CHAPTER

07

Determinants

Session 1

Definition of Determinants, Expansion of Determinant, Sarrus Rule for Expansion, Window Rule for Expansion

Determinants were invented independently by **Gabriel Cramer**, whose now well-known rule for solving linear system was published in 1750, although not in present day notation. The now-standard "Vertical line notation", i.e. "| |" was given in 1841 by **Arthur Cayley**. The working knowledge of determinants is a basic necessity for a student. Determinants have wide applications in Engineering, Science, Economics, Social science, etc.

Definition of Determinants

Consider the system of two homogeneous linear equations

$$a_1 x + b_1 y = 0 \quad \text{...(i)}$$

$$a_2 x + b_2 y = 0 \quad \text{...(ii)}$$

in the two variables x and y. From these equations, we obtain

$$-\frac{a_1}{b_1} = \frac{y}{x} = -\frac{a_2}{b_2} \Rightarrow \frac{a_1}{b_1} = \frac{a_2}{b_2}$$

$$\Rightarrow \quad a_1 b_2 - a_2 b_1 = 0$$

The result $a_1 b_2 - a_2 b_1$ is represented by $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$

which is known as determinant of order two. The quantities a_1, b_1, a_2 and b_2 are called constituents or elements of the determinant and $a_1 b_2 - a_2 b_1$ is called its value.

The horizontal lines are called rows and vertical lines are called columns. Here, this determinant consists two rows and two columns.

For example, The value of the determinant

$$\begin{vmatrix} 2 & 3 \\ 4 & -5 \end{vmatrix} = 2 \times (-5) - 3 \times 4 = -10 - 12 = -22$$

Now, let us consider the system of three homogeneous linear equations

$$a_1 x + b_1 y + c_1 z = 0 \quad \text{...(i)}$$

$$a_2 x + b_2 y + c_2 z = 0 \quad \text{...(ii)}$$

$$a_3 x + b_3 y + c_3 z = 0 \quad \text{...(iii)}$$

On solving Eqs. (ii) and (iii) for x, y and z by cross-multiplication, we get

$$\frac{x}{b_2 c_3 - b_3 c_2} = \frac{y}{c_2 a_3 - c_3 a_2} = \frac{z}{a_2 b_3 - a_3 b_2} = k \quad \text{[say]}$$

$$\Rightarrow \quad x = k(b_2 c_3 - b_3 c_2), y = k(c_2 a_3 - c_3 a_2)$$

and $\quad z = k(a_2 b_3 - a_3 b_2)$

On putting these values of x, y and z in Eq. (i), we get

$$a_1(b_2 c_3 - b_3 c_2) + b_1(c_2 a_3 - c_3 a_2) + c_1(a_2 b_3 - a_3 b_2) = 0$$

or $\quad a_1(b_2 c_3 - b_3 c_2) - b_1(c_3 a_2 - c_2 a_3) + c_1(a_2 b_3 - a_3 b_2) = 0 \quad \text{...(iv)}$

or $\quad a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} c_2 & a_2 \\ c_3 & a_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} = 0 \quad \text{...(v)}$

Usually this is written as $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$

Here, the expression $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ consisting of three rows and three columns, is called determinant of order three.

The quantities $a_1, b_1, c_1, a_2, b_2, c_2, a_3, b_3$ and c_3 are called constituents or elements of the determinant.

Remark

1. A determinant is generally denoted by D or Δ.
2. A determinant of the nth order consists of n rows and n columns and its expansion contains $n!$ terms.
3. A determinant of nth order consists of n rows and n columns.
   $\therefore$ Number of constituents in determinant = n^2
4. In a determinant the horizontal lines counting from top to bottom 1st, 2nd, 3rd, … respectively, known as rows and denoted by $R_1, R_2, R_3, \dots$ and vertical lines from left to right 1st, 2nd, 3rd, … respectively, known as columns and denoted by $C_1, C_2, C_3, \dots$.
5. Shape of every determinant is square.
6. Sign system for order 2, order 3, order 4, … are given by

$$\begin{vmatrix} + & - \\ - & + \end{vmatrix}, \begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}, \begin{vmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{vmatrix}, \dots$$

Expansion of Determinant

(i) Expansion of two order

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = \begin{vmatrix} a_1 & \\ & b_2 \end{vmatrix} - \begin{vmatrix} & b_1 \\ a_2 & \end{vmatrix} = a_1b_2 - b_1a_2$$

For example,
$$\begin{vmatrix} 5 & -4 \\ -3 & 2 \end{vmatrix} = \begin{vmatrix} 5 & \\ & 2 \end{vmatrix} - \begin{vmatrix} & -4 \\ -3 & \end{vmatrix}$$
$$= 10 - 12 = -2$$

(ii) Expansion of third order

(a) *With respect to first row.*

$$\begin{vmatrix} a_1 \cdots & b_1 \cdots & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

$$= a_1 (b_2 c_3 - b_3 c_2) - b_1 (a_2 c_3 - a_3 c_2) + c_1 (a_2b_3 - a_3b_2)$$

(b) *With respect to second column.*

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ & \vdots & \\ a_2 & b_2 & c_2 \\ & \vdots & \\ a_3 & b_3 & c_3 \end{vmatrix} = -b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + b_2 \begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix} - b_3 \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$$

$$= -b_1 (a_2c_3 - a_3c_2) + b_2 (a_1c_3 - a_3c_1) - b_3 (a_1 c_2 - a_2 c_1)$$

Remark

A determinant can be expanded along any of its row or column. Value of the determinant remains same in any of the cases.

Example 1. Find the value of the determinant
$$\begin{vmatrix} 1 & 2 & 4 \\ 3 & 4 & 9 \\ 2 & -1 & 6 \end{vmatrix}$$

Sol. Expanding the determinant along the first row

$$\Delta_1 = 1 \begin{vmatrix} 4 & 9 \\ -1 & 6 \end{vmatrix} - 2 \begin{vmatrix} 3 & 9 \\ 2 & 6 \end{vmatrix} + 4 \begin{vmatrix} 3 & 4 \\ 2 & -1 \end{vmatrix}$$
$$= 1 (24 + 9) - 2 (18 - 18) + 4 (-3 - 8)$$
$$= 33 - 0 - 44$$
$$= -11$$

and expanding the determinant along third column

$$\Delta_2 = 4 \begin{vmatrix} 3 & 4 \\ 2 & -1 \end{vmatrix} - 9 \begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix} + 6 \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}$$
$$= 4(-3 - 8) - 9 (-1 - 4) + 6(4 - 6)$$
$$= -44 + 45 - 12$$
$$= 1 - 12$$
$$= -11$$

and expanding the determinant along second column

$$\Delta_3 = -2 \begin{vmatrix} 3 & 9 \\ 2 & 6 \end{vmatrix} + 4 \begin{vmatrix} 1 & 4 \\ 2 & 6 \end{vmatrix} - (-1) \begin{vmatrix} 1 & 4 \\ 3 & 9 \end{vmatrix}$$
$$= -2 (18 - 18) + 4(6 - 8) + 1 (9 - 12)$$
$$= 0 - 8 - 3$$
$$= -11$$

Hence, $\Delta_1 = \Delta_2 = \Delta_3$

Example 2. ***If*** $\Delta = \begin{vmatrix} 1 & \sin\theta & 1 \\ -\sin\theta & 1 & \sin\theta \\ -1 & -\sin\theta & 1 \end{vmatrix}$, ***prove that*** $2 \le \Delta \le 4$.

Sol. Given, $\Delta = \begin{vmatrix} 1 & \sin\theta & 1 \\ -\sin\theta & 1 & \sin\theta \\ -1 & -\sin\theta & 1 \end{vmatrix}$

Expanding along first row, we get

$$\Delta = 1 \begin{vmatrix} 1 & \sin\theta \\ -\sin\theta & 1 \end{vmatrix} - \sin\theta \begin{vmatrix} -\sin\theta & \sin\theta \\ -1 & 1 \end{vmatrix} + 1 \begin{vmatrix} -\sin\theta & 1 \\ -1 & -\sin\theta \end{vmatrix}$$
$$= (1 + \sin^2\theta) - \sin\theta(-\sin\theta + \sin\theta) + (\sin^2\theta + 1)$$
$$= 2(1 + \sin^2\theta)$$

Again, $0 \le \sin^2\theta \le 1$

$\Rightarrow \quad 1 \le (1 + \sin^2\theta) \le 1 + 1$

$\Rightarrow \quad 2 \le 2 (1 + \sin^2\theta) \le 4$

$\therefore \quad 2 \le \Delta \le 4$

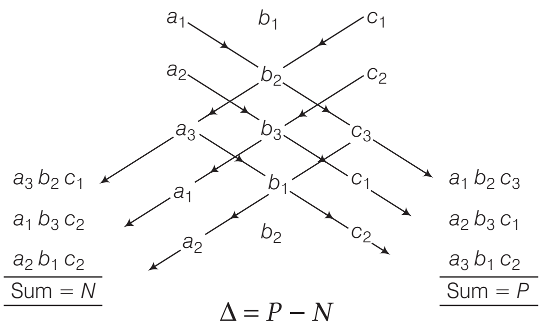
Sarrus Rule for Expansion

Sarrus gave a rule for a determinant of order 3.

Rule Write down the three rows of the Δ and rewrite the first two rows. The three diagonals sloping down to the right given the three terms and the three diagonals sloping down to the left also given the three terms.

If $\quad \Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

Rule

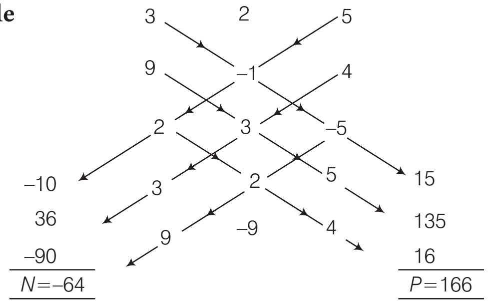


$\therefore \qquad \Delta = P - N$

Example 3. ***Expand*** $\begin{vmatrix} 3 & 2 & 5 \\ 9 & -1 & 4 \\ 2 & 3 & -5 \end{vmatrix}$ ***by Sarrus rule.***

Sol. Let $\Delta = \begin{vmatrix} 3 & 2 & 5 \\ 9 & -1 & 4 \\ 2 & 3 & -5 \end{vmatrix}$

Rule

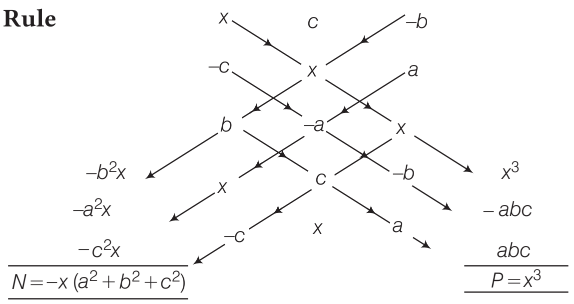


$\therefore \qquad \Delta = P - N = 166 - (-64) = 230$

Example 4. ***If*** $a, b, c \in R$***, find the number of real roots of the equation*** $\begin{vmatrix} x & c & -b \\ -c & x & a \\ b & -a & x \end{vmatrix} = 0$

Sol. Let $\Delta = \begin{vmatrix} x & c & -b \\ -c & x & a \\ b & -a & x \end{vmatrix}$

Rule



$\therefore \qquad \Delta = P - N$

$= x^3 + x(a^2 + b^2 + c^2) = 0$ [given]

$\therefore \quad x = 0$ or $x^2 = -(a^2 + b^2 + c^2)$

$\Rightarrow \quad x = 0$ or $x = \pm i\sqrt{(a^2 + b^2 + c^2)}$, where $i = \sqrt{-1}$

Hence, number of real roots is one.

Window Rule for Expansion

Window rule valid only for third order determinant.

Let $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

In this method, rewrite first two elements of second row and third row, then

Rule

$$\begin{matrix} a_1 & b_1 & c_1 & & \\ a_2 & b_2 & c_2 & a_2 & b_2 \\ a_3 & b_3 & c_3 & a_3 & b_3 \end{matrix}$$

Now, taking positive sign with a_1, b_1 and c_1.

$$\Delta = a_1 (b_2 c_3 - b_3 c_2) + b_1 (c_2 a_3 - c_3 a_2) + c_1 (a_2 b_3 - a_3 b_2)$$

Example 5. Expand $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 6 & 2 \\ 5 & 9 & 4 \end{vmatrix}$ by window rule.

Sol. Let $\Delta = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 6 & 2 \\ 5 & 9 & 4 \end{vmatrix}$

Rule:

$$\begin{matrix} 1 & 2 & 3 & & \\ 4 & 6 & 2 & 4 & 6 \\ 5 & 9 & 4 & 5 & 9 \end{matrix}$$

$\therefore \quad \Delta = 1(24 - 18) + 2(10 - 16) + 3(36 - 30)$

$= 6 - 12 + 18 = 12$

Example 6. Find the value of the determinant $\begin{vmatrix} -1 & 2 & 1 \\ 3 + 2\sqrt{2} & 2 + 2\sqrt{2} & 1 \\ 3 - 2\sqrt{2} & 2 - 2\sqrt{2} & 1 \end{vmatrix}$.

Sol. Let $\Delta = \begin{vmatrix} -1 & 2 & 1 \\ 3 + 2\sqrt{2} & 2 + 2\sqrt{2} & 1 \\ 3 - 2\sqrt{2} & 2 - 2\sqrt{2} & 1 \end{vmatrix}$ and let $2\sqrt{2} = \lambda$,

then $\Delta = \begin{vmatrix} -1 & 2 & 1 \\ 3 + \lambda & 2 + \lambda & 1 \\ 3 - \lambda & 2 - \lambda & 1 \end{vmatrix}$

Rule

$$\begin{matrix} -1 & 2 & 1 & & \\ 3 + \lambda & 2 + \lambda & 1 & 3 + \lambda & 2 + \lambda \\ 3 - \lambda & 2 - \lambda & 1 & 3 - \lambda & 2 - \lambda \end{matrix}$$

Now, $\Delta = -1(2 + \lambda - 2 + \lambda) + 2(3 - \lambda - 3 - \lambda) + 1[(3 + \lambda)(2 - \lambda) - (3 - \lambda)(2 + \lambda)]$

$= -2\lambda - 4\lambda + (-2\lambda) = -8\lambda = -16\sqrt{2}$

$[\because \lambda = 2\sqrt{2}]$

Exercise for Session 1

1 Sum of real roots of the equation $\begin{vmatrix} 1 & 4 & 20 \\ 1 & -2 & 5 \\ 1 & 2x & 5x^2 \end{vmatrix} = 0$ is

(a) –2 (b) –1 (c) 0 (d) 1

2 If $\begin{vmatrix} 6i & -3i & 1 \\ 4 & 3i & -1 \\ 20 & 3 & i \end{vmatrix} = x + iy, i = \sqrt{-1}$, then

(a) $x = 3, y = 1$ (b) $x = 1, y = 3$
(c) $x = 0, y = 3$ (d) $x = 0, y = 0$

3 If $p\lambda^4 + q\lambda^3 + r\lambda^2 + s\lambda + t = \begin{vmatrix} \lambda^2 + 3\lambda & \lambda - 1 & \lambda + 3 \\ \lambda^2 + 1 & 2 - \lambda & \lambda - 3 \\ \lambda^2 - 3 & \lambda + 4 & 3\lambda \end{vmatrix}$, then t is equal to

(a) 7 (b) 14
(c) 21 (d) 28

4 If one root of the equation $\begin{vmatrix} 7 & 6 & x^2 - 13 \\ 2 & x^2 - 13 & 2 \\ x^2 - 13 & 3 & 7 \end{vmatrix} = 0$ is $x = 2$, the sum of all other five roots is

(a) $2\sqrt{15}$ (b) –2
(c) $\sqrt{20} + \sqrt{15} - 2$ (d) None of these

5 If A, B and C are the angles of a non-right angled ΔABC, the value of $\begin{vmatrix} \tan A & 1 & 1 \\ 1 & \tan B & 1 \\ 1 & 1 & \tan C \end{vmatrix}$ is

(a) 0 (b) 1 (c) 2 (d) 3

6 If $\Delta = \begin{vmatrix} 1 & 3\cos\theta & 1 \\ \sin\theta & 1 & 3\cos\theta \\ 1 & \sin\theta & 1 \end{vmatrix}$, the maximum value of Δ is

(a) –10 (b) $-\sqrt{10}$ (c) $\sqrt{10}$ (d) 10

7 If the value of the determinant $\begin{vmatrix} a & 1 & 1 \\ 1 & b & 1 \\ 1 & 1 & c \end{vmatrix}$ is positive, then $(a, b, c > 0)$

(a) $abc > 1$ (b) $abc > -8$ (c) $abc < -8$ (d) $abc > -2$

Answers

Exercise for Session 1

1. (d) **2.** (d) **3.** (c) **4.** (b) **5.** (c) **6.** (d)
7. (b)