{k})$.

12. $\dfrac{1}{\sqrt{11}}(-\hat{i} - \hat{j} + 3\hat{k})$.

13. $\dfrac{1}{5\sqrt{5}}(-5\hat{i} - 6\hat{j} + 8\hat{k})$.

14. $-\dfrac{14}{\sqrt{245}}\hat{j} - \dfrac{7}{\sqrt{245}}\hat{k}$.

15. (*a*) $\dfrac{\sqrt{6}}{6}(\hat{i} - 2\hat{j} + \hat{k})$

(*b*) (*i*) $\dfrac{5}{\sqrt{6}}(-\hat{i} + 2\hat{j} - \hat{k})$ (*ii*) $\dfrac{40}{3}(2\hat{i} - 2\hat{j} - \hat{k})$

16. $3(-\hat{i} + 2\hat{j} + 2\hat{k})$.

18. (*i*) 6 (*ii*) 9 (*iii*) $\sqrt{42}$ (*iv*) $15\sqrt{2}$ (*v*) $\sqrt{90}$.

19. (*i*) $4\sqrt{3}$ (*ii*) $5\sqrt{3}$ (*iii*) 3.

21. (*iv*) $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$.

22. (*i*) 60 (*ii*) 25.

23. No. Take any two non-zero collinear vectors.

24. $\lambda = 3, \ \mu = \dfrac{27}{2}$.

25. (*i*) $< \dfrac{3}{5}, \dfrac{2}{5}, \dfrac{-2\sqrt{3}}{5} >$ (*ii*) $\dfrac{7}{4}(2\hat{i} + \sqrt{3}\hat{j} + 3\hat{k})$

(*iii*) $\cos^{-1}\left(\dfrac{3 - 2\sqrt{3}}{10}\right)$

(*iv*) $(-36 + 6\sqrt{3})\hat{i} + (12\sqrt{3} - 28)\hat{j} + (28 - 12\sqrt{3})\hat{k}$.

26. (*i*) $\frac{\pi}{3}$ (*ii*) $\frac{1}{2}, \frac{1}{\sqrt{2}}, \frac{1}{2}$.

29. (*i*) $\frac{1}{2}\sqrt{61}$ sq. units (*ii*) $\frac{1}{2}\sqrt{274}$ sq. units

(*iii*) $\frac{5}{2}\sqrt{10}$ sq. units (*iv*) $\sqrt{5}$ sq. units

(*v*) $-\frac{1}{2}\sqrt{61}$ sq. units

30. $x = -2$. **31.** $\frac{1}{2} 5\sqrt{6}$ sq. units.

35. $\frac{1}{3}(160\hat{i} - 5\hat{j} - 70\hat{k})$. **36.** $\frac{1}{3}(5\hat{i} + 2\hat{j} + 2\hat{k})$.

39. $\frac{1}{7}(3\hat{i} + 6\hat{j} + 2\hat{k})$; $7\sqrt{2}$ sq. units.

41. (*b*) $\frac{1}{\sqrt{14}}(3\hat{i} + 2\hat{j} - \hat{k})$.

Hints to Selected Questions

16. Find $9\left(\frac{\vec{a} \times \vec{b}}{|\vec{a}||\vec{b}|}\right)$.

17. $|\vec{a} - \vec{b}|^2 = (\vec{a} - \vec{b})^2 = \vec{a}^2 + \vec{b}^2 - 2\vec{a}.\vec{b}$

$= 1 + 1 - 2\cos\theta = 2(1 - \cos\theta)$

$= 4\sin^2\frac{\theta}{2}$.

19. Area of || gm $= \frac{1}{2}|\vec{a} \times \vec{b}|$.

21. (*ii*) $\left|\vec{a} \times \vec{b}\right| = ab\sin\theta = ab\cos\theta . \tan\theta$

$= \left|\vec{a}.\vec{b}\right|\tan\theta$.

22. (*i*) Use $(\vec{a} \times \vec{b})^2 = \vec{a}^2\, \vec{b}^2 - (\vec{a}.\vec{b})^2$.

27. Here $\hat{n} = \frac{\vec{a} \times \vec{b}}{\left|\vec{a} \times \vec{b}\right|} = \frac{-3\hat{i} + 5\hat{j} + 11\hat{k}}{\sqrt{155}}$.

Now $ab\sin\theta = \left|\vec{a} \times \vec{b}\right|$

$\Rightarrow \sin\theta = \frac{\left|\vec{a} \times \vec{b}\right|}{ab} = \frac{\sqrt{155}}{\sqrt{6}\sqrt{26}} = \frac{\sqrt{155}}{\sqrt{156}}$.

28. If the given vectors are $\vec{A}, \vec{B}, \vec{C}$, then :

$\vec{A} \times \vec{B} + \vec{B} \times \vec{C} + \vec{C} \times \vec{A} = \vec{0}$

$\Rightarrow$ area of triangle, the position vectors of whose vertices are $\vec{A}, \vec{B}, \vec{C}$ is zero

$\Rightarrow$ given points are collinear.

29. Area of Δ ABC $= \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{BC}\right|$.

31. Reqd. area $= \frac{1}{2}|\overrightarrow{OA} \times \overrightarrow{OB}|$.

34. $\vec{a} \times \vec{b} = (1)(1)\sin 90°\,\hat{n} = \hat{n} = \vec{c}$.

$\therefore\ \vec{c} \times \vec{a} = (1)(1)\sin 90°\,\vec{b} = \vec{b}$.

38. $\left|\vec{a} \times \vec{b}\right|^2 = \left|\vec{a}\right|^2\left|\vec{b}\right|^2\sin^2\theta = \left|\vec{a}\right|^2\left|\vec{b}\right|^2(1 - \cos^2\theta)$

$= \left|\vec{a}\right|^2\left|\vec{b}\right|^2 - \left(\vec{a}.\vec{b}\right)^2 = \begin{vmatrix} \vec{a}.\vec{a} & \vec{a}.\vec{b} \\ \vec{a}.\vec{b} & \vec{b}.\vec{b} \end{vmatrix}$.

42. As in Ex. 19.

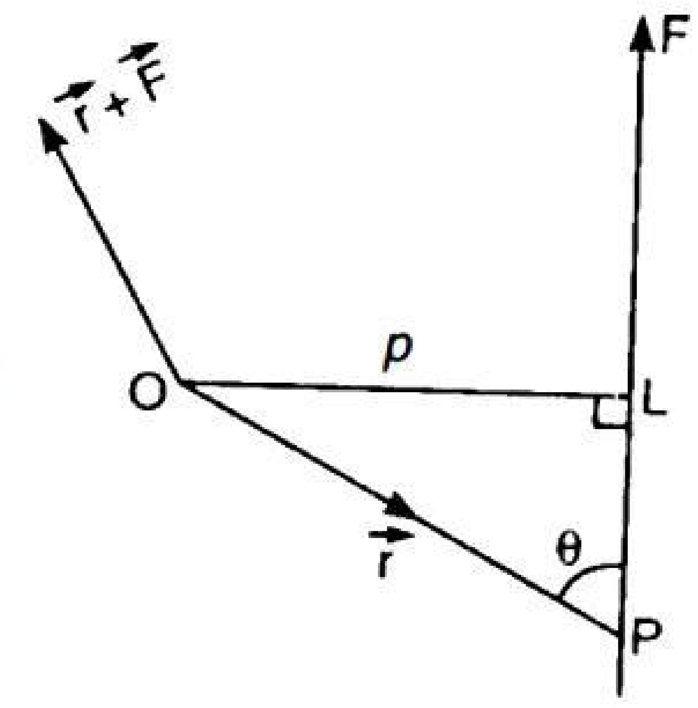
10.30. MOMENT OF A FORCE ABOUT A POINT (VECTOR MOMENT OR TORQUE)

Definition

Let a force $\vec{F}$ act at P whose position vector is $\vec{r}$ w.r.t. the origin O such that $\overrightarrow{OP} = \vec{r}$.

Then the Vector Moment or Torque of the force $\vec{F}$ about the point O is defined as the vector :

$\vec{m} = \vec{r} \times \vec{F} = r\,F\sin\theta\,\hat{n}$



$= r \sin\theta\ F\ \hat{n}$, where $\hat{n}$ is a unit vector perpendicular to the plane of $\vec{r}$ and $\vec{F}$ and its direction is towards the reader

$= p\ F\ \hat{n}$, where p = OL = Length of perpendicular from O on the line of action of $\vec{F}$

$\therefore\ |\vec{m}|$ = Magnitude of the vector moment of $pF = Fp$

= (Force) (Perpendicular distance of O from the line of action of $\vec{F}$)

Fig.

Convention of Signs. The moment is positive or negative according as the rotation of the force about the point is anticlockwise or clockwise.

Note : Moment of a force about a point is a *vector.*

10.31. MOMENT OF A FORCE ABOUT A LINE

The moment of a force $\vec{F}$, acting at a point A, about a line is a scalar, given by :

$$(\vec{r} \times \vec{F}) \cdot \hat{a},$$

where $\overrightarrow{OA} = \vec{r}$, $\hat{a}$ is a unit vector in the direction of the line and O is any point on the line.

The moment of a force $\vec{F}$ about a line = Resolved part of the moment of $\vec{F}$ about any point on the line along this line.

KEY POINT

Moment of a force about a point is vector while moment of a force about a line is scalar quantity.

10.32. MOMENT OF A COUPLE ABOUT A POINT

Let two unlike parallel forces $\vec{F}$ and $\vec{F}$ form a couple*. Let A be the given point.

$\therefore$ Mt. of the couple about A

$= $ Mt. of $\vec{F}$ about A + Mt. of $(-\vec{F})$ about A

$= \overrightarrow{AP} \times \vec{F} + \overrightarrow{AQ} \times (-\vec{F})$, where P and Q are any points on the lines of action of forces

$= (\overrightarrow{AP} + \overrightarrow{QA}) \times \vec{F} = \overrightarrow{QP} \times \vec{F}$,

which is known as **moment of the couple**, which is independent of A.

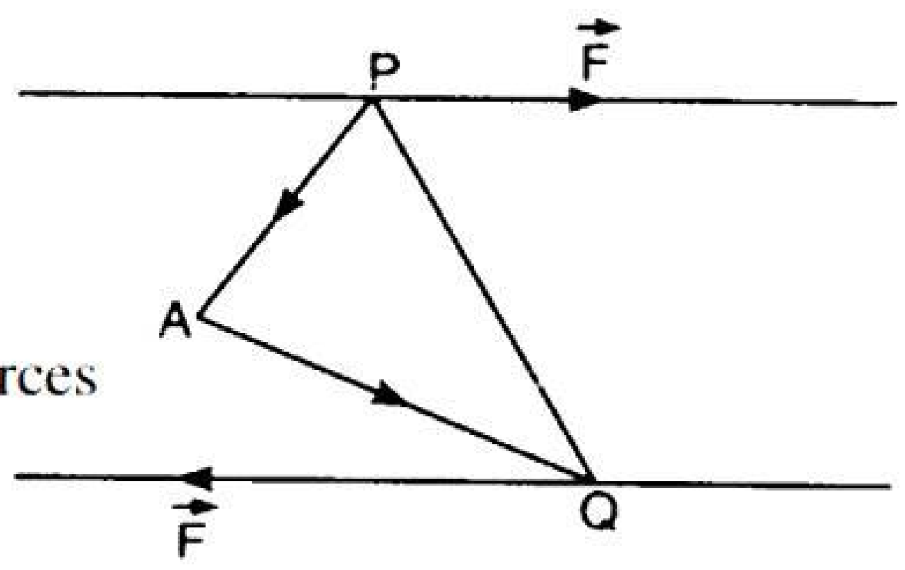


Fig.

ILLUSTRATIVE EXAMPLES

Example 1. Find the moment (torque) about the point $\hat{i} + 2\hat{j} + 3\hat{k}$ of a force represented by $\hat{i} + \hat{j} + \hat{k}$ acting through the point $-2\hat{i} + 3\hat{j} + \hat{k}$.

Solution. Let O be the point $\hat{i} + 2\hat{j} + 3\hat{k}$ and P the point $-2\hat{i} + 3\hat{j} + \hat{k}$.

Then $\overrightarrow{OP} = (-2\hat{i} + 3\hat{j} + \hat{k}) - (\hat{i} + 2\hat{j} + 3\hat{k})$

$$= -3\hat{i} + \hat{j} - 2\hat{k}.$$

$\therefore$ The moment about O of the force $\vec{F}\ (= \hat{i} + \hat{j} + \hat{k})$ acting through P $= \overrightarrow{OP} \times \vec{F}$.

* Two unlike equal forces, not acting at the same point, form a **couple.**

$\therefore \quad \vec{M} = (-3\hat{i} + \hat{j} - 2\hat{k}) \times (\hat{i} + \hat{j} + \hat{k})$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -3 & 1 & -2 \\ 1 & 1 & 1 \end{vmatrix} = 3\hat{i} + \hat{j} - 4\hat{k}.$$

$\therefore \quad |\vec{M}| = \sqrt{9+1+16} = \sqrt{26}$ units.

Example 2. Two unlike forces of equal magnitudes $3\hat{i} + \hat{k}$ and $-3\hat{i} - \hat{k}$ acting at the points $\hat{i} + 2\hat{j} - \hat{k}$ and $2\hat{i} - \hat{j} + 3\hat{k}$ respectively. Find the moment of the couple formed by these forces.

Solution. Moment of the couple $= (\vec{r_1} - \vec{r_2}) \times \vec{F}$, where $\vec{r_1}$ and $\vec{r_2}$ are position vectors of the points of application of the forces $\vec{F}$ and $-\vec{F}$.

Here $\vec{r_1} - \vec{r_2} = (\hat{i} + 2\hat{j} - \hat{k}) - (2\hat{i} - \hat{j} + 3\hat{k})$

$$= -\hat{i} + 3\hat{j} - 4\hat{k}.$$

Also $\quad \vec{F} = 3\hat{i} + \hat{k}.$

$\therefore$ Vector moment of the couple $= \vec{M}$

$$= (-\hat{i} + 3\hat{j} - 4\hat{k}) \times (3\hat{i} + \hat{k})$$

$$\therefore \vec{M} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 3 & -4 \\ 3 & 0 & 1 \end{vmatrix} = 3\hat{i} - 11\hat{j} - 9\hat{k}.$$

Hence, the moment of the couple

$$= |\vec{M}| = \sqrt{9+121+81} = \sqrt{211} \textit{ units.}$$

Example 3. Find the moment of the couple consisting of the force $\vec{F} = 3\hat{i} + 2\hat{j} - \hat{k}$ acting through the point $\hat{i} - \hat{j} + \hat{k}$ and $-\vec{F}$ acting through the point $2\hat{i} - 3\hat{j} - \hat{k}$.

Solution. Let A and B be the points $\hat{i} - \hat{j} + \hat{k}$ and $2\hat{i} - 3\hat{j} - \hat{k}$ respectively.

Then $\quad \vec{r} = \vec{BA} = (\hat{i} - \hat{j} + \hat{k}) - (2\hat{i} - 3\hat{j} - \hat{k}).$

$$= -\hat{i} + 2\hat{j} + 2\hat{k}.$$

If $\vec{M}$ be the vector moment of the couple, then :

$$\vec{M} = \vec{r} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 2 & 2 \\ 3 & 2 & -1 \end{vmatrix} = -6\hat{i} + 5\hat{j} - 8\hat{k}.$$

$\therefore$ Moment of the couple

$$= |\vec{M}| = \sqrt{36+25+64} = 5\sqrt{5} \textit{ units.}$$

Example 4. Find the moment about a line through (0, 0, 0) having the direction $2\hat{i} - 2\hat{j} + \hat{k}$ due to a 20 kg force acting at (– 4, 2, 5) in the direction of $12\hat{i} - 4\hat{j} - 3\hat{k}$.

Solution. Let $\vec{F}$ be the force.

Then $\quad \vec{F} = \dfrac{20\,(12\hat{i} - 4\hat{j} - 3\hat{k})}{\sqrt{(12)^2 + (-4)^2 + (-3)^2}}$

$$= \frac{20\,(12\hat{i} - 4\hat{j} - 3\hat{k})}{\sqrt{144+16+9}} = \frac{20}{13}(12\hat{i} - 4\hat{j} - 3\hat{k}).$$

Let $\vec{F}$ act at P (– 4 , 2 , 5).

$\therefore$ Moment of $\vec{F}$ acting at P about a line in the direction $2\hat{i} - 2\hat{j} + \hat{k}$

= Resolved part of the moment of $\vec{F}$ about a point on the line.

Since the line passes thro' (0, 0, 0), [*Given*]

$\therefore$ we select O as a point on the line.

Let $\quad \vec{OP} = \vec{r}.$

Then $\quad \vec{r} = \vec{OP} = (-4\hat{i} + 2\hat{j} + 5\hat{k}) - \hat{0}$

$$= -4\hat{i} + 2\hat{j} + 5\hat{k}.$$

If $\vec{M}$ be the moment of $\vec{F}$ about O,

then $\vec{M} = \vec{r} \times \vec{F}$

$$= (-4\hat{i} + 2\hat{j} + 5\hat{k}) \times \frac{20}{13}(12\hat{i} - 4\hat{j} - 3\hat{j})$$

$$= \frac{20}{13}\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -4 & 2 & 5 \\ 12 & -4 & -3 \end{vmatrix}$$

$$= \frac{20}{13}(14\hat{i} + 48\hat{j} - 8\hat{k}).$$

Let $\hat{a}$ be the unit vector in the direction of $2\hat{i} - 2\hat{j} + \hat{k}$.

Then $\quad \hat{a} = \dfrac{2\hat{i} - 2\hat{j} + \hat{k}}{\sqrt{4+4+1}} = \dfrac{1}{3}(2\hat{i} - 2\hat{j} + \hat{k}).$

Hence, the moment of $\vec{F}$ about the given line

$$= \vec{M} \,.\, \vec{a}$$

$$= \frac{20}{13}(14\hat{i} + 48\hat{j} - 8\hat{k})\,(2\hat{i} - 2\hat{j} + \hat{k}).$$

$$= \frac{20}{39}(28 - 96 - 8) = \frac{20}{39}(-76) = -\frac{1520}{39} \text{ units.}$$

EXERCISE 10 (i)

Very Short Answer Type Questions

VSATQ

1. Find the moment about (1, –1, –1) of the force $3\hat{i}+4\hat{j}-5\hat{k}$ acting at (1, 0, –2).

2. The force represented by $3\hat{i}+2\hat{k}$ is acting through the point $5\hat{i}+4\hat{j}-3\hat{k}$. Find the moment about the point $\hat{i}+3\hat{j}+\hat{k}$.

3. Find the moment about the point $\hat{i}+2\hat{j}-\hat{k}$ of a force represented by $\hat{i}+2\hat{j}+\hat{k}$ acting through the point $2\hat{i}+3\hat{j}+\hat{k}$.

4. The force represented by $5\hat{i}+\hat{k}$ is acting through the point $9\hat{i}-\hat{j}+2\hat{k}$. Find the moment about the point $3\hat{i}-2\hat{j}+2\hat{k}$.

Short Answer Type Questions

SATQ

5. A force $\vec{F}=4\hat{i}+\hat{k}$ acts through point A (0, 2, 0). Find the moment $\vec{m}$ of $\vec{F}$ about the point B (4, 0, 4). *(H.B. 2011)*

6. Let $\vec{F}=2\hat{i}+4\hat{j}+3\hat{k}$ at the point P with position vector $\hat{i}-\hat{j}+3\hat{k}$. Find the moment of $\vec{F}$ about the line through the origin O in the direction of the vector $\vec{a}=\hat{i}+2\hat{j}+2\hat{k}$.

7. A force $\vec{F}=3\hat{i}+2\hat{j}-4\hat{k}$ is applied at the point (1, – 1, 2). Find the moment of $\vec{F}$ about the point (2, – 1, 3).

8. Two unlike forces of equal magnitudes $\hat{j}+2\hat{k}$ and $-\hat{j}-2\hat{k}$ are acting at the points whose position vectors are given by $\hat{i}+\hat{j}+\hat{k}$ and $\hat{i}+2\hat{j}+3\hat{k}$ respectively. Find the moment of the couple formed by these forces.

9. A force of 3 units acts through the point (4, – 1, 7) in the direction of the vector $9\hat{i}+6\hat{j}-2\hat{k}$. Find the moment of the force about the point (1, – 3, 2) and the moment about the axes, parallel to the co-ordinate axes, which pass through (1, – 3, 2).

10. Find the moment about the point (3, 4, 5) of the force through the point (1, 2, –3) having components equal to –2, 3, –4. What is the moment of the same force about the line through the origin having direction–ratios $< 4, -2, 5 >$?

11. Find the moment of the couple formed by the forces $5\hat{i}+\hat{k}$ and $-5\hat{i}-\hat{k}$ acting at the points (9, –1, 2) and (3, –2, 1) respectively.

12. Find the vector moment of the forces :

$$\hat{i}+2\hat{j}-3\hat{k};\ 2\hat{i}+3\hat{j}+4\hat{k} \text{ and } -\hat{i}-\hat{j}+\hat{k}$$

acting on a particle at a point P (0, 1, 2) about the point A (1, – 2, 0).

Answers

1. $-\hat{i}-3\hat{j}-3\hat{k}; \sqrt{19}$. **2.** $2\hat{i}-20\hat{j}-3\hat{k}; \sqrt{413}$.

3. $-3\hat{i}+\hat{j}+\hat{k}; \sqrt{11}$.

4. $-\hat{i}+\hat{j}+5\hat{k}; \sqrt{27}$. **5.** $\vec{m}=-2\hat{i}+12\hat{j}+8\hat{k}$.

6. 1.

7. $2\hat{i}-7\hat{j}-2\hat{k}; \sqrt{57}$. **8.** $\vec{0}, 0$.

9. $\frac{3}{11}(-34\hat{i}+51\hat{j}); \frac{-102}{11}, \frac{153}{11}, 0$.

10. $32\hat{i}+8\hat{j}-10\hat{k}; \frac{19}{\sqrt{45}}$.

11. $\hat{i}-\hat{j}-5\hat{k}; \sqrt{27}$.

12. $-2\hat{i}+6\hat{j}-10\hat{k}$.

Hints to Selected Questions

8. Here $\vec{F}=\hat{j}+2\hat{k}$ and

$\vec{r_1}-\vec{r_2}=(\hat{i}+\hat{j}+\hat{k})-(\hat{i}+2\hat{j}+3\hat{k})=-\hat{j}-2\hat{k}$.

$\therefore$ Moment of the couple $=(\vec{r_1}-\vec{r_2})\times\vec{F}$.

SUB CHAPTER

10.3 Product of Three and Four Vectors

10.33. SCALAR TRIPLE PRODUCT

Definition

If $\vec{a}, \vec{b}, \vec{c}$ are three vectors, then the scalar product of $\vec{a} \times \vec{b}$ with $\vec{c}$ is called scalar triple product.

(Kashmir B. 2013)

Notation. The scalar triple product of $\vec{a}, \vec{b}, \vec{c}$ is written as : $\left(\vec{a} \times \vec{b}\right) . \vec{c}$ or $\left[\vec{a}\ \vec{b}\ \vec{c}\right]$ or $\left[\vec{a}, \vec{b}, \vec{c}\right]$.

Note. Since the scalar triple product involves both the signs of 'cross' and 'dot', therefore, it is also called **mixed product.**

Geometrical Interpretation. **(Kashmir B. 2011)**

Consider a parallelopiped whose coterminous edges OA, OB, OC have the lengths and directions of the vectors, $\vec{a}, \vec{b}, \vec{c}$ respectively.

Let V be its volume.

Now $\vec{a} \times \vec{b}$ represents a vector $\vec{n}$ whose magnitude is the area of the || gm OADB.

If 'θ' be the angle between $\vec{n}\left(i.e., \vec{a} \times \vec{b}\right)$ and $\vec{c}$,

then $\left(\vec{a} \times \vec{b}\right) . \vec{c} = \vec{n} . \vec{c}$ = (area OADB) (c) cos θ

= (area OADB) (c cos θ)

= V, which is + ve or – ve according as 'θ' is acute or obtuse.

Hence, $\left(\vec{a} \times \vec{b}\right) . \vec{c}$ represents numerically the volume of the parallelopiped having $\vec{a}, \vec{b}, \vec{c}$ as coterminous edges.

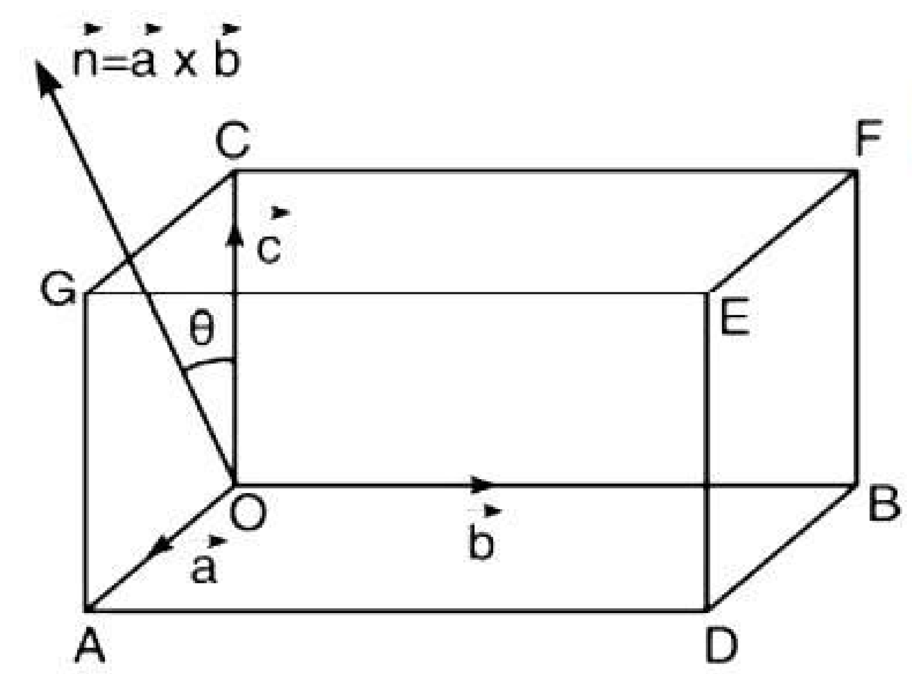


Fig.

10.34. CO-PLANARITY OF THREE VECTORS

Theorem. Three vectors $\vec{a}, \vec{b}$ and $\vec{c}$ are coplanar if and only if $\vec{a} . (\vec{b} \times \vec{c}) = 0$ *i.e.* $[\vec{a}\ \vec{b}\ \vec{c}] = 0$.

Proof. Suppose that $\vec{a}, \vec{b}$ and $\vec{c}$ are coplanar.

When $\vec{b}$ and $\vec{c}$ are parallel vectors, then

$$\vec{b} \times \vec{c} = \vec{0} \quad \Rightarrow \quad \vec{a} . (\vec{b} \times \vec{c}) = 0.$$

When $\vec{b}$ and $\vec{c}$ are not parallel vectors.

Since $\vec{a}, \vec{b}$ and $\vec{c}$ are coplanar,

$\therefore$ $\vec{b} \times \vec{c}$ is perpendicular to $\vec{a}$.

$\therefore$ $\vec{a} . (\vec{b} \times \vec{c}) = 0$.

Hence, $[\vec{a}\ \vec{b}\ \vec{c}] = 0$.

Conversely. Suppose that $\vec{a} . (\vec{b} \times \vec{c}) = 0$.

When $\vec{a}$ and $\vec{b} \times \vec{c}$ are both non-zero, then $\vec{a}$ and $\vec{b} \times \vec{c}$ are perpendicular vectors.

But $\vec{b} \times \vec{c}$ is perpendicular to both $\vec{b}$ and $\vec{c}$.

$\therefore$ $\vec{a}, \vec{b}$ and $\vec{c}$ must lie in the plane *i.e.* $\vec{a}, \vec{b}$ and $\vec{c}$ are coplanar.

When $\vec{a} = \vec{0}$, then $\vec{a}$ is coplanar with $\vec{b}$ and $\vec{c}$.

When $\vec{b} \times \vec{c} = \vec{0}$, then $\vec{b}$ and $\vec{c}$ are parallel vectors.

Thus $\vec{a}, \vec{b}$ and $\vec{c}$ are coplanar.

Cor. Coplanarity of Four Points

Four points A, B, C and D are coplanar if the vectors $\overrightarrow{AB}, \overrightarrow{AC}$ and $\overrightarrow{AD}$ are coplanar.

10.35. PROPERTIES

Property I. Condition for three vectors to be coplanar. *If $\vec{a}, \vec{b}, \vec{c}$ are all coplanar vectors, then $\left[\vec{a}\ \vec{b}\ \vec{c}\right] = 0$.*

Proof. Since $\left(\vec{a} \times \vec{b}\right)$ represents a vector, which is perpendicular to the plane containing $\vec{a}$ and $\vec{b}$ in which also lies the vector $\vec{c}$, therefore, $\vec{a} \times \vec{b}$ is perpendicular to $\vec{c}$.

Hence, $\left(\vec{a} \times \vec{b}\right) . \vec{c} = 0$ *i.e.* $\left[\vec{a}\ \vec{b}\ \vec{c}\right] = 0$.

Conversely. *If $\left[\vec{a}\ \vec{b}\ \vec{c}\right] = 0$, then $\vec{a}, \vec{b}, \vec{c}$ are all coplanar vectors.*

Since $\left[\vec{a}\ \vec{b}\ \vec{c}\right] = 0$ *i.e.* $\left(\vec{a} \times \vec{b}\right) . \vec{c} = 0$, $\therefore$ $\left(\vec{a} \times \vec{b}\right)$ is perpendicular to $\vec{c}$.

But $\vec{a} \times \vec{b}$ is perpendicular to the plane containing $\vec{a}$ and $\vec{b}$. $\therefore$ $\vec{c}$ lies in the plane of $\vec{a}$ and $\vec{b}$.

Hence, $\vec{a}, \vec{b}, \vec{c}$ are coplanar.

Property II. *If any two of the three vectors $\vec{a}, \vec{b}, \vec{c}$ are equal, then $\left[\vec{a}\ \vec{b}\ \vec{c}\right] = 0$.*

Proof. Without any loss of generality, let $\vec{b} = \vec{a}$. Then :

$$\left[\vec{a}\ \vec{b}\ \vec{c}\right] = \left[\vec{a}\ \vec{a}\ \vec{c}\right] = \left(\vec{a} \times \vec{a}\right) . \vec{c} = \vec{0} . \vec{c} = 0.$$

Similarly we can also prove when $\vec{c} = \vec{b}$ or $\vec{a} = \vec{c}$.

Property III. *If any two of the three vectors $\vec{a}, \vec{b}, \vec{c}$ are parallel, then $\left[\vec{a}\ \vec{b}\ \vec{c}\right] = 0$.*

Proof. Without any loss of generality, let $\vec{b} \parallel \vec{a}$.

Then $\vec{b} = k\,\vec{a}$, where k is some scalar.

Now
$$\left[\vec{a}\ \vec{b}\ \vec{c}\right] = \left(\vec{a} \times \vec{b}\right) . \vec{c}$$
$$= \left(\vec{a} \times k\,\vec{a}\right) . \vec{c} = k\left(\vec{a} \times \vec{a}\right) . \vec{c} = k\left(\vec{0} . \vec{c}\right) = k\,(0) = 0.$$

Similarly, we can also prove when $\vec{b}, \vec{c}$ are parallel and $\vec{a}, \vec{c}$ are parallel.

CONCLUSION

$\left[\vec{a}\ \vec{b}\ \vec{c}\right] = 0$ **when either** $\vec{a}, \vec{b}, \vec{c}$ **are all coplanar or any two of the three vectors are equal or parallel.**

Property IV. *In the scalar triple product, the position of dot and cross can be interchanged at pleasure provided the cyclic order of the vectors is maintained.*

Proof. Let $\vec{a}, \vec{b}, \vec{c}$ be a right-handed triad.

Then $\vec{b}, \vec{c}, \vec{a}$ and $\vec{c}, \vec{a}, \vec{b}$ are also right-handed triads.

We know that $\left(\vec{a} \times \vec{b}\right) . \vec{c}$ = Volume of the parallelopiped having $\vec{a}, \vec{b}, \vec{c}$ as coterminous edges.

Also $\left(\vec{b} \times \vec{c}\right) . \vec{a}$ = Volume of the parallelopiped having $\vec{b}, \vec{c}, \vec{a}$ as coterminous edges

and $\left(\vec{c} \times \vec{a}\right) . \vec{b}$ = Volume of the parallelopiped having $\vec{c}, \vec{a}, \vec{b}$ as coterminous edges.

$$\therefore \quad V = \left(\vec{a} \times \vec{b}\right) . \vec{c} = \left(\vec{b} \times \vec{c}\right) . \vec{a} = \left(\vec{c} \times \vec{a}\right) . \vec{b}$$

As we know that $\vec{a} . \vec{b} = \vec{b} . \vec{a}$ so as such :

$$\left(\vec{a} \times \vec{b}\right) . \vec{c} = \vec{c} . \left(\vec{a} \times \vec{b}\right) \qquad ....(1)$$

$$\left(\vec{b} \times \vec{c}\right) . \vec{a} = \vec{a} . \left(\vec{b} \times \vec{c}\right) \qquad ....(2)$$

$$\left(\vec{c} \times \vec{a}\right) . \vec{b} = \vec{b} . \left(\vec{c} \times \vec{a}\right) \qquad ....(3)$$

and each = Volume V.

From (1) and (2), $\left(\vec{a} \times \vec{b}\right) . \vec{c} = \vec{a} . \left(\vec{b} \times \vec{c}\right)$

Similarly, $\left(\vec{b} \times \vec{c}\right) . \vec{a} = \vec{b} . \left(\vec{c} \times \vec{a}\right)$ and $\left(\vec{c} \times \vec{a}\right) . \vec{b} = \vec{c} . \left(\vec{a} \times \vec{b}\right)$.

Hence, the result is established.

KEY POINT

$$[\vec{a}\ \vec{b}\ \vec{c}] = [\vec{b}\ \vec{c}\ \vec{a}] = [\vec{c}\ \vec{a}\ \vec{b}]$$

i.e. Cyclic permutable of three vectors does not change the value of the scalar triple product.

Property V. *In the scalar triple product, the change in the cyclic order of the vectors changes the sign of the product.*

Proof. As in (IV), $V = \left(\vec{a} \times \vec{b}\right) . \vec{c} = \left(\vec{b} \times \vec{c}\right) . \vec{a} = \left(\vec{c} \times \vec{a}\right) . \vec{b}$

When the cyclic order of $\vec{a}, \vec{b}, \vec{c}$ be changed, then $\vec{a}, \vec{c}, \vec{b}$ becomes a left-handed triad,

$\therefore \vec{a}, \vec{c}, \vec{b}; \vec{b}, \vec{a}, \vec{c}; \vec{c}, \vec{b}, \vec{a}$ are left-handed triads.

As we know that $\quad \vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$ so as such

$$V = \left(\vec{a} \times \vec{b}\right).\vec{c} = -\left(\vec{b} \times \vec{a}\right).\vec{c} = -\vec{c}.\left(\vec{b} \times \vec{a}\right) = \left(\vec{b} \times \vec{c}\right).\vec{a}$$

$$= -\left(\vec{c} \times \vec{b}\right).\vec{a} = -\vec{a}.\left(\vec{c} \times \vec{b}\right)$$

$$= \left(\vec{c} \times \vec{a}\right).\vec{b} = -\left(\vec{a} \times \vec{c}\right).\vec{b} = -\vec{b}.\left(\vec{a} \times \vec{c}\right).$$

Thus by changing the cyclic order, a sign of scalar triple product is changed.

Property VI. Scalar triple product is distributive

i.e. $\quad \left[\vec{a}, \vec{b} + \vec{c}, \vec{d} - \vec{e}\right] = \left[\vec{a}\ \vec{b}\ \vec{d}\right] - \left[\vec{a}\ \vec{b}\ \vec{e}\right] + \left[\vec{a}\ \vec{c}\ \vec{d}\right] - \left[\vec{a}\ \vec{c}\ \vec{e}\right].$

Proof. $\quad$ **LHS** $= \left[\vec{a}, \vec{b} + \vec{c}, \vec{d} - \vec{e}\right] = \vec{a} \times \left(\vec{b} + \vec{c}\right).\left(\vec{d} - \vec{e}\right)$

$$= \left(\vec{a} \times \vec{b} + \vec{a} \times \vec{c}\right).\left(\vec{d} - \vec{e}\right) = \left(\vec{a} \times \vec{b} + \vec{a} \times \vec{c}\right).\vec{d} - \left(\vec{a} \times \vec{b} + \vec{a} \times \vec{c}\right).\vec{e}$$

$$= \vec{a} \times \vec{b}.\vec{d} + \vec{a} \times \vec{c}.\vec{d} - \vec{a} \times \vec{b}.\vec{e} - \vec{a} \times \vec{c}.\vec{e}$$

$$= \left[\vec{a}\ \vec{b}\ \vec{d}\right] + \left[\vec{a}\ \vec{c}\ \vec{d}\right] - \left[\vec{a}\ \vec{b}\ \vec{e}\right] - \left[\vec{a}\ \vec{c}\ \vec{e}\right]$$

$$= \left[\vec{a}\ \vec{b}\ \vec{d}\right] - \left[\vec{a}\ \vec{b}\ \vec{e}\right] + \left[\vec{a}\ \vec{c}\ \vec{d}\right] - \left[\vec{a}\ \vec{c}\ \vec{e}\right] = \textbf{RHS.}$$

10.36. EXPRESSION IN TERMS OF RECTANGULAR COMPONENTS

To express $\left[\vec{\mathbf{a}}\ \vec{\mathbf{b}}\ \vec{\mathbf{c}}\right]$ in terms of rectangular components of $\vec{\mathbf{a}}, \vec{\mathbf{b}}, \vec{\mathbf{c}}$.

Let $\quad \vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k},\ \vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k},\ \vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}.$

Now $\quad \vec{a} \times \vec{b} = \left(a_1\hat{i} + a_2\hat{j} + a_3\hat{k}\right) \times \left(b_1\hat{i} + b_2\hat{j} + b_3\hat{k}\right)$

$$= a_1b_1\,\hat{i} \times \hat{i} + a_1b_2\,\hat{i} \times \hat{j} + a_1b_3\,\hat{i} \times \hat{k} + a_2b_1\,\hat{j} \times \hat{i} + a_2b_2\,\hat{j} \times \hat{j}$$
$$+ a_2b_3\,\hat{j} \times \hat{k} + a_3b_1\,\hat{k} \times \hat{i} + a_3b_2\,\hat{k} \times \hat{j} + a_3b_3\,\hat{k} \times \hat{k}$$

$$= a_1b_1\left(\vec{0}\right) + a_1b_2\,\hat{k} + a_1b_3\left(-\hat{j}\right) + a_2b_1\left(-\hat{k}\right) + a_2b_2\left(\vec{0}\right)$$
$$+ a_2b_3\,\hat{i} + a_3b_1\,\hat{j} + a_3b_2\left(-\hat{i}\right) + a_3b_3\left(\vec{0}\right)$$

$$= \left(a_2b_3 - a_3b_2\right)\hat{i} + \left(a_3b_1 - a_1b_3\right)\hat{j} + \left(a_1b_2 - a_2b_1\right)\hat{k}.$$

$\therefore \qquad \left[\vec{a}\ \vec{b}\ \vec{c}\right] = \left(\vec{a}\times\vec{b}\right).\vec{c}$

$$= \left[\left(a_2b_3 - a_3b_2\right)\hat{i} + \left(a_3b_1 - a_1b_3\right)\hat{j} + \left(a_1b_2 - a_2b_1\right)\hat{k}\right].\left(c_1\hat{i} + c_2\hat{j} + c_3\hat{k}\right)$$

$$= \left(a_2b_3 - a_3b_2\right)c_1\,\hat{i}.\hat{i} + \left(a_2b_3 - a_3b_2\right)c_2\,\hat{i}.\hat{j}$$

$$+\left(a_2b_3 - a_3b_2\right)c_3\,\hat{i}.\hat{k} + \left(a_3b_1 - a_1b_3\right)c_1\,\hat{j}.\hat{i} + \left(a_3b_1 - a_1b_3\right)c_2\,\hat{j}.\hat{j} + \left(a_3b_1 - a_1b_3\right)c_3\,\hat{j}.\hat{k}$$

$$+\left(a_1b_2 - a_2b_1\right)c_1\,\hat{k}.\hat{i} + \left(a_1b_2 - a_2b_1\right)c_2\,\hat{k}.\hat{j} + \left(a_1b_2 - a_2b_1\right)c_3\,\hat{k}.\hat{k}$$

$$= (a_2b_3 - a_3b_2)\,c_1 + (a_3b_1 - a_1b_3)\,c_2 + (a_1b_2 - a_2b_1)\,c_3$$

$$= \begin{vmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} \mathbf{a_1} & \mathbf{a_2} & \mathbf{a_3} \\ \mathbf{b_1} & \mathbf{b_2} & \mathbf{b_3} \\ \mathbf{c_1} & \mathbf{c_2} & \mathbf{c_3} \end{vmatrix}. \qquad \text{[By Properties of Determinants]}$$

10.37. RECIPROCAL SYSTEM OF VECTORS

Let $\vec{a}, \vec{b}, \vec{c}$ be three non-coplanar vectors, such that $\left[\vec{a}\ \vec{b}\ \vec{c}\right] \neq 0$. Then the vectors :

$$\vec{a'} = \frac{\vec{b}\times\vec{c}}{\left[\vec{a}\ \vec{b}\ \vec{c}\right]},\ \vec{b'} = \frac{\vec{c}\times\vec{a}}{\left[\vec{a}\ \vec{b}\ \vec{c}\right]},\ \vec{c'} = \frac{\vec{a}\times\vec{b}}{\left[\vec{a}\ \vec{b}\ \vec{c}\right]}$$

are said to form a *reciprocal system* of vectors for the vectors $\vec{a}, \vec{b}, \vec{c}$.

Properties :

Property I. *The system of orthonormal triad of unit vectors* $\hat{i}, \hat{j}, \hat{k}$ *is self orthogonal*

i.e. $\qquad \hat{i'} = \hat{i},\ \hat{j'} = \hat{j},\ \hat{k'} = \hat{k}.$

Proof. Here $\qquad \hat{i'} = \dfrac{\hat{j}\times\hat{k}}{\left[\hat{i}\ \hat{j}\ \hat{k}\right]} = \dfrac{\hat{i}}{1} = \hat{i}.$

Similarly, $\qquad \hat{j'} = \hat{j}$ and $\hat{k'} = \hat{k}$.

Hence, the system $\hat{i}, \hat{j}, \hat{k}$ is self-orthogonal.

Property II. *If* $\vec{a}, \vec{b}, \vec{c}$ *and* $\vec{a'}, \vec{b'}, \vec{c'}$ *form a reciprocal system of vectors, then :*

(*i*) $\quad \vec{a}.\vec{a'} = \vec{b}.\vec{b'} = \vec{c}.\vec{c'} = 1$

(*ii*) $\quad \vec{a}.\vec{b'} = \vec{a}.\vec{c'} = 0;\ \vec{b}.\vec{c'} = \vec{b}.\vec{a'} = 0;\ \vec{c}.\vec{a'} = \vec{c}.\vec{b'} = 0$

(*iii*) $\quad \left[\vec{a}\ \vec{b}\ \vec{c}\right]\left[\vec{a'}\ \vec{b'}\ \vec{c'}\right] = 1$

(*iv*) $\vec{a}, \vec{b}, \vec{c}$ *are non-coplanar iff so are* $\vec{a'}, \vec{b'}, \vec{c'}$.

Proof. We have : $\vec{a}' = \dfrac{\vec{b} \times \vec{c}}{[\vec{a}\ \vec{b}\ \vec{c}]}$, $\vec{b}' = \dfrac{\vec{c} \times \vec{a}}{[\vec{a}\ \vec{b}\ \vec{c}]}$, and $\vec{c}' = \dfrac{\vec{a} \times \vec{b}}{[\vec{a}\ \vec{b}\ \vec{c}]}$.

(*i*) $$\vec{a} . \vec{a}' = \vec{a} . \frac{\vec{b} \times \vec{c}}{[\vec{a}\ \vec{b}\ \vec{c}]} = \frac{\vec{a} . (\vec{b} \times \vec{c})}{[\vec{a}\ \vec{b}\ \vec{c}]} = \frac{[\vec{a}\ \vec{b}\ \vec{c}]}{[\vec{a}\ \vec{b}\ \vec{c}]} = 1.$$

Similarly, $\vec{b} . \vec{b}' = \vec{c} . \vec{c}' = 1.$

(*ii*) $$\vec{a} . \vec{b}' = \vec{a} . \frac{\vec{c} \times \vec{a}}{[\vec{a}\ \vec{b}\ \vec{c}]} = \frac{\vec{a} . (\vec{c} \times \vec{a})}{[\vec{a}\ \vec{b}\ \vec{c}]} = \frac{[\vec{a}\ \vec{c}\ \vec{a}]}{[\vec{a}\ \vec{b}\ \vec{c}]} = 0.$$

Similarly, $\vec{a} . \vec{c}' = \vec{b} . \vec{c}' = \vec{b} . \vec{a}' = \vec{c} . \vec{a}' = \vec{c} . \vec{b}' = 0.$

(*iii*) $$[\vec{a}'\ \vec{b}'\ \vec{c}'] = \left\{\left(\frac{(\vec{b} \times \vec{c})}{[\vec{a}\ \vec{b}\ \vec{c}]} . \frac{\vec{c} \times \vec{a}}{[\vec{a}\ \vec{b}\ \vec{c}]}\right) \times \frac{\vec{a} \times \vec{b}}{[\vec{a}\ \vec{b}\ \vec{c}]}\right\}$$

$$= \frac{1}{[\vec{a}\ \vec{b}\ \vec{c}]^3}\left[\left\{(\vec{b} \times \vec{c}) . (\vec{c} \times \vec{a}) \times (\vec{a} \times \vec{b})\right\}\right]$$

$$= \frac{1}{[\vec{a}\ \vec{b}\ \vec{c}]^3}\left[\vec{b} \times \vec{c}\ \ \vec{c} \times \vec{a}\ \ \vec{a} \times \vec{b}\right]$$

$$= \frac{[\vec{a}\ \vec{b}\ \vec{c}]^2}{[\vec{a}\ \vec{b}\ \vec{c}]^3} = \frac{1}{[\vec{a}\ \vec{b}\ \vec{c}]}.$$

Hence, $[\vec{a}\ \vec{b}\ \vec{c}][\vec{a}'\ \vec{b}'\ \vec{c}'] = 1.$

(*iv*) $\vec{a}, \vec{b}, \vec{c}$ are non-coplanar $\Leftrightarrow [\vec{a}\ \vec{b}\ \vec{c}] \neq 0$

$\Leftrightarrow$ $\dfrac{1}{[\vec{a}'\ \vec{b}'\ \vec{c}']} \neq 0 \Leftrightarrow [\vec{a}'\ \vec{b}'\ \vec{c}'] \neq 0$ [*Part* (*iii*)]

$\Leftrightarrow$ $\vec{a}', \vec{b}', \vec{c}'$ are non-coplanar.

Frequently Asked Questions

FAQs

Example 1. If $\vec{a} \times \vec{b} = \lambda \vec{c}$ for a non-zero scalar 'λ' and non-zero vectors $\vec{a}, \vec{b}$ and $\vec{c}$, then find $\vec{a} \cdot \vec{c}$. *(Meghalaya B. 2015)*

Solution. We have : $\vec{a} \times \vec{b} = \lambda \vec{c}$.

Multiplying scalarly by $\vec{a}$ on the left, we get :

$$\vec{a}.(\vec{a} \times \vec{b}) = \vec{a}.(\lambda \vec{c})$$

$$\Rightarrow \quad [\vec{a}\ \vec{a}\ \vec{b}] = \lambda(\vec{a}.\vec{c})$$

$$\Rightarrow \quad 0 = \lambda(\vec{a}.\vec{c})$$

Hence, $$\vec{a}.\vec{c} = 0. \qquad [\because \lambda \neq 0]$$

Example 2. Find $\vec{a}.(\vec{b} \times \vec{c})$, if :

$$\vec{a} = 2\hat{i} + \hat{j} + 3\hat{k}, \quad \vec{b} = -\hat{i} + 2\hat{j} + \hat{k} \text{ and}$$

$$\vec{c} = 3\hat{i} + \hat{j} + 2\hat{k}.$$ *(A.I.C.B.S.E. 2014)*

Solution. We have :

$\vec{a} = 2\hat{i} + \hat{j} + 3\hat{k}$, $\vec{b} = -\hat{i} + 2\hat{j} + \hat{k}$ and

$\vec{c} = 3\hat{i} + \hat{j} + 2\hat{k}$.

$$\therefore \quad \vec{a}.(\vec{b} \times \vec{c}) = [\vec{a}\quad \vec{b}\quad \vec{c}]$$

$$= \begin{vmatrix} 2 & 1 & 3 \\ -1 & 2 & 1 \\ 3 & 1 & 2 \end{vmatrix}$$

$$= 2(4-1) - (1)(-2-3) + 3(-1-6)$$
$$= 6 + 5 - 21 = 11 - 21 = -10.$$

Example 3. Find the value of 'λ' such that the vectors : $3\hat{i} + \lambda\hat{j} + 5\hat{k}$, $\hat{i} + 2\hat{j} - 3\hat{k}$ and $2\hat{i} - \hat{j} + \hat{k}$ are coplanar. *(Mizoram B. 2017; P.B. 2014 S)*

Solution. The given vectors are coplanar if their scalar triple product is zero

i.e. if $\begin{vmatrix} 3 & \lambda & 5 \\ 1 & 2 & -3 \\ 2 & -1 & 1 \end{vmatrix} = 0$

i.e. if $3(2-3) - \lambda(1+6) + 5(-1-4) = 0$ *[Expanding by R_1]*

if $-3 - 7\lambda - 25 = 0$

if $7\lambda = -28$ if $\lambda = -4$.

Example 4. Find the value of 'λ' if four points with position vectors $3\hat{i} + 6\hat{j} + 9\hat{k}$, $\hat{i} + 2\hat{j} + 3\hat{k}$, $2\hat{i} + 3\hat{j} + \hat{k}$ and $4\hat{i} + 6\hat{j} + \lambda\hat{k}$ are coplanar. *(A.I.C.B.S.E. 2017)*

Solution. Let A, B, C and D be the points with position vectors $3\hat{i} + 6\hat{j} + 9\hat{k}$, $\hat{i} + 2\hat{j} + 3\hat{k}$, $2\hat{i} + 3\hat{j} + \hat{k}$ and $4\hat{i} + 6\hat{j} + \lambda\hat{k}$ respectively.

$$\therefore \quad \overrightarrow{AB} = \left(\hat{i} + 2\hat{j} + 3\hat{k}\right) - \left(3\hat{i} + 6\hat{j} + 9\hat{k}\right)$$

$$= -2\hat{i} - 4\hat{j} - 6\hat{k},$$

$$\overrightarrow{BC} = \left(2\hat{i} + 3\hat{j} + \hat{k}\right) - \left(\hat{i} + 2\hat{j} + 3\hat{k}\right)$$

$$= \hat{i} + \hat{j} - 2\hat{k}$$

and $$\overrightarrow{CD} = \left(4\hat{i} + 6\hat{j} + \lambda\hat{k}\right) - \left(2\hat{i} + 3\hat{j} + \hat{k}\right)$$

$$= 2\hat{i} + 3\hat{j} + (\lambda - 1)\hat{k}.$$

The four points are coplanar if $\overrightarrow{AB}, \overrightarrow{BC}, \overrightarrow{CD}$ are coplanar

if $[\overrightarrow{AB}, \overrightarrow{BC}, \overrightarrow{CD}] = 0$

if $\begin{vmatrix} -2 & -4 & -6 \\ 1 & 1 & -2 \\ 2 & 3 & \lambda - 1 \end{vmatrix} = 0$

if $-2(\lambda - 1 + 6) + 4(\lambda - 1 + 4) - 6(3 - 2) = 0$

if $-2\lambda - 10 + 4\lambda + 12 - 6 = 0$

if $2\lambda - 4 = 0$ if $\lambda = 2$.

Hence, $\lambda = 2$.

Example 5. Let $\vec{a} = \hat{i} + \hat{j} + \hat{k}$, $\vec{b} = \hat{i}$ and $\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$. Then :

(a) Let $c_1 = 1$ and $c_2 = 2$, find c_3, which makes $\vec{a}, \vec{b}$ and $\vec{c}$ coplanar.

(b) If $c_2 = -1$ and $c_3 = 1$, show that no value of c_1 can make $\vec{a}, \vec{b}$ and $\vec{c}$ coplanar. *(C.B.S.E. 2017)*

Solution. We have : $\vec{a} = \hat{i} + \hat{j} + \hat{k},\ \vec{b} = \hat{i}$ and

$$\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}.$$

(*a*) Here $c_1 = 1$ and $c_2 = 2$.

$$\therefore \quad \vec{c} = \hat{i} + 2\hat{j} + c_3\hat{k}.$$

Since $\vec{a}, \vec{b}$ and $\vec{c}$ are coplanar, [*Given*]

$$\therefore \quad \begin{vmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 2 & c_3 \end{vmatrix} = 0$$

$\Rightarrow \quad (-1)\,[c_3 - 2] = 0 \Rightarrow c_3 - 2 = 0 \Rightarrow c_3 = 2.$

(*b*) Here $c_2 = -1,\ c_3 = 1$.

$$\therefore \quad \vec{c} = c_1\hat{i} - \hat{j} + \hat{k}.$$

Since $\vec{a}, \vec{b}$ and $\vec{c}$ are coplanar, [*Given*]

$$\therefore \quad \begin{vmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ c_1 & -1 & 1 \end{vmatrix} = 0$$

$\Rightarrow \quad (-1)\,[1 + 1] = 0 \quad \Rightarrow \quad -2 = 0$, which is false.

Hence, no value of c_1 can make $\vec{a}, \vec{b}$ and $\vec{c}$ coplanar.

Example 6. Find the volume of the parallelopiped whose sides are given by vectors :

$$\mathbf{2\hat{i} - 3\hat{j} + 4\hat{k},\ \hat{i} + 2\hat{j} - \hat{k}\ \text{and}\ 3\hat{i} - \hat{j} + 2\hat{k}.}$$

Solution. Let

$$\vec{a} = 2\hat{i} - 3\hat{j} + 4\hat{k},\quad \vec{b} = \hat{i} + 2\hat{j} - \hat{k}$$

and $\qquad \vec{c} = 3\hat{i} - \hat{j} + 2\hat{k}.$

$\therefore$ Volume of the parallelopiped $= \left[\vec{a}\ \vec{b}\ \vec{c}\right]$

$$= \begin{vmatrix} 2 & -3 & 4 \\ 1 & 2 & -1 \\ 3 & -1 & 2 \end{vmatrix}$$

$= 2\,(4 - 1) + 3\,(2 + 3) + 4\,(-1 - 6)$ [*Expanding by* R_1]

$= 6 + 15 - 28 = -7 = 7$ cub. units. [*Rejecting – ve sign*]

Example 7. Prove that :

$$\mathbf{\hat{i}\cdot\left(\hat{j}\times\hat{k}\right) = \hat{j}\cdot\left(\hat{k}\times\hat{i}\right) = \hat{k}\cdot\left(\hat{i}\times\hat{j}\right) = 1.}$$

Solution. $\hat{i}\cdot\left(\hat{j}\times\hat{k}\right) = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$

$= 1\,.\,(1 - 0) - 0 + 0 = 1.$

Similarly, $\hat{j}\cdot\left(\hat{k}\times\hat{i}\right) = \hat{k}\cdot\left(\hat{i}\times\hat{j}\right) = 1.$

Hence, $\quad \hat{i}\cdot\left(\hat{j}\times\hat{k}\right) = \hat{j}\cdot\left(\hat{k}\times\hat{i}\right) = \hat{k}\cdot\left(\hat{i}\times\hat{j}\right) = 1.$

Example 8. Show that the vectors $\vec{a}, \vec{b}$ and $\vec{c}$ are coplanar if $\vec{a} + \vec{b},\ \vec{b} + \vec{c}$ and $\vec{c} + \vec{a}$ are coplanar.

(N.C.E.R.T. (Supplement); C.B.S.E. 2016; Nagaland 2016)

Solution. Since $\vec{a} + \vec{b}, \vec{b} + \vec{c}$ and $\vec{c} + \vec{a}$ are coplanar,

$\therefore \ (\vec{a} + \vec{b}).[(\vec{b} + \vec{c}) \times (\vec{c} + \vec{a})] = 0$

$\Rightarrow (\vec{a} + \vec{b}).[\vec{b}\times\vec{c} + \vec{b}\times\vec{a} + \vec{c}\times\vec{c} + \vec{c}\times\vec{a}] = 0$

$\Rightarrow (\vec{a} + \vec{b}).(\vec{b}\times\vec{c} + \vec{b}\times\vec{a} + \vec{0} + \vec{c}\times\vec{a}) = 0$

$\Rightarrow \vec{a}.(\vec{b}\times\vec{c}) + \vec{a}.(\vec{b}\times\vec{a}) + \vec{a}.(\vec{c}\times\vec{a}) + \vec{b}.(\vec{b}\times\vec{c})$

$\qquad + \vec{b}.(\vec{b}\times\vec{a}) + \vec{b}.(\vec{c}\times\vec{a}) = 0$

$\Rightarrow 2[\vec{a}.(\vec{b}\times\vec{c})] = 0 \quad \Rightarrow \quad \vec{a}.(\vec{b}\times\vec{c}) = 0$

$\Rightarrow [\vec{a}\ \vec{b}\ \vec{c}] = 0.$

Hence, $\vec{a}, \vec{b}$ and $\vec{c}$ are coplanar.

Example 9. If $\vec{p} = \hat{i} + \hat{j} + \hat{k}$ and $\vec{q} = \hat{i} - 2\hat{j} + \hat{k}$, then find a vector of magnitude $5\sqrt{3}$ units perpendicular to the vector $\vec{q}$ and coplanar with $\vec{p}$ and $\vec{q}$.

(C.B.S.E. Sample Paper 2019)

Solution. Here, $\vec{p} = \hat{i} + \hat{j} + \hat{k}$ and $\vec{q} = \hat{i} - 2\hat{j} + \hat{k}$.

Let $\vec{r} = a\hat{i} + b\hat{j} + c\hat{k}$ be the required vector.

Since $\vec{r} \perp \vec{q}$,

$\therefore \ \vec{r}.\vec{q} = 0$

$\Rightarrow \quad (a\hat{i} + b\hat{j} + c\hat{k}).(\hat{i} - 2\hat{j} + \hat{k}) = 0$

$\Rightarrow \quad a(1) + b(-2) + c(1) = 0$

$\Rightarrow \quad a - 2b + c = 0 \qquad ...(1)$

And, $\vec{p}, \vec{q}$ and $\vec{r}$ are coplanar,

$$\therefore \quad [\vec{p} \ \vec{q} \ \vec{r}] = 0$$

$$\Rightarrow \quad \begin{vmatrix} 1 & 1 & 1 \\ 1 & -2 & 1 \\ a & b & c \end{vmatrix} = 0$$

$$\Rightarrow \quad \begin{vmatrix} 1 & 0 & 0 \\ 1 & -3 & 3 \\ a & b-a & c-b \end{vmatrix} = 0$$

[*Operating* $C_2 \to C_2 - C_1$ & $C_3 \to C_3 - C_2$]

$\Rightarrow$ $(1)\,[-3(c-b) - (b-a)(3)] = 0$

$\Rightarrow$ $3[-c+b-b+a] = 0$

$\Rightarrow$ $-c+a = 0$

$\Rightarrow$ $a - c = 0$

$\Rightarrow$ $a + 0.\,b - c = 0$ ...(2)

Solve (1) and (2), $\dfrac{a}{2-0} = \dfrac{b}{1+1} = \dfrac{c}{0+2}$

$\Rightarrow$ $\dfrac{a}{1} = \dfrac{b}{1} = \dfrac{c}{1}$ ...(3)

Thus, $\vec{r} = 1.\hat{i} + 1.\hat{j} + 1.\hat{k}$

$= \hat{i} + \hat{j} + \hat{k}$.

$\therefore$ $|\vec{r}| = \sqrt{1+1+1} = \sqrt{3}$.

$\therefore$ Unit vector, $\hat{r} = \dfrac{\vec{r}}{|\vec{r}|} = \dfrac{\hat{i}+\hat{j}+\hat{k}}{\sqrt{3}}$.

Hence, the required vector

$= 5\sqrt{3}\,(\hat{r})$

$= 5\sqrt{3}\left(\dfrac{\hat{i}+\hat{j}+\hat{k}}{\sqrt{3}}\right)$

$= 5(\hat{i}+\hat{j}+\hat{k})$.

Example 10. Prove that $\vec{\mathbf{a}} \cdot \left(\vec{\mathbf{b}} + \vec{\mathbf{c}}\right) \times \left(\vec{\mathbf{a}} + \vec{\mathbf{b}} + \vec{\mathbf{c}}\right) = 0$.

Solution. $\vec{a} \cdot \left(\vec{b} + \vec{c}\right) \times \left(\vec{a} + \vec{b} + \vec{c}\right)$

$= \vec{a} \cdot \left(\vec{b}\times\vec{a} + \vec{b}\times\vec{b} + \vec{b}\times\vec{c} + \vec{c}\times\vec{a} + \vec{c}\times\vec{b} + \vec{c}\times\vec{c}\right)$

$= \vec{a} \cdot \left(-\vec{a}\times\vec{b} + 0 + \vec{b}\times\vec{c} + \vec{c}\times\vec{a} - \vec{b}\times\vec{c} + 0\right)$

$= -\vec{a}\cdot\left(\vec{a}\times\vec{b}\right) + \vec{a}\cdot\left(\vec{b}\times\vec{c}\right) + \vec{a}\cdot\left(\vec{c}\times\vec{a}\right) - \vec{a}\cdot\left(\vec{b}\times\vec{c}\right)$

$= -\left[\vec{a}\ \vec{a}\ \vec{b}\right] + \left[\vec{a}\ \vec{b}\ \vec{c}\right] + \left[\vec{a}\ \vec{c}\ \vec{a}\right] - \left[\vec{a}\ \vec{b}\ \vec{c}\right]$

$= -0 + \left[\vec{a}\ \vec{b}\ \vec{c}\right] + 0 - \left[\vec{a}\ \vec{b}\ \vec{c}\right] = 0$.

Example 11. It is given that :

$$\vec{\mathbf{x}} = \frac{\vec{\mathbf{b}}\times\vec{\mathbf{c}}}{\left[\vec{\mathbf{a}}\ \vec{\mathbf{b}}\ \vec{\mathbf{c}}\right]},\quad \vec{\mathbf{y}} = \frac{\vec{\mathbf{c}}\times\vec{\mathbf{a}}}{\left[\vec{\mathbf{a}}\ \vec{\mathbf{b}}\ \vec{\mathbf{c}}\right]} \text{ and } \vec{\mathbf{z}} = \frac{\vec{\mathbf{a}}\times\vec{\mathbf{b}}}{\left[\vec{\mathbf{a}}\ \vec{\mathbf{b}}\ \vec{\mathbf{c}}\right]},$$

where $\vec{\mathbf{a}}, \vec{\mathbf{b}}, \vec{\mathbf{c}}$ are non-coplanar vectors.

Find the value of $\vec{\mathbf{x}} \cdot \left(\vec{\mathbf{a}} + \vec{\mathbf{b}}\right) + \vec{\mathbf{y}} \cdot \left(\vec{\mathbf{b}} + \vec{\mathbf{c}}\right) + \vec{\mathbf{z}} \cdot \left(\vec{\mathbf{c}} + \vec{\mathbf{a}}\right)$.

Solution. $\vec{x}\cdot\left(\vec{a}+\vec{b}\right) + \vec{y}\cdot\left(\vec{b}+\vec{c}\right) + \vec{z}\cdot\left(\vec{c}+\vec{a}\right)$

$$= \frac{\vec{b}\times\vec{c}}{\left[\vec{a}\ \vec{b}\ \vec{c}\right]}\cdot\left(\vec{a}+\vec{b}\right) + \frac{\vec{c}\times\vec{a}}{\left[\vec{a}\ \vec{b}\ \vec{c}\right]}\cdot\left(\vec{b}+\vec{c}\right) + \frac{\vec{a}\times\vec{b}}{\left[\vec{a}\ \vec{b}\ \vec{c}\right]}\cdot\left(\vec{c}+\vec{a}\right)$$

$$= \frac{1}{\left[\vec{a}\ \vec{b}\ \vec{c}\right]}\left[\left(\vec{b}\times\vec{c}\right)\cdot\vec{a} + \left(\vec{b}\times\vec{c}\right)\cdot\vec{b} + \left(\vec{c}\times\vec{a}\right)\cdot\vec{b} + \left(\vec{c}\times\vec{a}\right)\cdot\vec{c} + \left(\vec{a}\times\vec{b}\right)\cdot\vec{c} + \left(\vec{a}\times\vec{b}\right)\cdot\vec{a}\right]$$

$$= \frac{1}{\left[\vec{a}\ \vec{b}\ \vec{c}\right]}\left[\left[\vec{b}\ \vec{c}\ \vec{a}\right] + \left[\vec{b}\ \vec{c}\ \vec{b}\right] + \left[\vec{c}\ \vec{a}\ \vec{b}\right] + \left[\vec{c}\ \vec{a}\ \vec{c}\right] + \left[\vec{a}\ \vec{b}\ \vec{c}\right] + \left[\vec{a}\ \vec{b}\ \vec{a}\right]\right]$$

$$= \frac{1}{[\vec{a}\ \vec{b}\ \vec{c}]}\left[[\vec{b}\ \vec{c}\ \vec{a}] + 0 + [\vec{c}\ \vec{a}\ \vec{b}] + 0 + [\vec{a}\ \vec{b}\ \vec{c}] + 0\right]$$

$$= \frac{3[\vec{a}\ \vec{b}\ \vec{c}]}{[\vec{a}\ \vec{b}\ \vec{c}]} = 3.$$

Example 12. Prove that :

$$[\vec{\mathbf{a}}+\vec{\mathbf{b}}, \vec{\mathbf{b}}+\vec{\mathbf{c}}, \vec{\mathbf{c}}+\vec{\mathbf{a}}] = 2[\vec{\mathbf{a}}\ \vec{\mathbf{b}}\ \vec{\mathbf{c}}].$$

(N.C.E.R.T. (Supplement); Kerala B. 2018 ; C.B.S.E. 2014)

Solution. LHS $= [\vec{a}+\vec{b}, \vec{b}+\vec{c}, \vec{c}+\vec{a}]$

$$= \left[(\vec{a}+\vec{b}) \times (\vec{b}+\vec{c})\right] . (\vec{c}+\vec{a})$$

$$= [\vec{a}\times\vec{b} + \vec{a}\times\vec{c} + \vec{b}\times\vec{b} + \vec{b}\times\vec{c}] . (\vec{c}+\vec{a})$$

$$= [\vec{a}\times\vec{b} + \vec{a}\times\vec{c} + \vec{0} + \vec{b}\times\vec{c}] . (\vec{c}+\vec{a})$$

$$= [\vec{a}\times\vec{b} + \vec{a}\times\vec{c} + \vec{b}\times\vec{c}] . (\vec{c}+\vec{a})$$

$$= (\vec{a}\times\vec{b}) . \vec{c} + (\vec{a}\times\vec{b}) . \vec{a} + (\vec{a}\times\vec{c}) . \vec{c} + (\vec{a}\times\vec{c}) . \vec{a} + (\vec{b}\times\vec{c}) . \vec{c} + (\vec{b}\times\vec{c}) . \vec{a}$$

$$= [\vec{a}\,\vec{b}\,\vec{c}] + [\vec{a}\,\vec{b}\,\vec{a}] + [\vec{a}\,\vec{c}\,\vec{c}] + [\vec{a}\,\vec{c}\,\vec{a}] + [\vec{b}\,\vec{c}\,\vec{c}] + [\vec{b}\,\vec{c}\,\vec{a}]$$

$$= [\vec{a}\ \vec{b}\ \vec{c}] + \vec{0} + \vec{0} + \vec{0} + \vec{0} + [\vec{b}\ \vec{c}\ \vec{a}]$$

$$= [\vec{a}\ \vec{b}\ \vec{c}] + [\vec{a}\ \vec{b}\ \vec{c}] = 2[\vec{a}\ \vec{b}\ \vec{c}] = \textbf{RHS.}$$

Example 13. Prove that :

$$[\vec{\mathbf{a}}\ \vec{\mathbf{b}}\ \vec{\mathbf{c}}+\vec{\mathbf{d}}] = [\vec{\mathbf{a}}\ \vec{\mathbf{b}}\ \vec{\mathbf{c}}] + [\vec{\mathbf{a}}\ \vec{\mathbf{b}}\ \vec{\mathbf{d}}].$$

(N.C.E.R.T. (Supplement); Karnataka B. 2014)

Solution : $[\vec{a}\ \vec{b}\ \vec{c}+\vec{d}] = \vec{a} . \left[\vec{b} \times (\vec{c}+\vec{d})\right]$

$$= \vec{a} . [\vec{b}\times\vec{c} + \vec{b}\times\vec{d}] = \vec{a} . (\vec{b}\times\vec{c}) + \vec{a} . (\vec{b}\times\vec{d})$$

$$= [\vec{a}\ \vec{b}\ \vec{c}] + [\vec{a}\ \vec{b}\ \vec{d}].$$

EXERCISE 10 (j)

Very Short Answer Type Questions

VSATQ

1. Classify the following quantities as vector or scalar : $\vec{a} . \vec{b}$, $[\vec{a}\ \vec{b}\ \vec{c}]$, $3\vec{a}$, $\vec{a}\times\vec{b}$.

2. Find $\vec{a} . (\vec{b}\times\vec{c})$ if :
$\vec{a} = 2\hat{i} + \hat{j} + 3\hat{k}$, $\vec{b} = -\hat{i} + 2\hat{j} + \hat{k}$ and
$\vec{c} = 3\hat{i} + \hat{j} + 2\hat{k}$. *(N.C.E.R.T. (Supplement))*

3. Show that if $\vec{a}+\vec{b}, \vec{b}+\vec{c}, \vec{c}+\vec{a}$ are coplanar, then $\vec{a}, \vec{b}, \vec{c}$ are also coplanar. *(Kerala B. 2018)*

4. If $\vec{a} = 7\hat{i} - 2\hat{j} + 3\hat{k}$, $\vec{b} = \hat{i} - \hat{j} + 2\hat{k}$, $\vec{c} = 2\hat{i} + 8\hat{j}$, then find $\vec{a} . (\vec{b}\times\vec{c})$ and $(\vec{b}\times\vec{c}) . \vec{a}$.

5. Show that the following vectors are coplanar :

(*i*) $\vec{a} = \hat{i} - 2\hat{j} + 3\hat{k}$, $\vec{b} = -2\hat{i} + 3\hat{j} - 4\hat{k}$
and $\vec{c} = \hat{i} - 3\hat{j} + 5\hat{k}$
(N.C.E.R.T. (Supplement); H.P.B. 2016)

(*ii*) $-2\hat{i} - 2\hat{j} + 4\hat{k}$, $-2\hat{i} + 4\hat{j} - 2\hat{k}$, $4\hat{i} - 2\hat{k}$ and $\hat{i} - \hat{j} + \hat{k}$.

6. For what value of 'λ' are the following vectors coplanar ? *(C.B.S.E. 2015)*

(i) $\vec{a} = \hat{i} + 3\hat{j} + \hat{k}, \vec{b} = 2\hat{i} - \hat{j} - \hat{k}$ and $\vec{c} = \lambda\hat{j} + 3\hat{k}$

(ii) $\vec{a} = \hat{i} - \hat{j} + \hat{k}, \vec{b} = 3\hat{i} + \hat{j} + 2\hat{k}$ and $\vec{c} = \hat{i} + \lambda\hat{j} - 3\hat{k}$

(H.P.B. 2018, 16, 14, 13 S)

(iii) $\vec{a} = 2\hat{i} - 4\hat{j} + 5\hat{k}, \vec{b} = \hat{i} - \lambda\hat{j} + \hat{k}$ and $\vec{c} = 3\hat{i} + 2\hat{j} - 5\hat{k}$.

(Kerala B. 2016)

(iv) $\vec{a} = \hat{i} + 3\hat{j} + \hat{k}, \vec{b} = 2\hat{i} - \hat{j} - \hat{k}$ and $\vec{c} = \lambda\hat{i} + 7\hat{j} + 3\hat{k}$.

(N.C.E.R.T. (Supplement)

7. (i) Show that the four points A, B, C and D with position vectors $4\hat{i} + 5\hat{j} + \hat{k}, -(\hat{j} + \hat{k}), 3\hat{i} + 9\hat{j} + 4\hat{k}$ and $4(-\hat{i} + \hat{j} + \hat{k})$ respectively are coplanar.

(N.C.E.R.T. (Supplement); H.P.B. Model Paper 2018; H.P.B. 2017; P.B. 2016; A.I.C.B.S.E. 2014)

(ii) The position vectors of four points A, B, C and D are $4\hat{i} + 8\hat{j} + 12\hat{k}, 2\hat{i} + 4\hat{j} + 6\hat{k}, 3\hat{i} + 5\hat{j} + 4\hat{k}$ and $5\hat{i} + 8\hat{j} + 5\hat{k}$ respectively. Prove that the four points A, B, C and D are coplanar.

(N.C.E.R.T. (Supplement); W. Bangal B. 2017)

8. Find 'x' such that the four points : A (3, 2, 1), B (4, x, 5), C (4, 2, – 2) and D (6, 5, – 1) are coplanar.

(N.C.E.R.T. (Supplement); H.P.B. 2018, 17; A.I.C.B.S.E. 2017; Karnataka B. 2017; Nagaland B. 2015)

9. Show that the four points having position vectors $6\hat{i} - 7\hat{j}, 16\hat{i} - 19\hat{j} - 4\hat{k}, 3\hat{j} - 6\hat{k}, 2\hat{i} + 5\hat{j} + 10\hat{k}$ are not coplanar.

10. Find the volume of the parallelopiped whose sides are given by the vectors :

(i) $11\hat{i}, 2\hat{j}, 13\hat{k}$ (ii) $3\hat{i} + 4\hat{j}, 2\hat{i} + 3\hat{j} + 4\hat{k}, 5\hat{k}$.

11. Let $\vec{a} = \hat{i} + \hat{j} + \hat{k}, \vec{b} = \hat{i}$ and $\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$. Then :

If $c_1 = 1$ and $c_2 = 2$, find c_3 which makes $\vec{a}, \vec{b}$ and $\vec{c}$ coplanar.

12. Find the volume of the parallelopiped with coterminous edges AB, AC and AD, where A ≡ (3, 2, 1), B ≡ (4, 2, 1), C ≡ (0, 1, 4) and D ≡ (0, 0, 7).

Short Answer Type Questions

SATQ

13. Prove that for any two vectors $\vec{a}$ and $\vec{b}$, $\vec{a}.\left(\vec{a} \times \vec{b}\right) = 0$.

Is $\vec{b}.\left(\vec{a} \times \vec{b}\right) = 0$?

14. If $\vec{a}, \vec{b}$ and $\vec{c}$ are perpendicular to each other, show that $\left[\vec{a}.\left(\vec{b} \times \vec{c}\right)\right]^2 = a^2b^2c^2$.

15. What can you conclude about four non-zero vectors $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$, given that :

$$\left[\left(\vec{a} \times \vec{b}\right).\vec{c}\right] + \left[\left(\vec{b} \times \vec{c}\right).\vec{d}\right] = 0 \; ?$$

16. (i) Simplify $\left(\vec{b} + \vec{c}\right).\left\{\left(\vec{c} + \vec{a}\right) \times \left(\vec{a} + \vec{b}\right)\right\}$

(ii) Prove that :

$$\left(\vec{b} + \vec{c}\right).\left\{\left(\vec{c} + \vec{a}\right) \times \left(\vec{a} + \vec{b}\right)\right\} = 2\left[\vec{a} \; \vec{b} \; \vec{c}\right].$$

17. Prove that :

(i) $\left\{\left(\vec{b} + \vec{c}\right) \times \left(\vec{c} + \vec{a}\right)\right\}.\left(\vec{a} + \vec{b}\right) = 2\left[\vec{a} \; \vec{b} \; \vec{c}\right]$

(ii) $\left\{\left(\vec{b} - \vec{c}\right) \times \left(\vec{c} - \vec{a}\right)\right\}.\left(\vec{a} - \vec{b}\right) = 0$.

18. Simplify : $\left[\vec{a} - \vec{b}, \vec{b} - \vec{c}, \vec{c} - \vec{a}\right]$.

19. For any three vectors $\vec{a}, \vec{b}, \vec{c}$, show that $\vec{a} - \vec{b}, \vec{b} - \vec{c}, \vec{c} - \vec{a}$ are coplanar.

20. If $\vec{a}.\vec{b} \times \vec{c} \neq 0$ and $\vec{a'} = \dfrac{\vec{b} \times \vec{c}}{\vec{a}.\vec{b} \times \vec{c}}$,

$\vec{b'} = \dfrac{\vec{c} \times \vec{a}}{\vec{a}.\vec{b} \times \vec{c}}, \vec{c'} = \dfrac{\vec{a} \times \vec{b}}{\vec{a}.\vec{b} \times \vec{c}}$, show that :

(i) $\vec{a}.\vec{a'} + \vec{b}.\vec{b'} + \vec{c}.\vec{c'} = 3$

(ii) $\vec{a'}.\left(\vec{b'} \times \vec{c'}\right) = \dfrac{1}{\vec{a}.\left(\vec{b} \times \vec{c}\right)}$

(iii) $\vec{a}.\vec{a'} + \vec{b} \times \vec{b'} + \vec{c} \times \vec{c'} = 0$.

Long Answer Type Questions

LATQ

21. If $\vec{a}, \vec{b}, \vec{c}$ are three vectors such that $\vec{a} + \vec{b} + \vec{c} = 0$, the prove that : $\vec{a} \times \vec{b} = \vec{b} \times \vec{c} = \vec{c} \times \vec{a}$ and hence, show that $[\vec{a} \; \vec{b} \; \vec{c}] = 0$.

(C.B.S.E. Sample Paper 2018)

Answers

1. Scalar, scalar, vector, vector.
2. -10. **4.** $-90, -90$.
6. (*i*) $\lambda = 7$ (*ii*) $\lambda = 15$ (*iii*) $\lambda = \dfrac{26}{25}$ (*iv*) $\lambda = 0$.
8. $x = 5$. **10.** (*i*) 286 (*ii*) 5.
11. $c_3 = 2$. **12.** 0. **13.** Yes.
15. Vectors are coplanar.
16. (*i*) $2\left[\vec{a}\ \vec{b}\ \vec{c}\right]$. **18.** 0.

Hints to Selected Questions

5. Show that $\left[\vec{a}\ \vec{b}\ \vec{c}\right] = 0$.

6. (*ii*) Given vectors are coplanar if $\begin{vmatrix} 1 & -1 & 1 \\ 3 & 1 & 2 \\ 1 & \lambda & -3 \end{vmatrix} = 0$.

7. **As in Ex. 4.**

9. If A, B, C, D, are four given points, then show that $\left[\overrightarrow{AB}, \overrightarrow{BC}, \overrightarrow{CD}\right] \neq 0$.

10. Volume $= \left[\vec{a}\ \vec{b}\ \vec{c}\right]$.

16. (*i*) **LHS** $= \left(\vec{b}+\vec{c}\right).\left[\left(\vec{c}+\vec{a}\right)\times\left(\vec{a}+\vec{b}\right)\right]$

$$= \left(\vec{b}+\vec{c}\right).\left[\vec{c}\times\vec{a}+\vec{c}\times\vec{b}+\vec{a}\times\vec{a}+\vec{a}\times\vec{b}\right]$$

$$= \left(\vec{b}+\vec{c}\right).\left[\vec{c}\times\vec{a}+\vec{c}\times\vec{b}+\vec{a}\times\vec{b}\right]$$

$$= \vec{b}.\left(\vec{c}\times\vec{a}\right)+\vec{b}.\left(\vec{c}\times\vec{b}\right)+\vec{b}.\left(\vec{a}\times\vec{b}\right)+\ldots\ldots.$$

$$= \left[\vec{b}\,\vec{c}\,\vec{a}\right]+\left[\vec{b}\,\vec{c}\,\vec{b}\right]+\left[\vec{b}\,\vec{a}\,\vec{b}\right]+\ldots..$$

$$= \left[\vec{b}\,\vec{c}\,\vec{a}\right]+\left[\vec{c}\,\vec{a}\,\vec{b}\right] = 2\left[\vec{a}\,\vec{b}\,\vec{c}\right].$$

20. (*i*) $\vec{a}.\vec{a}' = \vec{a}.\left(\dfrac{\vec{b}\times\vec{c}}{\vec{a}.\vec{b}\times\vec{c}}\right) = \dfrac{\vec{a}.\left(\vec{b}\times\vec{c}\right)}{\vec{a}.\left(\vec{b}\times\vec{c}\right)} = 1$.

10.38. VECTOR TRIPLE PRODUCT

Definition

If $\vec{a}, \vec{b}$ and $\vec{c}$ are any three vectors, then the vector product of $\vec{a}\times\vec{b}$ with $\vec{c}$ is called the vector triple product.

Notation. *The vector triple product is written as* $(\vec{a}\times\vec{b})\times\vec{c}$.

PROPERTIES

Property. *If $\vec{a}, \vec{b}, \vec{c}$ be any three vectors, then (I)* $(\vec{a}\times\vec{b})\times\vec{c} = (\vec{a}.\vec{c})\,\vec{b}-(\vec{b}.\vec{c})\,\vec{a}$.

(II) $\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\,.\,\vec{c})\,\vec{b}-(\vec{a}\,.\,\vec{b})\,\vec{c}$

Aid to Memory

(I) (First × Second) × Third = (First . Third) Second – (Second . Third) First

(II) First × (Second × Third) = (First . Third) Second – (First . Second) Third

10.39. VECTOR PRODUCT OF FOUR VECTORS

Definition

If $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$ be any four vectors, then the vector product of $(\vec{a}\times\vec{b})$ and $(\vec{c}\times\vec{d})$ is called the vector product of four vectors and is written as $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d})$.

Theorem. (*i*) $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d}) = [\vec{a}\ \vec{b}\ \vec{d}]\,\vec{c}-[\vec{a}\ \vec{b}\ \vec{c}]\,\vec{d}$ (*ii*) $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d}) = [\vec{a}\ \vec{c}\ \vec{d}]\,\vec{b}-[\vec{b}\ \vec{c}\ \vec{d}]\,\vec{a}$.

NCERT-FILE

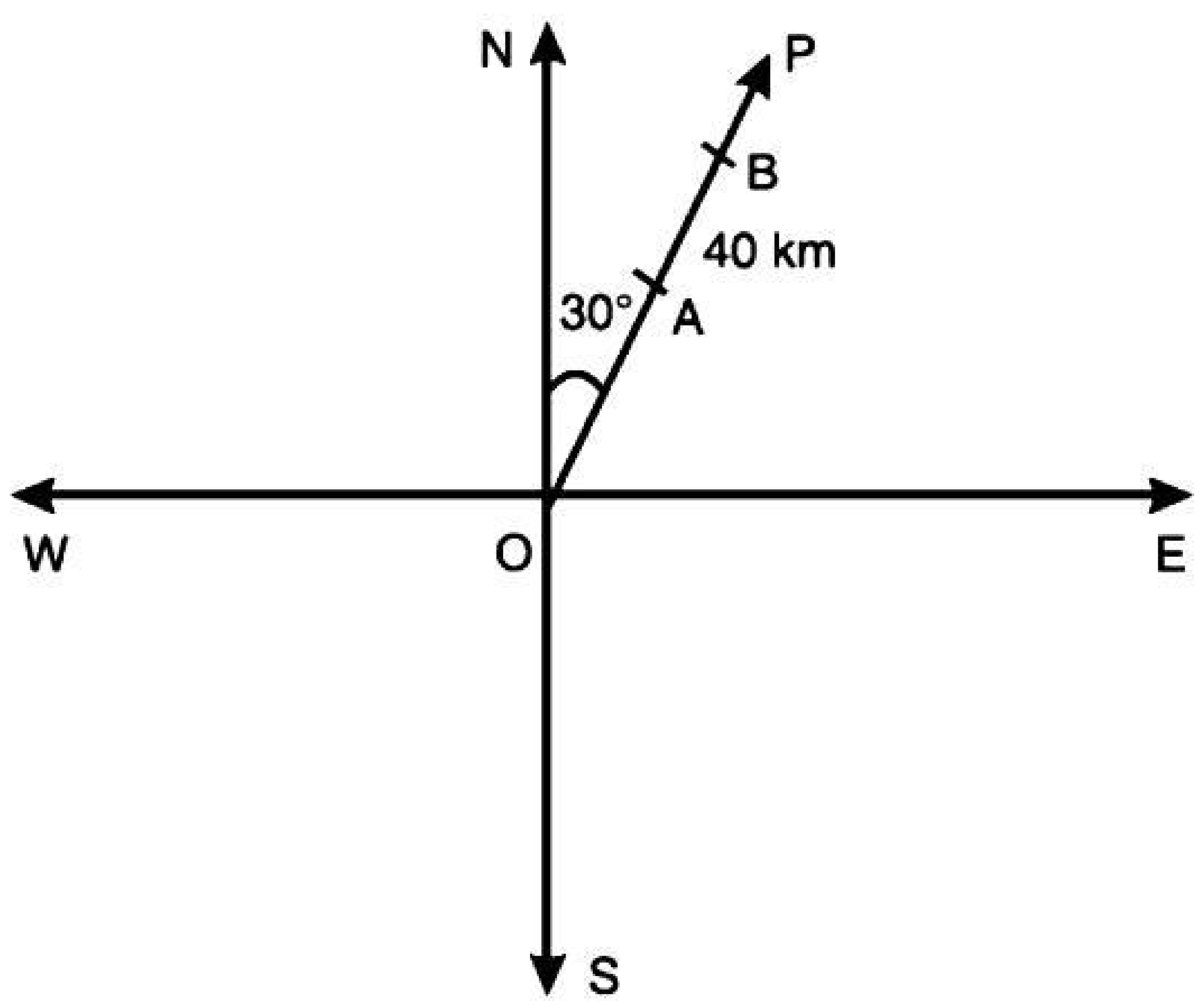
Questions from NCERT Book

(For each unsolved question, refer : "Solution of Modern's abc of Mathematics")

Exercise 10.1

1. Represent graphically a displacement of 40 km, 30° east of north.

Solution : The graphical representation of 40 km, 30° east of north is as below :



2. Classify the following measures as scalars and vectors :

(*i*) 10 kg (*ii*) 2 meters north-west
(*iii*) 40° (*iv*) 40 watt
(*v*) 10^{-19} coulomb (*vi*) 20 m/s^2.

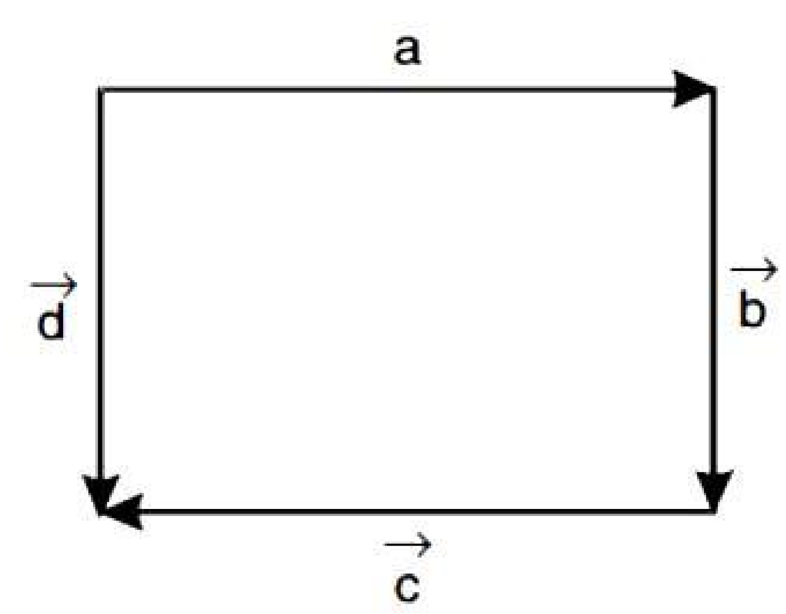
[**Solution.** Refer Q. 2 ; Ex. 10(*a*)]

3. Classify the following as scalar and vector quantities :

(*i*) time period (*ii*) distance
(*iii*) force (*iv*) velocity
(*v*) work done.

[**Solution :** Refer Q. 3 ; Ex. 10(*a*)]

4. In the Fig., identify the following vectors :



(*i*) Coinitial
(*ii*) Equal
(*iii*) Collinear but not equal

[**Solution.** Refer Q. 4 ; Ex. 10(*a*)]

5. Answer the following as true or false.

(*i*) $\vec{a}$ and $-\vec{a}$ are collinear.

(*ii*) Two collinear vectors are always equal in magnitude.

(*iii*) Two vectors having same magnitude are collinear.

(*iv*) Two collinear vectors having the same magnitude are equal.

[**Solution.** Refer Q. 6 ; Ex. 10(*a*)]

Exercise 10.2

1. Compute the magnitude of the following vectors :

$$\vec{a} = \hat{i} + \hat{j} + \hat{k};\ \vec{b} = 2\hat{i} - 7\hat{j} - 3\hat{k};$$

$$\vec{c} = \frac{1}{\sqrt{3}}\hat{i} + \frac{1}{\sqrt{3}}\hat{j} - \frac{1}{\sqrt{3}}\hat{k}.$$

Solution : We have :

$$\vec{a} = \hat{i} + \hat{j} + \hat{k} \text{ and } \vec{b} = 2\hat{i} - 7\hat{j} - 3\hat{k}.$$

$$\therefore \quad |\vec{a}| = |\hat{i} + \hat{j} + \hat{k}|$$

$$= \sqrt{(1)^2 + (1)^2 + (1)^2} = \sqrt{1+1+1} = \sqrt{3}$$

and $\quad |\vec{b}| = |2\hat{i} - 7\hat{j} - 3\hat{k}| = \sqrt{4+49+9} = \sqrt{62}$

and $\quad |\vec{c}| = \left|\frac{1}{\sqrt{3}}\hat{i} + \frac{1}{\sqrt{3}}\hat{j} - \frac{1}{\sqrt{3}}\hat{k}\right|$

$$= \sqrt{\frac{1}{3} + \frac{1}{3} + \frac{1}{3}} = \sqrt{1} = 1.$$

2. Write two different vectors having same magnitude.

3. Write two different vectors having same direction.

[**Solutions :1 (2-3)** Refer Q. 2 ; Ex. 10(*c*)]

4. Find the values of '*x*' and '*y*' so that the vectors $2\hat{i} + 3\hat{j}$ and $x\hat{i} + y\hat{j}$ are equal.

[**Solution.** Refer Q. 8(*i*) ; Ex. 10(*c*)]

5. Find the scalar and vector components of the vector with initial point (2, 1) and terminal point (–5, 7).

Solution : Let A (2, 1) and B (– 5, 7) be the initial and terminal points respectively.

$$\therefore \quad \overrightarrow{AB} = \text{P.V. of B} - \text{P.V. of A}$$

$$= (-5\hat{i} + 7\hat{j}) - (2\hat{i} + \hat{j})$$

$$= -7\hat{i} + 6\hat{j}.$$

Hence, – 7 and 6 are scalar components and $-7\hat{i}$ and $6\hat{j}$ are vector components of $\overrightarrow{AB}$.

6. Find the sum of the vectors

$\vec{a} = \hat{i} - 2\hat{j} + \hat{k}, \vec{b} = -2\hat{i} + 4\hat{j} + 5\hat{k}$, and

$\vec{c} = \hat{i} - 6\hat{j} - 7\hat{k}$.

[**Solution.** Refer Q. 3 ; Ex. 10(*c*)]

7. Find the unit vector in the direction of the vector

$\vec{a} = \hat{i} + \hat{j} + 2\hat{k}$.

[**Solution.** Refer Q. 6(*a*) ; Ex. 10(*c*)]

8. Find the unit vector in the direction of vector $\overrightarrow{PQ}$, where P and Q are the points (1, 2, 3) and (4, 5, 6), respectively.

[**Solution.** Refer Q. 7 ; Ex. 10(*c*)]

9. For given vectors $\vec{a} = 2\hat{i} - \hat{j} + 2\hat{k}$ and $\vec{b} = -\hat{i} + \hat{j} - \hat{k}$, find the unit vector in the direction of the vector $\vec{a} + \vec{b}$.

[**Solution.** Refer Q. 12 ; Ex. 10(*c*)]

10. Find a vector in the direction of vector $5\hat{i} - \hat{j} + 2\hat{k}$, which has magnitude 8 units.

Solution. The given vector is :

$$\vec{a} = 5\hat{i} - \hat{j} + 2\hat{k}.$$

$$\therefore |\vec{a}| = \sqrt{5^2 + (-1)^2 + 2^2}$$

$$= \sqrt{25+1+4}$$

$$= \sqrt{30}.$$

$\therefore$ Unit vector in the direction of given vector $\vec{a}$

$$= \frac{\vec{a}}{|\vec{a}|} = \frac{1}{\sqrt{30}}(5\hat{i} - \hat{j} + 2\hat{k}).$$

$\therefore$ Vector of magnitude 8 in the direction of vector $\vec{a}$

$$= 8\frac{\vec{a}}{|\vec{a}|} = 8.\frac{1}{\sqrt{30}}(5\hat{i} - \hat{j} + 2\hat{k})$$

$$= \frac{40}{\sqrt{30}}\hat{i} - \frac{8}{\sqrt{30}}\hat{j} + \frac{16}{\sqrt{30}}\hat{k}.$$

11. Show that the vectors $2\hat{i} - 3\hat{j} + 4\hat{k}$ and $-4\hat{i} + 6\hat{j} - 8\hat{k}$ are collinear.

[**Solution.** Refer Q. 26(*b*) ; Ex. 10(*c*)]

12. Find the direction cosines of the vector $\hat{i} + 2\hat{j} + 3\hat{k}$.

[**Solution.** Refer Q. 10 ; Ex. 10(*c*)]

13. Find the direction cosines of the vector joining the points A (1, 2, –3) and B(–1, –2, 1), directed from A to B.

Solution : Vector joining the points A and B

$$= \overrightarrow{AB} = \text{P.V. of B} - \text{P.V. of A}$$

$$= (-\hat{i} - 2\hat{j} + \hat{k}) - (\hat{i} + 2\hat{j} - 3\hat{k})$$

$$= -2\hat{i} - 4\hat{j} + 4\hat{k}.$$

$\therefore$ Direction–cosines of $\overrightarrow{AB}$ are :

$$< \frac{-2}{\sqrt{4+16+16}}, \frac{-4}{\sqrt{4+16+16}}, \frac{4}{\sqrt{4+16+16}} >$$

i.e. $< \frac{-2}{6}, \frac{-4}{6}, \frac{4}{6} >$ *i.e.* $< \frac{-1}{3}, \frac{-2}{3}, \frac{2}{3} >$.

14. Show that the vector $\hat{i} + \hat{j} + \hat{k}$ is equally inclined to the axes OX, OY and OZ.

[**Solution.** Refer Q. 11(*ii*) ; Ex. 10(*c*)]

15. Find the position vector of a point R, which divides the line joining two points P and Q whose position vectors are $\hat{i} + 2\hat{j} - \hat{k}$ and $-\hat{i} + \hat{j} + \hat{k}$ respectively, in the ratio 2 : 1

(*i*) internally (*ii*) externally.

Solution : (*i*) The position vector of a point R, which divides the line joining internally in the ratio 2 : 1 is given by :

$$R(\vec{r}) = \frac{2(-\hat{i} + \hat{j} + \hat{k}) + 1.(\hat{i} + 2\hat{j} - \hat{k})}{2+1}$$

$$= \frac{-\hat{i} + 4\hat{j} + \hat{k}}{3}$$

$$= \frac{-\hat{i}}{3} + \frac{4}{3}\hat{j} + \frac{1}{3}\hat{k}.$$

(*ii*) The position vector of a point S, which divides the line joining P and Q externally in the ratio 2 : 1 is given by :

$$S(\vec{r}) = \frac{2(-\hat{i} + \hat{j} + \hat{k}) - 1.(\hat{i} + 2\hat{j} - \hat{k})}{2-1}$$

$$= -3\hat{i} + 3\hat{k}.$$

16. Find the position vector of the mid-point of the vector joining the points P(2, 3, 4) and Q(4, 1, –2).

[**Solution :** Refer Q. 14(*i*) ; Ex. 10(*c*)]

17. Show that the points A, B and C with position vectors, $\vec{a} = 3\hat{i} - 4\hat{j} - 4\hat{k}$, $\vec{b} = 2\hat{i} - \hat{j} + \hat{k}$ and $\vec{c} = \hat{i} - 3\hat{j} - 5\hat{k}$, respectively form the vertices a right angled triangle.

Solution : Here

$$\overrightarrow{AB} = \vec{b} - \vec{a}$$

$$= (2\hat{i} - \hat{j} + \hat{k}) - (3\hat{i} - 4\hat{j} - 4\hat{k})$$

$$= -\hat{i} + 3\hat{j} + 5\hat{k},$$

$$\overrightarrow{BC} = \vec{c} - \vec{b} = (\hat{i} - 3\hat{j} - 5\hat{k}) - (2\hat{i} - \hat{j} + \hat{k})$$

$$= -\hat{i} - 2\hat{j} - 6\hat{k}$$

and $$\overrightarrow{CA} = \vec{a} - \vec{c}$$

$$= (3\hat{i} - 4\hat{j} - 4\hat{k}) - (\hat{i} - 3\hat{j} - 5\hat{k})$$

$$= 2\hat{i} - \hat{j} + \hat{k}.$$

$$\therefore \quad |\overrightarrow{AB}| = \sqrt{(-1)^2 + 3^2 + 5^2}$$

$$= \sqrt{1+9+25} = \sqrt{35},$$

$$|\overrightarrow{BC}| = \sqrt{(-1)^2 + (-2)^2 + (-6)^2}$$

$$= \sqrt{1+4+36} = \sqrt{41}$$

and $$|\overrightarrow{CA}| = \sqrt{2^2 + (-1)^2 + 1^2}$$

$$= \sqrt{4+1+1} = \sqrt{6}.$$

Clearly $|\overrightarrow{BC}|^2 = |\overrightarrow{AB}|^2 + |\overrightarrow{CA}|^2$. $[\because 41 = 35 + 6]$

Hence, ΔABC is a right–angled triangle.

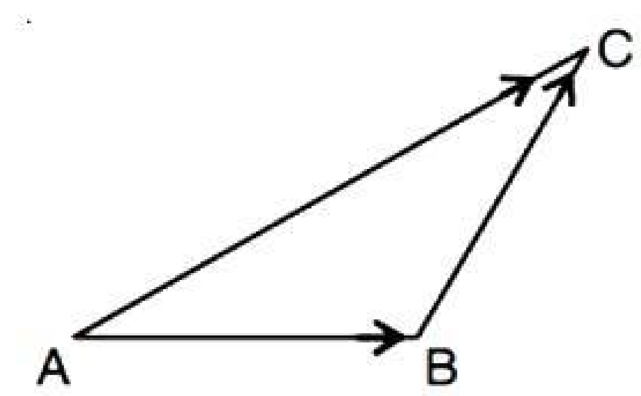
18. In triangle ABC, which of the following is not true :

(A) $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = \vec{0}$

(B) $\overrightarrow{AB} + \overrightarrow{BC} - \overrightarrow{AC} = \vec{0}$

(C) $\overrightarrow{AB} + \overrightarrow{BC} - \overrightarrow{CA} = \vec{0}$

(D) $\overrightarrow{AB} - \overrightarrow{CB} + \overrightarrow{CA} = \vec{0}$.



[**Ans.** (C)]

19. If $\vec{a}$ and $\vec{b}$ are two collinear vectors, then which of the following are incorrect :

(A) $\vec{b} = \lambda \vec{a}$, for some scalar λ

(B) $\vec{a} = \pm \vec{b}$

(C) the respective components of $\vec{a}$ and $\vec{b}$ are not proportional

(D) both the vectors $\vec{a}$ and $\vec{b}$ have same direction, but different magnitudes.

[**Ans.** (D)]

Exercise 10.3

1. Find the angle between two vectors $\vec{a}$ and $\vec{b}$ with magnitudes $\sqrt{3}$ and 2, respectively having $\vec{a} \cdot \vec{b} = \sqrt{6}$.

Solution : If 'θ' be the angle between $\vec{a}$ and $\vec{b}$, then

$$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|} = \frac{\sqrt{6}}{\sqrt{3}(2)} = \frac{(\sqrt{3})(\sqrt{2})}{(\sqrt{3})(2)}$$

$$= \frac{1}{\sqrt{2}} = \cos\frac{\pi}{4}.$$

Hence, $\quad$ 'θ' $= \dfrac{\pi}{4}$.

2. Find the angle between the vectors $\hat{i} - 2\hat{j} + 3\hat{k}$ and $3\hat{i} - 2\hat{j} + \hat{k}$.

[**Solution.** Refer Q. 8(*iii*); Ex. 10(*e*)]

3. Find the projection of the vector $\hat{i} - \hat{j}$ on the vector $\hat{i} + \hat{j}$.

[**Solution.** Refer Q. 3(*b*) ; Ex. 10(*e*)]

4. Find the projection of the vector $\hat{i}+3\hat{j}+7\hat{k}$ on the vector $7\hat{i}-\hat{j}+8\hat{k}$.

Solution : Let $\vec{a} = \hat{i}+3\hat{j}+7\hat{k}$ and

$$\vec{b} = 7\hat{i}-\hat{j}+8\hat{k}.$$

$$\therefore \quad |\vec{b}| = \sqrt{7^2+(-1)^2+8^2}$$

$$= \sqrt{49+1+64} = \sqrt{114}$$

and $\vec{a}.\vec{b} = (\hat{i}+3\hat{j}+7\hat{k}).(7\hat{i}-\hat{j}+8\hat{k})$

$$= (1)(7)+(3)(-1)+(7)(8)$$

$$= 7-3+56 = 60.$$

$$\therefore \text{ Projection of } \vec{a} \text{ on } \vec{b} = \frac{\vec{a}.\vec{b}}{|\vec{b}|} = \frac{60}{\sqrt{114}}.$$

5. Show that each of the given three vectors is a unit vector :

$$\frac{1}{7}(2\hat{i}+3\hat{j}+6\hat{k}),\ \frac{1}{7}(3\hat{i}-6\hat{j}+2\hat{k}),\ \frac{1}{7}(6\hat{i}+2\hat{j}-3\hat{k})$$

Also, show that they are mutually perpendicular to each other. *(Jammu B. 2015 W)*

Solution : Let $\vec{a} = \frac{1}{7}(2\hat{i}+3\hat{j}+6\hat{k})$,

$$\vec{b} = \frac{1}{7}(3\hat{i}-6\hat{j}+2\hat{k}) \text{ and}$$

$$\vec{c} = \frac{1}{7}(6\hat{i}+2\hat{j}-3\hat{k}).$$

$$\therefore \quad |\vec{a}| = \sqrt{\left(\frac{2}{7}\right)^2+\left(\frac{3}{7}\right)^2+\left(\frac{6}{7}\right)^2}$$

$$= \sqrt{\frac{4}{49}+\frac{9}{49}+\frac{36}{49}} = \sqrt{\frac{49}{49}} = \sqrt{1} = 1,$$

$$|\vec{b}| = \sqrt{\left(\frac{3}{7}\right)^2+\left(\frac{-6}{7}\right)^2+\left(\frac{2}{7}\right)^2}$$

$$= \sqrt{\frac{9}{49}+\frac{36}{49}+\frac{4}{49}} = \sqrt{\frac{49}{49}} = \sqrt{1} = 1$$

and $|\vec{c}| = \sqrt{\left(\frac{6}{7}\right)^2+\left(\frac{2}{7}\right)^2+\left(\frac{-3}{7}\right)^2}$

$$= \sqrt{\frac{36}{49}+\frac{4}{49}+\frac{9}{49}} = \sqrt{\frac{49}{49}} = \sqrt{1} = 1.$$

Hence $\vec{a}, \vec{b}, \vec{c}$ are unit vectors.

Now $\vec{a}.\vec{b} = \frac{1}{49}[(2)(3)+(3)(-6)+(6)(2)]$

$$= \frac{1}{49}[6-18+12] = 0.$$

Then $\vec{a}$ is perp. to $\vec{b}$.

Again $\vec{b}.\vec{c} = \frac{1}{49}[(3)(6)+(-6)(2)+(2)(-3)]$

$$= \frac{1}{49}[18-12-6] = 0.$$

Thus $\vec{b}$ is perp. to $\vec{c}$.

Lastly $\vec{c}.\vec{a} = \frac{1}{49}[(6)(2)+(2)(3)+(-3)(6)]$

$$= \frac{1}{49}[12+6-18] = 0.$$

Thus $\vec{c}$ is perp. to $\vec{a}$.

Hence, $\vec{a}$, $\vec{b}$ and $\vec{c}$ are three mutually perpendicular unit vectors.

6. Find $|\vec{a}|$ and $|\vec{b}|$, if $(\vec{a}+\vec{b}).(\vec{a}-\vec{b}) = 8$ and $|\vec{a}| = 8|\vec{b}|$.

Solution : We have : $(\vec{a}+\vec{b}).(\vec{a}-\vec{b}) = 8$

$$\Rightarrow \quad \vec{a}.\vec{a}-\vec{a}.\vec{b}+\vec{b}.\vec{a}-\vec{b}.\vec{b} = 8$$

$$\Rightarrow \quad \vec{a}.\vec{a}-\vec{b}.\vec{b} = 8 \qquad [\because \vec{a}.\vec{b} = \vec{b}.\vec{a}]$$

$$\Rightarrow \quad |\vec{a}|^2-|\vec{b}|^2 = 8$$

$$\Rightarrow \quad 64|\vec{b}|^2-|\vec{b}|^2 = 8 \qquad [\because |\vec{a}| = 8|\vec{b}|]$$

$$\Rightarrow \quad 63|\vec{b}|^2 = 8 \quad \Rightarrow \quad |\vec{b}|^2 = \frac{8}{63}$$

$$\Rightarrow \quad |\vec{b}| = \sqrt{\frac{8}{63}} \qquad [\text{Taking +ve value}]$$

and $|\vec{a}| = 8|\vec{b}| = 8\sqrt{\frac{8}{63}}$.

Hence, $|\vec{a}| = 8\sqrt{\frac{8}{63}}$ and $|\vec{b}| = \sqrt{\frac{8}{63}}$.

7. Evaluate the product $(3\vec{a}-5\vec{b}).(2\vec{a}+7\vec{b})$.

[**Solution.** Refer Q. 5(*a*) ; Ex. 10(*e*)]

8. Find the magnitude of two vectors $\vec{a}$ and $\vec{b}$, having the same magnitude and such that the angle between them is 60° and their scalar product is $\frac{1}{2}$.

[**Solution.** Refer Q. 11 ; Ex. 10(*e*)]

9. Find $|\vec{x}|$, if for a unit vector $\vec{a}$, $(\vec{x}-\vec{a}).(\vec{x}+\vec{a})=12$.

[**Solution.** Refer Q. 7 ; Ex. 10(*e*)]

10. If $\vec{a}=2\hat{i}+2\hat{j}+3\hat{k}$, $\vec{b}=-\hat{i}+2\hat{j}+\hat{k}$ and $\vec{c}=3\hat{i}+\hat{j}$ are such that $\vec{a}+\lambda\vec{b}$ is perpendicular to $\vec{c}$, then find the value of λ.

Solution : Here $\vec{a} = 2\hat{i}+2\hat{j}+3\hat{k}$,

$\vec{b} = -\hat{i}+2\hat{j}+\hat{k}$ and $\vec{c} = 3\hat{i}+\hat{j}$.

Now $\vec{a}+\lambda\vec{b}$

$= (2\hat{i}+2\hat{j}+3\hat{k})+\lambda(-\hat{i}+2\hat{j}+\hat{k})$

$= (2-\lambda)\hat{i}+(2+2\lambda)\hat{j}+(3+\lambda)\hat{k}$.

Since $(\vec{a}+\lambda\vec{b})$ is perpendicular to $\vec{c}$,

$\therefore \quad (\vec{a}+\lambda\vec{b}).\vec{c} = 0$

$\Rightarrow (2-\lambda)(3)+(2+2\lambda)(1)+(3+\lambda)(0) = 0$

$\Rightarrow \quad 6-3\lambda+2+2\lambda = 0.$

Hence, $\lambda = 8$.

11. Show that $|\vec{a}|\vec{b}+|\vec{b}|\vec{a}$ is perpendicular to $|\vec{a}|\vec{b}-|\vec{b}|\vec{a}$, for any two non-zero vectors $\vec{a}$ and $\vec{b}$.

[**Solution.** Refer Q. 22 ; Ex. 10(*e*)]

12. If $\vec{a}.\vec{a}=0$ and $\vec{a}.\vec{b}=0$, then what can be concluded about the vector $\vec{b}$?

[**Solution.** Refer Q. 12 (*a*) ; Ex. 10(*e*)]

13. If $\vec{a}, \vec{b}, \vec{c}$ are unit vectors such that $\vec{a}+\vec{b}+\vec{c}=\vec{0}$, find the value of $\vec{a}.\vec{b}+\vec{b}.\vec{c}+\vec{c}.\vec{a}$.

Solution : We have : $|\vec{a}| = |\vec{b}| = |\vec{c}| = 1$ ...(1)

Now $\vec{a}+\vec{b}+\vec{c} = \vec{0}$ ...(2) [*Given*]

Squaring, $(\vec{a}+\vec{b}+\vec{c})^2 = 0$

$\Rightarrow |\vec{a}|^2+|\vec{b}|^2+|\vec{c}|^2+2(\vec{a}.\vec{b}+\vec{b}.\vec{c}+\vec{c}.\vec{a}) = 0$

$\Rightarrow (1)^2+(1)^2+(1)^2+2(\vec{a}.\vec{b}+\vec{b}.\vec{c}+\vec{c}.\vec{a}) = 0.$

Hence $\vec{a}.\vec{b}+\vec{b}.\vec{c}+\vec{c}.\vec{a} = -\frac{3}{2}$.

14. If either vector $\vec{a}=\vec{0}$ or $\vec{b}=\vec{0}$, then $\vec{a}.\vec{b}=\vec{0}$. But the converse need not be true. Justify your answer with an example.

[**Solution.** Refer Q. 12 (*b*) ; Ex. 10(*e*)]

15. If the vertices A, B, C of a triangle ABC are (1, 2, 3), (–1, 0, 0), (0, 1, 2), respectively, then find ∠ABC. [∠ABC is the angle between the vectors $\overrightarrow{BA}$ and $\overrightarrow{BC}$].

(Kashmir B. 2017)

Solution : If O be the origin, then :

$\overrightarrow{OA} = \hat{i}+2\hat{j}+3\hat{k}$, $\overrightarrow{OB} = -\hat{i}$ and $\overrightarrow{OC} = \hat{j}+2\hat{k}$.

$$\therefore \quad \overrightarrow{BC} = \overrightarrow{OC}-\overrightarrow{OB} = (\hat{j}+2\hat{k})-(-\hat{i}) = \hat{i}+\hat{j}+2\hat{k}$$

and
$$\overrightarrow{BA} = \overrightarrow{OA}-\overrightarrow{OB} = (\hat{i}+2\hat{j}+3\hat{k})-(-\hat{i}) = 2\hat{i}+2\hat{j}+3\hat{k}.$$

$$\therefore \cos\angle ABC = \frac{\overrightarrow{BC}.\overrightarrow{BA}}{|\overrightarrow{BC}||\overrightarrow{BA}|}$$

$$= \frac{(\hat{i}+\hat{j}+2\hat{k}).(2\hat{i}+2\hat{j}+3\hat{k})}{\sqrt{1^2+1^2+2^2}\sqrt{2^2+2^2+3^2}}$$

$$= \frac{(1)(2)+(1)(2)+(2)(3)}{\sqrt{1+1+4}\sqrt{4+4+9}}$$

$$= \frac{2+2+6}{\sqrt{6}\sqrt{17}} = \frac{10}{\sqrt{102}}.$$

Hence, $\angle ABC = \cos^{-1}\left(\frac{10}{\sqrt{102}}\right)$.

16. Show that the points A(1, 2, 7), B(2, 6, 3) and C (3, 10, –1) are collinear.

Solution : If O be the origin,

then $\overrightarrow{OA} = \hat{i}+2\hat{j}+7\hat{k}$,

$\overrightarrow{OB} = 2\hat{i}+6\hat{j}+3\hat{k}$

and $\overrightarrow{OC} = 3\hat{i}+10\hat{j}-\hat{k}$.

$$\therefore \quad \overrightarrow{AB} = \overrightarrow{OB}-\overrightarrow{OA}$$

$$= (2\hat{i}+6\hat{j}+3\hat{k})-(\hat{i}+2\hat{j}+7\hat{k})$$

$$= \hat{i}+4\hat{j}-4\hat{k},$$

$$\overrightarrow{BC} = \overrightarrow{OC}-\overrightarrow{OB}$$

$$= (3\hat{i}+10\hat{j}-\hat{k})-(2\hat{i}+6\hat{j}+3\hat{k})$$

$$= \hat{i}+4\hat{j}-4\hat{k}$$

and $\overrightarrow{AC} = \overrightarrow{OC}-\overrightarrow{OA}$

$$= (3\hat{i}+10\hat{j}-\hat{k})-(\hat{i}+2\hat{j}+7\hat{k})$$

$$= 2\hat{i}+8\hat{j}-8\hat{k}$$

$$= 2(\hat{i}+4\hat{j}-4\hat{k}).$$

$$\therefore \quad AB = |\overrightarrow{AB}| = \sqrt{1^2+4^2+(-4)^2}$$

$$= \sqrt{1+16+16} = \sqrt{33},$$

$$BC = |\overrightarrow{BC}| = \sqrt{1^2+4^2+(-4)^2}$$

$$= \sqrt{1+16+16} = \sqrt{33}$$

and $AC = |\overrightarrow{AC}| = \sqrt{2^2+8^2+(-8)^2}$

$$= \sqrt{4+64+64}$$

$$= \sqrt{132}$$

$$= 2\sqrt{33}.$$

Clearly AB + BC = AC.

Hence, A, B, C are collinear.

17. Show that the vectors $2\hat{i}-\hat{j}+\hat{k}, \hat{i}-3\hat{j}-5\hat{k}$ and $3\hat{i}-4\hat{j}-4\hat{k}$ form the vertices of a right angled triangle.

Solution : Let the position vectors of vertices A, B and C be $2\hat{i}-\hat{j}+\hat{k}, \hat{i}-3\hat{j}-5\hat{k}$ and $3\hat{i}-4\hat{j}-4\hat{k}$ respectively.

$$\therefore \quad \overrightarrow{AB} = \overrightarrow{OB}-\overrightarrow{OA}$$

$$= (\hat{i}-3\hat{j}-5\hat{k})-(2\hat{i}-\hat{j}+\hat{k})$$

$$= -\hat{i}-2\hat{j}-6\hat{k}.$$

$$\therefore \quad AB = |\overrightarrow{AB}|$$

$$= \sqrt{(-1)^2+(-2)^2+(-6)^2}$$

$$= \sqrt{1+4+36} = \sqrt{41},$$

$$\overrightarrow{BC} = \overrightarrow{OC}-\overrightarrow{OB}$$

$$= (3\hat{i}-4\hat{j}-4\hat{k})-(\hat{i}-3\hat{j}-5\hat{k})$$

$$= 2\hat{i}-\hat{j}+\hat{k}.$$

$$\therefore \quad BC = |\overrightarrow{BC}| = \sqrt{2^2+(-1)^2+1^2}$$

$$= \sqrt{4+1+1} = \sqrt{6}$$

and $\overrightarrow{AC} = \overrightarrow{OC}-\overrightarrow{OA}$

$$= (3\hat{i}-4\hat{j}-4\hat{k})-(2\hat{i}-\hat{j}+\hat{k})$$

$$= \hat{i}-3\hat{j}-5\hat{k}.$$

$$\therefore \quad AC = |\overrightarrow{AC}| = \sqrt{1^2+(-3)^2+(-5)^2}$$

$$= \sqrt{1+9+25} = \sqrt{35}.$$

Now $BC^2 + AC^2 = 6 + 35 = 41 = AB$.

Hence, ΔABC is a right angled triangle.

18. If $\vec{a}$ is a non-zero vector of magnitude '*a*' and λ a nonzero scalar, then $\lambda\vec{a}$ is unit vector if :

(A) $\lambda = 1$ (B) $\lambda = -1$

(C) $a = |\lambda|$ (D) $a = 1/|\lambda|$.

[**Ans.** (D).]

Exercise 10.4

1. Find $|\vec{a}\times\vec{b}|$, if $\vec{a}=\hat{i}-7\hat{j}+7\hat{k}$ and $\vec{b}=3\hat{i}-2\hat{j}+2\hat{k}$.

Solution : $\vec{a}\times\vec{b}=(\hat{i}-7\hat{j}+7\hat{k})\times(3\hat{i}-2\hat{j}+2\hat{k})$

$$=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & -7 & 7\\ 3 & -2 & 2\end{vmatrix}$$

$$=\hat{i}(-14+14)-\hat{j}(2-21)+\hat{k}(-2+21)$$

$$=19\hat{j}+19\hat{k}.$$

$\therefore\ |\vec{a}\times\vec{b}|=\sqrt{0^2+(19)^2+(19)^2}=19\sqrt{2}.$

2. Find a unit vector perpendicular to each of the vector $\vec{a}+\vec{b}$ and $\vec{a}-\vec{b}$, where $\vec{a}=3\hat{i}+2\hat{j}+2\hat{k}$ and $\vec{b}=\hat{i}+2\hat{j}-2\hat{k}$.

Solution : We have :

$$\vec{a}=3\hat{i}+2\hat{j}+2\hat{k} \text{ and } \vec{b}=\hat{i}+2\hat{j}-2\hat{k}.$$

$$\therefore\quad \vec{a}+\vec{b}=(3\hat{i}+2\hat{j}+2\hat{k})+(\hat{i}+2\hat{j}-2\hat{k})$$

$$=4\hat{i}+4\hat{j}$$

and $$\vec{a}-\vec{b}=(3\hat{i}+2\hat{j}+2\hat{k})-(\hat{i}+2\hat{j}-2\hat{k})$$

$$=2\hat{i}+4\hat{k}.$$

$$\therefore\ (\vec{a}+\vec{b})\times(\vec{a}-\vec{b})=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 4 & 4 & 0\\ 2 & 0 & 4\end{vmatrix}$$

$$=(16-0)\hat{i}-(16-0)\hat{j}+(0-8)\hat{k}$$

$$=16\hat{i}-16\hat{j}-8\hat{k}.$$

$\therefore$Unit vector perpendicular to both $(\vec{a}+\vec{b})$ and $(\vec{a}-\vec{b})$ is given by :

$$\hat{n}=\pm\frac{(\vec{a}+\vec{b})\times(\vec{a}-\vec{b})}{|(\vec{a}+\vec{b})\times(\vec{a}-\vec{b})|}$$

$$=\pm\frac{16\hat{i}-16\hat{j}-8\hat{k}}{\sqrt{(16)^2+(-16)^2+(-8)^2}}$$

$$=\pm\frac{8(2\hat{i}-2\hat{j}-\hat{k})}{8\sqrt{4+4+1}}=\pm\frac{2\hat{i}-2\hat{j}-\hat{k}}{3}$$

$$=\pm\frac{2}{3}\hat{i}\mp\frac{2}{3}\hat{j}\mp\frac{1}{3}\hat{k}.$$

3. If a unit vector $\vec{a}$ make angles $\frac{\pi}{3}$ with $\hat{i}$, $\frac{\pi}{4}$ with $\hat{j}$ and an acute angle θ with $\hat{k}$, then find 'θ' and hence, the components of $\vec{a}$.

[**Solution.** Refer Q. 26 ; Ex. 10(*h*)]

4. Show that

$$(\vec{a}-\vec{b})\times(\vec{a}+\vec{b})=2(\vec{a}\times\vec{b}).$$

[**Solution.** Refer Q. 21(*iii*) ; Ex. 10(*h*)]

5. Find 'λ' and 'μ' if $(2\hat{i}+6\hat{j}+27\hat{k})\times(\hat{i}+\lambda\hat{j}+\mu\hat{k})=\vec{0}$.

Solution : Let $\vec{a}=2\hat{i}+6\hat{j}+27\hat{k}$ and $\vec{b}=\hat{i}+\lambda\hat{j}+\mu\hat{k}$.

$$\therefore\quad \vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & 6 & 27\\ 1 & \lambda & \mu\end{vmatrix}$$

$$=\hat{i}(6\mu-27\lambda)-\hat{j}(2\mu-27)+\hat{k}(2\lambda-6)$$

$$=(6\mu-27\lambda)\hat{i}+(27-2\mu)\hat{j}+(2\lambda-6)\hat{k}.$$

By the question, $\vec{a}\times\vec{b}=\vec{0}$

$\Rightarrow (6\mu-27\lambda)\hat{i}+(27-2\mu)\hat{j}+(2\lambda-6)\hat{k}=0$

$\Rightarrow 6\mu-27\lambda=0$ ...(1) $\quad 27-2\mu=0$ ...(2)

and $(2\lambda-6)=0$ ...(3)

From (3), $\lambda=3$. From (2), $\mu=\frac{27}{2}$.

These satisfy (1). $\quad [\because\ 6\left(\frac{27}{2}\right)-27(3)=0]$

Hence, $\lambda=3$ and $\mu=\frac{27}{2}$.

6. Given that $\vec{a}.\vec{b}=0$ and $\vec{a}\times\vec{b}=\vec{0}$. What can you conclude about the vectors $\vec{a}$ and $\vec{b}$?
[**Solution.** Refer Q. 21(*iv*) ; Ex. 10(*h*)]

7. Let the vectors $\vec{a},\vec{b},\vec{c}$ be given as :

$a_1\hat{i}+a_2\hat{j}+a_3\hat{k},\ b_1\hat{i}+b_2\hat{j}+b_3\hat{k},\ c_1\hat{i}+c_2\hat{j}+c_3\hat{k}$.

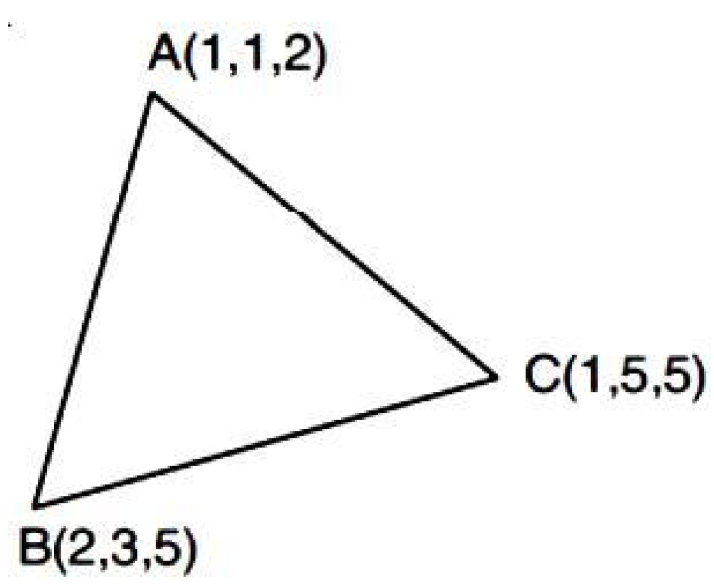
Then show that $\vec{a}\times(\vec{b}+\vec{c})=\vec{a}\times\vec{b}+\vec{a}\times\vec{c}$.
[**Solution.** Refer Q. 7 ; Ex. 10(*h*)]

8. If either $\vec{a}=\vec{0}$ or $\vec{b}=\vec{0}$, then $\vec{a}\times\vec{b}=\vec{0}$. Is the converse true ? Justify your answer with an example.
[**Solution.** Refer Q. 23 ; Ex. 10(*h*)]

9. Find the area of the triangle with vertices A(1, 1, 2), B(2, 3, 5) and C(1, 5, 5).

Solution : Here :

$$\overrightarrow{BC} = (\hat{i}+5\hat{j}+5\hat{k})-(2\hat{i}+3\hat{j}+5\hat{k})$$
$$= -\hat{i}+2\hat{j}$$



$$\overrightarrow{BA} = (\hat{i}+\hat{j}+2\hat{k})-(2\hat{i}+3\hat{j}+5\hat{k})$$
$$= -\hat{i}-2\hat{j}-3\hat{k}.$$

$$\therefore\ \overrightarrow{BC}\times\overrightarrow{BA} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 2 & 0 \\ -1 & -2 & -3 \end{vmatrix}$$

$$= (-6+0)\hat{i}-(3+0)\hat{j}+(2+2)\hat{k}$$
$$= -6\hat{i}-3\hat{j}+4\hat{k}.$$

$$\therefore\ |\overrightarrow{BC}\times\overrightarrow{BA}| = \sqrt{36+9+16} = \sqrt{61}.$$

Hence, area of ΔABC $= \frac{1}{2}|\overrightarrow{BC}\times\overrightarrow{BA}|$

$$= \frac{1}{2}(\sqrt{61}) \text{ sq. units.}$$

10. Find the area of the parallelogram whose adjacent sides are determined by the vectors $\vec{a}=\hat{i}-\hat{j}+3\hat{k}$ and $\vec{b}=2\hat{i}-7\hat{j}+\hat{k}$.
[**Solution.** Refer Q. 18(*iv*) ; Ex. 10(*h*)]

11. Let the vectors $\vec{a}$ and $\vec{b}$ be such that $|\vec{a}|=3$ and $|\vec{b}|=\frac{\sqrt{2}}{3}$, then $\vec{a}\times\vec{b}$ is a unit vector, if the angle between $\vec{a}$ and $\vec{b}$ is :

(A) $\pi/6$ (B) $\pi/4$
(C) $\pi/3$ (D) $\pi/2$.

[**Ans.** (B)]

12. Area of a rectangle having vertices A, B, C and D with position vectors :

$-\hat{i}+\frac{1}{2}\hat{j}+4\hat{k},\ \hat{i}+\frac{1}{2}\hat{j}+4\hat{k},\ \hat{i}\frac{1}{2}\hat{j}+4\hat{k}$ and

$-\hat{i}-\frac{1}{2}\hat{j}+4\hat{k}$, respectively is :

(A) $\frac{1}{2}$ (B) 1
(C) 2 (D) 4.

[**Ans.** (C)]

Exercise 10.5 (Supplement)

1. Find $[\vec{a}\,\vec{b}\,\vec{c}]$ if $\vec{a}=\hat{i}-2\hat{j}+3\hat{k}$, $\vec{b}=2\hat{i}-3\hat{j}+\hat{k}$ and $\vec{c}=3\hat{i}+\hat{j}-2\hat{k}$.

Solution. We have : $\vec{a}=\hat{i}-2\hat{j}+3\hat{k}$, $\vec{b}=2\hat{i}-3\hat{j}+\hat{k}$ and $\vec{c}=3\hat{i}+\hat{j}-2\hat{k}$.

$$\therefore\ [\vec{a}\,\vec{b}\,\vec{c}] = \begin{vmatrix} 1 & -2 & 3 \\ 2 & -3 & 1 \\ 3 & 1 & -2 \end{vmatrix}$$

$$= 1.(6-1)+2(-4-3)+3(2+9)$$
$$= 5-14+33 = 24.$$

2. Show that the vectors :
$\vec{a}=\hat{i}-2\hat{j}+3\hat{k}$, $\vec{b}=-2\hat{i}+3\hat{j}-4\hat{k}$ and $\vec{c}=\hat{i}-3\hat{j}+5\hat{k}$ are coplanar.

Solution. Here $[\vec{a}\,\vec{b}\,\vec{c}] = \begin{vmatrix} 1 & -2 & 3 \\ -2 & 3 & -4 \\ 1 & -3 & 5 \end{vmatrix}$

$= 1.(15 - 12) + 2(-10 + 4) + 3(6 - 3)$
$= 3 - 12 + 9 = 0.$

Hence, the given vectors are coplanar.

3. Find 'λ' if the vectors :

$\hat{i} - \hat{j} + \hat{k},\ 3\hat{i} + \hat{j} + 2\hat{k}$ and $\hat{i} + \lambda\hat{j} - 3\hat{k}$ are coplanar.

Solution. Let $\vec{a} = \hat{i} - \hat{j} + \hat{k},\ \vec{b} = 3\hat{i} + \hat{j} + 2\hat{k}$ and

$\vec{c} = \hat{i} + \lambda\hat{j} - 3\hat{k}.$

The given vectors are coplanar if $[\vec{a}\ \vec{b}\ \vec{c}] = 0$

if $\begin{vmatrix} 1 & -1 & 1 \\ 3 & 1 & 2 \\ 1 & \lambda & -3 \end{vmatrix} = 0$

if $1.(-3 - 2\lambda) + 1.(-9 - 2) + 1.(3\lambda - 1) = 0$

if $-3 - 2\lambda - 11 + 3\lambda - 1 = 0$ if $\lambda = 15.$

4. Let $\vec{a} = \hat{i} + \hat{j} + \hat{k},\ \vec{b} = \hat{i}$ and $\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}.$ Then :

(*a*) If $c_1 = 1$ and $c_2 = 2$, find c_3, which makes $\vec{a}, \vec{b}$ and $\vec{c}$ coplanar [*Refer Q. 11; Ex. 10 (j)*]

(*b*) If $c_2 = -1$ and $c_3 = 1$, show that no value of c_1 can make $\vec{a}, \vec{b}$ and $\vec{c}$ coplanar.

Solution. Here $[\vec{a}\ \vec{b}\ \vec{c}] = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ c_1 & c_2 & c_3 \end{vmatrix}$

$= (-1)(c_3 - c_2)$
$= c_2 - c_3.$

(*a*) When $c_1 = 1$ and $c_2 = 2$, then

$c_2 - c_3 = 0 \Rightarrow 2 - c_3 = 0 \Rightarrow c_3 = 2.$

(*b*) When $c_2 = -1$ and $c_3 = 1$,

then $c_2 - c_3 = -1 - 1 = -2 \neq 0.$

Here no value of c_1 can make $\vec{a}, \vec{b}$ and $\vec{c}$ coplanar.

5. Show that four points with position vectors :

$4\hat{i} + 8\hat{j} + 12\hat{k},\ 2\hat{i} + 4\hat{j} + 6\hat{k},\ 3\hat{i} + 5\hat{j} + 4\hat{k}$ and $5\hat{i} + 8\hat{j} + 5\hat{k}$ are coplanar.

[**Solution.** Refer Q. 7 (*ii*) ; Ex. 10 (*j*)].

6. Find 'x' such that the four points :

A(3, 2, 1), B(4, x, 5), C(4, 2, –2) and D(6, 5, –1) are coplanar.

[**Solution.** Refer Q. 8; Ex. 10 (*j*)].

7. Show that the vectors $\vec{a}, \vec{b}$ and $\vec{c}$ are coplanar if $\vec{a} + \vec{b}, \vec{b} + \vec{c}$ and $\vec{c} + \vec{a}$ are coplanar.

[**Solution.** Refer Ex. 8; Page 10/81].

Miscellaneous Exercise on Chapter 10

1. Write down a unit vector in XY-plane, making an angle of 30° with the positive direction of x-axis.

[**Solution.** Refer Q. 2 ; Rev. Ex.]

2. Find the scalar components and magnitude of the vector joining the points P(x_1, y_1, z_1) and Q(x_2, y_2, z_2).

[**Solution.** Refer Q. 16 ; Ex. 10(*c*)]

3. A girl walks 4 km towards west, then she walks 3 km in a direction 30° east of north and stops. Determine the girl's displacement from her initial point of departure.

[**Solution.** Refer Q. 5 ; Ex. 10(*a*)]

4. If $\vec{a} = \vec{b} + \vec{c}$, then is it true that $|\vec{a}| = |\vec{b}| + |\vec{c}|$? Justify your answer.

Solution : We have : $\vec{a} = \vec{b} + \vec{c}$

$\therefore \quad |\vec{a}| = |\vec{b} + \vec{c}|$

Squaring, $\quad |\vec{a}|^2 = |\vec{b} + \vec{c}|^2$

$= (\vec{b} + \vec{c}).(\vec{b} + \vec{c})$

$= \vec{b}.\vec{b} + \vec{b}.\vec{c} + \vec{c}.\vec{b} + \vec{c}.\vec{c}$

$= |\vec{b}|^2 + 2\vec{b}.\vec{c} + |\vec{c}|^2 \qquad [\because \vec{b}.\vec{c} = \vec{c}.\vec{b}]$

$= |\vec{b}|^2 + |\vec{c}|^2 + 2|\vec{b}||\vec{c}|\cos\theta.$

where 'θ' is the angle between $\vec{b}$ and $\vec{c}$.

When $\theta = 0°$, then

$|\vec{a}|^2 = |\vec{b}|^2 + |\vec{c}|^2 + 2|\vec{b}||\vec{c}|$

$\Rightarrow \quad |\vec{a}| = |\vec{b}| + |\vec{c}|.$

When $\theta \neq 0°$, then $|\vec{a}| \neq |\vec{b}| + |\vec{c}|.$

5. Find the value of 'x' for which $x(\hat{i}+\hat{j}+\hat{k})$ is a unit vector.

Solution : Let $\vec{a} = x\hat{i}+x\hat{j}+x\hat{k}$.

By the question, $|\vec{a}| = 1$

$\Rightarrow \sqrt{x^2+x^2+x^2} = 1$

$\Rightarrow \sqrt{3}x = \pm 1 \Rightarrow x = \pm\dfrac{1}{\sqrt{3}}$.

6. Find a vector of magnitude 5 units, and parallel to the resultant of the vectors $\vec{a} = 2\hat{i}+3\hat{j}-\hat{k}$ and $\vec{b} = \hat{i}-2\hat{j}+\hat{k}$.

[**Solution.** Refer Q. 3 ; Rev. Ex.]

7. If $\vec{a} = \hat{i}+\hat{j}+\hat{k}$, $\vec{b} = 2\hat{i}-\hat{j}+3\hat{k}$ and $\vec{c} = \hat{i}-2\hat{j}+\hat{k}$, find a unit vector parallel to the vector $2\vec{a}-\vec{b}+3\vec{c}$.

[**Solution.** Refer Q. 4 ; Rev. Ex.]

8. Show that the points A(1, –2, –8), B (5, 0, –2) and C(11, 3, 7) are collinear, and find the ratio in which B divides AC.

[**Solution.** Refer Q. 6 ; Rev. Ex.]

9. Find the position vector of a point R, which divides the line joining two points P and Q whose position vectors are $(2\vec{a}+\vec{b})$ and $(\vec{a}-3\vec{b})$ externally in the ratio 1 : 2. Also, show that P is the mid-point of the line segment RQ.

Solution : The position vector of R is given by :

$$\vec{r} = \frac{1(\vec{a}-3\vec{b})-2(2\vec{a}+\vec{b})}{1-2}$$

$$= \frac{-3\vec{a}-5\vec{b}}{-1} = 3\vec{a}+5\vec{b}.$$

(*ii*) Mid-point of [RQ] is

$$= \left(\frac{(3\vec{a}+5\vec{b})+(\vec{a}-3\vec{b})}{2}\right)$$

$$= 2\vec{a}+\vec{b}, \text{ which is P.}$$

Hence P is the middle point of [RQ].

10. The two adjacent sides of a parallelogram are $2\hat{i}-4\hat{j}+5\hat{k}$ and $\hat{i}-2\hat{j}-3\hat{k}$. Find the unit vector parallel to its diagonal. Also, find its area.

[**Solution.** Refer Q. 7 ; Rev. Ex.]

11. Show that the direction cosines of a vector equally inclined to the axes OX, OY and OZ are $\dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}$.

[**Solution.** Refer Q. 11(*i*) ; Ex. 10(*c*)]

12. Let $\vec{a} = \hat{i}+4\hat{j}+2\hat{k}$, $\vec{b} = 3\hat{i}-2\hat{j}+7\hat{k}$ and $\vec{c} = 2\hat{i}-\hat{j}+4\hat{k}$. Find a vector $\vec{d}$ which is perpendicular to both $\vec{a}$ and $\vec{b}$, and $\vec{c}\cdot\vec{d} = 15$.

Solution : Let $\vec{d} = x\hat{i}+y\hat{j}+z\hat{k}$ ...(1)

Now $\vec{d}$ is perp. to $\vec{a}$.

$\therefore \quad \vec{d}.\vec{a} = 0$

$\Rightarrow (x\hat{i}+y\hat{j}+z\hat{k}).(\hat{i}+4\hat{j}+2\hat{k}) = 0$

$\Rightarrow \quad x+4y+2z = 0$ ...(2)

and $\vec{d}$ is perp to $\vec{b}$,

$\therefore \quad \vec{d}.\vec{b} = 0$

$\therefore (x\hat{i}+y\hat{j}+z\hat{k}).(3\hat{i}-2\hat{j}+7\hat{k}) = 0$

$\Rightarrow \quad 3x-2y+7z = 0$ ...(3)

Also $\quad \vec{c}.\vec{d} = 15$

$\Rightarrow (2\hat{i}-\hat{j}+4\hat{k}).(x\hat{i}+y\hat{j}+z\hat{k}) = 15$

$\Rightarrow \quad 2x-y+4z = 15$ ....(4)

(3) – 3(2) gives : $-14y+z = 0$ ...(5)

(4) – 2(2) gives : $-9y = 15$ ...(6)

From (6), $\quad y = -\dfrac{5}{3}$.

Putting in (5), $-14\left(\dfrac{-5}{3}\right)+z = 0$

$\Rightarrow z = -\dfrac{70}{3}$.

Putting in (2), $x-\dfrac{20}{3}-\dfrac{140}{3} = 0 \Rightarrow x = \dfrac{160}{3}$.

Putting in (1),

$$\vec{d} = \frac{160}{3}\hat{i}-\frac{5}{3}\hat{j}-\frac{70}{3}\hat{k}$$

$$= \frac{5}{3}(32\hat{i}-\hat{j}-14\hat{k}).$$

13. The scalar product of the vector $\hat{i}+\hat{j}+\hat{k}$ with a unit vector along the sum of vectors $2\hat{i}+4\hat{j}-5\hat{k}$ and $\lambda\hat{i}+2\hat{j}+3\hat{k}$ is equal to one. Find the value of 'λ'.
[**Solution.** Refer Q. 8 ; Rev. Ex.]

14. If $\vec{a}, \vec{b}, \vec{c}$ are mutually perpendicular vectors of equal magnitudes, show that the vector $\vec{a}+\vec{b}+\vec{c}$ is equally inclined to $\vec{a}, \vec{b}$ and $\vec{c}$.

Solution : Here $|\vec{a}|=|\vec{b}|=|\vec{c}|$

and $\quad \vec{a}.\vec{b} = \vec{b}.\vec{c} = \vec{c}.\vec{a} = 0 \qquad \ldots(1)$

Let α, β, γ be the angles, which $\vec{a}+\vec{b}+\vec{c}$ makes with $\vec{a}, \vec{b}, \vec{c}$ respectively.

$$\therefore \cos\alpha = \frac{\vec{a}.(\vec{a}+\vec{b}+\vec{c})}{|\vec{a}||\vec{a}+\vec{b}+\vec{c}|}$$

$$= \frac{\vec{a}.\vec{a}+\vec{a}.\vec{b}+\vec{a}.\vec{c}}{|\vec{a}||\vec{a}+\vec{b}+\vec{c}|}$$

$$= \frac{|\vec{a}|^2+0+0}{|\vec{a}||\vec{a}+\vec{b}+\vec{c}|} \qquad [Using\ (1)]$$

$$= \frac{|\vec{a}|}{|\vec{a}+\vec{b}+\vec{c}|} \qquad \ldots(2)$$

Similarly $\cos\beta = \dfrac{|\vec{b}|}{|\vec{a}+\vec{b}+\vec{c}|} \qquad \ldots(3)$

and $\cos\gamma = \dfrac{|\vec{c}|}{|\vec{a}+\vec{b}+\vec{c}|} \qquad \ldots(4)$

From (2), (3) and (4), $\cos\alpha = \cos\beta = \cos\gamma$

$[\because |\vec{a}|=|\vec{b}|=|\vec{c}|]$

$\Rightarrow \quad \alpha = \beta = \gamma.$

Hence, the vector $(\vec{a}+\vec{b}+\vec{c})$ is equally inclined to $\vec{a}, \vec{b}$ and $\vec{c}$.

15. Prove that $(\vec{a}+\vec{b}).(\vec{a}+\vec{b}) = |\vec{a}|^2+|\vec{b}|^2$, if and only if $\vec{a}, \vec{b}$ are perpendicular, given $\vec{a} \neq \vec{0}, \vec{b} \neq \vec{0}$.

Solution : $(\vec{a}+\vec{b}).(\vec{a}+\vec{b}) = \vec{a}.(\vec{a}+\vec{b})+\vec{b}.(\vec{a}+\vec{b})$

$= \vec{a}.\vec{a}+\vec{a}.\vec{b}+\vec{b}.\vec{a}+\vec{b}.\vec{b}$

$= |\vec{a}|^2+\vec{a}.\vec{b}+\vec{b}.\vec{a}+|\vec{b}|^2$

$= |\vec{a}|^2+|\vec{b}|^2+2\vec{a}.\vec{b} \qquad \ldots(1)$

$[\because \vec{a}.\vec{b} = \vec{b}.\vec{a}]$

When $\vec{a}, \vec{b}$ are perpendicular $\Rightarrow \vec{a}.\vec{b} = 0$

$\therefore$ From (1), $(\vec{a}+\vec{b}).(\vec{a}+\vec{b}) = |\vec{a}|^2+|\vec{b}|^2$.

Conversely.

$(\vec{a}+\vec{b}).(\vec{a}+\vec{b}) = |\vec{a}|^2+|\vec{b}|^2$

$\Rightarrow \vec{a}.\vec{b} = 0 \Rightarrow \vec{a}, \vec{b}$ are perpendicular.

Choose the correct answer in Exercises 16 to 19.

16. If 'θ' is the angle between the vectors $\vec{a}$ and $\vec{b}$, then $\vec{a}.\vec{b} \geq 0$ only when :

(A) $0 < \theta < \dfrac{\pi}{2}$ (B) $0 \leq \theta \leq \dfrac{\pi}{2}$

(C) $0 < \theta < \pi$ (D) $0 < \theta < \pi$

[Ans. (B)]

17. Let $\vec{a}$ and $\vec{b}$ be two unit vectors and 'θ' is the angle between them. Then $\vec{a}+\vec{b}$ is a unit vector if :

(A) $\theta = \dfrac{\pi}{4}$ (B) $\theta = \dfrac{\pi}{3}$

(C) $\theta = \dfrac{\pi}{2}$ (D) $\theta = \dfrac{2\pi}{3}$.

[Ans. (D)]

18. The value of $\hat{i}.(\hat{j}\times\hat{k})+\hat{j}.(\hat{i}\times\hat{k})+\hat{k}.(\hat{i}\times\hat{k})$ is :

(A) 0 (B) −1

(C) 1 (D) 3.

[Ans. (C)**]**

19. If 'θ' is the angle between any two vectors $\vec{a}$ and $\vec{b}$, then : $|\vec{a}.\vec{b}|=|\vec{a}\times\vec{b}|$ when 'θ' is equal to :

(A) 0 (B) $\frac{\pi}{4}$

(C) $\frac{\pi}{2}$ (D) π

[Ans. (B)**]**

Questions From NCERT Exemplar

Example 1. Find a vector of magnitude 11 in the direction opposite to that of $\overrightarrow{PQ}$, where P and Q are the points (1, 3, 2) and (–1 ,0, 8) respectively.

Solution. Here $\overrightarrow{PQ}$ = P.V. of Q – P.V. of P

$$= \left(-\hat{i}+0\hat{j}+8\hat{k}\right)-\left(\hat{i}+3\hat{j}+2\hat{k}\right)$$

$$= -2\hat{i}-3\hat{j}+6\hat{k}.$$

$$\therefore\ \overrightarrow{QP} = -\overrightarrow{PQ}=2\hat{i}+3\hat{j}-6\hat{k}.$$

$$\therefore\ \left|\overrightarrow{QP}\right| = \sqrt{2^2+3^2+(-6)^2}$$

$$=\sqrt{4+9+36}=\sqrt{49}=7.$$

$\therefore$ Unit vector in the direction of $\overrightarrow{QP}$ is given by :

$$\hat{QP} = \frac{\overrightarrow{QP}}{\left|\overrightarrow{QP}\right|}=\frac{2\hat{i}+3\hat{j}-6\hat{k}}{7}.$$

Hence, the reqd. vector of magnitude 11 in the direction of $\overrightarrow{QP}$ is :

$$11\,\overrightarrow{QP} = 11\ \frac{2\hat{i}+3\hat{j}-6\hat{k}}{7}=\frac{22}{7}\hat{i}+\frac{33}{7}\hat{j}-\frac{66}{7}\hat{k}.$$

Example 2. Find a vector $\vec{r}$ of magnitude $3\sqrt{2}$ units, which makes an angle of $\frac{\pi}{4}$ and $\frac{\pi}{2}$ with y and z-axes respectively.

Solution. $m=\cos\frac{\pi}{4}=\frac{1}{\sqrt{2}}$ and $n=\cos\frac{\pi}{2}=0$.

Now $l^2+m^2+n^2=1$ gives : $l^2+\frac{1}{2}+0=1$

$$\Rightarrow l^2=\frac{1}{2}\quad\Rightarrow l=\pm\frac{1}{\sqrt{2}}.$$

Hence, the reqd. vector is $\vec{r} = 3\sqrt{2}\left(l\hat{i}+m\hat{j}+n\hat{k}\right)$

i.e. $\vec{r} = 3\sqrt{2}\left(\pm\frac{1}{\sqrt{2}}\hat{i}+\frac{1}{\sqrt{2}}\hat{j}+0\hat{k}\right)$ *i.e.* $\vec{r}=\pm3\hat{i}+3\hat{j}$.

Example 3. Find all vectors of magnitude $10\sqrt{3}$ that are perpendicular to the plane of

$$\hat{i}+2\hat{j}+\hat{k}\ \text{and}\ -\hat{i}+3\hat{j}+4\hat{k}.$$

Solution. Let $\vec{a}=\hat{i}+2\hat{j}+\hat{k}$ and $\vec{b}=-\hat{i}+3\hat{j}+4\hat{k}$.

Then $\vec{a}\times\vec{b} = \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & 2 & 1\\ -1 & 3 & 4\end{vmatrix}$

$$= \hat{i}(8-3)-\hat{j}(4+1)+\hat{k}(3+2)$$

$$= 5\hat{i}-5\hat{j}+5\hat{k}.$$

$$\therefore\ \left|\vec{a}\times\vec{b}\right| = \sqrt{(5)^2+(-5)^2+(5)^2}=\sqrt{3(5)^2}=5\sqrt{3}.$$

$\therefore$ The unit vector perpendicular to the plane of $\vec{a}$ and $\vec{b}$ is given by :

$$\frac{\vec{a}\times\vec{b}}{\left|\vec{a}\times\vec{b}\right|} = \frac{5\hat{i}-5\hat{j}+5\hat{k}}{5\sqrt{3}}.$$

Hence, the vectors of magnitude $10\sqrt{3}$ that are perpendicular to the plane of $\vec{a}$ and $\vec{b}$ are

$$\pm10\sqrt{3}\left(\frac{5\hat{i}-5\hat{j}+5\hat{k}}{5\sqrt{3}}\right)\ \textit{i.e.}\ \pm10\left(\hat{i}-\hat{j}+\hat{k}\right).$$

Exercise

1. Using vectors, find the value of 'k' such that the points $(k, -10, 3)$, $(1, -1, 3)$ and $(3, 5, 3)$ are collinear.

2. If A, B, C, D are the points with position vectors :

$\hat{i}+\hat{j}-\hat{k}, 2\hat{i}-\hat{j}+3\hat{k}, 2\hat{i}-3\hat{k}, 3\hat{i}-2\hat{j}+\hat{k}$ respectively, find the projection of $\overrightarrow{AB}$ along CD.

3. Using vectors, prove that parallelograms on the same base and between the same parallels are equal in area.

Answers

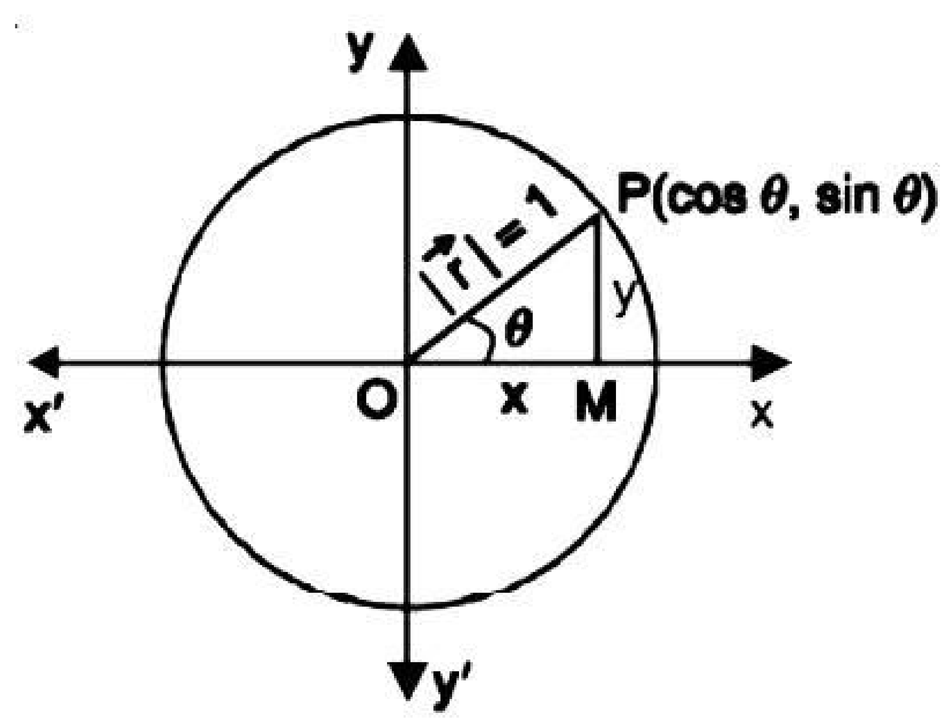
1. $k = -2$.

2. $\sqrt{21}$.

Revision Exercise

1. Write all the unit vectors in XY-plane. *(N.C.E.R.T.)*

Solution. Let $\vec{r} = x\hat{i} + y\hat{j}$ be a unit vector in XY-plane.



In the figure, $x = \cos\theta$, $y = \sin\theta$ $\quad [\because |\vec{r}| = 1]$

$\therefore \quad \vec{r} = \overrightarrow{OP}$

$= \cos\theta\,\hat{i} + \sin\theta\,\hat{j} \quad \ldots(1)$

Here $|\vec{r}| = \sqrt{\cos^2\theta + \sin^2\theta} = \sqrt{1} = 1.$

As θ varies from 0 to 2π, the point P traces $x^2 + y^2 = 1$ in the counter-clockwise direction and covers each possible direction.

Hence, (1) represents each unit vector in the XY-plane.

2. Write down a unit vector in XY-plane, making an angle of 30° with the positive direction of x-axis. *(N.C.E.R.T.)*

3. Find a vector of magnitude 5 units, and parallel to the resultant of the vectors :

$\vec{a} = 2\hat{i} + 3\hat{j} - \hat{k}$, and $\vec{b} = \hat{i} - 2\hat{j} + \hat{k}$. *(N.C.E.R.T.)*

4. If $\vec{a} = \hat{i} + \hat{j} + \hat{k}$, $\vec{b} = 2\hat{i} - \hat{j} + 3\hat{k}$ and $\vec{c} = \hat{i} - 2\hat{j} + \hat{k}$, find a unit vector parallel to the vector $2\vec{a} - \vec{b} + 3\vec{c}$. *(N.C.E.R.T.)*

5. If $\vec{a} = 2\hat{i} - \hat{j} + \hat{k}$, $\vec{b} = \hat{i} + 3\hat{j} - \hat{k}$, $\vec{c} = -2\hat{i} + \hat{j} - 3\hat{k}$ and $\vec{d} = 3\hat{i} + 2\hat{j} + 5\hat{k}$, find the scalars α, β and γ such that $\vec{d} = \alpha\vec{a} + \beta\vec{b} + \gamma\vec{c}$.

6. Show that the points A $(1, -2, -8)$, B $(5, 0, -2)$ and C $(11, 3, 7)$ are collinear and find the ratio in which B divides AC. *(N.C.E.R.T.)*

7. The two adjacent sides of a parallelogram are :

$2\hat{i} - 4\hat{j} + 5\hat{k}$ and $\hat{i} - 2\hat{j} - 3\hat{k}$.

Find the unit vector parallel to its diagonal. Also, find its area. *(N.C.E.R.T.; Jammu B. 2012)*

8. The scalar product of the vector $\hat{i} + \hat{j} + \hat{k}$ with a unit vector along the sum of the vectors :

$2\hat{i} + 4\hat{j} - 5\hat{k}$ and $\lambda\hat{i} + 2\hat{j} + 3\hat{k}$

is equal to one. Find the value of 'λ'. *(N.C.E.R.T.; Jammu B. 2012)*

9. If $\vec{a}, \vec{b}, \vec{c}$ are mutually perpendicular vectors of equal magnitudes, show that the vector $\vec{a} + \vec{b} + \vec{c}$ is equally inclined to $\vec{a}, \vec{b}$ and $\vec{c}$. *(N.C.E.R.T. ; Jammu B. 2012)*

10. If $\vec{a}, \vec{b}, \vec{c}$ are three non–coplanar vectors and $\vec{d}\cdot\vec{a} = \vec{d}\cdot\vec{b} = \vec{d}\cdot\vec{c} = 0$, then show that $\vec{d}$ is a zero vector.

Solution. $\vec{d}\cdot\vec{a} = \vec{d}\cdot\vec{b} = \vec{d}\cdot\vec{c} = 0$

$\Rightarrow \vec{d} \perp \vec{a}, \vec{d} \perp \vec{b}, \vec{d} \perp \vec{c}$

$\Rightarrow \vec{d} \perp$ plane containing $\vec{a}\,\vec{b}\,\vec{c}$.

But $\vec{a}, \vec{b}, \vec{c}$ are coplanar.

$\therefore$ The only possibility is that $\vec{d} = \vec{0}$.

11. If $\vec{c}$ is normal to $\vec{a}$ and $\vec{b}$, show that $\vec{c}$ is normal to $\vec{a}+\vec{b}$ and $\vec{a}-\vec{b}$.

12. Prove that $|\vec{a}|\vec{b}+|\vec{b}|\vec{a}$ is orthogonal to $|\vec{a}|\vec{b}-|\vec{b}|\vec{a}$, for any vectors $\vec{a}$ and $\vec{b}$. *(N.C.E.R.T.)*

13. Find the area of the parallelogram having diagonals $\vec{a}+\vec{b}$ and $\vec{b}+\vec{c}$, where :

$$\vec{a}=2\hat{i}-3\hat{j}+\hat{k},\ \vec{b}=-\hat{i}+\hat{k},\ \vec{c}=2\hat{j}-\hat{k}.$$

14. Prove that the quadrilateral obtained by joining mid-points of adjacent sides of a rectangle is a rhombus.

15. Prove that $\frac{1}{2}\overrightarrow{AC}\times\overrightarrow{BD}$ represents the vector area of the plane quadrilateral ABCD.

Solution. Vector area of quad. ABCD = Vector area of ΔABC + Vector area of ΔACD

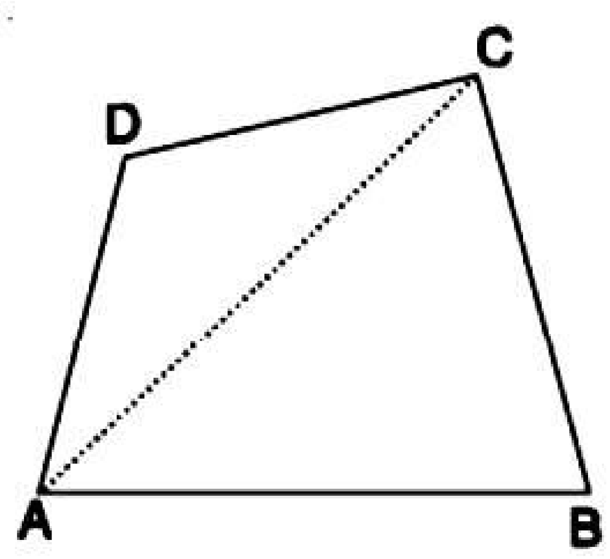


Fig.

$$=-\frac{1}{2}\overrightarrow{AC}\times\overrightarrow{AB}+\frac{1}{2}\overrightarrow{AC}\times\overrightarrow{AD}$$

$$=-\frac{1}{2}\overrightarrow{AC}\times(\overrightarrow{AB}-\overrightarrow{AD})$$

$$=\frac{1}{2}\overrightarrow{AC}\times(\overrightarrow{AD}+\overrightarrow{BA})=\frac{1}{2}\overrightarrow{AC}\times(\overrightarrow{BA}+\overrightarrow{AD})$$

$$=\frac{1}{2}\overrightarrow{AC}\times\overrightarrow{BD}.$$

16. Given $\vec{a}=\frac{1}{7}(2\hat{i}+3\hat{j}+6\hat{k})$, $\vec{b}=\frac{1}{7}(3\hat{i}-6\hat{j}+2\hat{k})$, $\vec{c}=\frac{1}{7}(6\hat{i}+2\hat{j}-3\hat{k})$; $\hat{i},\hat{j},\hat{k}$ being a right handed orthogonal system of unit vectors in space. Show that $\vec{a},\vec{b},\vec{c}$ is also another such system.

17. Show that $\vec{a},\vec{b},\vec{c}$ are non-coplanar if and only if $\vec{a}+\vec{b},\vec{b}+\vec{c},\vec{c}+\vec{a}$ are non-coplanar.

18. Prove that the vector $\vec{\beta}-\left(\dfrac{\vec{\alpha}\cdot\vec{\beta}}{|\vec{\alpha}|^2}\vec{\alpha}\right)$ is perpendicular to the vector $\vec{\alpha}$. *(W. Bengal B. 2017)*

19. Show that the vector of magnitude $\sqrt{51}$, which makes equal angles with the vectors $\vec{a}=\frac{1}{3}(\hat{i}-2\hat{j}+2\hat{k})$, $\vec{b}=\frac{1}{5}(-4\hat{i}-3\hat{k})$ and $\vec{c}=\hat{j}$ is $-5\hat{i}+\hat{j}+5\hat{k}$. *(Assam B. 2017)*

Answers

2. $\frac{\sqrt{3}}{2}\hat{i}+\frac{1}{2}\hat{j}$. **3.** $\sqrt{\frac{5}{2}}(3\hat{i}+\hat{j})$.

4. $\frac{1}{\sqrt{22}}(3\hat{i}-3\hat{j}+2\hat{k})$.

5. $\alpha=\frac{-13}{7},\beta=\frac{9}{7},\gamma=\frac{-19}{7}$.

6. 2 : 3. **7.** $\frac{1}{7}(3\hat{i}-6\hat{j}+2\hat{k})$; $11\sqrt{5}$ sq. units.

8. $\lambda=1$. **13.** $\frac{1}{2}\sqrt{21}$ sq. units.

Hints to Selected Questions

11. Here $\vec{c}\cdot\vec{a}=0$ ...(1)

and $\vec{c}\cdot\vec{b}=0$ ...(2)

(*i*) $\vec{c}\cdot(\vec{a}+\vec{b})=\vec{c}\cdot\vec{a}+\vec{c}\cdot\vec{b}=0+0=0$

$\Rightarrow \vec{c}$ is normal to $\vec{a}+\vec{b}$.

13. Area of parallelogram $=\frac{1}{2}|(\vec{a}+\vec{b})\times(\vec{b}+\vec{c})|$.

16. Here $\vec{a}\cdot\vec{b}=\vec{b}\cdot\vec{c}=\vec{c}\cdot\vec{a}=0$.

And $|\vec{a}|=|\vec{b}|=|\vec{c}|=1$.

17. $\vec{a},\vec{b},\vec{c}$ are non-coplanar $\Rightarrow \left[\vec{a}\,\vec{b}\,\vec{c}\right]\neq 0$

$\Rightarrow 2\left[\vec{a}\,\vec{b}\,\vec{c}\right]\neq 0$

$\Rightarrow \left[\vec{a}+\vec{b},\vec{b}+\vec{c},\vec{c}+\vec{a}\right]\neq 0$; etc.

CHECK YOUR UNDERSTANDING

1. Are vectors $\vec{a}$ and $-\vec{a}$ collinear ?

Ans. Yes.

2. Write two different vectors having same magnitude.

Ans. $\hat{i}+\hat{j}+\hat{k}$ and $\hat{i}-\hat{j}+\hat{k}$.

3. What is the unit vector in the direction of $\vec{a}=\vec{i}+\vec{j}+2\vec{k}$?

Ans. $\frac{1}{\sqrt{6}}(\hat{i}+\hat{j}+2\hat{k})$.

4. What is the position vector of the mid-point of the line segment joining the points A(2, 3, 4) and B(4, 1, – 2).

Ans. $3\vec{i}+2\hat{j}+\hat{k}$.

5. The scalar product of two given vectors $\vec{a}$ and $\vec{b}$ having angle 'θ' between them is defined as $\vec{a}.\vec{b}=$ ....

Ans. $\vec{a}.\vec{b}=|\vec{a}|.|\vec{b}|\cos\theta$.

6. For mutually orthogonal unit vectors $\hat{i},\hat{j},\hat{k}$, we have :

$\hat{i}\times\hat{i}=\hat{j}\times\hat{j}=\hat{k}\times\hat{k}$. **(True/False)**

Ans. True.

7. What is the value of :

$\hat{i}\cdot(\hat{j}\times\hat{k})+\hat{j}(\hat{k}\times\hat{i})+\hat{k}\times(\hat{j}\times\hat{i})$.

Ans. 1.

8. If $|\vec{a}\,.\,\vec{b}|\times|\vec{a}\,.\,\vec{b}|$, find the angle between $\vec{a}$ and $\vec{b}$.

(Tripura B. 2016) **[Ans.** $\frac{\pi}{4}$ **]**

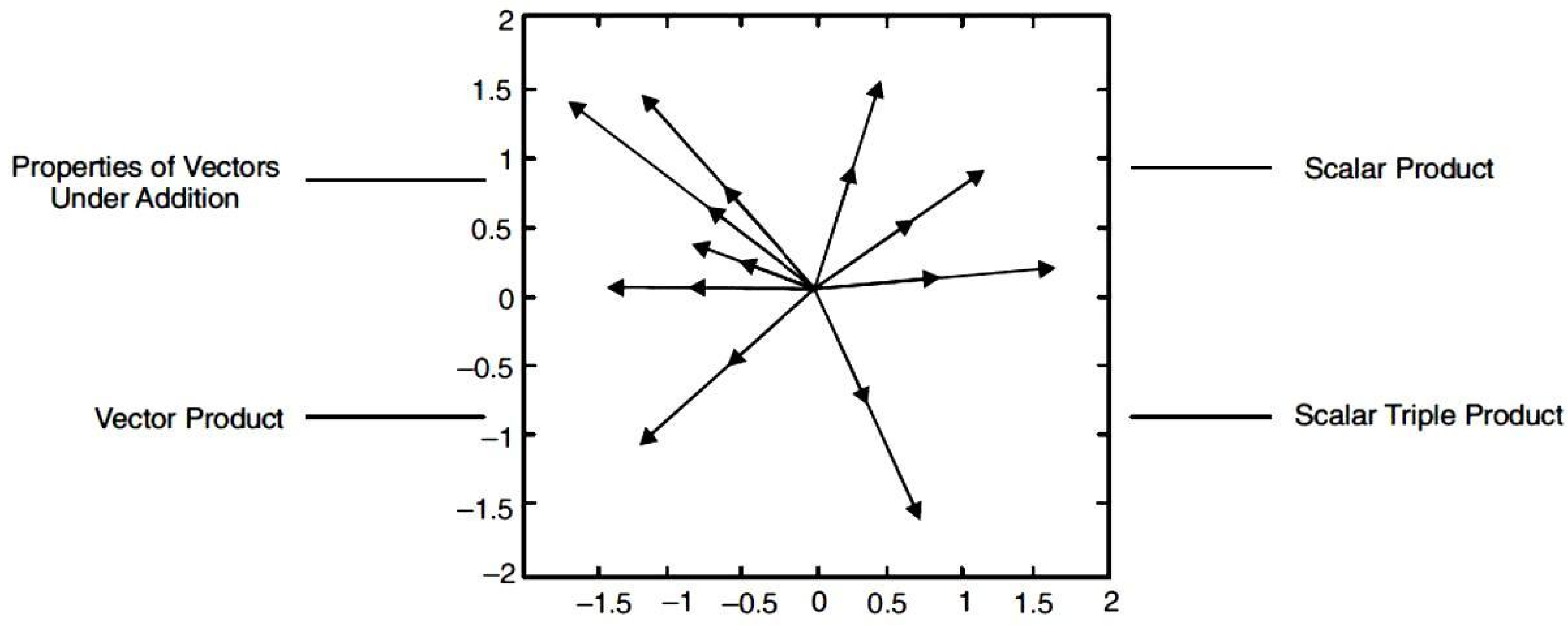
9. Classify the following as vector or scalar :

$\vec{a}.\vec{b},[\vec{a}\,\vec{b}\,\vec{c}],3\vec{a},\vec{a}\times\vec{b}$.

Ans. Scalar, scalar, vector, vector.

10. What is $[\vec{a}\,\vec{b}\,\vec{c}]$ when $\vec{a},\vec{b},\vec{c}$ are coplanar ?

Ans. 0.



DEFINITIONS AND IMPORTANT RESULTS

1. PROPERTIES OF VECTORS UNDER ADDITION

(*i*) **Commutative Law.**

$$\vec{a}+\vec{b}=\vec{b}+\vec{a}.$$

(*ii*) **Associative Law.**

$$\vec{a}+(\vec{b}+\vec{c})=(\vec{a}+\vec{b})+\vec{c}.$$

(*iii*) **Additive Identity.**

$$\vec{a}+\vec{0}=\vec{a}=\vec{0}+\vec{a}.$$

(*iv*) **Additive Inverse.**

$$\vec{a}+(-\vec{a})=\vec{0}=(-\vec{a})+\vec{a}.$$

2. $\overrightarrow{AB}$ = (Position vector of B) – (Position vector of A).

3. SCALAR PRODUCT

(*i*) **Def.** The scalar product of $\vec{a}$ and $\vec{b}$ is defined as $\vec{a}\cdot\vec{b}=|\vec{a}||\vec{b}|\cos\theta$, where '$\theta$' is the angle between $\vec{a}$ and $\vec{b}$.

(*ii*) **Condition of Perpendicularity.** If $\vec{a}$ and $\vec{b}$ are perpendicular, then $\vec{a}\cdot\vec{b}=0$.

(*iii*) If $\hat{i},\hat{j},\hat{k}$ are unit vectors, which are perpendicular to each other, then :

$$\hat{i}\cdot\hat{j}=\hat{j}\cdot\hat{k}=\hat{k}\cdot\hat{i}=0 \quad\text{and}\quad \hat{i}^2=\hat{j}^2=\hat{k}^2=1.$$

(*iv*) **Properties.** (I) **Commutative Law.** $\vec{a}\cdot\vec{b}=\vec{b}\cdot\vec{a}$

(II) **Associative Law** does not hold.

(III) **Distributive Laws.** $\vec{a}\cdot(\vec{b}+\vec{c})=\vec{a}\cdot\vec{b}+\vec{a}\cdot\vec{c}$ and

$$(\vec{b}+\vec{c})\cdot\vec{a}=\vec{b}\cdot\vec{a}+\vec{c}\cdot\vec{a}$$

(IV) $(m\vec{a})\cdot\vec{b}=m(\vec{a}\cdot\vec{b})=\vec{a}\cdot(m\vec{b})$, m being a scalar.

(V) $\vec{a}\cdot\vec{a}=a^2$.

4. VECTOR PRODUCT.

(*i*) **Definition.** The vector product of $\vec{a}$ and $\vec{b}$ is given by :

$\vec{a} \times \vec{b} = |\vec{a}||\vec{b}| \sin\theta$, where '$\theta$' is the angle between $\vec{a}$ and $\vec{b}$ and $\hat{n}$ is a unit vector perpendicular to the plane of $\vec{a}$ and $\vec{b}$.

(*ii*) When two vectors are parallel, then $\vec{a} \times \vec{b} = \vec{0}$.

(*iii*) If $\hat{i}, \hat{j}, \hat{k}$ are unit vectors, which are perpendicular to each other, then $\hat{i} \times \hat{j} = \hat{k} = -\hat{j} \times \hat{i}$; etc.

$\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = \vec{0}$.

(*iv*) **Properties.** (I) **Commutative Law** does not hold.

(II) $(m\vec{a}) \times \vec{b} = m(\vec{a} \times \vec{b}) = \vec{a} \times (m\vec{b})$, m being a scalar.

(III) If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, then

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}.$$

(IV) **Distribution Law.** $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$.

5. SCALAR TRIPLE PRODUCT

(*i*) **Def.** If $\vec{a}, \vec{b}, \vec{c}$ are three vectors, then the scalar product of $\vec{a} \times \vec{b}$ with $\vec{c}$ is called scalar triple product.

(*ii*) **Properties.** (I) **Condition for three vectors to be coplanar.** If $\vec{a}, \vec{b}, \vec{c}$ are all coplanar vectors, then

$$\begin{bmatrix} \vec{a} & \vec{b} & \vec{c} \end{bmatrix} = 0.$$

(II) If any two of the three vectors $\vec{a}, \vec{b}, \vec{c}$ are equal,

then $\begin{bmatrix} \vec{a} & \vec{b} & \vec{c} \end{bmatrix} = 0.$

(III) If any two of the three vectors $\vec{a}, \vec{b}, \vec{c}$ are parallel,

then $\begin{bmatrix} \vec{a} & \vec{b} & \vec{c} \end{bmatrix} = 0.$

(IV) In the scalar triple product, the position of dot and cross can be interchanged at pleasure provided the cyclic order of the vectors is maintained.

(V) In the scalar triple product, the change in the cyclic order of the vectors changes the sign of the product.

MULTIPLE CHOICE QUESTIONS

For Board Examinations

1. The inequality $|\vec{a}.\vec{b}| \le |\vec{a}||\vec{b}|$ is called :

(A) Cauchy-Schwartz
(B) Triangle Inequality
(C) Rolle's Theorem
(D) Lagrange's Mean Value Theorem. *(P.B. 2018)*

2. The vectors $\vec{a}$ and $\vec{b}$ are perpendicular if :

(A) $\vec{a}.\vec{b}=0$ (B) $\vec{a}.\vec{b}\neq 0$

(C) $\vec{a}\times\vec{b}=\vec{0}$ (D) $\vec{a}\times\vec{b}\neq\vec{0}$. *(H.P.B. 2018)*

3. Angle between two vectors $\vec{a}$ and $\vec{b}$ with magnitudes :

(*i*) 1 and 2 respectively and when $\vec{a}.\vec{b}=1$:

(A) 3 (B) $\frac{\pi}{2}$

(C) $\frac{\pi}{3}$ (D) $\frac{\pi}{4}$. *(H.P.B. 2018)*

4. Find $|\vec{a}-\vec{b}|$, if $|\vec{a}|=2, |\vec{b}|=3$ and $\vec{a}.\vec{b}=4$:

(A) $\sqrt{3}$ (B) $\sqrt{2}$

(C) $\sqrt{5}$ (D) $\sqrt{7}$. *(H.P.B. 2018)*

5. The angle between the vectors :

$\vec{a}=\hat{i}+2\hat{j}-3\hat{k}$ and $\vec{b}=3\hat{i}-\hat{j}+2\hat{k}$ is :

(A) $\cos^{-1}\left(\frac{5}{14}\right)$ (B) $\cos^{-1}\left(\frac{9}{14}\right)$

(C) $\cos^{-1}\left(-\frac{5}{14}\right)$ (D) None of these. *(H.B. 2018)*

6. The D.C.'s of the vector $\hat{i}+2\hat{j}+3\hat{k}$ are :

(A) $\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{3}{\sqrt{6}}$ (B) $\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}$

(C) 1, 2, 3 (D) None of these.

(Jammu B. 2018)

7. If $\vec{a}=2\hat{i}+2\hat{j}+3\hat{k}$, then its magnitude is :

(A) 17 (B) $\sqrt{17}$

(C) 34 (D) None of these.

(Kashmir B. 2018)

8. If $\vec{a}$ and $\vec{b}$ are unlike vectors, then the angle between them is :

(A) 0 (B) $\frac{\pi}{2}$

(C) $-\pi$ (D) π. *(Mizoram B. 2018)*

9. The angle between the vectors $\hat{i}-\hat{j}$ and $\hat{j}+\hat{k}$ is :

(A) $\frac{\pi}{6}$ (B) $\frac{\pi}{4}$

(C) $\frac{\pi}{3}$ (D) $\frac{2\pi}{3}$. *(Nagaland B. 2018)*

10. If $\vec{a}.\vec{b}=|\vec{a}\times\vec{b}|$, then angle between vector $\vec{a}$ and vector $\vec{b}$ is :

(A) $\frac{\pi}{2}$ (B) $\frac{\pi}{6}$

(C) $\frac{\pi}{4}$ (D) $\frac{\pi}{3}$. *(P.B. 2017)*

11. The projection of the vector $\hat{i}+3\hat{j}+7\hat{k}$ on the vector $7\hat{i}-\hat{j}+8\hat{k}$ is :

(A) $\frac{60}{\sqrt{114}}$ (B) $\frac{60}{114}$

(C) $\frac{66}{\sqrt{114}}$ (D) None of the above.

(H.P.B. 2017)

12. If the angle between two vectors $\vec{a}$ and $\vec{b}$ is zero, then :

(A) $\vec{a}.\vec{b}=|\vec{a}||\vec{b}|$ (B) $\vec{a}.\vec{b}=0$

(C) $|\vec{a}||\vec{b}|=1$ (D) None of the above.

(H.P.B. 2017)

13. The projection of vector $\vec{a}=2\hat{i}+3\hat{j}+2\hat{k}$ on $\vec{b}=\hat{i}+2\hat{j}+\hat{k}$ is :

(A) $\frac{\sqrt{5}}{6}$ (B) $\frac{2}{3}\sqrt{6}$

(C) $\frac{\sqrt{3}}{2}$ (D) $\frac{5}{3}\sqrt{6}$. *(H.B. 2017)*

14. If the vectors $5\hat{i}+2\hat{j}-\hat{k}$ and $\lambda\hat{i}-\hat{j}+5\hat{k}$ are orthogonal vectors, then the value of λ is :

(A) $\frac{3}{5}$ (B) $\frac{5}{7}$

(C) $\frac{7}{5}$ (D) $\frac{2}{5}$. *(H.B. 2017)*

15. Let the vectors $\vec{a}$ and $\vec{b}$ be such that $|\vec{a}|=3$ and $|\vec{b}|=\frac{\sqrt{2}}{3}$. Then $\vec{a}\times\vec{b}$ is a unit vector, if the angle between $\vec{a}$ and $\vec{b}$ is :

(A) $\frac{\pi}{6}$ (B) $\frac{\pi}{4}$

(C) $\frac{\pi}{3}$ (D) $\frac{\pi}{2}$. *(Jammu B. 2017)*

16. If $\vec{a}=2\hat{i}-\hat{j}+\hat{k}$ and $\vec{b}=-2\hat{i}+\hat{j}+2\hat{k}$, then the vector in the direction of $\vec{a}+\vec{b}$ with magnitude 9 is :

(A) $9\hat{k}$ (B) $3\hat{k}$

(C) $\hat{k}$ (D) $6\hat{k}$. *(P.B. 2016)*

17. If $\vec{a}$ and $\vec{b}$ are two collinear vectors, then which of the following is incorrect ?

(A) $\vec{b}=\lambda\vec{a}$ for some scalar λ

(B) $\vec{a}=\pm\vec{b}$

(C) The respective components of $\vec{a}$ and $\vec{b}$ are proportional

(D) Both $\vec{a}$ and $\vec{b}$ have same direction but different magnitude. *(H.P.B. 2016)*

18. If $\vec{a}\times\vec{b}=|\vec{a}||\vec{b}|\sin\theta\,\hat{n}$, which one is correct ?

(A) $\hat{n}$ is a unit vector perpendicular to both $\vec{a}$ and $\vec{b}$

(B) $\hat{n}$ is a unit vector parallel to both $\vec{a}$ and $\vec{b}$

(C) $\hat{n}$ is a unit vector neither perpendicular nor parallel to both $\vec{a}$ and $\vec{b}$

(D) None of these. *(H.P.B. 2016)*

19. The projection of the vector $\vec{a}=2\hat{i}+3\hat{j}+2\hat{k}$ on the vector $\vec{b}=\hat{i}+2\hat{j}+\hat{k}$ is :

(A) $\frac{5}{\sqrt{3}}\sqrt{6}$ (B) $\frac{2}{3}\sqrt{5}$

(C) $\frac{3}{5}\sqrt{6}$ (D) $\frac{5}{6}\sqrt{3}$. *(H.P.B. 2016, 14)*

20. The magnitude of :

$\vec{a}=3\hat{i}+2\hat{j}$ is :

(A) $\sqrt{5}$ (B) $\sqrt{13}$

(C) $\sqrt{7}$ (D) None of these.

(H.B. 2016)

21. The value of $\hat{i}.(\hat{j}\times\hat{k})+\hat{j}.(\hat{i}\times\hat{k})+\hat{k}.(\hat{i}\times\hat{j})$ is :

(A) 0 (B) – 1

(C) 1 (D) 3. *(Jammu B. 2015)*

RCQ Pocket

(Single Correct Answer Type)

(JEE-Main and Advanced)

22. The non-zero vectors $\vec{a}, \vec{b}$ and $\vec{c}$ are related by $\vec{a}=8\vec{b}$ and $\vec{c}=-7\vec{b}$. Then the angle between $\vec{a}$ and $\vec{c}$ is :

(A) π (B) 0

(C) $\frac{\pi}{4}$ (D) $\frac{\pi}{2}$.

(A.I.E.E.E. 2008)

23. If $\vec{u}, \vec{v}, \vec{w}$ are non-coplanar vectors and p, q are real numbers, then the equality.

$$[3\vec{u}\ \ p\vec{v}\ \ p\vec{w}]-[p\vec{v}\ \ \vec{w}\ \ q\vec{u}]-[2\vec{w}\ \ q\vec{v}\ \ q\vec{u}]=0$$

holds for :

(A) exactly two values of (p, q)

(B) more than two but not all values (p, q)

(C) all values of (p, q)

(D) exactly one value of (p, q). *(A.I.E.E.E. 2009)*

24. Let $\vec{a}=\hat{j}-\hat{k}$ and $\vec{c}=\hat{i}-\hat{j}-\hat{k}$. Then the vector $\vec{b}$ satisfying $\vec{a}\times\vec{b}+\vec{c}=\vec{0}$ and $\vec{a}.\vec{b}=3$ is :

(A) $-\hat{i}+\hat{j}-2\hat{k}$ (B) $2\hat{i}-\hat{j}+2\hat{k}$

(C) $\hat{i}-\hat{j}-2\hat{k}$ (D) $\hat{i}+\hat{j}-2\hat{k}$.

(A.I.E.E.E. 2010)

25. If the vectors $\vec{a} = \hat{i} - \hat{j} + 2\hat{k}$, $\vec{b} = 2\hat{i} + 4\hat{j} + \hat{k}$ and $\vec{c} = \lambda\hat{i} + \hat{j} + \mu\hat{k}$ are mutually orthogonal, then $(\lambda, \mu) =$

(A) $(-3, 2)$ (B) $(2, -3)$
(C) $(-2, 3)$ (D) $(3, -2)$.

(A.I.E.E.E. 2010)

26. The vectors $\vec{a}$ and $\vec{b}$ are not perpendicular and $\vec{c}$ and $\vec{d}$ are two vectors satisfying $\vec{b} \times \vec{c} = \vec{b} \times \vec{d}$ and $\vec{a}.\vec{d} = 0$. Then the vector $\vec{d}$ is equal to :

(A) $\vec{b} - \left(\dfrac{\vec{b}.\vec{c}}{\vec{a}.\vec{b}}\right)\vec{c}$ (B) $\vec{c} + \left(\dfrac{\vec{a}.\vec{c}}{\vec{a}.\vec{b}}\right)\vec{b}$

(C) $\vec{b} + \left(\dfrac{\vec{b}.\vec{c}}{\vec{a}.\vec{b}}\right)\vec{c}$ (D) $\vec{c} - \left(\dfrac{\vec{a}.\vec{c}}{\vec{a}.\vec{b}}\right)\vec{b}$.

(A.I.E.E.E. 2011)

27. If the vectors $p\hat{i} + \hat{j} + \hat{k}, \hat{i} + q\hat{j} + \hat{k}$ and $\hat{i} + \hat{j} + r\hat{k}$ $(p \neq q \neq r \neq 1)$ are coplanar, then the value of $pqr - (p + q + r)$ is :

(A) 2 (B) 0
(C) -1 (D) -2.

(A.I.E.E.E. 2011 S)

28. Let $\vec{a}, \vec{b}, \vec{c}$ be three non-zero vectors, which are pair–wise non-collinear. If $\vec{a} + 3\vec{b}$ is collinear with $\vec{c}$ and $\vec{b} + 2\vec{c}$ is collinear with $\vec{a}$, then $\vec{a} + 3\vec{b} + 6\vec{c}$ is :

(A) $\vec{a}$ (B) $\vec{b}$
(C) $\vec{0}$ (D) $\vec{a} + \vec{c}$.

(A.I.E.E.E. 2011 S)

29. Let $\vec{a}$ and $\vec{b}$ he two unit vectors. If the vectors : $\vec{c} = \vec{a} + 2\vec{b}$ and $\vec{d} = 5\vec{a} - 4\vec{b}$ are perpendicular to each other, then the angle between $\vec{a}$ and $\vec{b}$ is :

(A) $\dfrac{\pi}{6}$ (B) $\dfrac{\pi}{2}$
(C) $\dfrac{\pi}{3}$ (D) $\dfrac{\pi}{4}$.

(A.I.E.E.E. 2012)

30. If the vectors $\overrightarrow{AB} = 3\hat{i} + 4\hat{k}$ and $\overrightarrow{AC} = 5\hat{i} - 2\hat{j} + 4\hat{k}$ are the sides of a triangle ABC, then the length of the median through A is :

(A) $\sqrt{72}$ (B) $\sqrt{33}$
(C) $\sqrt{45}$ (D) $\sqrt{18}$.

(J. E. E. (Main) 2013)

31. Let $\overrightarrow{PR} = 3\hat{i} + \hat{j} - 2\hat{k}$ and $\overrightarrow{SQ} = \hat{i} - 3\hat{j} - 4\hat{k}$ determine diagonals of a parallelogram PQRS and $\overrightarrow{PR} = \hat{i} + 2\hat{j} + 3\hat{k}$ be another vector. Then the volume of the parallelopiped determined by the vectors $\overrightarrow{PR}, \overrightarrow{PQ}$ and $\overrightarrow{PS}$ is :

(A) 5 (B) 20
(C) 10 (D) 30.

(J. E. E. (Advanced) 2013)

32. If $[\vec{a}\times\vec{b} \quad \vec{b}\times\vec{c} \quad \vec{c}\times\vec{a}] = \lambda\,[\vec{a} \quad \vec{b} \quad \vec{c}]^2$, then λ is equal to :

(A) 3 (B) 0
(C) 1 (D) 2.

(J.E.E. (Main) 2014)

33. Let $\vec{a}, \vec{b}$ and $\vec{c}$ be three non-zero vectors such that no two of them are collinear and

$$(\vec{a}\times\vec{b})\times\vec{c} = \frac{1}{3}|\vec{b}||\vec{c}|\,\vec{a}.$$

If 'θ' is the angle between the vectors $\vec{b}$ and $\vec{c}$, then a value of $\sin\theta$ is :

(A) $\dfrac{2\sqrt{2}}{3}$ (B) $\dfrac{-\sqrt{2}}{3}$

(C) $\dfrac{2}{3}$ (D) $\dfrac{-2\sqrt{3}}{3}$.

(J.E.E. (Main) 2015)

34. Let $\vec{a}, \vec{b}$ and $\vec{c}$ be three unit vectors such that :

$$\vec{a} \times (\vec{b} \times \vec{c}) = \frac{\sqrt{3}}{2}(\vec{b} + \vec{c}).$$

If $\vec{b}$ is not parallel to $\vec{c}$, then the angle between $\vec{a}$ and $\vec{b}$ is :

(A) $\dfrac{\pi}{2}$ (B) $\dfrac{2\pi}{3}$

(C) $\dfrac{5\pi}{6}$ (D) $\dfrac{3\pi}{4}$.

(J.E.E. (Main) 2016)

35. Let $\vec{a} = 2\hat{i} + \hat{j} - 2\hat{k}$ and $\vec{b} = \hat{i} + \hat{j}$. Let $\vec{c}$ be a vector such that $|\vec{c} - \vec{a}| = 3$ and $|(\vec{a} \times \vec{b}) \times \vec{c}| = 3$ and the angle between $\vec{c}$ and $\vec{a} \times \vec{b}$ is 30°. Then $\vec{a} \cdot \vec{c}$ is equal to :

(A) 5 (B) $-\frac{1}{8}$

(C) $\frac{25}{8}$ (D) 2.

(J.E.E. (Main) 2017)

36. Let O be the origin and let PQR be an arbitrary triangle. The point S is such that :

$$\overrightarrow{OP} \cdot \overrightarrow{OQ} + \overrightarrow{OR} \cdot \overrightarrow{OS} = \overrightarrow{OR} \cdot \overrightarrow{OP} + \overrightarrow{OQ} \cdot \overrightarrow{OS}$$

$$= \overrightarrow{OQ} \cdot \overrightarrow{OR} + \overrightarrow{OP} \cdot \overrightarrow{OS}.$$

Then the triangle PQR has S as its :

(A) orthocentre (B) centroid

(C) circumcentre (D) incentre.

(J.E.E. (Advanced) 2017)

37. Let $\vec{u}$ be a vector coplanar with the vectors :

$$\vec{a} = 2\hat{i} + 3\hat{j} - \hat{k} \text{ and } \vec{b} = \hat{j} + \hat{k}.$$

If $\vec{u}$ is perpendicular to $\vec{a}$ and $\vec{a} - \vec{b} = 24$, then $|\vec{u}|$ is equal to :

(A) 336 (B) 315

(C) 256 (D) 84.

(J.E.E. (Main) 2018)

Answers

1. (A)	**2.** (A)	**3.** (C)	**4.** (C)	**5.** (C)	**6.** (B)	**7.** (B)	**8.** (D)	**9.** (D)	**10.** (B)
11. (A)	**12.** (A)	**13.** (D)	**14.** (C)	**15.** (B)	**16.** (A)	**17.** (D)	**18.** (A)	**19.** (A)	**20.** (B)
21. (C)	**22.** (A)	**23.** (D)	**24.** (A)	**25.** (A)	**26.** (D)	**27.** (D)	**28.** (A)	**29.** (C)	**30.** (B)
31. (C)	**32.** (C)	**33.** (A)	**34.** (C)	**35.** (D)	**36.** (A)	**37.** (A).			

Hints/Solutions

RCQ Pocket

22. (A) $\vec{a} = 8\vec{b}$ and $\vec{c} = -7\vec{b}$.

Clearly $\vec{a}$ and $\vec{b}$ are parallel and $\vec{b}$ and $\vec{c}$ are anti-parallel.

$\therefore$ $\vec{a}$ and $\vec{c}$ are anti–parallel.

Hence, angle between $\vec{a}$ and $\vec{c}$ is π.

23. (A) $[3\vec{u}, p\vec{v}, p\vec{w}] - [p\vec{v}, \vec{w}, q\vec{u}] - [2\vec{w}, q\vec{v}, q\vec{u}] = 0$

$\Rightarrow 3p^2 [\vec{u}, \vec{v}, \vec{w}] - pq[\vec{v}, \vec{w}, \vec{u}] + 2q^2 [\vec{u}, \vec{v}, \vec{w}] = 0$

$\Rightarrow (3p^2 - pq + 2q^2)[\vec{u}, \vec{v}, \vec{w}] = 0$

$\Rightarrow 3p^2 - pq + 2q^2 = 0$

$[\because \vec{u}, \vec{v}, \vec{w}$ *are non-coplanar* $\Rightarrow [\vec{u}, \vec{v}, \vec{w}] \neq 0]$

$\Rightarrow 3p^2 - pq + 2q^2 = 0,\ p, q \in \mathbf{R}$.

Since this is quad. in p, roots are real

$\Rightarrow q^2 - 24q^2 \geq 0 \Rightarrow -23q^2 \geq 0$

$\Rightarrow q^2 \leq 0 \Rightarrow q = 0$.

Consequently, $p = 0$.

Hence, $(p, q) = (0, 0)$ is the only possibility.

24. (A) Here $\vec{a} \times \vec{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 1 & -1 \\ 1 & -1 & -1 \end{vmatrix}$

$= \hat{i}(-1-1) - \hat{j}(-0+1) + \hat{k}(-0-1)$

$= -2\hat{i} - \hat{j} - \hat{k}$.

Also $(\vec{a}.\vec{a}) = (\hat{j} - \hat{k}).(\hat{j} - \hat{k})$

$= (1)(1) + (-1)(-1) = 2$.

Given : $\vec{a} \times \vec{b} + \vec{c} = \vec{0}$

$\Rightarrow \vec{a} \times (\vec{a} \times \vec{b}) + \vec{a} \times \vec{c} = \vec{0}$

$\Rightarrow (\vec{a}.\vec{b})\vec{a} - (\vec{a}.\vec{a})\vec{b} + \vec{a} \times \vec{c} = \vec{0}$

$\Rightarrow 3\vec{a} - 2\vec{b} + (-2\hat{i} - \hat{j} - \hat{k}) = \vec{0}$

$\Rightarrow 2\vec{b} = 3(\hat{j} - \hat{k}) + (-2\hat{i} - \hat{j} - \hat{k})$

$= -2\hat{i} + 2\hat{j} - 4\hat{k}$.

Hence, $\vec{b} = -\hat{i} + \hat{j} - 2\hat{k}$.

25. (A) $\vec{a}, \vec{b}$ and $\vec{c}$ are mutually orthogonal

$\Rightarrow \quad \vec{b}.\vec{c} = 0$ and $\vec{a}.\vec{c} = 0$

$\Rightarrow \quad (2\hat{i} + 4\hat{j} + \hat{k}).(\lambda\hat{i} + \hat{j} + \mu\hat{k}) = 0$

$\Rightarrow \quad 2\lambda + 4 + \mu = 0 \Rightarrow 2\lambda + \mu = -4 \qquad ...(1)$

and $(\hat{i} - \hat{j} + 2\hat{k}).(\lambda\hat{i} + \hat{j} + \mu\hat{k}) = 0$

$\Rightarrow \quad \lambda - 1 + 2\mu = 0 \Rightarrow \lambda + 2\mu = 1 \qquad ....(2)$

Solving (1) and (2), $\lambda = -3$ and $\mu = 2$.

26. (D) $\vec{a}$ and $\vec{b}$ are not perpendicular $\Rightarrow \vec{a}.\vec{b} \neq 0$.

Also $\quad \vec{a}.\vec{d} = 0$

and $\vec{b} \times \vec{c} = \vec{b} \times \vec{d} \Rightarrow \vec{b} \times \left(\vec{c} - \vec{d}\right) = \vec{0}$

$\Rightarrow \quad \vec{b} \parallel \left(\vec{c} - \vec{d}\right) \Rightarrow \vec{c} - \vec{d} = \lambda\vec{b} \qquad ....(1)$

Taking dot product with $\vec{a}$, $\vec{a}.\vec{c} - \vec{a}.\vec{d} = \lambda\vec{a}.\vec{b}$

$\Rightarrow \quad \vec{a}.\vec{c} - 0 = \lambda\vec{a}.\vec{b} \Rightarrow \lambda = \dfrac{\vec{a}.\vec{c}}{\vec{a}.\vec{b}}$.

From (1), $\vec{d} = \vec{c} - \left(\dfrac{\vec{a}.\vec{c}}{\vec{a}.\vec{b}}\right)\vec{b}$.

27. (D) The given vectors are coplanar if :

$$\begin{vmatrix} p & 1 & 1 \\ 1 & q & 1 \\ 1 & 1 & r \end{vmatrix} = 0$$

$\Rightarrow \quad p(qr - 1) - (r - 1) + (1 - q) = 0$

$\Rightarrow \quad pqr - p - r + 1 + 1 - q = 0$

$\Rightarrow \quad pqr - (p + q + r) = -2$.

28. (C) Since $\vec{a} + 3\vec{b}$ is collinear with $\vec{c}$,

$\therefore \quad \vec{a} + 3\vec{b} = \lambda\vec{c} \qquad ....(1)$

Again since $\vec{b} + 2\vec{c}$ is collinear with $\vec{a}$,

$\therefore \quad \vec{b} + 2\vec{c} = \mu\vec{a} \qquad ....(2)$

(1) – 3 (2) gives : $\vec{a} - 6\vec{c} = \lambda\vec{c} - 3\mu\vec{a}$

$\Rightarrow \quad (1 + 3\mu)\vec{a} - (\lambda + 6)\vec{c} = \vec{0}$.

Since $\vec{a}$ and $\vec{c}$ are non-collinear,

$\therefore \quad 1 + 3\mu = 0$ and $\lambda + 6 = 0 \Rightarrow \mu = -\dfrac{1}{3}$ and $\lambda = -6$.

Putting (1), $\vec{a} + 3\vec{b} = -6\vec{c} \Rightarrow \vec{a} + 3\vec{b} + 6\vec{c} = \vec{0}$.

29. (C) We have : $\vec{c} = \hat{a} + 2\hat{b}$ and $\vec{d} = 5\hat{a} - 4\hat{b}$

Since $\vec{c}$ and $\vec{d}$ are perpendicular, $\therefore \vec{c}.\vec{d} = 0$

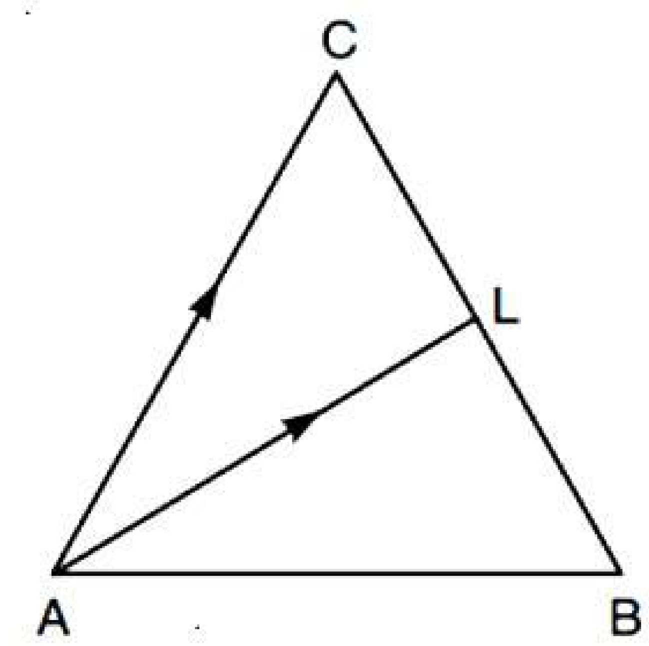
$\Rightarrow \quad (\hat{a} + 2\hat{b}).(5\hat{a} - 4\hat{b}) = 0$

$\Rightarrow \quad 5 + 6\hat{a}.\hat{b} - 8 = 0 \Rightarrow \hat{a}.\hat{b} = \dfrac{1}{2}$

$\Rightarrow \quad (1)(1)\cos\theta = \dfrac{1}{2} \Rightarrow \cos\theta = \dfrac{1}{2}$.

Hence, $\theta = \dfrac{\pi}{3}$.

30. (B) Here $\vec{AL} = \vec{AB} + \dfrac{\vec{AC}}{2}$



$= \dfrac{(3\hat{i} + 4\hat{k}) + (5\hat{i} - 2\hat{j} + 4\hat{k})}{2}$

$= 4\hat{i} - \hat{j} + 4\hat{k}$.

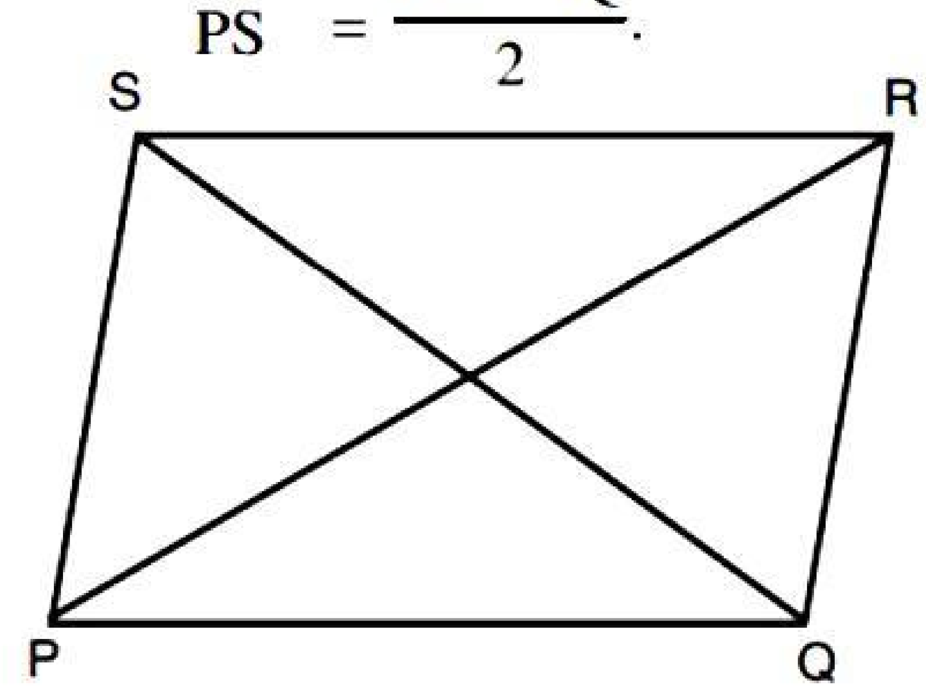
$\therefore \quad \left|\vec{AL}\right| = \sqrt{16 + 1 + 16} = \sqrt{33}$.

31. (C) We have :

$\vec{PT} = \hat{i} + 2\hat{j} + 3\hat{k}$

$\vec{PQ} = \dfrac{\vec{PR} + \vec{SQ}}{2}$

and $\quad \vec{PS} = \dfrac{\vec{PR} - \vec{SQ}}{2}$.



$\therefore \quad V = |[\vec{PQ}\ \vec{PS}\ \vec{PR}]|$

$= \dfrac{1}{4}|[\vec{PQ} + \vec{SQ}\ \ \vec{PS} - \vec{SQ}\ \ \vec{PR}]|$

$= \dfrac{1}{2}[\vec{PR}\ \ \vec{SQ}\ \ \vec{PR}]$

$$= \frac{1}{2}\begin{vmatrix} 3 & 1 & -2 \\ 1 & -3 & -4 \\ 1 & 2 & 3 \end{vmatrix}$$

$$= \frac{1}{2}[3(-1)-1.(7)-2(5)]$$

$$= \frac{1}{2}|-3-7-10| = \frac{1}{2}|-20| = \frac{20}{2} = 10.$$

32. (C) As usual, we will have :

$$[\vec{a}\times\vec{b} \;\; \vec{b}\times\vec{c} \;\; \vec{c}\times\vec{a}] = [\vec{a} \; \vec{b} \; \vec{c}]^2$$

Given : $[\vec{a}\times\vec{b} \;\; \vec{b}\times\vec{c} \;\; \vec{c}\times\vec{a}] = [\vec{a} \; \vec{b} \; \vec{c}]^2$.

Hence, $\lambda = 1$.

33. (A) $(\vec{a}\times\vec{b})\times\vec{c} = \frac{1}{3}|\vec{b}||\vec{c}|\vec{a}$

$$\Rightarrow (\vec{a}.\vec{c})\vec{b}-(\vec{b}.\vec{c})\vec{a} = \frac{1}{3}|\vec{b}||\vec{c}|\vec{a}$$

$$\Rightarrow (\vec{c}.\vec{a})\vec{b}-(\vec{c}.\vec{b})\vec{a} = \frac{1}{3}|\vec{b}||\vec{c}|\vec{a}$$

Comparing, $-(\vec{c}.\vec{b}) = \frac{1}{3}|\vec{b}||\vec{c}|$

$$\Rightarrow -|\vec{c}||\vec{b}|\cos\theta = \frac{1}{3}|\vec{b}||\vec{c}|$$

$$\Rightarrow -|\vec{c}||\vec{b}|\cos\theta = \frac{1}{3}|\vec{b}||\vec{c}|$$

$$\Rightarrow \cos\theta = -\frac{1}{3}.$$

Hence, $\sin\theta = \sqrt{1-\cos^2\theta} = \sqrt{1-\frac{1}{9}} = \frac{2\sqrt{2}}{3}$.

34. (C) We have : $\vec{a}\times(\vec{b}\times\vec{c}) = \frac{\sqrt{3}}{2}(\vec{b}+\vec{c})$

$$\Rightarrow (\vec{a}.\vec{c})\vec{b}-(\vec{a}.\vec{b})\vec{c} = \frac{\sqrt{3}}{2}\vec{b}+\frac{\sqrt{3}}{2}\vec{c}.$$

Comparing, $\vec{a}.\vec{b} = -\frac{\sqrt{3}}{2} \Rightarrow \cos\theta = -\frac{\sqrt{3}}{2} \Rightarrow \theta = \frac{5\pi}{6}$.

35. (D) Here $\vec{a} = 2\hat{i}+\hat{j}-2\hat{k}$ and $\vec{b} = \hat{i}+\hat{j}$.

$$\therefore \quad \vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & -2 \\ 1 & 1 & 0 \end{vmatrix}$$

$$= \hat{i}(0+2)-\hat{j}(0+2)+\hat{k}(2-1)$$

$$= 2\hat{i}-2\hat{j}+\hat{k}.$$

$$\therefore \quad |\vec{a}\times\vec{b}| = \sqrt{4+4+1} = \sqrt{9} = 3 \qquad ...(1)$$

Now, $|(\vec{a}\times\vec{b})\times\vec{c}| = 3$

$$\Rightarrow |\vec{a}\times\vec{b}||\vec{c}|\sin 30^\circ = 3$$

$$\Rightarrow \quad 3|\vec{c}| = 6 \qquad [\textit{Using (1)}]$$

$$\Rightarrow \quad |\vec{c}| = 2 \qquad ...(2)$$

Now, $|\vec{c}-\vec{a}| = 3$.

Squaring, $|\vec{c}|^2+|\vec{a}|^2-2\vec{a}.\vec{c} = 9$

$$\Rightarrow 4+(\sqrt{4+1+4})^2-2\vec{a}.\vec{c} = 9 \qquad [\textit{Using (2)}]$$

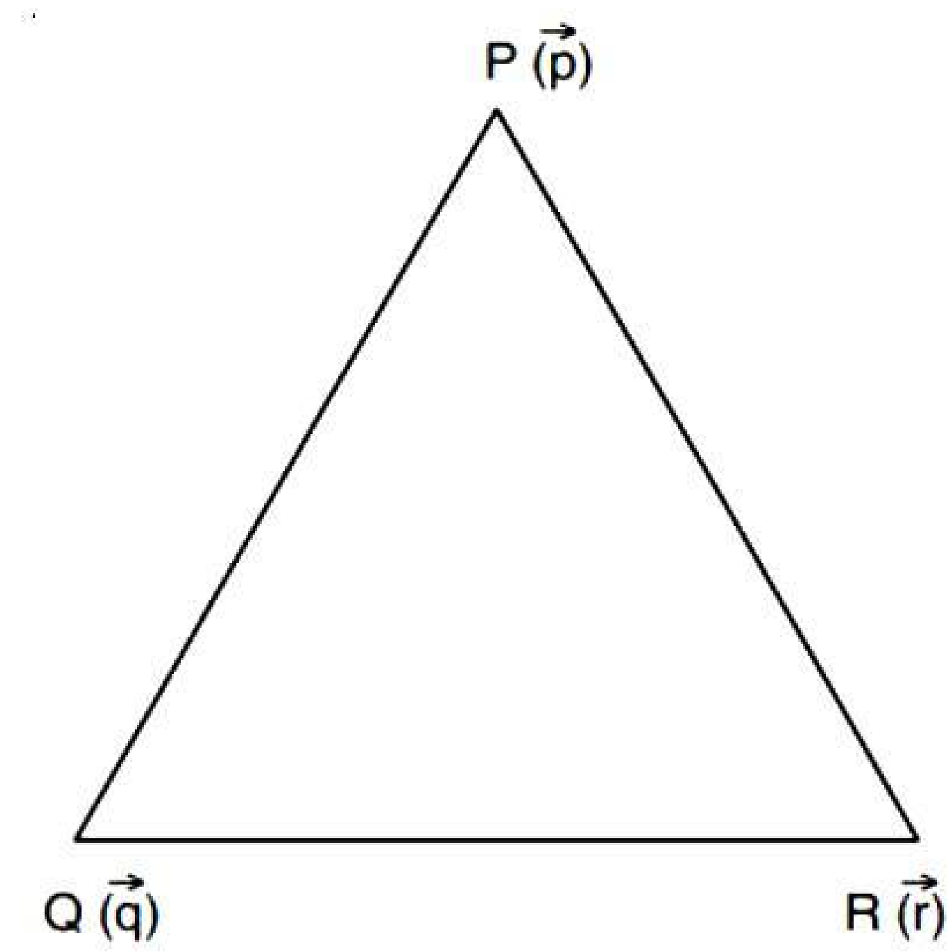
$$\Rightarrow 4+9-2\vec{a}.\vec{c} = 9 \Rightarrow 2\vec{a}.\vec{c} = 4.$$

Hence, $\vec{a}.\vec{c} = 2$.

36. (A) Let $\vec{p}, \vec{q}, \vec{r}$ and $\vec{s}$ be the position vector of P, Q, R and S respectively *w.r.t.* O.

Now $\overrightarrow{OP}.\overrightarrow{OQ}+\overrightarrow{OR}.\overrightarrow{OS}$

$$= \overrightarrow{OR}.\overrightarrow{OP}+\overrightarrow{OQ}.\overrightarrow{OS}$$



$$\Rightarrow \vec{p}.\vec{q}+\vec{r}.\vec{s} = \vec{r}.\vec{p}+\vec{q}.\vec{s}$$

$$\Rightarrow (\vec{p}-\vec{s}).(\vec{q}-\vec{r}) = 0 \qquad ...(1)$$

Similarly $(\vec{r}-\vec{s}).(\vec{p}-\vec{q}) = 0 \qquad ...(2)$

and $(\vec{q}-\vec{s}).(\vec{p}-\vec{r}) = 0 \qquad ...(3)$

Hence, triangle PQR has S as its orthocentre.

37. (B) Here, $\vec{u} = \lambda(\vec{a}\times(\vec{a}\times\vec{b}))$

$$\Rightarrow \quad \vec{u} = \lambda((\vec{a}.\vec{b})\vec{a}-|\vec{a}|^2\times\vec{b})$$

$\Rightarrow \quad \vec{u} = \lambda(2\vec{a} - 14\vec{b})$

$\Rightarrow \quad \vec{u} = 2\lambda((2\hat{i} + 3\hat{j} - \hat{k}) - 7(\hat{j} + \hat{k}))$

$\Rightarrow \quad \vec{u} = 2\lambda(2\hat{i} - 4\hat{j} - 8\hat{k})$.

Since, $\vec{u} \cdot \vec{b} = 24$,

$\therefore \; 2\lambda(2\hat{i} - 4\hat{j} - 8\hat{k}) \cdot (\hat{j} + \hat{k}) = 24$

$\Rightarrow -8\lambda - 16\lambda = 24 \Rightarrow \lambda = -1$.

$\therefore \quad -\vec{u} = -4(\hat{i} - 2\hat{j} - 4\hat{k})$

Hence, $|\vec{u}|^2 = (\sqrt{16+64+256})^2$

$= (\sqrt{336})^2 = 336$.

UNIT TEST 10

Time Allowed : 1 Hour **Max. Marks : 34**

Notes : 1. *All questions are compulsory.*

2. *Marks have been indicated against each question.*

1. Find the values of 'x' for which $x(\hat{i} + \hat{j} + \hat{k})$ is a unit vector. **(1)**

2. Find the angle between the vectors $\hat{i} - \hat{j}$ and $\hat{j} - \hat{k}$. **(1)**

3. Find $|\vec{a} \times \vec{b}|$, if $\vec{a} = 2\hat{i} + \hat{j} + 3\hat{k}$ and $\vec{b} = 3\hat{i} + 5\hat{j} - 2\hat{k}$. **(2)**

4. Show that the following vectors are collinear :

$2\hat{i} - 3\hat{j} + 4\hat{k}$ and $-4\hat{i} + 6\hat{j} - 8\hat{k}$. **(2)**

5. Show that the vectors : $2\hat{i} - \hat{j} + \hat{k}$, $\hat{i} - 3\hat{j} - 5\hat{k}$ and $3\hat{i} - 4\hat{j} - 4\hat{k}$.

form the vertices of a right angled triangle. **(4)**

6. If $|\vec{a}| = a$ and $|\vec{b}| = b$, prove that $\left(\frac{\vec{a}}{a^2} - \frac{\vec{b}}{b^2}\right)^2 = \left(\frac{\vec{a} - \vec{b}}{ab}\right)^2$. **(4)**

7. If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, find : $(r \times \hat{i}).(r \times \hat{j}) + xy$. **(4)**

8. Find the value of 'λ' such that the vectors :

$3\hat{i} + \lambda\hat{j} + 5\hat{k}$, $\hat{i} + 2\hat{j} - 3\hat{k}$ and $2\hat{i} - \hat{j} + \hat{k}$ are coplanar. **(4)**

9. If $|\vec{a}| = 3$, $|\vec{b}| = 4$ and $|\vec{c}| = 5$, such that each is perpendicular to the sum of the other two, prove that $|\vec{a} + \vec{b} + \vec{c}| = 5\sqrt{2}$. **(6)**

10. Establish, using vectors :

sin (A – B) = sin A cos B – cos A sin B. **(6)**

Answers

1. $\pm\frac{1}{\sqrt{3}}$. **2.** 120°. **3.** $\sqrt{507}$. **7.** 0. **8.** $\lambda = -4$.