Session 3

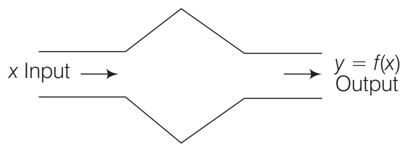
Definition of Functions, Domain, Codomain and Range, Composition of Mapping, Equivalence Classes, Partition of Set, Congruences

Functions

Introduction

If two variable quantities x and y according to some law are so related that corresponding to each value of x (considered only real), which belongs to set E, there corresponds one and only one finite value of the quantity y (i.e., unique value of y). Then, y is said to be a function (single valued) of x, defined by $y = f(x)$, where x is the **argument** or **independent variable** and y is the **dependent variable** defined on the set E.

For example, If r is the radius of the circle and A its area, then r and A are related by $A = \pi r^2$ or $A = f(r)$. Then, we say that the area A of the circle is the function of the radius r. **Graphically**,



Where, y is the image of x and x is the pre-image of y under f.

Remark

1. If to each value of x, which belongs to set E there corresponds one or more than one values of the quantity y. Then, y is called the multiple valued function of x defined on the set E.
2. The word 'FUNCTION' is used only for single valued function. *For example,* $y = \sqrt{x}$ is single valued functions but $y^2 = x$ is a multiple valued function.
   $\therefore$ $y^2 = x \Rightarrow y = \pm\sqrt{x}$ for one value of x, y gives two values.

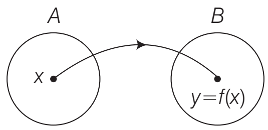
Definition of Functions

If A and B be two non-empty sets, then a function from A to B associates to each element x in A, a unique element $f(x)$ in B and is written as

$$f : A \to B \text{ or } A \xrightarrow{f} B$$

which is read as f is a mapping from A to B.

The other terms used for functions are **operators** or **transformations.**



Remark

1. If $x \in A$, $y = [f(x)] \in B$, then $(x, y) \in f$.
2. If $(x_1, y_1) \in f$ and $(x_2, y_2) \in f$, then $y_1 = y_2$.

Domain, Codomain and Range

Domain The set of A is called the domain of f (denoted by D_f).

Codomain The set of B is called the codomain of f (denoted by C_f).

Range The range of f denoted by R_f is the set consisting of all the images of the elements of the domain A.

Range of $f = [f(x) : x \in A]$

The range of f is always a subset of codomain B.

Onto and Into Mappings

In the mapping $f : A \to B$ such

$$f(A) = B$$

i.e., $\quad$ Range = Codomain

Then, the function is **Onto** and if $f(A) \subset B$, i.e. Range $\subset$ Codomain, then the function is **Into**.

Remark

Onto functions is also known as **surjective.**

Method to Test Onto or Into Mapping

Let $f : A \to B$ be a mapping. Let y be an arbitrary element in B and then $y = f(x)$, where $x \in A$. Then, express x in terms of y.

Now, if $x \in A, \forall y \in B$, then f is onto

and if $x \notin A, \forall y \in B$, then f is into.

For into mapping Find an element of B which is not f-image of any element of A.

One-one and Many-one Mapping

(i) The mapping $f : A \to B$ is called one-one mapping, if no two different elements of A have the same image in B. Such a mapping is also known as **injective mapping** or an **injection** or **monomorphism.**

Method to Test One-one If $x_1, x_2 \in A$,

then $$f(x_1) = f(x_2)$$

$$\Rightarrow \quad x_1 = x_2 \text{ and } x_1 \neq x_2$$

$$\Rightarrow \quad f(x_1) \neq f(x_2)$$

(ii) The mapping $f : A \to B$ is called many-one mapping, if two or more than two different elements in A have the same image in B.

Method to Test Many-one

If $x_1, x_2 \in A$, then $f(x_1) = f(x_2)$

$$\Rightarrow \quad x_1 \neq x_2$$

From above classification, we conclude that function is of four types

(i) One-one onto (bijective)
(ii) One-one into
(iii) Many-one onto
(iv) Many-one into

Number of Functions (Mappings) at One Place in a Table

Let $f : A \to B$ be a mapping such that A and B are finite sets having m and n elements respectively, then

	Description of mappings
(i)	Total number of mappings from A to B
(ii)	Total number of one-one mappings from A to B
(iii)	Total number of many-one mappings from A to B
(iv)	Total number of onto (surjective) mappings from A to B
(v)	Total number of one-one onto (bijective) mappings from A to B
(vi)	Total number of into mappings from A to B

Example 21. Let N be the set of all natural numbers. Consider $f : N \to N : f(x) = 2x, \forall x \in N$. Show that f is one-one into**.**

Sol. Let $x_1, x_2 \in N$, then

$$f(x_1) = f(x_2)$$

$$\Rightarrow \quad 2x_1 = 2x_2 \Rightarrow x_1 = x_2$$

$\therefore$ f is one-one.

Let $\quad y = 2x$, then $x = \frac{y}{2}$

Now, if we put $y = 5$, then $x = \frac{5}{2} \notin N$.

This show that $5 \in N$ has no pre-image in N. So, f is into.

Hence, f is one-one and into.

Example 22. Show that the mapping $f : R \to R : f(x) = \cos x, \forall x \in R$ is neither one-one nor onto.

Sol. Let $x_1, x_2 \in R$.

Then, $\quad f(x_1) = f(x_2) \Rightarrow \cos x_1 = \cos x_2$

$$\Rightarrow \quad x_1 = 2n\pi \pm x_2 \Rightarrow x_1 \neq x_2$$

$\therefore$ f is not one-one.

Let $\quad y = \cos x$, but $-1 \leq \cos x \leq 1$

$$\therefore \quad y \in [-1, 1]$$

$$[-1, 1] \subset R$$

So, f is into (not onto).

Hence, f is neither one-one nor onto.

Constant Mapping

The mapping $f : A \to B$ is known as a constant mapping, if the range of B has only one element.

For all $x \in A$, $f(x) = a$, where as $a \in B$.

Identity Mapping

The mapping $f : A \to B$ is known as an identity mapping, if $f(a) = a, \forall a \in A$ and it is denoted by I_A.

Remark

I_A is bijective or bijection.

Equal Mapping

Let A and B be two mappings are $f : A \to B$ and $g : A \to B$ such that

$$f(x) = g(x), \forall x \in A$$

Then, the mappings f and g are equal and written as $f = g$.

Inclusion Mapping

The mapping $f : A \to B$ is known as inclusion mapping.

If $A \subseteq B$, then $f(a) = a, \forall a \in A$.

Equivalent or Equipotent or Equinumerous Set

The mapping $f : A \to B$ is known as equivalent sets, if A and B are both one-one and onto and written as $A \sim B$ which is read as 'A wiggle B'.

Inverse Mapping

If $f : A \to B$ be one-one and onto mapping, let $b \in B$, then there exist exactly one element $a \in A$ such that $f(a) = b$, so we may define

$$f^{-1} : B \to A : f^{-1}(b) = a$$

$$\Leftrightarrow \qquad f(a) = b$$

The function f^{-1} is called the inverse of f. A functions is invertible iff f is one-one onto.

Remark

1. $f^{-1}(b) \subseteq A$
2. If $f : A \to B$ and $g : B \to A$, then f and g are said to be invertible.

Example 23. Let $f : R \to R$ be defined by $f(x) = \cos(5x + 2)$. Is f invertible? Justify your answer.

Sol. For invertible of f, f must be bijective (i.e., one-one onto).

If $\quad x_1, x_2 \in R$,

then $\quad f(x_1) = f(x_2)$

$\Rightarrow \quad \cos(5x_1 + 2) = \cos(5x_2 + 2)$

$\Rightarrow \quad 5x_1 + 2 = 2n\pi \pm (5x_2 + 2)$

$\Rightarrow \quad x_1 \neq x_2$

$\therefore$ f is not one-one.

But $\quad -1 \leq \cos(5x + 2) \leq 1$

$\therefore \quad -1 \leq f(x) \leq 1$

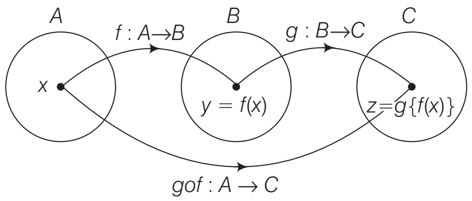
Range $= [-1, 1] \subset R$

$\therefore$ f is into mapping.

Hence, the function $f(x)$ is no bijective and so it is not invertible.

Composition of Mapping

Let A, B and C be three non-empty sets. Let $f : A \to B$ and $g : B \to C$ be two mappings, then $gof : A \to C$. This function is called the product or composite of f and g, given by $(gof)x = g\{f(x)\}, \forall\ x \in A$.



Important Remarks

1. (i) $(fog)x = f\{g(x)\}$ (ii) $(fof)x = f\{f(x)\}$
   (iii) $(gog)x = g\{g(x)\}$ (iv) $(fg)x = f(x) \cdot g(x)$
   (v) $(f \pm g)x = f(x) \pm g(x)$ (vi) $\left(\dfrac{f}{g}\right)x = \dfrac{f(x)}{g(x)}$; $g(x) \neq 0$
2. Let $h : A \to B$, $g : B \to C$ and $f : C \to D$ be any three functions. Then, $(fog)oh = fo(goh)$.
3. Let $f : A \to B$, $g : B \to C$ be two functions, then
   (i) f and g are injective $\Rightarrow gof$ is injective.
   (ii) f and g are surjective $\Rightarrow gof$ is surjective.
   (iii) f and g are bijective $\Rightarrow gof$ is bijective.
4. An injective mapping from a finite set to itself in bijective.

Example 24. If $f : R \to R$ and $g : R \to R$ be two mapping such that $f(x) = \sin x$ and $g(x) = x^2$, then

(i) prove that $fog \neq gof$.

(ii) find the values of $(fog)\dfrac{\sqrt{\pi}}{2}$ and $(gof)\left(\dfrac{\pi}{3}\right)$.

Sol. (i) Let $x \in R$

$\therefore \quad (fog)x = f\{g(x)\}$ $\qquad [\because g(x) = x^2]$

$= f\{x^2\} = \sin x^2$ $\qquad$ ...(i)

$[\because f(x) = \sin x]$

and $(gof)x = g\{f(x)\}$

$= g(\sin x)$ $\qquad [\because f(x) = \sin x]$

$= \sin^2 x$ $\qquad$ ...(ii)

$[\because g(x) = x^2]$

From Eqs. (i) and (ii), we get $(fog)x \neq (gof)x, \forall\ x \in R$

Hence, $fog \neq gof$

(ii) From Eq. (i), $(fog)x = \sin x^2$

$\therefore \quad (fog)\dfrac{\sqrt{\pi}}{2} = \sin\dfrac{\pi}{4} = \dfrac{1}{\sqrt{2}}$

and from Eq. (ii), $(gof)x = \sin^2 x$

$\therefore \quad (gof)\dfrac{\pi}{3} = \sin^2\dfrac{\pi}{3} = \left(\dfrac{\sqrt{3}}{2}\right)^2 = \dfrac{3}{4}$

Example 25. If the mapping f and g are given by

$f = \{(1, 2), (3, 5), (4, 1)\}$

and $\quad g = \{(2, 3), (5, 1), (1, 3)\}$,

write down pairs in the mapping fog and gof.

Sol. Domain $f = \{1, 3, 4\}$, Range $f = \{2, 5, 1\}$

Domain $g = \{2, 5, 1\}$, Range $g = \{1, 3\}$

$\because \quad$ Range f = Dom $g = \{(2, 5, 1)\}$

$\therefore$ gof mapping is defined.

Then, gof mapping defined following way

$$\{1, 3, 4\} \xrightarrow{f} \{2, 5, 1\} \xrightarrow{g} \{1, 3\}$$

gof

We see that, $\quad f(1) = 2, f(3) = 5, f(4) = 1$

and $\quad g(2) = 3, g(5) = 1, g(1) = 3$

$\therefore \quad (gof)(1) = g\{f(1)\} = g(2) = 3$

$(gof)(3) = g\{f(3)\} = g(5) = 1$

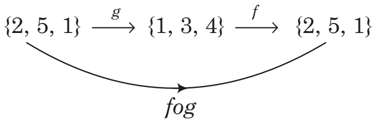
$$(gof)(4) = g\{f(4)\} = g(1) = 3$$

Hence, $gof = \{(1, 3), (3, 1), (4, 3)\}$

Now, since Range of $f \subset$ Dom f

$\therefore$ fog is defined.

Then, fog mapping defined following way



We see that, $g(2) = 3,\ g(5) = 1,\ g(1) = 3$

$f(1) = 2,\ f(3) = 5,\ f(4) = 1$

$\therefore$ $(fog)(2) = f\{g(2)\} = f(3) = 5$

$(fog)(5) = f\{g(5)\} = f(1) = 2$

$(fog)(1) = f\{g(1)\} = f(3) = 5$

Hence, $fog = \{(2,5),(5,2),(1,5)\}$

Equivalence Classes

If R be an equivalence relation on a set A, then $[a]$ is equivalence class of a with respect to R.

Symbolically, X_a or $[a] = \{x : x \in X,\ x\,R\,a\}$.

Remark

1. Square brackets[ ] are used to denote the equivalence classes.
2. $a \in [a]$ and $a \in [b] \Rightarrow [a] = [b]$
3. Either $[a] = [b]$ or $[a] \cap [b] = \phi$
4. Equivalence class of a also denoted by $E(a)$ or $\bar{a}$.
5. If $a \sim b$, $\dfrac{(a-b)}{m} = k$, the total number of equivalence class is m.

Example 26. Let $I = \{0, \pm 1, \pm 2, \pm 3, \pm 4, \dots\}$ and $R = \{(a,b) : (a-b)/4 = k, k \in I\}$ is an equivalence relation, find equivalence class.

Sol. Given, $\dfrac{a-b}{4} = k$

$\Rightarrow$ $a = 4k + b$, where $0 \le b < 4$

It is clear b has only value in 0, 1, 2, 3.

(i) Equivalence class of $[0] = \{x : x \in I \text{ and } x \sim 0\}$
$= \{x : x - 0 = 4k\} = \{0, \pm 4, \pm 8, \pm 12, \dots)$
where, $k = 0, \pm 1, \pm 2, \pm 3, \dots$

(ii) Equivalence class of $[1] = \{x : x \in I \text{ and } x \sim 1\}$
$= \{x : x - 1 = 4k\} = \{x : x = 4k + 1\}$
$= \{\dots, -11, -7, -3, 1, 5, 9, \dots\}$

(iii) Equivalence class of $[2] = \{x : x \in I \text{ and } x \sim 2\}$
$= \{x : x - 2 = 4k\} = \{x : x = 4k + 2\}$
$= \{\dots, -10, -6, -2, 2, 6, 10, \dots\}$

(iv) Equivalence class of $[3] = \{x : x \in I \text{ and } x \sim 3\}$
$= \{x : x - 3 = 4k\} = \{x : x = 4k + 3\}$
$= \{\dots, -9, -5, -1, 5, 9, 13, \dots\}$

Continue this process, we see that the equivalence class

$$[4] = [0], [5] = [1], [6] = [2], [7] = [3], [8] = [0]$$

Hence, total equivalence relations are [0], [1], [2], [3] and also clear

(i) $I = [0] \cup [1] \cup [2] \cup [3]$

(ii) every equivalence is a non-empty.

(iii) for any two equivalence classes $[a] \cap [b] = \phi$.

Partition of a Set

If A be a non-empty set, then a partition of A, if

(i) A is a collection of non-empty disjoint subsets of A.

(ii) union of collection of non-empty sets is A.

i.e., If A be a non-empty set and A_1, A_2, A_3, A_4 are subsets of A, then the set $\{A_1, A_2, A_3, A_4\}$ is called partition, if

(i) $A_1 \cup A_2 \cup A_3 \cup A_4 = A$

(ii) $A_1 \cap A_2 \cap A_3 \cap A_4 = \phi$

For example,

If $A = \{0, 1, 2, 3, 4\}$ and $A_1 = \{0\}, A_2 = \{1\}, A_3 = \{4\}$ and $A_4 = \{2, 3\}$, then we see that for $P = \{A_1, A_2, A_3, A_4\}$

(i) all A_1, A_2, A_3, A_4 are non-empty subset of A

(ii) $A_1 \cup A_2 \cup A_3 \cup A_4 = \{0, 1, 2, 3, 4\} = A$ and

(iii) $A_i \cap A_j \ne \phi, \forall\ i \ne j\ (i, j = 1, 2, 3, 4)$
Hence, from definition $P = \{A_1, A_2, A_3, A_4\}$ is partition of A.

Congruences

Let m be a positive integer, then two integers a and b are said to be congruent modulo m, if $a - b$ is divisible by m.

i.e., $m\overline{)\,a - b\,}(\lambda$
$\quad a - b$
$\quad -\ +$
$\quad\quad 0$

$\therefore$ $a - b = m\lambda$, where λ is a positive integer.

The congruent modulo 'm' is defined on all $a\ b \in I$ by $a \equiv b$ (mod m), if $a - b = m\lambda$, $\lambda \in I_+$.

Example 27. Find congruent solutions of $155 \equiv 7$ (mod 4).

Sol. Since, $\left(\dfrac{155 - 7}{4} = \dfrac{148}{4} = 37\right)$

and $a = 155, b = 7, m = 4$

$\therefore$ $\lambda = \dfrac{a-b}{4} = \dfrac{155-7}{4} = \dfrac{148}{4}$ [here, $a = 155, b = 7$]

$= 37$ (integer)

Example 28. Find all congruent solutions of $8x \equiv 6$ (mod 14).

Sol. Given, $8x \equiv 6 \pmod{14}$

$\therefore \quad \lambda = \dfrac{8x - 6}{14}$, where $\lambda \in I_+$

$\therefore \quad 8x = 14\lambda + 6$

$\Rightarrow \quad x = \dfrac{14\lambda + 6}{8}$

$\Rightarrow \quad x = \dfrac{7\lambda + 3}{4}$

$= \dfrac{4\lambda + 3(\lambda + 1)}{4}$

$x = \lambda + \dfrac{3}{4}(\lambda + 1)$, where $\lambda \in I_+$

and here greatest common divisor of 8 and 14 is 2, so there are two required solutions.

For $\lambda = 3$ and 7, $x = 6$ and 13.

Exercise for Session 3

1. The values of b and c for which the identity $f(x+1) - f(x) = 8x + 3$ is satisfied, where $f(x) = bx^2 + cx + d$, are

(a) $b = 2, c = 1$ (b) $b = 4, c = -1$ (c) $b = -1, c = 4$ (d) $b = -1, c = 1$

2. If $f(x) = \dfrac{x-1}{x+1}$, then $f(ax)$ in terms of $f(x)$ is equal to

(a) $\dfrac{f(x) + a}{1 + af(x)}$ (b) $\dfrac{(a-1)f(x) + a + 1}{(a+1)f(x) + a - 1}$ (c) $\dfrac{(a+1)f(x) + a - 1}{(a-1)f(x) + a + 1}$ (d) None of these

3. If f be a function satisfying $f(x+y) = f(x) + f(y)$, $\forall\, x, y \in R$. If $f(1) = k$, then $f(n)$, $n \in N$ is equal to

(a) k^n (b) nk (c) k (d) None of these

4. If $g = \{(1,1), (2,3), (3,5), (4,7)\}$ is a function described by the formula $g(x) = \alpha x + \beta$, what values should be assigned to α and β?

(a) $\alpha = 1, \beta = 1$ (b) $\alpha = 2, \beta = -1$ (c) $\alpha = 1, \beta = -2$ (d) $\alpha = -2, \beta = -1$

5. The values of the parameter α for which the function $f(x) = 1 + \alpha x$, $\alpha \neq 0$ is the inverse of itself, is

(a) -2 (b) -1 (c) 1 (d) 2

6. If $f(x) = (a - x^n)^{1/n}$, where $a > 0$ and $n \in N$, then $fof(x)$ is equal to

(a) a (b) x (c) x^n (d) a^n

7. If $f(x) = (ax^2 + b)^3$, the function g such that $f(g(x)) = g(f(x))$, is given by

(a) $g(x) = \left(\dfrac{b - x^{1/3}}{a}\right)^{1/2}$ (b) $g(x) = \dfrac{1}{(ax^2 + b)^3}$ (c) $g(x) = (ax^2 + b)^{1/3}$ (d) $g(x) = \left(\dfrac{x^{1/3} - b}{a}\right)^{1/2}$

8. Which of the following functions from I to itself are bijections?

(a) $f(x) = x^3$ (b) $f(x) = x + 2$ (c) $f(x) = 2x + 1$ (d) $f(x) = x^2 + x$

9. Let $f: R - \{n\} \to R$ be a function defined by $f(x) = \dfrac{x - m}{x - n}$, where $m \neq n$. Then,

(a) f is one-one onto (b) f is one-one into (c) f is many-one onto (d) f is many-one into

10. If $f(x + 2y, x - 2y) = xy$, then $f(x, y)$ equals

(a) $\dfrac{x^2 - y^2}{8}$ (b) $\dfrac{x^2 - y^2}{4}$ (c) $\dfrac{x^2 + y^2}{4}$ (d) $\dfrac{x^2 - y^2}{2}$

Answers

Exercise for Session 3

1. (b) **2.** (c) **3.** (b) **4.** (b) **5.** (b) **6.** (b)
7. (d) **8.** (b) **9.** (b) **10.** (a)