# JEE Type Solved Examples :
## Single Option Correct Type Questions

- This section contains **5 multiple choice examples**. Each example has four choices (a), (b), (c) and (d), out of which **ONLY ONE** is correct.

**Ex. 1** *When $P$ is a natural number, then $P^{n+1}+(P+1)^{2n-1}$ is divisible by*

(a) $P$ (b) $P^2+P$ (c) $P^2+P+1$ (d) $P^2-1$

**Sol.** (*c*) For $n=1$, we get

$$P^{n+1}+(P+1)^{2n-1}=P^2+(P+1)^1=P^2+P+1,$$

which is divisible by $P^2+P+1$, so result is true for $n=1$.

Let us assume that the given result is true for $n=m\in N$.

i.e., $P^{m+1}+(P+1)^{2m-1}$ is divisible by $P^2+P+1$.

i.e., $P^{m+1}+(P+1)^{2m-1}=k(P^2+P+1), \forall\ k\in N$ ...(i)

Now, $P^{(m+1)+1}+(P+1)^{2(m+1)-1}$

$$= P^{m+2}+(P+1)^{2m+1}$$

$$= P^{m+2}+(P+1)^2(P+1)^{2m-1}$$

$$= P^{m+2}+(P+1)^2[k(P^2+P+1)-P^{m+1}]$$

[by using Eq. (i)]

$$= P^{m+2}+(P+1)^2\cdot k(P^2+P+1)-(P+1)^2(P)^{m+1}$$

$$= P^{m+1}[P-(P+1)^2]+(P+1)^2\cdot k(P^2+P+1)$$

$$= P^{m+1}[P-P^2-2P-1]+(P+1)^2\cdot k(P^2+P+1)$$

$$= -P^{m+1}[P^2+P+1]+(P+1)^2\cdot k(P^2+P+1)$$

$$= (P^2+P+1)[k\cdot(P+1)^2-P^{m+1}]$$

which is divisible by $P^2+P+1$, so the result is true for $n=m+1$. Therefore, the given result is true for all $n\in N$ by induction.

**Ex. 2** *Let $P(n)$ denote the statement that $n^2+n$ is odd. It is seen that $P(n)\Rightarrow P(n+1)$, $P(n)$ is true for all*

(a) $n>1$ (b) $n$

(c) $n>2$ (d) None of these

**Sol.** (*d*) $P(n)=n^2+n$. It is always odd (statement) but square of any odd number is always odd and also, sum of two odd numbers is always even. So, for no any '$n$' for which this statement is true.

**Ex. 3** *For a positive integer $n$,*

*let* $a(n)=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots+\frac{1}{2^n-1}$. *Then*

(a) $a(100)>100$

(b) $a(100)<200$

(c) $a(200)\le 100$

(d) $a(200)>100$

**Sol.** (*d*) It can be proved with the help of mathematical induction that

$$\frac{n}{2}<a(n)\le n$$

$$\Rightarrow \quad \frac{200}{2}<a(200)$$

$$\Rightarrow \quad a(200)>100$$

**Ex. 4** *Let $S(k)=1+3+5+\ldots+(2k-1)=3+k^2$. Which of the following is true?*

(a) Principle of mathematical induction can be used to prove the formula

(b) $S(k)\Rightarrow S(k+1)$

(c) $S(k)\not\Rightarrow S(k+1)$

(d) $S(1)$ is correct

**Sol.** (*c*) We have, $S(k)=1+3+5+\ldots+(2k-1)=3+k^2$,

$S(1)\Rightarrow 1=4$, which is not true

and $S(2)\Rightarrow 4=7$, which is not true.

Hence, induction cannot be applied and $S(k)\ne S(k+1)$.

**Ex. 5** $10^n+3(4^{n+2})+5$ *is divisible by* ($n\in N$)

(a) 7 (b) 5

(c) 9 (d) 17

**Sol.** (*c*) $10^n+3(4^{n+2})+5$

Taking $n=2$,

$$10^2+3\times4^4+5=100+768+5=873$$

Therefore, this is divisible by 9.

## JEE Type Solved Examples :
## Statement I and II Type Questions

- **Directions** Example numbers 6 and 7 are Assertion-Reason type Examples. Each of these Examples contains two statements :
**Statement-1** (Assertion) and
**Statement-2** (Reason)
Each of these examples also has four alternative choices, only one of which is the correct answer. You have to select the correct choice as given below:
(a) Statement-1 is true, Statement-2 is true Statement-2 is correct explanation for Statement-1
(b) Statement-1 is true, Statement-2 is true, Statement-2 is not the correct explanation for Statement-1
(c) Statement-1 is true, Statement-2 is false
(d) Statement-1 is false, Statement-2 is true

**Ex. 6** **Statement-1** *For all natural number n,*

$$1+2+\ldots+n<(2n+1)^2$$

**Statement-2** *For all natural numbers,* $(2n+3)^2<7(n+1)$

**Sol.** (*b*) Let $P(n): 1+2+3+\ldots+n<(2n+1)^2$

***Step* I** For $n=1$,

$P(1): 1<(2+1)^2 \Rightarrow 1<9$

which is true.

***Step* II** Assume $P(n)$ is true for $n=k$, then

$P(k): 1+2+\ldots+k<(2k+1)^2$

***Step* III** For $n=k+1$, we shall prove that

$P(k+1): 1+2+3+\ldots+k+(k+1)<(2k+3)^2$

From assumption step

$1+2+3+\ldots+k+(k+1)<(2k+1)^2+k+1$

$=4k^2+5k+2$

$=(2k+3)^2-7(k+1)<(2k+3)^2 \qquad [\because 7(k+1)>0]$

$\therefore\ P(k+1)$ is true.

Here, both Statements are true but Statement-2 is not correct explanation of Statement-1.

**Ex. 7** **Statement-1** *For all natural numbers n,*

$$7+77+777+\ldots\underbrace{777\ldots7}_{n \text{ digits}}=\frac{7}{81}(10^{n+1}-9n-10)$$

**Statement-2** *For all natural numbers,*

$$\underbrace{777\ldots7}_{n \text{ digits}}=7+7\times10+7\times10^2+\ldots+7\times10^n$$

**Sol.** (*c*) $\because\ \underbrace{777\ldots7}_{n \text{ digits}}=7(\underbrace{111\ldots1}_{n \text{ digits}})$

$=7(1+10+10^2+\ldots+10^{n-1})$

$=7+7\times10+7\times10^2+\ldots+7\times10^{n-1}$

$\neq 7+7\times10+7\times10^2+\ldots+7\times10^n$

$\therefore$ Statement-2 is false.

Now, let $P(n): 7+77+777+\ldots+\underbrace{777\ldots7}_{n \text{ digits}}=\frac{7}{81}$

$(10^{n+1}-9n-10)$

***Step* I** For $n=1$,

LHS $=7$ and RHS $=\frac{7}{81}(10^2-9-10)=7$

$\therefore$ LHS = RHS

which is true for $n=1$.

***Step* II** Assume $P(n)$ is true for $n=k$, then

$P(k): 7+77+777+\ldots+\underbrace{777\ldots7}_{k \text{ digits}}$

$$=\frac{7}{81}(10^{k+1}-9k-10)$$

***Step* III** For $n=k+1$,

$P(k+1): 7+77+777+\ldots+\underbrace{777\ldots7}_{k \text{ digits}}+\underbrace{777\ldots7}_{(k+1) \text{ digits}}$

$$=\frac{7}{81}[10^{k+2}-9(k+1)-10]$$

LHS $=7+77+777+\ldots+\underbrace{777\ldots7}_{k \text{ digits}}+\underbrace{777\ldots7}_{(k+1) \text{ digits}}$

$=\frac{7}{81}(10^{k+1}-9k-10)+7(1+10+10^2+\ldots+10^k)$

[from assumption step]

$=\frac{7}{81}(10^{k+1}-9k-10)+\frac{7(10^{k+1}-1)}{10-1}$

$=\frac{7}{81}(10^{k+1}-9k-10+9\cdot10^{k+1}-9)$

$=\frac{7}{81}[10^{k+1}(1+9)-9(k+1)-10]$

$=\frac{7}{81}[10^{k+2}-9(k+1)-10]$

= RHS

Therefore, $P(k+1)$ is true. Hence, by mathematical induction $P(n)$ is true for all natural numbers.

Hence, Statement-1 is true and Statement-2 is false.

## Subjective Type Examples

- In this section, there are **8 subjective** solved examples.

**Ex. 8** *Prove by induction that the integer next greater than $(3+\sqrt{5})^n$ is divisible by $2^n$ for all $n \in N$.*

**Sol.** Let $\alpha = 3+\sqrt{5}$ and $\beta = 3-\sqrt{5}$

$\therefore \quad 0 < \beta^n < 1, \forall n \in N$

$\Rightarrow \quad \alpha + \beta = 6, \alpha\beta = 4 \quad \text{...(i)}$

Then, $\alpha$ and $\beta$ are the roots of

$$x^2 - 6x + 4 = 0$$

$$\Rightarrow \left.\begin{aligned} \alpha^2 &= 6\alpha - 4 \\ \beta^2 &= 6\beta - 4 \end{aligned}\right] \quad \text{...(ii)}$$

$$\therefore \quad \alpha^2 + \beta^2 = 6(\alpha+\beta) - 8 = 28 \quad \text{...(iii)}$$

$$\therefore \quad \alpha^n + \beta^n = (3+\sqrt{5})^n + (3-\sqrt{5})^n$$

$$= 2[3^n + {}^nC_2 3^{n-2}\cdot 5 + {}^nC_4 3^{n-4}\cdot 5^2 + \dots]$$

= Even integer.

As, $0 < \beta^n < 1$, $\alpha^n + \beta^n$ is the even integer next greater than $\alpha^n$.

***Step* I** For $n = 1$,

$\alpha + \beta = 6$ [from Eq. (i)]

divisible by $2^1$

and for $n = 2$, $\alpha^2 + \beta^2 = 28$ [from Eq. (iii)]

divisible by $2^2$

which is true for $n = 1, 2$.

***Step* II** Assume it is true for $n = k$.

i.e., $\alpha^k + \beta^k$ is divisible by $2^k$.

***Step* III** For $n = k+1$,

the integer next greater than $\alpha^{k+1}$ is $\alpha^{k+1} + \beta^{k+1}$

$$= \alpha^2 \cdot \alpha^{k-1} + \beta^2 \cdot \beta^{k-1}$$

$$= (6\alpha - 4)\cdot\alpha^{k-1} + (6\beta - 4)\cdot\beta^{k-1} \quad \text{[from Eq. (ii)]}$$

$$= 6(\alpha^k + \beta^k) - 4(\alpha^{k-1} + \beta^{k-1})$$

= 3 (divisible by $2^{k+1}$) – (divisible by $2^{k+1}$)

= Divisible by $2^{k+1}$

This shows that the result is true for $n = k+1$. Hence, the integer next greater than $\alpha^{k+1}$ is divisible by $2^{k+1}$.

**Ex. 9** *Using mathematical induction, show that*

$$\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\dots\left(1-\frac{1}{(n+1)^2}\right) = \frac{n+2}{2(n+1)}, \forall n \in N.$$

**Sol.** Let $P(n): \left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\dots\left(1-\frac{1}{(n+1)^2}\right)$

$$= \frac{n+2}{2(n+1)} \quad \text{...(i)}$$

***Step* I** For $n = 1$,

LHS of Eq. (i) $= 1 - \frac{1}{2^2} = \frac{3}{4}$ and RHS of Eq. (i)

$$= \frac{3}{2\cdot 2} = \frac{3}{4}.$$

Therefore, $P(1)$ is true.

***Step* II** Assume it is true for $n = k$, then

$$P(k): \left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\dots\left(1-\frac{1}{(k+1)^2}\right)$$

$$= \frac{k+2}{2(k+1)}$$

***Step* III** For $n = k+1$,

$$P(k+1): \left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\dots \left(1-\frac{1}{(k+1)^2}\right)\left(1-\frac{1}{(k+2)^2}\right) = \frac{k+3}{2(k+2)}$$

$$\therefore \quad \text{LHS} = \left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\dots \left(1-\frac{1}{(k+1)^2}\right)\left(1-\frac{1}{(k+2)^2}\right)$$

$$= \frac{k+2}{2(k+1)}\left(1-\frac{1}{(k+2)^2}\right) \quad \text{[by assumption step]}$$

$$= \frac{(k+2)}{2(k+1)}\cdot\frac{[(k+2)^2-1)]}{(k+2)^2} = \frac{k^2+4k+3}{2(k+1)(k+2)}$$

$$= \frac{(k+1)(k+3)}{2(k+1)(k+2)} = \frac{(k+3)}{2(k+2)} = \text{RHS}$$

This shows that the result is true for $n = k+1$. Hence, by the principle of mathematical induction, the result is true for all $n \in N$.

**Ex. 10** *Using the principle of mathematical induction to show that*

$$\tan^{-1}\left(\frac{x}{1+1\cdot 2\cdot x^2}\right) + \tan^{-1}\left(\frac{x}{1+2\cdot 3\cdot x^2}\right) + \dots + \tan^{-1}\left(\frac{x}{1+n(n+1)x^2}\right)$$

$$= \tan^{-1}(n+1)x - \tan^{-1}x, \forall x \in N.$$

**Sol.** Let $P(n): \tan^{-1}\left(\frac{x}{1+1\cdot 2\cdot x^2}\right)+\tan^{-1}\left(\frac{x}{1+2\cdot 3\cdot x^2}\right)$

$$+\ldots+\tan^{-1}\left(\frac{x}{1+n(n+1)x^2}\right)$$

$$=\tan^{-1}(n+1)x-\tan^{-1}x \quad \ldots\text{(i)}$$

***Step* I** For $n=1$,

LHS of Eq. (i) $=\tan^{-1}\left(\frac{x}{1+1\cdot 2\cdot x^2}\right)$

$$=\tan^{-1}\left(\frac{2x-x}{1+2x\cdot x}\right)=\tan^{-1}2x-\tan^{-1}x$$

= RHS of Eq. (i)

Therefore, $P(1)$ is true.

***Step* II** Assume it is true for $n=k$.

$$P(k): \tan^{-1}\left(\frac{x}{1+1\cdot 2\,x^2}\right)+\tan^{-1}\left(\frac{x}{1+2\cdot 3x^2}\right)+\ldots$$

$$+\tan^{-1}\left(\frac{x}{[1+k(k+1)x^2]}\right)$$

$$=\tan^{-1}(k+1)x-\tan^{-1}x$$

***Step* III** For $n=k+1$,

$$P(k+1): \tan^{-1}\left(\frac{x}{1+1\cdot 2\cdot x^2}\right)+\tan^{-1}\left(\frac{x}{1+2\cdot 3\cdot x^2}\right)$$

$$+\ldots+\tan^{-1}\left(\frac{x}{1+k(k+1)x^2}\right)$$

$$+\tan^{-1}\left(\frac{x}{1+(k+1)(k+2)x^2}\right)$$

$$=\tan^{-1}(k+2)x-\tan^{-1}x$$

$$\therefore \text{LHS}=\tan^{-1}\left(\frac{x}{1+1\cdot 2\cdot x^2}\right)$$

$$+\tan^{-1}\left(\frac{x}{1+2\cdot 3\cdot x^2}\right)+\ldots+\tan^{-1}\left(\frac{x}{1+k(k+1)x^2}\right)$$

$$+\tan^{-1}\left(\frac{x}{1+(k+1)(k+2)x^2}\right)$$

$$=\tan^{-1}(k+1)x-\tan^{-1}x$$

$$+\tan^{-1}\left(\frac{x}{1+(k+1)(k+2)x^2}\right)$$

[by assumption step]

$$=\tan^{-1}(k+1)x-\tan^{-1}x$$

$$+\tan^{-1}\left(\frac{(k+2)x-(k+1)x}{1+(k+2)x(k+1)x}\right)$$

$$=\tan^{-1}(k+1)x-\tan^{-1}x+\tan^{-1}(k+2)x$$

$$-\tan^{-1}(k+1)x$$

$$=\tan^{-1}(k+2)x-\tan^{-1}x=\text{RHS}$$

This shows that the result is true for $n=k+1$. Hence, by the principle of mathematical induction, the result is true for all $n\in N$.

**Ex. 11** *Use the principle of mathematical induction to prove that for all $n\in N$.*

$$\sqrt{2+\sqrt{2+\sqrt{2+\ldots+\ldots+\sqrt{2}}}}=2\cos\left(\frac{\pi}{2^{n+1}}\right)$$

*when the LHS contains n radical sign.*

**Sol.** Let $P(n)=\sqrt{2+\sqrt{2+\sqrt{2+\ldots+\ldots+\sqrt{2}}}}$

$$=2\cos\left(\frac{\pi}{2^{n+1}}\right) \quad \ldots\text{(i)}$$

***Step* I** For $n=1$,

LHS of Eq. (i) $=\sqrt{2}$ and RHS Eq. (i) $=2\cos\left(\frac{\pi}{2^2}\right)$

$$=2\cos\left(\frac{\pi}{4}\right)=2\cdot\frac{1}{\sqrt{2}}=\sqrt{2}$$

Therefore, $P(1)$ is true.

***Step* II** Assume it is true for $n=k$,

$$P(k)=\underbrace{\sqrt{2+\sqrt{2+\sqrt{2+\ldots+\ldots+\sqrt{2}}}}}_{k\text{ radical sign}}=2\cos\left(\frac{\pi}{2^{k+1}}\right)$$

***Step* III** For $n=k+1$,

$$\therefore\ P(k+1)=\underbrace{\sqrt{2+\sqrt{2+\sqrt{2+\ldots+\ldots+\sqrt{2}}}}}_{(k+1)\text{ radical sign}}$$

$$=\sqrt{\{2+P(k)\}}$$

$$=\sqrt{2+2\cos\left(\frac{\pi}{2^{k+1}}\right)} \quad \text{[by assumption step]}$$

$$=\sqrt{2\left(1+\cos\left(\frac{\pi}{2^{k+1}}\right)\right)}$$

$$=\sqrt{2\left(1+2\cos^2\left(\frac{\pi}{2^{k+2}}\right)-1\right)}$$

$$=\sqrt{4\cos^2\left(\frac{\pi}{2^{k+2}}\right)}=2\cos\left(\frac{\pi}{2^{k+2}}\right)$$

This shows that the result is true for $n=k+1$. Hence, by the principle of mathematical induction, the result is true for all $n\in N$.

**Ex. 12** *Prove by mathematical induction that*

$$\frac{1}{1+x}+\frac{2}{1+x^2}+\frac{4}{1+x^4}+\ldots+\frac{2^n}{1+x^{2^n}}$$

$$=\frac{1}{x-1}+\frac{2^{n+1}}{1-x^{2^{n+1}}}$$

*where, $|x|\neq 1$ and n is non-negative integer.*

**Sol.** Let $P(n): \frac{1}{1+x}+\frac{2}{1+x^2}+\frac{4}{1+x^4}+\ldots+\frac{2^n}{1+x^{2^n}}$

$$=\frac{1}{x-1}+\frac{2^{n+1}}{1-x^{2^{n+1}}} \quad \ldots\text{(i)}$$

**Step I** For $n = 1$,

$$\text{LHS of Eq. (i)} = \frac{1}{1+x} + \frac{2}{1+x^2}$$

$$= \frac{1}{x-1} - \frac{1}{x-1} + \frac{1}{1+x} + \frac{2}{1+x^2}$$

$$= \frac{1}{x-1} + \left(\frac{1}{1-x} + \frac{1}{1+x}\right) + \frac{2}{1+x^2}$$

$$= \frac{1}{x-1} + \frac{2}{1-x^2} + \frac{2}{1+x^2}$$

$$= \frac{1}{x-1} + 2\left(\frac{2}{(1-x^2)(1+x^2)}\right) = \frac{1}{x-1} + \frac{2^2}{1-x^2}$$

= RHS of Eq. (i)

**Step II** Assume it is true for $n = k$, then

$$P(k): \frac{1}{1+x} + \frac{2}{1+x^2} + \frac{4}{1+x^4} + \ldots + \frac{2^k}{1+x^{2^k}}$$

$$= \frac{1}{x-1} + \frac{2^{k+1}}{1-x^{2^{k+1}}}$$

**Step III** For $n = k+1$,

$$P(k+1): \frac{1}{1+x} + \frac{2}{1+x^2} + \frac{4}{1+x^4} + \ldots + \frac{2^k}{1+x^{2^k}} + \frac{2^{k+1}}{1+x^{2^{k+1}}}$$

$$= \frac{1}{x-1} + \frac{2^{k+2}}{1-x^{2^{k+1}}}$$

$$\text{LHS} = \frac{1}{1+x} + \frac{2}{1+x^2} + \frac{4}{1+x^4} + \ldots + \frac{2k}{1+x^{2^k}} + \frac{2^{k+1}}{1+x^{2^{k+1}}}$$

$$= \frac{1}{x-1} + \frac{2^{k+1}}{1-x^{2^{k+1}}} + \frac{2^{k+1}}{1+x^{2^{k+1}}}$$

[by assumption step]

$$= \frac{1}{x-1} + 2^{k+1}\left(\frac{2}{(1-x^{2^{k+1}})(1+x^{2^{k+1}})}\right)$$

$$= \frac{1}{x-1} + \frac{2^{k+2}}{1-x^{2^{k+2}}} = \text{RHS}$$

This shows that the result is true for $n = k+1$. Hence, by the principle of mathematical induction, the result is true for all $n \in N$.

● ***Ex. 13*** *Using the principle of mathematical induction to prove that* $\int_0^{\pi/2} \frac{\sin^2 nx}{\sin x}\, dx = 1 + \frac{1}{3} + \frac{1}{5} + \ldots + \frac{1}{2n-1}$

***Sol.*** Let $P(n): \int_0^{\pi/2} \frac{\sin^2 nx}{\sin x}\, dx = 1 + \frac{1}{3} + \frac{1}{5} + \ldots + \frac{1}{2n-1}$ ...(i)

**Step I** For $n = 1$,

$$\text{LHS of Eq. (i)} = \int_0^{\pi/2} \frac{\sin^2 x}{\sin x}\, dx = \int_0^{\pi/2} \sin x\, dx$$

$$= -[\cos x]_0^{\pi/2} = -(0-1) = 1$$

and RHS of Eq. (i) = 1

Therefore, $P(1)$ is true.

**Step II** Assume it is true for $n = k$, then

$$P(k): \int_0^{\pi/2} \frac{\sin^2 kx}{\sin x}\, dx = 1 + \frac{1}{3} + \frac{1}{5} + \ldots + \frac{1}{2k-1}$$

**Step III** For $n = k+1$,

$$P(k+1): \int_0^{\pi/2} \frac{\sin^2 (k+1)x}{\sin x}\, dx = 1 + \frac{1}{3} + \frac{1}{5} + \ldots + \frac{1}{2k-1} + \frac{1}{2k+1}$$

$$\text{LHS} = \int_0^{\pi/2} \frac{\sin^2 (k+1)x}{\sin x}\, dx$$

$$= \int_0^{\pi/2} \frac{\sin^2 (k+1)x - \sin^2 kx + \sin^2 kx}{\sin x}\, dx$$

$$= \int_0^{\pi/2} \frac{\sin^2 (k+1)x - \sin^2 kx}{\sin x}\, dx + \int_0^{\pi/2} \frac{\sin^2 kx}{\sin x}\, dx$$

$$= \int_0^{\pi/2} \frac{\sin (2k+1)x \sin x}{\sin x}\, dx + P(k)$$

[by assumption step]

$$= \int_0^{\pi/2} \sin (2k+1)x\, dx + P(k)$$

$$= -\left[\frac{\cos (2k+1)x}{2k+1}\right]_0^{\pi/2} + P(k)$$

$$= -\frac{1}{(2k+1)}\left[\cos\left(\pi k + \frac{\pi}{2}\right) - 1\right] + P(k)$$

$$= -\frac{1}{(2k+1)}[-\sin \pi k - 1] + P(k)$$

$$= -\frac{1}{2k+1}[-0-1] + P(k)$$

$$= \frac{1}{(2k+1)} + 1 + \frac{1}{3} + \frac{1}{5} + \ldots + \frac{1}{(2k-1)}$$

[by assumption step]

$$= 1 + \frac{1}{3} + \frac{1}{5} + \ldots + \frac{1}{(2k-1)} + \frac{1}{(2k+1)}$$

= RHS

This shows that the result is true for $n = k+1$. Hence, by the principle of mathematical induction, the result is true for all $n \in N$.

**Ex. 14** *Use induction to show that for all $n \in N$.*

$$\sqrt{a+\sqrt{a+\sqrt{a+\ldots+\sqrt{a}}}} < \frac{1+\sqrt{(4a+1)}}{2}$$

*where 'a' is fixed positive number and n radical signs are taken on LHS.*

**Sol.** Let $P(n): \sqrt{a+\sqrt{a+\sqrt{a+\ldots+\sqrt{a}}}} < 1+\sqrt{\frac{(4a+1)}{2}}$

**Step I** For $n=1$, then $\sqrt{a} < \dfrac{1+\sqrt{(4a+1)}}{2}$

$\Rightarrow \quad 2\sqrt{a} < 1+\sqrt{(4a+1)}$

$\Rightarrow \quad 4a < 1+4a+1+2\sqrt{(4a+1)}$

$\Rightarrow \quad 2\sqrt{(4a+1)}+2 > 0$ which is true.

Therefore, $P(1)$ is true.

**Step II** Assume it is true for $n = k$,

$$P(k): \underbrace{\sqrt{a+\sqrt{a+\sqrt{a+\ldots+\sqrt{a}}}}}_{k \text{ radical signs}} < \frac{1+\sqrt{(4a+1)}}{2}$$

**Step III** For $n = k+1$,

$$P(k+1): \underbrace{\sqrt{a+\sqrt{a+\sqrt{a+\ldots+\sqrt{a}}}}}_{(k+1) \text{ radical signs}} < \frac{1+\sqrt{(4a+1)}}{2}$$

From assumption step

$$\underbrace{\sqrt{a+\sqrt{a+\sqrt{a+\ldots+\sqrt{a}}}}}_{k \text{ radical signs}} < \frac{1+\sqrt{4a+1}}{2}$$

$$a+\underbrace{\sqrt{a+\sqrt{a+\sqrt{a+\ldots+\sqrt{a}}}}}_{k \text{ radical signs}} < a+\frac{1+\sqrt{(4a+1)}}{2}$$

$$\Rightarrow \underbrace{\sqrt{a+\sqrt{a+\sqrt{a+\sqrt{a+\ldots+\sqrt{a}}}}}}_{(k+1) \text{ radical signs}} < \sqrt{a+\frac{1+\sqrt{(4a+1)}}{2}}$$

$$= \sqrt{\frac{2a+1+\sqrt{(4a+1)}}{2}} = \sqrt{\frac{4a+2+2\sqrt{(4a+1)}}{4}}$$

$$= \sqrt{\frac{(\sqrt{(4a+1)})^2+1+2\sqrt{(4a+1)}}{4}}$$

$$= \sqrt{\left(\frac{1+\sqrt{(4a+1)}}{2}\right)^2} = \frac{1+\sqrt{(4a+1)}}{2}$$

$$\therefore \underbrace{\sqrt{a+\sqrt{a+\sqrt{a+\sqrt{a+\ldots+\sqrt{a}}}}}}_{(k+1) \text{ radical signs}} < \frac{1+\sqrt{(4a+1)}}{2}$$

which is true for $n = k+1$.

Hence, by the principle of mathematical induction, the result is true for all $n \in N$.

**Ex. 15** *Prove by induction that*

$$\left\{\prod_{r=0}^{n} f_r(x)\right\}' = \sum_{i=1}^{n}\{f_1(x)f_2(x)\ldots f_i'(x)\ldots f_n(x)\},$$

*where dash denotes derivative with respect to x.*

**Sol.** Let $P(n): \left\{\prod_{r=0}^{n} f_r(x)\right\}' = \sum_{i=1}^{n}\{f_1(x)f_2(x)\ldots f_{i'}(x)\ldots f_n(x)\}$ ...(i)

**Step I** For $n = 1$,

$$\text{LHS of Eq. (i)} = \left\{\prod_{r=1}^{1} f_r(x)\right\}' = \{f_1(x)\}' = f_1'(x)$$

$$\text{RHS of Eq. (i)} = \sum_{i=1}^{1}\{f_1(x)f_2(x)\ldots f_i'(x)\ldots f_1(x)\}$$

$$= f_1'(x)$$

which is true for $n = 1$.

**Step II** Assume it is true for $n = k$, then

$$P(k): \left\{\prod_{r=1}^{k} f_r(x)\right\}' = \sum_{i=1}^{k}\{f_1(x)\,f_2(x)\ldots f_{i'}(x)\ldots f_k(x)\}$$

**Step III** For $n = k+1$,

$$P(k+1): \left\{\prod_{r=1}^{k+1} f_r(x)\right\}' = \sum_{i=1}^{k+1}\{f_1(x)\,f_2(x)\ldots f_{i'}(x)\ldots f_k(x)\}$$

$$\text{LHS} = \left\{\prod_{r=1}^{(k+1)} f_r(x)\right\}' = \left\{\prod_{r=1}^{k} f_r(x)\cdot f_{k+1}(x)\right\}'$$

$$= \prod_{r=1}^{k} f_r(x)\cdot f'_{k+1}(x) + f_{k+1}(x)\left\{\prod_{r=1}^{k} f_r(x)\right\}'$$

$$= \prod_{r=1}^{k} f_r(x)\cdot f'_{k+1}(x) + f_{k+1}(x) \cdot\sum_{i=1}^{k}\{f_1(x)\cdot f_2(x)\ldots f_{i'}(x)\ldots f_k(x)\}$$

[by assumption step]

$$= \{f_1(x)\,f_2(x)\ldots f_k(x)\}\,f'_{k+1}(x) + f_{k+1}(x)\sum_{i=1}^{k}\{f_1(x)\,f_2(x)\ldots f_{i'}(x)\ldots f_k(x)\}$$

$$= \sum_{i=1}^{k+1}\{f_1(x)\,f_2(x)\ldots f_{i'}(x)\ldots f_{k+1}(x)\}$$

$= \text{RHS}$

This shows that the result is true for $n = k+1$. Hence, by the principle of mathematical induction, the result is true for all $n \in N$.