8 APPLICATIONS OF THE INTEGRALS

Objectives

After studying the material of this chapter, you should be able to :

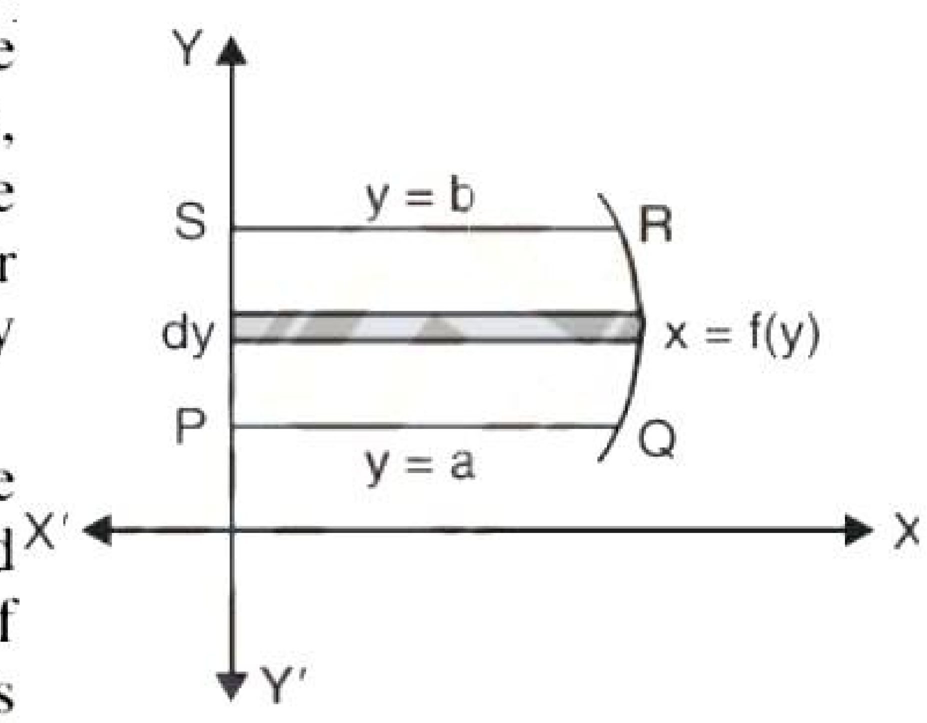
- Understand to find the area under simple curve.
- Understand to find the area of the region bounded by the curve and line.
- Understand to find the area between two curves.

—*Birkhoff*

One should study Mathematics because it is only through Mathematics that nature can be conceived in harmonious form

INTRODUCTION

The formulae, which we learnt in elementary geometry, were used to calculate the areas of many simple curves like triangles, rectangles, circles and many simple plane figures. But these are inadequate for calculating the areas, which are enclosed by curves.



The area enclosed by simple curves like circles, parabolas and ellipses in their standard forms are calculated by using the concept of definite integral. The definite integral acts as a practical tool for Science and Engineering.

In this chapter we will study the following concepts :

- Area under Simple Curves
- Area between Two Curves.

Chapter at Glance

8.1. AREA UNDER SIMPLE CURVE

(*a*) To find the area, bounded by the curve y = f (x), x-axis and the ordinates x = a and x = b.

The required area is composed of a number of thin vertical strips. Let the arbitrary strip be of height y and width dx.

Then dA, area of elementary strip $= y\,dx$, where $y = f(x)$.

This area is known as **elementary area.**

$\therefore$ A, total area of the region ALMB

$$= \int_{x=a}^{b} dA .$$

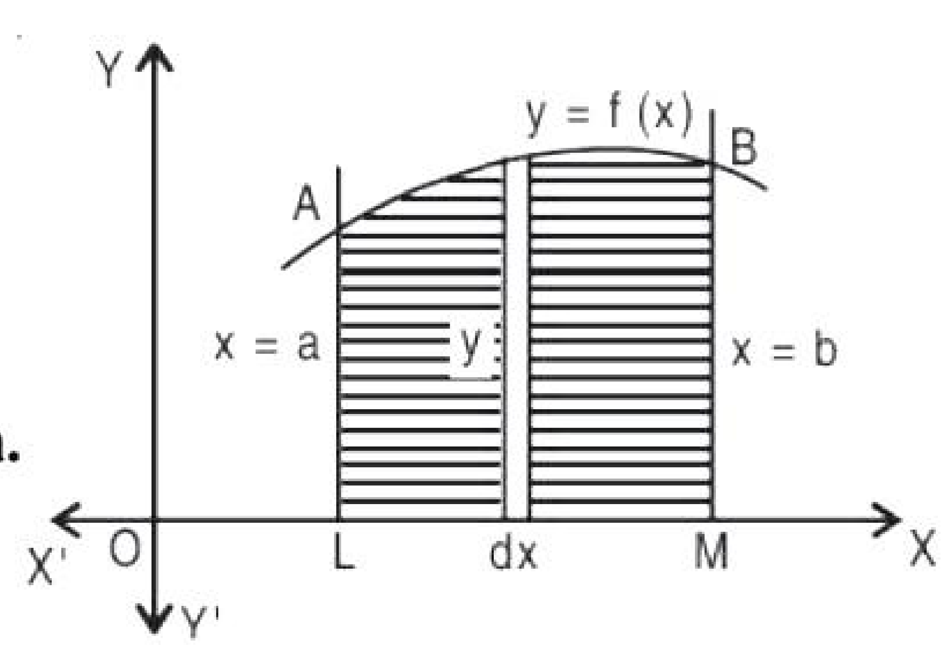


Fig.

Hence, $$\mathbf{A} = \int_{x=a}^{b} y\,dx = \int_{a}^{b} f(x)\,dx .$$

(*b*) To find the area, bounded by the curve x = g (y), y-axis and the lines y = c and y = d.

The required area is composed of a number of thin horizontal strips.

Let the arbitrary strip be of length x and width dy.

Then dA, area of elementary strip $= x\,dy$, where $x = g(y)$.

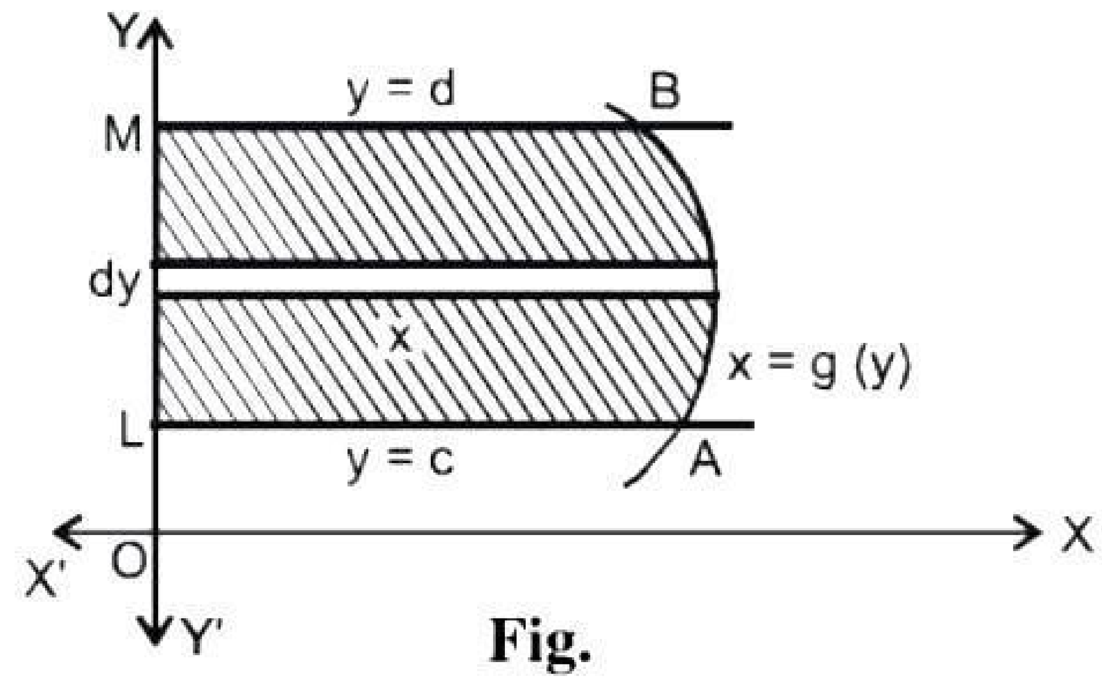


Fig.

$\therefore$ A, total area of the region ABML $= \displaystyle\int_{y=c}^{d} dA.$

Hence, $\quad \mathbf{A} = \displaystyle\int_{y=c}^{d} x\,dy = \int_{c}^{d} g(y)\,dy.$

Remarks : 1. When the curve, under consideration, is below x-axis.

Here $f(x) < 0$ from $x = a$ to $x = b$ as in the adjoining figure.

$\therefore$ The area bounded by the curve, x-axis and the ordinates $x = a$, $x = b$ will be negative.

So we consider the numerical value in this case.

Thus reqd. area $= \left| \displaystyle\int_{a}^{b} f(x)\,dx \right|$, taking absolute value.

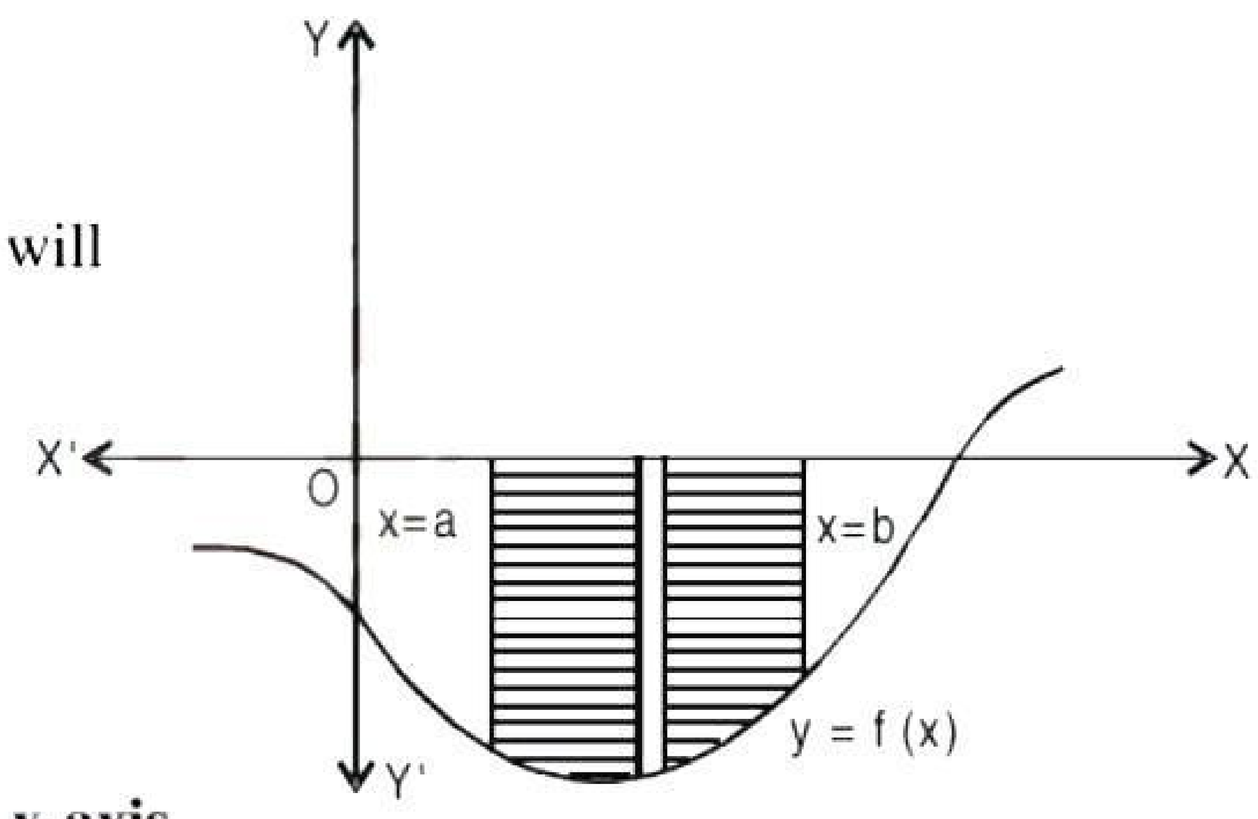
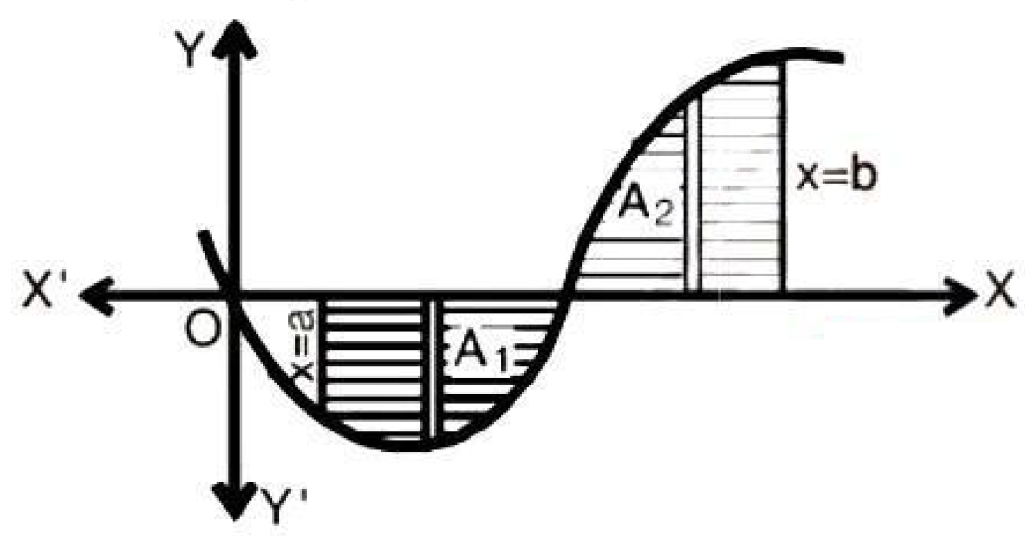


Fig.

2. When the curve, under consideration, lies above as well as below x-axis.



Here $A_1 < 0$ and $A_2 > 0$.

Fig.

Then A, area of the region, is given by :

$$A = |A_1| + A_2.$$

8.2. AREA OF THE REGION BOUNDED BY A CURVE AND A LINE

Here we shall find the area of the region bounded by a line and a circle (or a parabola or an ellipse). The equations of the circle, parabola and ellipse will be in their standard forms only. The other forms of the curves are beyond the scope of this text.

8.3. CURVE SKETCHING

Here we find some points (x, y), which satisfy the equation of the given curve. Then we plot these points and join them with a free hand so as to obtain the rough sketch of the given curve.

For **symmetry**, we remember the following points :

SYMMETRY

(I) **Symmetry about x-axis.** *If on changing y to – y, there is no change in the equation, then the curve is symmetrical about x-axis.*

For Ex. $y^2 = 4ax$ is symmetrical about x-axis.

(II) **Symmetry about y-axis.** *If on changing x to –x, there is no change in the equation, then the curve is symmetrical about y-axis.*

For Ex. $x^2 = 4ay$ is symmetrical about y-axis.

(III) **Symmetry about both axes.** *If on changing x to $-x$ and y to $-y$, there is no change in the equation, then the curve is symmetrical about both axes.*

For Ex. $x^2 + y^2 = r^2$ is symmetrical about both axes.

(IV) **Symmetry about the line y = x.** *If on interchanging x and y, there is no change in the equation, then the curve is symmetrical about the line $y = x$.*

For Ex. $x^2 + y^2 = r^2$ is symmetrical about the line $y = x$.

Frequently Asked Questions

FAQs

Example 1. Find the area enclosed by the circle : $x^2 + y^2 = a^2$. *(N.C.E.R.T.; Jammu B. 2018; Kashmir B. 2011)*

Solution. The given circle is $x^2 + y^2 = a^2$ ....(1)

This is a circle whose centre is (0, 0) and radius 'a'.

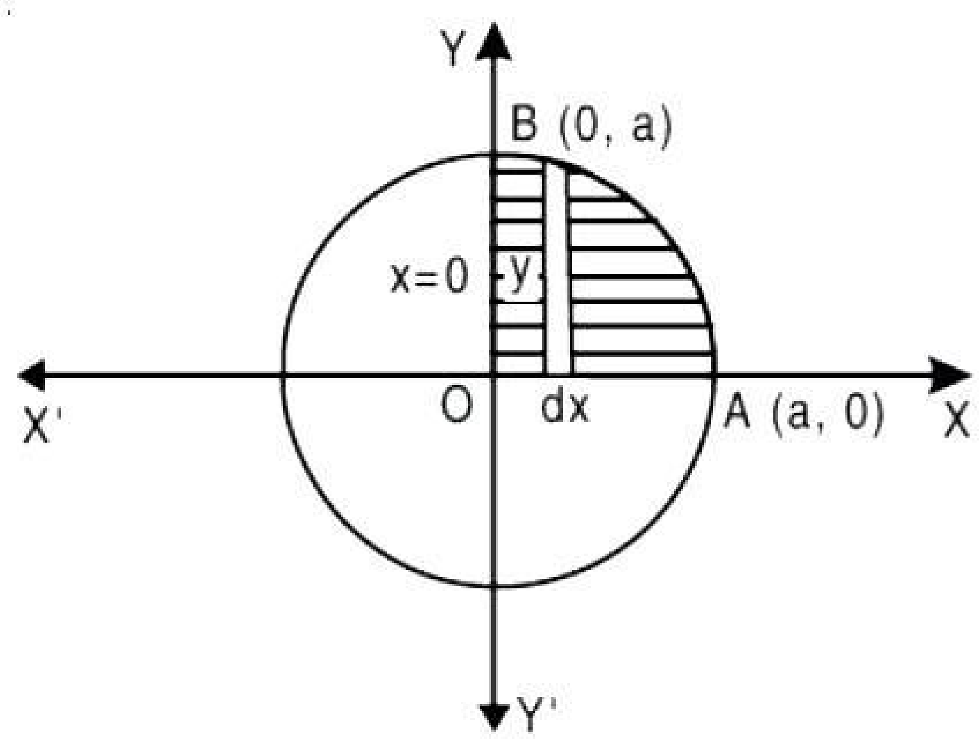


Fig.

Area of the circle = 4 (area of the region OABO, bounded by the curve, x-axis and ordinates $x = 0$, $x = a$)

[$\because$ *Circle is symmetrical about both the axes*]

$$= 4\int_0^a y\,dx \qquad \text{[Taking vertical strips]}$$

$$= 4\int_0^a \sqrt{a^2 - x^2}\,dx \qquad [\because (1) \Rightarrow y = \pm\sqrt{a^2 - x^2}.$$

But region OABO lies in 1st quadrant, $\therefore$ *y is + ve*]

$$= 4\left[\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a$$

$$= 4\left[\left\{\frac{a}{2}(0) + \frac{a^2}{2}\sin^{-1}(1)\right\} - \{0 + 0\}\right]$$

$$= 4\left(\frac{a^2}{2}\cdot\frac{\pi}{2}\right) = \pi a^2 \text{ sq. units.}$$

Example 2. Find the area of the parabola $y^2 = 4ax$, bounded by the latus-rectum. *(H.P.B. 2017; H.B. 2017, 16, 13)*

Solution. $y^2 = 4ax$ is a parabola with focus S(a, 0).

Here L L′ ($x = a$) is the latus-rectum.

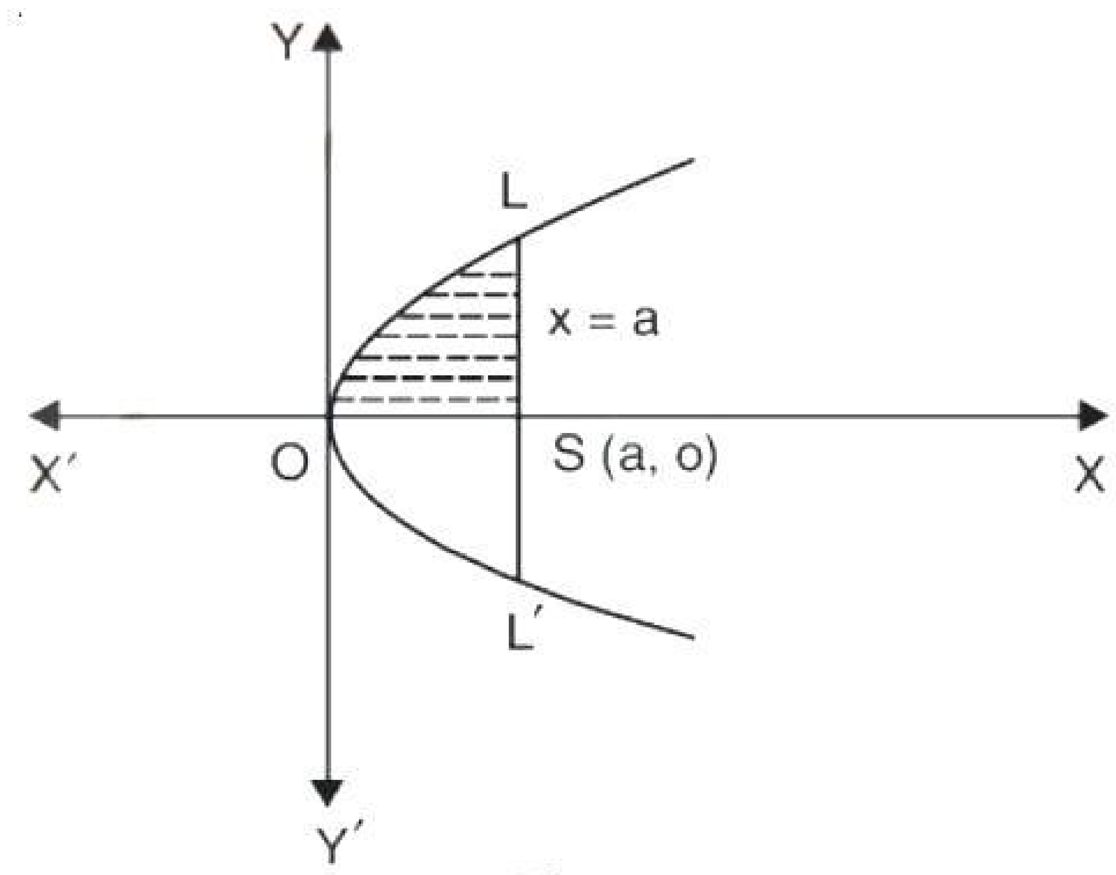


Fig.

$\therefore$ Reqd. area = 2 (area OSL)

$$= 2\int_0^a \sqrt{4ax}\,dx = 4\sqrt{a}\int_0^a x^{1/2}\,dx$$

$$= 4\sqrt{a}\left[\frac{x^{3/2}}{3/2}\right]_0^a$$

$$= \frac{8}{3}\sqrt{a}\left[a^{3/2} - 0\right] = \frac{8}{3}a^2 \text{ sq.units.}$$

Example 3. Find the area of the region bounded by : $y^2 = 4x$, $x = 1$, $x = 4$ and x-axis in the first quadrant.

Solution. $y^2 = 4x$ is right-handed parabola.

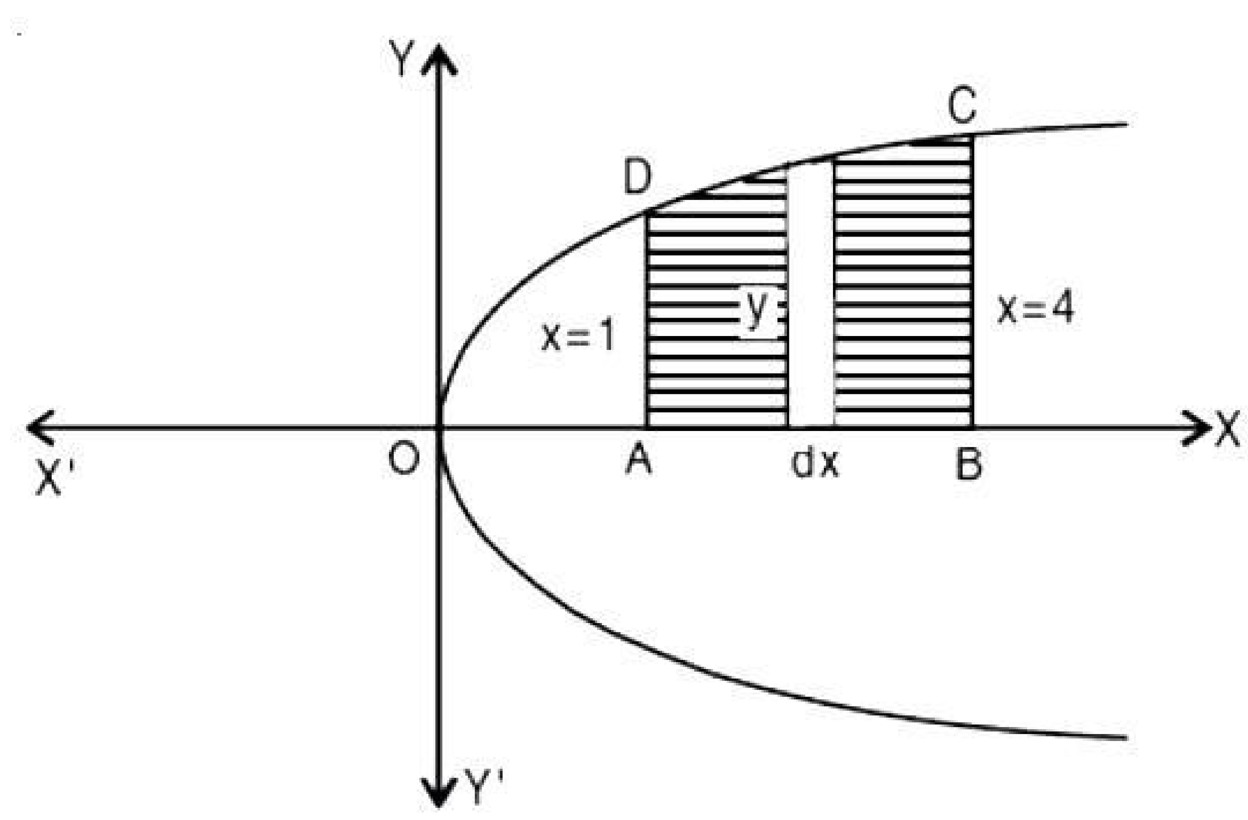


Fig.

$\therefore$ Reqd. area, ABCD $= \int_1^4 y\,dx$ [*Taking vertical strips*]

$$= \int_1^4 2\sqrt{x}\,dx$$

[$y^2 = 4x \Rightarrow y = \pm 2\sqrt{x}$. *But region ABCD lies in 1st quadrant,* $\therefore$ *y is +ve*]

$$= 2\int_1^4 x^{1/2}\,dx = 2\left[\frac{x^{3/2}}{3/2}\right]_1^4 = \frac{4}{3}[4^{3/2} - 1]$$

$$= \frac{4}{3}[8-1] = \frac{28}{3} = 9\frac{1}{3} \text{ sq. units.}$$

Example 4. Find the area of the region bounded by the curve $y = x^2$ and the line $y = 4$.

(Jharkhand B. 2016; Kerala B. 2014)

Solution. The given curve is $y = x^2$ ...(1)

and the given line is $y = 4$ ...(2)

(1) is an upward parabola, which is symmetrical about y-axis only.

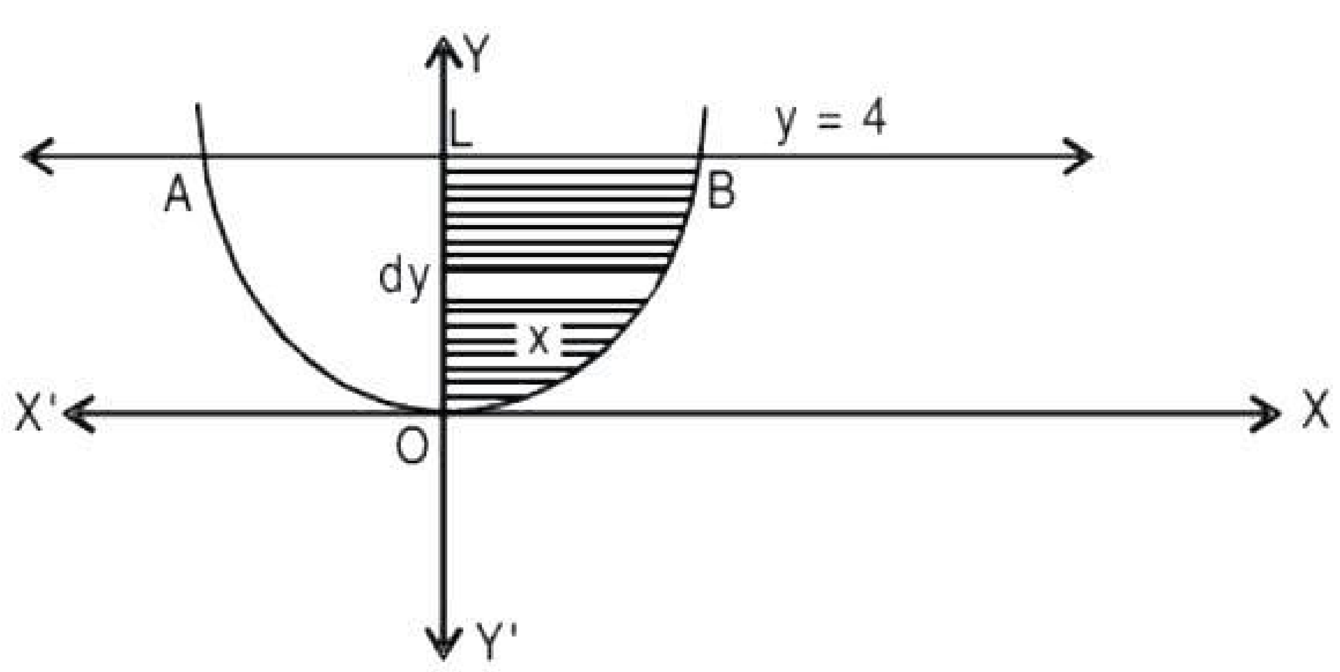


Fig.

$\therefore$ Reqd. area = area of the region OBAO

$$= 2\int_0^4 x\,dy \text{ [Taking horizontal strips]}$$

[$\because$ *Parabola is symmetrical about y-axis*]

$$= 2\int_0^4 \sqrt{y}\,dy$$

[$\because$ *From (1),* $x = \sqrt{y}$, *because x lies in first quadrant*]

$$= 2\left[\frac{y^{3/2}}{3/2}\right]_0^4$$

$$= \frac{4}{3}\left[4^{3/2} - 0\right] = \frac{4}{3}[8] = \frac{32}{3} \text{ sq. units.}$$

Note. The above example can also be calculated by taking vertical strips.

Example 5. Find the area enclosed by the ellipse :

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,\ a > b.$$

(N.C.E.R.T.; H.P.B. Model Question Paper 2018; Mizoram B. 2016; H.P.B. 2013 S, 10 ; Jammu B. 2013 ; J. & K.B. 2011)

Solution. The given ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ ...(1)

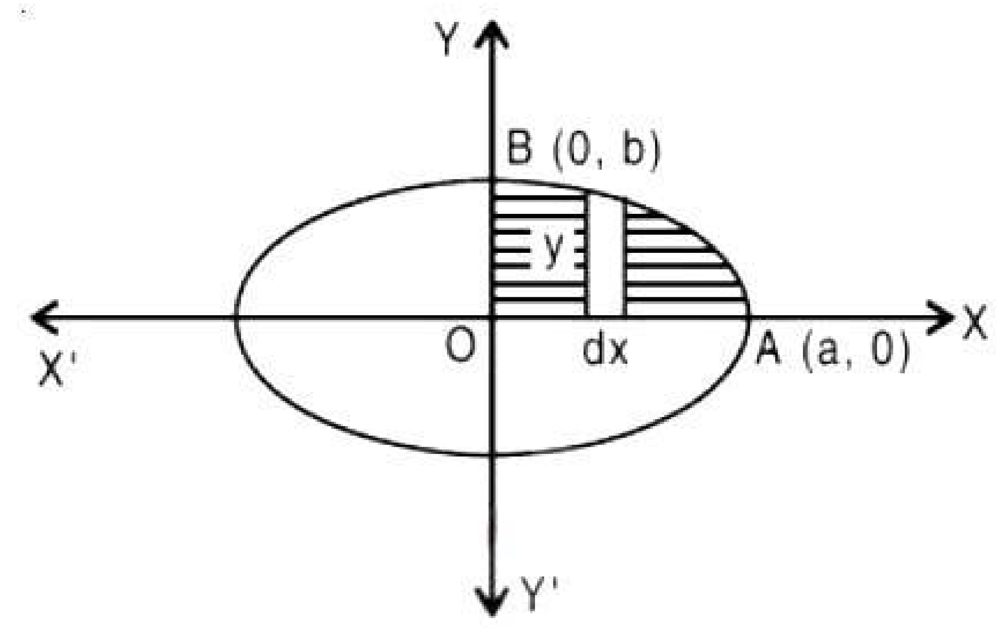


Fig.

Area of the ellipse = 4 (area of the region OABO, bounded by the curve, x-axis and the ordinates $x = 0$, $x = a$)

[$\because$ *Ellipse is symmetrical about both the axes*]

$$= 4\int_0^a y\,dx \qquad \text{[Taking vertical strips]}$$

$$= 4\int_0^a \frac{b}{a}\sqrt{a^2 - x^2}\,dx$$

[$\because$ *(1)* $\Rightarrow y = \pm\frac{b}{a}\sqrt{a^2 - x^2}$. *But region OABO lies in 1st quadrant,* $\therefore$ *y is +ve*]

$$= \frac{4b}{a}\int_0^a \sqrt{a^2 - x^2}\,dx$$

$$= \frac{4b}{a}\left[\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a$$

$$= \frac{4b}{a}\left[\left\{\frac{a}{2}(0) + \frac{a^2}{2}\sin^{-1}(1)\right\} - \{0 + 0\}\right]$$

$$= \frac{4b}{a}\left[\frac{a^2}{2}.\frac{\pi}{2}\right] = \pi ab \text{ sq. units.}$$

ALTERNATIVELY :

The given ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ ....(1)

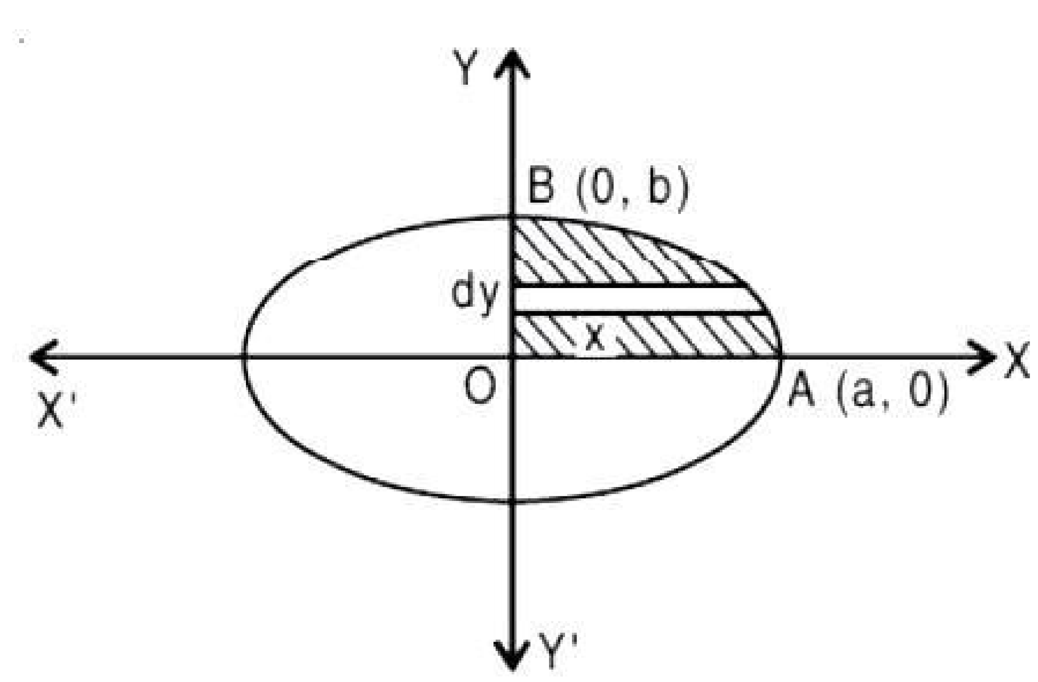


Fig.

Area of the ellipse = 4 (area of the region OABO, bounded by the curve, y-axis and $y = 0$, $y = b$)

[$\because$ *Ellipse is symmetrical about both the axes*]

$$= 4\int_0^b x\,dy \qquad [\textit{Taking horizontal strips}]$$

$$= 4\int_0^b \frac{a}{b}\sqrt{b^2 - y^2}\,dy$$

[$\because$ *(1)* $\Rightarrow x = \pm \frac{a}{b}\sqrt{b^2 - y^2}$. *But region OABO lies in Ist quadrant,* $\therefore$ *x is + ve*]

$$= \frac{4a}{b}\int_0^b \sqrt{b^2 - y^2}\,dy$$

$$= \frac{4a}{b}\left[\frac{y}{2}\sqrt{b^2 - y^2} + \frac{b^2}{2}\sin^{-1}\frac{y}{b}\right]_0^b$$

$$= \frac{4a}{b}\left[\left\{\frac{b}{2}(0) + \frac{b^2}{2}\sin^{-1}(1)\right\} - (0 + 0)\right]$$

$$= \frac{4a}{b}\left[\frac{b^2}{2}\cdot\frac{\pi}{2}\right] = \pi ab \text{ sq. units.}$$

Example 6. Find the area bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and the ordinates x = ae and x = 0, where $b^2 = a^2(1 - e^2)$ and e < 1.

Solution. The given ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ ...(1)

The reqd. area of the region BOB′QPB is enclosed by the ellipse (1) and the lines $x = 0$ and $x = ae$.

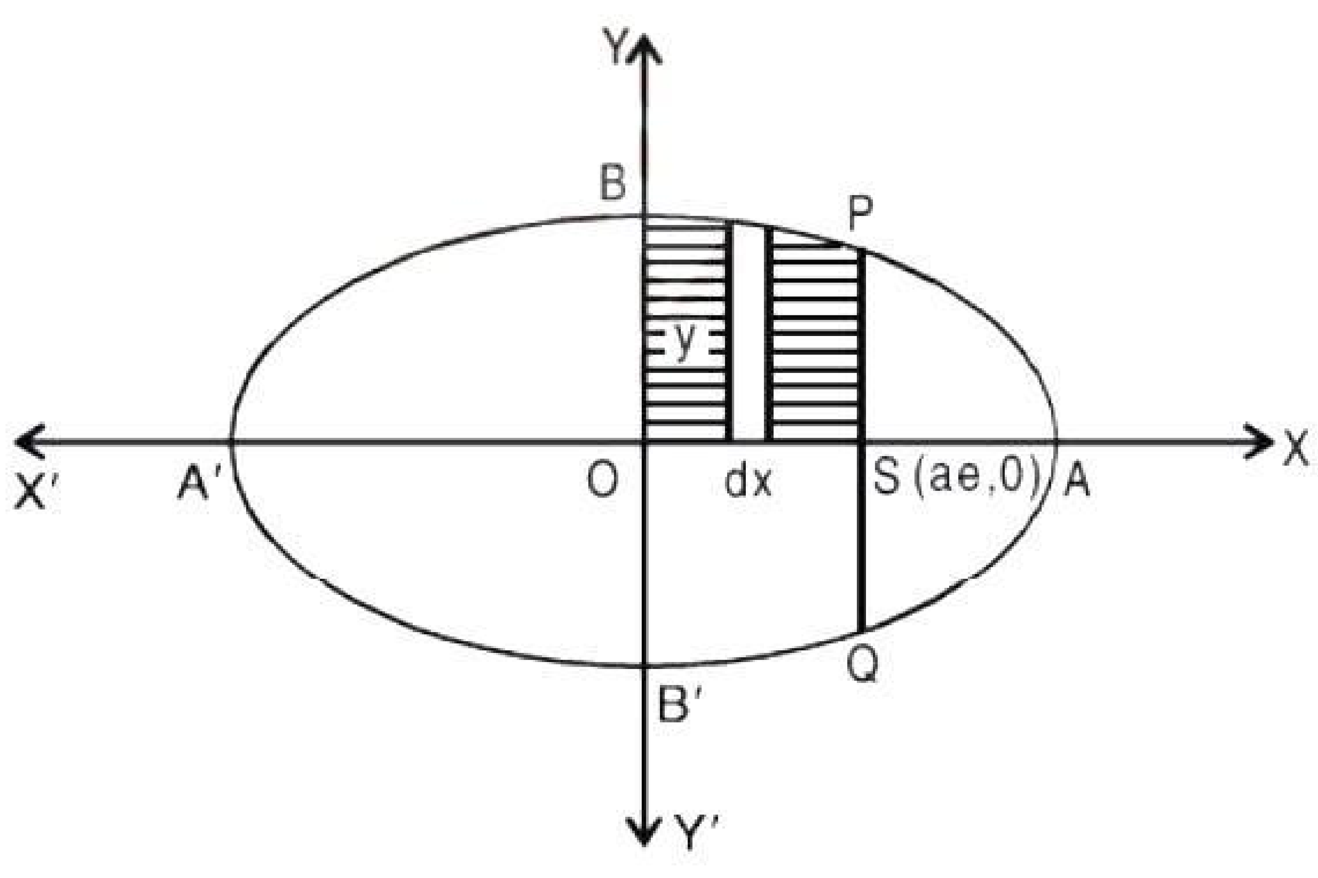


Fig.

$\therefore$ Reqd. area

$$= 2\int_0^{ae} y\,dx \qquad [\textit{Taking vertical strips}]$$

$$= 2\int_0^{ae} \frac{b}{a}\sqrt{a^2 - x^2}\,dx$$

$$\left[\because \textit{From (1), } y = \frac{b}{a}\sqrt{a^2 - x^2}\right]$$

$$= \frac{2b}{a}\left[\frac{x\sqrt{a^2 - x^2}}{2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^{ae}$$

$$= \frac{b}{a}\left[\left\{(ae)\sqrt{a^2 - a^2e^2} + a^2\sin^{-1}\frac{ae}{a}\right\} - (0 + 0)\right]$$

$$= ab\,[e\sqrt{1 - e^2} + \sin^{-1} e] \text{ sq. units.}$$

Example 7. Find the area bounded by the region given by :

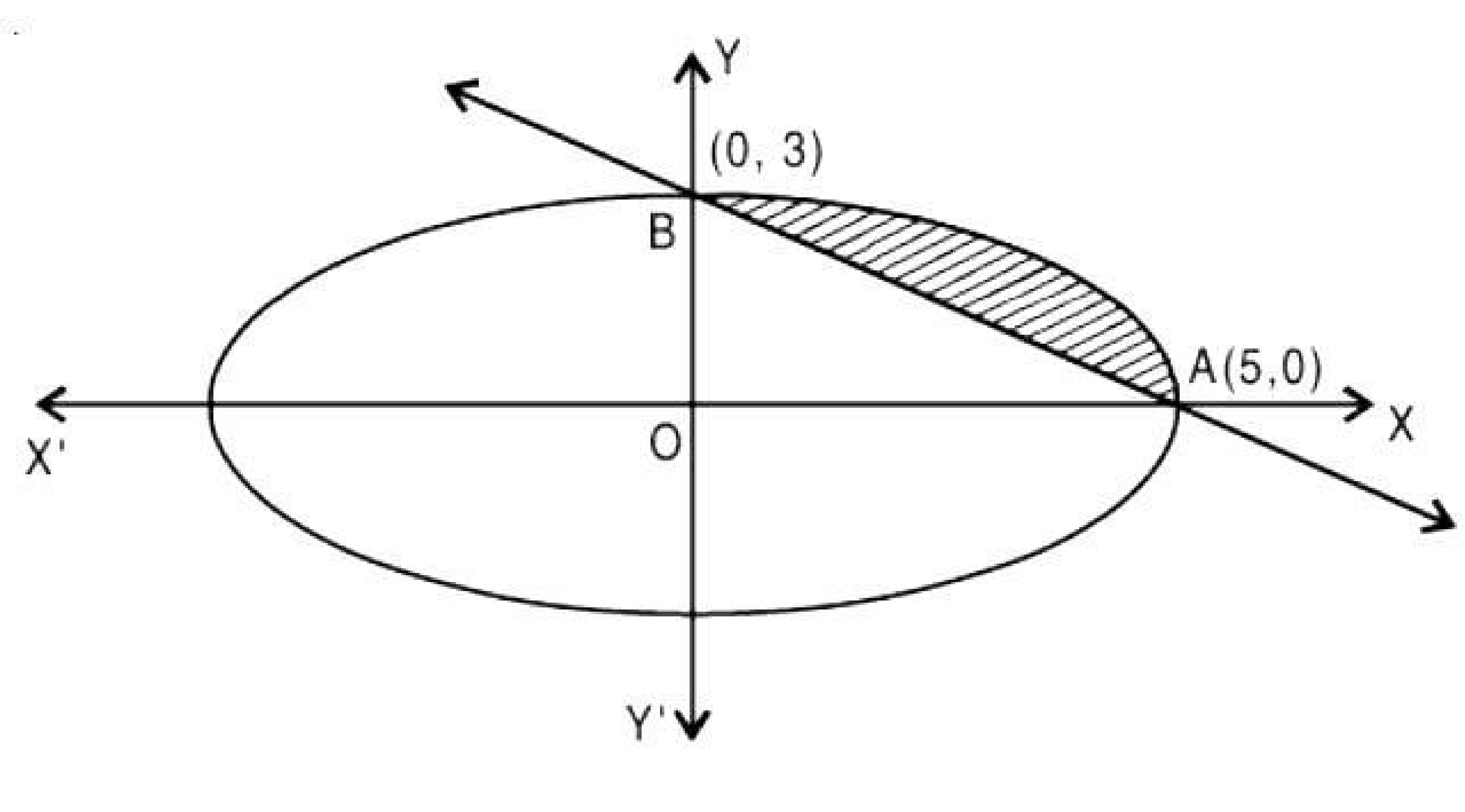
$$\mathbf{A = \left\{(x, y) : (x, y) : \frac{x^2}{25} + \frac{y^2}{9} \le 1 \le \frac{x}{5} + \frac{y}{3}\right\}.}$$

(P.B. 2016)

Solution. We have : $\frac{x^2}{25} + \frac{y^2}{9} = 1$ ....(1)

and $\frac{x}{5} + \frac{y}{3} = 1$ ....(2)

Reqd. area is the shaded area, as shown in the figure :



$\therefore$ Shaded area

$$= \int_0^5 \frac{3}{5}\sqrt{25 - x^2}\,dx - \int_0^5 \frac{3}{5}(5 - x)\,dx$$

$$= \frac{3}{5}\int_0^5 \sqrt{5^2 - x^2} - \frac{3}{5}\int_0^5 (5-x)\,dx$$

$$= \frac{3}{5}\left[\frac{x}{2}\sqrt{25-x^2} + \frac{25}{2}\sin^{-1}\frac{x}{5} - 5x + \frac{x^2}{2}\right]_0^5$$

$$= \frac{3}{5}\left[\left\{0 + \frac{25}{2}(\sin^{-1}(1)) - 5(5) + \frac{25}{2}\right\} - 0\right]$$

$$= \frac{3}{5}\left[\frac{25}{2}.\frac{\pi}{2} - 25 + \frac{25}{2}\right]$$

$$= \frac{3}{5}\left[\frac{25\pi}{4} - \frac{25}{2}\right]$$

$$= \frac{15\pi}{4} - \frac{15}{2}$$

$$= \frac{15}{4}(\pi - 2) \text{ sq. units.}$$

Example 8. Find the area of the region in the first quadrant enclosed by the x-axis, the line y = x and the circle $x^2 + y^2 = 32$.

(N.C.E.R.T.; C.B.S.E. 2018, 14, Type : Kerala B. 2018 ; Assam B. 2016)

Solution. We have :

$$y = x \quad ....(1)$$

and $$x^2 + y^2 = 32 \quad ....(2)$$

(1) is a st. line, passing thro' (0, 0) and (2) is a circle with centre (0, 0) and radius $4\sqrt{2}$.

Solving (1) and (2) :

Putting the value of y from (1) in (2), we get :

$$x^2 + x^2 = 32 \Rightarrow 2x^2 = 32$$

$\Rightarrow$ $x^2 = 16 \Rightarrow x = 4$, because region lies in first quadrant.

Also $y = 4$.

Thus the line (1) and the circle (2) meet each other at B (4, 4), in the first quadrant.

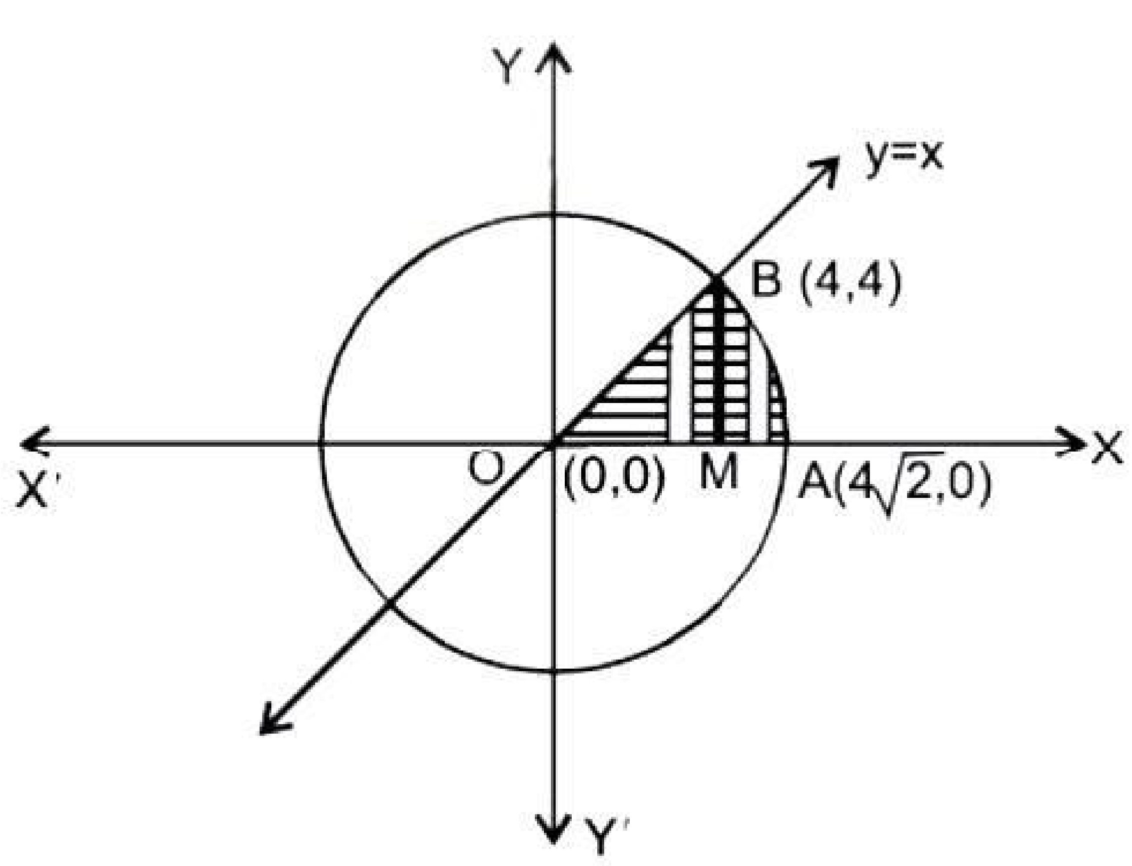


Fig.

Draw BM perp. to x-axis.

$\therefore$ Reqd. area = area of the region OMBO + area of the region BMAB ....(3)

Now, area of the region OMBO

$$= \int_0^4 y\,dx \qquad [\textit{Taking vertical strips}]$$

$$= \int_0^4 x\,dx$$

$$= \left[\frac{x^2}{2}\right]_0^4$$

$$= \frac{1}{2}(16 - 0) = 8.$$

Again, area of the region BMAB

$$= \int_4^{4\sqrt{2}} y\,dx \qquad [\textit{Taking vertical strips}]$$

$$= \int_4^{4\sqrt{2}} \sqrt{32 - x^2}\,dx$$

[$\because$ $y^2 = 32 - x^2 \Rightarrow y = \sqrt{32 - x^2}$, *taking +ve sign, as it lies in Ist quadrant*]

$$= \int_4^{4\sqrt{2}} \sqrt{\left(4\sqrt{2}\right)^2 - x^2}\,dx$$

$$= \left[\frac{x\sqrt{32 - x^2}}{2} + \frac{32}{2}\sin^{-1}\frac{x}{4\sqrt{2}}\right]_4^{4\sqrt{2}}$$

$$= \left\{\frac{1}{2}4\sqrt{2} \times 0 + \frac{32}{2}\sin^{-1}(1)\right\}$$

$$- \left\{\frac{4}{2}\sqrt{32 - 16} + \frac{32}{2}\sin^{-1}\frac{1}{\sqrt{2}}\right\}$$

$$= 0 + 16\left(\frac{\pi}{2}\right) - \left(2 \times 4 + 16 \times \frac{\pi}{4}\right)$$

$$= 8\pi - (8 + 4\pi) = 4\pi - 8.$$

$\therefore$ From (3), Reqd. area = $8 + (4\pi - 8) = 4\pi$ sq. units.

EXERCISE 8 (a)

Short Answer Type Questions

SATQ

1. Using integration,

(*i*) find the area of the quadrant of the circle : $x^2 + y^2 = 4$ **(H.B. 2015)**

(*ii*) find the area of the circle : $x^2 + y^2 = 4$. **(Kerala B. 2016)**

2. Find the area of the region bounded by the curve $y^2 = 4x$ and the line $x = 3$. **(H.B. 2016, 11)**

3. Using integration, find the area bounded between the parabola $x^2 = 4y$ and the line $y = 4$. **(Uttrakhand B. 2015; Kashmir B. 2011)**

4. Find the area bounded by the curve $y^2 = 2y - x$ and the y–axis. **(H.B. 2011)**

5. (*i*) Find the area bounded by $y = 2x + 3$, the x-axis and the ordinates $x = -2$ and $x = 2$. **(H.B. 2014)**

(*ii*) Find the area bounded by $y = x$, the x-axis and the lines $x = -1$ and $x = 2$. **(H.B. 2011)**

Find the area of the region bounded by (6 – 8) :

6. (*i*) $y = x^4$; $x = 1, x = 5$ and x-axis **(N.C.E.R.T. ; Kashmir B. 2012)**

(*ii*) $y = x^2, x = 0, x = 2$ and x-axis **(H.B. 2016)**

(*iii*) $y = x^2 - 4, x = 0, x = 3$ and x-axis **(H.B. 2016)**

(*iv*) $y = x^3, x = 2, x = 4$ and x-axis. **(H.B. 2016)**

7. $y^2 = 9x$, $x = 2$, $x = 4$ and the x-axis in the first quadrant. **(N.C.E.R.T. ; Jammu B. 2014; H.P.B. 2012)**

8. $y = 4x^2, x = 0, y = 1, y = 4$ in the first quadrant. **(N.C.E.R.T.)**

9. Calculate the area under the curve :

(*i*) $y = 2\sqrt{x}$ between the ordinates $x = 0$ and $x = 1$

(*ii*) $y = (x^2 + 2)^2 + 2x$ between the ordinates $x = 0$ and $x = 2$.

10. Prove that the area of the region in the first quadrant enclosed by x-axis and $x = \sqrt{3}\,y$ by the circle $x^2 + y^2 = 4$ is $\frac{\pi}{3}$ sq. units. **(N.C.E.R.T.)**

11. Prove that area of the smaller part of the circle $x^2 + y^2 = a^2$ cut off by the line $x = \frac{a}{\sqrt{2}}$ is $\frac{a^2}{4}(\pi - 2)$ sq. units. **(N.C.E.R.T.)**

12. (*i*) Determine the area under the curve $y = \sqrt{a^2 - x^2}$ included between the lines $x = 0$ and $x = a$.

(*ii*) Using definite integrals, find the area of the circle $(x - 1)^2 + y^2 = 1$.

13. Determine the area enclosed between the curve $y = \cos 2x, 0 \le x \le \frac{\pi}{4}$ and the co-ordinate axes.

14. Calculate the area bounded by the curve :

$f(x) = \sin^2 \frac{x}{2}$, axis of x and the ordinates :

$x = 0, x = \frac{\pi}{2}$.

15. Draw a rough sketch of the curve $y = \cos^2 x$ in $[0, \pi]$ and find the area enclosed by the curve, the lines $x = 0, x = \pi$ and the x-axis.

16. (*i*) Make a rough sketch of the graph of the function $y = \sin x, 0 \le x \le \frac{\pi}{2}$ and determine the area enclosed between the curve, the x-axis and the line $x = \frac{\pi}{2}$.

(*ii*) Find the area bounded by the curve :

(I) $y = \sin x$ (II) $y = \cos x$

between $x = 0$ and $x = 2\pi$. **(N.C.E.R.T.; H.B. 2014)**

17. Make a rough sketch of the graph of the function $y = 2 \sin x, 0 \le x \le \frac{\pi}{2}$ and determine the area enclosed between the curve, the x–axis and the line $x = \frac{\pi}{2}$.

18. (*i*) Draw a rough sketch of $y = \sin 2x$ and determine the area enclosed by the curve, x– axis and the lines $x = \frac{\pi}{4}$ and $x = \frac{3\pi}{4}$.

(*ii*) Draw the graph of $y = \cos 3x, 0 \le x \le \frac{\pi}{6}$ and find the area between the curve and the axes.

19. Make a rough sketch of the graph of $y = \cos^2 x, 0 \le x \le \frac{\pi}{2}$ and find the area enclosed between the curve and the axes.

20. Find the area bounded by the circle $x^2 + y^2 = 16$ and the line $y = x$ in the first quadrant.

Long Answer Type Questions

LATQ

21. Find the area of the smaller part of the circle $x^2 + y^2 = a^2$ cut off by the line $x = \frac{a}{\sqrt{2}}$. **(N.C.E.R.T.)**

22. Find the area under the given curve and the given lines :

$y = x^2$, $x = 1$, $x = 2$ and x-axis. **(N.C.E.R.T.)**

23. Draw a rough sketch of the curve $y^2 + 1 = x$, $x \leq 2$. Find the area enclosed by the curve and the line $x = 2$.

24. Find the area of the region bounded by the ellipse :

(*a*) $\frac{x^2}{9} + \frac{y^2}{4} = 1$ **(N.C.E.R.T.; H.B. 2011, 10)**

(*b*) (*i*) $16x^2 + 9y^2 = 144$ (*ii*) $4x^2 + 25y^2 = 1$.

25. Find the area between the curve $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and the x-axis between $x = 0$ and $x = a$. Draw a rough sketch of the curve also.

26. Sketch the region of the ellipse and find its area, using integration :

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1.$$

27. Sketch the region $\{(x, y) : 4x^2 + 9y^2 = 36\}$ and find its area, using integration.

28. Find the area bounded by the circle $x^2 + y^2 = 16$ and the line $\sqrt{3}y = x$ in the first quadrant, using integration. **(C.B.S.E. 2017)**

Answers

1. (*i*) π (*ii*) 4π. **2.** $8\sqrt{3}$. **3.** $21\frac{1}{3}$.

4. $\frac{4}{3}$. **5.** (*i*) $\frac{25}{2}$ (*ii*) $\frac{5}{2}$.

6. (*i*) 624·8 (*ii*) $\frac{8}{3}$ (*iii*) 3 (*iv*) 60.

7. $16 - 4\sqrt{2}$.

8. $\frac{7}{3}$. **9.** (*i*) $\frac{4}{3}$ (*ii*) 29·07.

12. (*i*) $\frac{\pi a^2}{4}$ (*ii*) π.

13. $\frac{1}{2}$. **14.** $\frac{\pi - 2}{4}$. **15.** $\frac{\pi}{2}$.

16. (*i*) 1 (*ii*) (I) –(II) 4.

17. 2. **18.** (*i*) 1 (*ii*) $\frac{1}{3}$. **19.** $\frac{\pi}{4}$.

20. 2π. **21.** $\frac{a^2}{2}\left(\frac{\pi}{2} - 1\right)$.

22. $\frac{7}{3}$. **23.** $\frac{4}{3}$.

24. (*a*) 6π (*b*) (*i*) 12π (*ii*) $\frac{\pi}{10}$.

25. $\frac{\pi ab}{4}$. **26** πab.

27. 6π **28.** $\frac{4\pi}{3}$ sq units.

Hints to Selected Questions

9. (*i*) Reqd. area $= \int_0^1 2\sqrt{x}\, dx$.

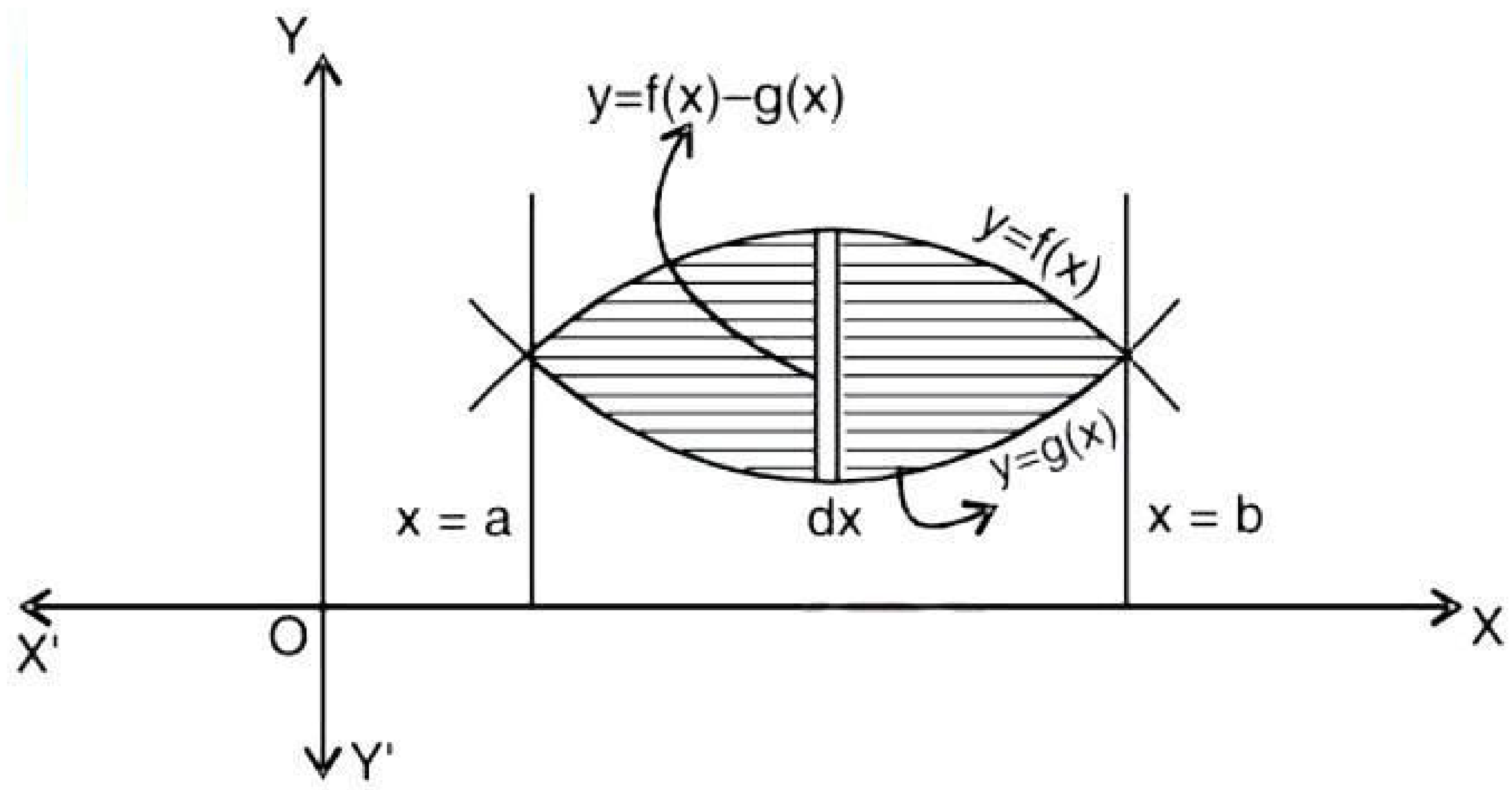
27. Given region is ellipse : $\frac{x^2}{9} + \frac{y^2}{4} = 1$.

Its area $= 4\int_0^3 \frac{2}{3}(\sqrt{9 - x^2})\, dx$.

8.4. AREA BETWEEN TWO CURVES

To find the area enclosed by the curves :

y = f (x), y = g (x), where f (x) ≥ g (x) in [a, b].



The given curves are : $y = f(x)$ ....(1)

and $y = g(x)$ ....(2),

where $f(x) \geq g(x)$.

(1) and (2) intersect at $x = a$ and $x = b$.

Taking vertical strips, elementary strip has height $(f(x) - g(x))$ and width dx.

$\therefore \quad dA = [f(x) - g(x)]\, dx.$

$\therefore \quad$ Total area, $A = \int_a^b [f(x) - g(x)]\, dx.$

ALTERNATIVELY :

A = [Area bounded by $y = f(x)$ between $x = a$ and $x = b$]

− [Area bounded by $y = g(x)$ between $x = a$ and $x = b$]

$$= \int_a^b f(x)\, dx - \int_a^b g(x)\, dx$$

$$= \int_a^b [f(x) - g(x)]\, dx, \text{ where } f(x) \geq g(x) \text{ in } [a, b].$$

Remark. When f (x) ≥ g (x) in [a, c] and f (x) ≤ g (x) in [c, b], where a < c < b.

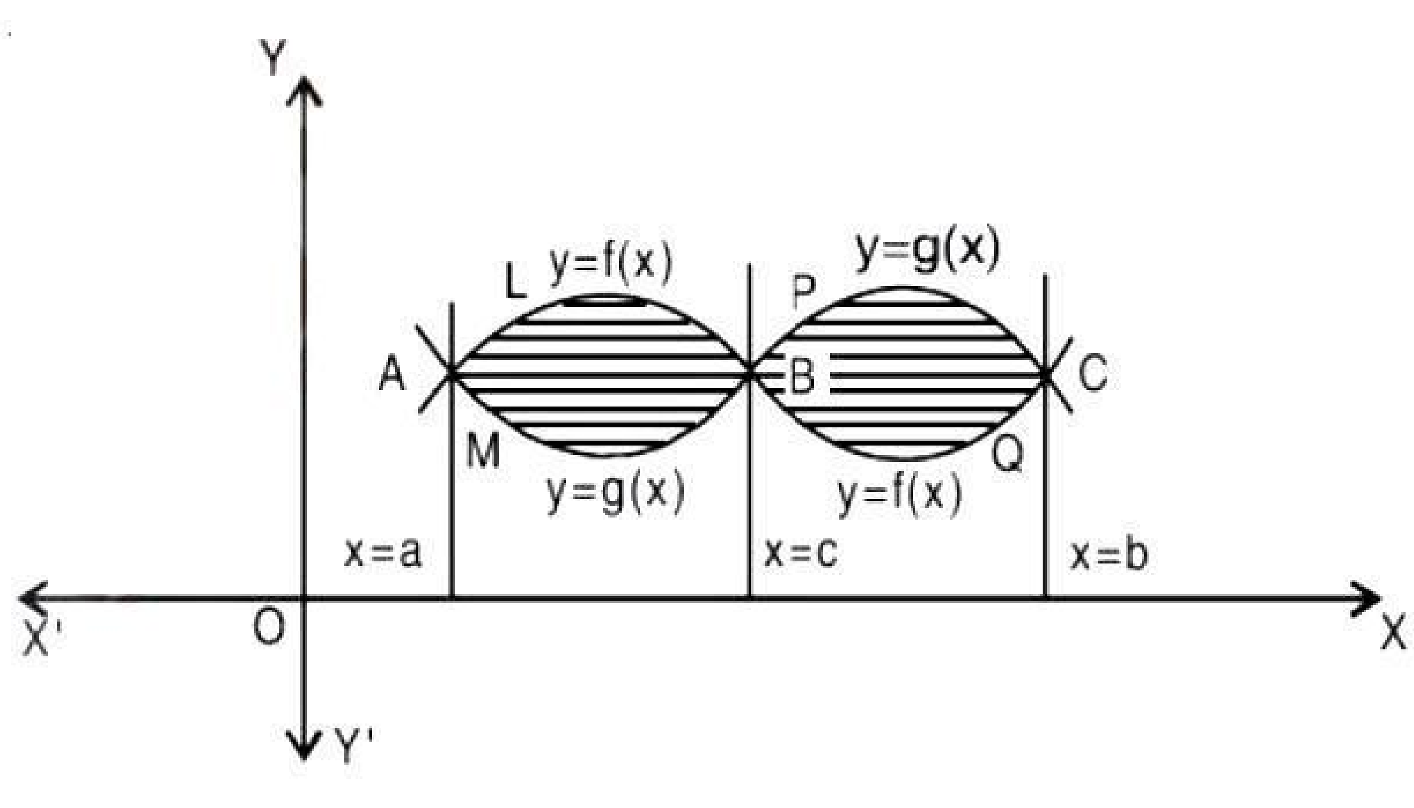


Fig.

Here area of the regions bounded by curves can be written as :

Total area = Area of the region ALBMA + Area of the region BQCPB

$$= \int_a^c [f(x) - g(x)]\, dx + \int_c^b [g(x) - f(x)]\, dx.$$

Frequently Asked Questions

Example 1. Using integration, find the area of the region bounded by :

(– 1, 1), (0, 5) and (3, 2). *(P.B. 2010)*

Solution. Let A (– 1, 1), B (0, 5) and C (3, 2) be the vertices of the triangle as shown in the following figure :

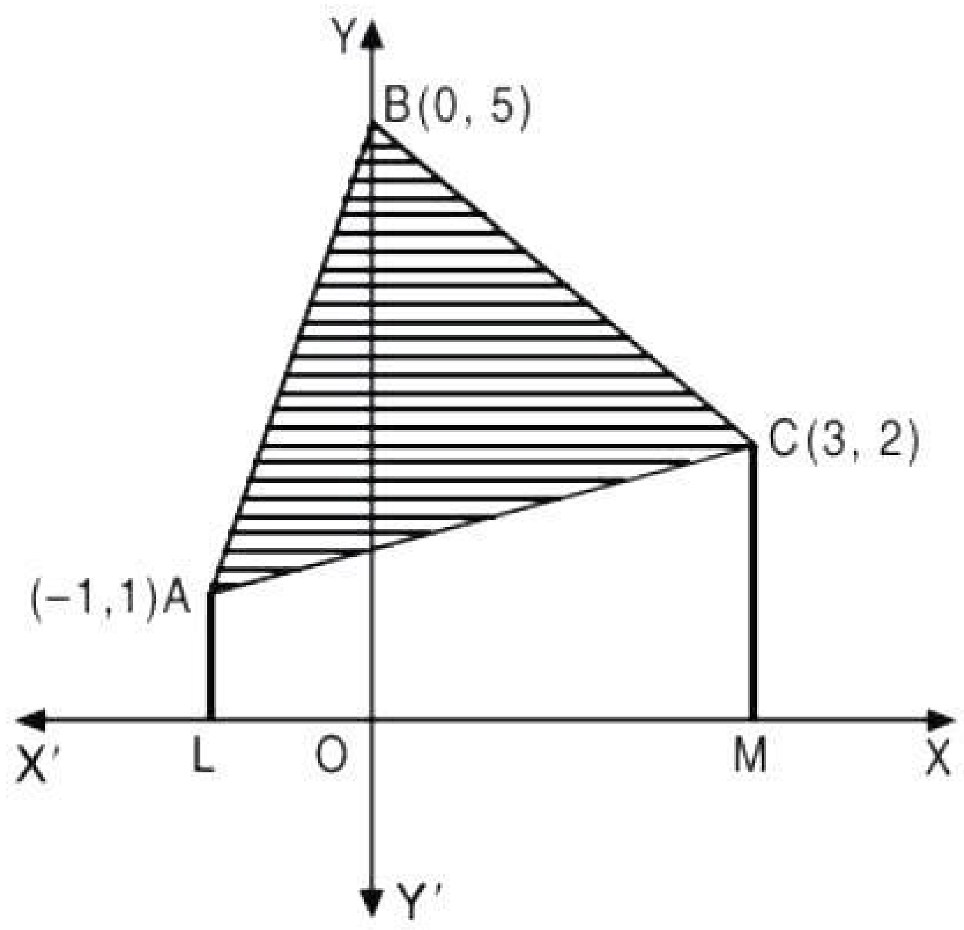


Fig.

Equation of AB is :

$$y - 1 = \frac{5-1}{0-(-1)}(x+1)$$

$$\left[Using\ y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)\right]$$

$\Rightarrow \quad y - 1 = 4(x + 1)$

$\Rightarrow \quad y = 4x + 5 \quad ...(1)$

Equation of BC is :

$$y - 5 = \frac{2-5}{3-0}(x-0)$$

$\Rightarrow \quad 3y - 15 = -3x$

$\Rightarrow \quad 3y = 15 - 3x$

$\Rightarrow \quad y = 5 - x \quad ...(2)$

Equation of AC is :

$$y - 1 = \frac{2-1}{3-(-1)}(x+1)$$

$\Rightarrow \quad y - 1 = \frac{1}{4}(x+1)$

$\Rightarrow \quad 4y - 4 = x + 1$

$\Rightarrow \quad 4y = x + 5$

$\Rightarrow \quad y = \frac{x}{4} + \frac{5}{4} \quad ...(3)$

Now ar (ΔABC) = ar (trap ALOB) + ar (trap OMCB) – ar (trap ALMC)

$$= \int_{-1}^{0} (4x+5)\,dx + \int_{0}^{3} (5-x)\,dx - \int_{-1}^{3} \left(\frac{1}{4}x + \frac{5}{4}\right) dx$$

$$= \left[\frac{4x^2}{2} + 5x\right]_{-1}^{0} + \left[5x - \frac{x^2}{2}\right]_{0}^{3} - \left[\frac{x^2}{8} + \frac{5}{4}x\right]_{-1}^{3}$$

$$= \left[2x^2 + 5x\right]_{-1}^{0} + \left[5x - \frac{x^2}{2}\right]_{0}^{3} - \left[\frac{x^2}{8} + \frac{5}{4}x\right]_{-1}^{3}$$

$$= \left[(0+0)-(2-5)\right] + \left[\left(15 - \frac{9}{2}\right) - (0-0)\right]$$

$$- \left[\left(\frac{9}{8} + \frac{15}{4}\right) - \left(\frac{1}{8} - \frac{5}{4}\right)\right]$$

$$= 3 + \frac{21}{2} - \left(\frac{9+30-1+10}{8}\right) = 3 + \frac{21}{2} - \frac{48}{8}$$

$$= 3 + \frac{21}{2} - 6 = \frac{21}{2} - 3 = \frac{15}{2} = 7{\cdot}5 \text{ sq. units.}$$

Example 2. Using the method of integration, find the area of the triangular region whose vertices are (2, –2), (4, 3) and (1, 2). *(A.I.C.B.S.E. 2016)*

Solution. Let A(2, –2), B(4, 3) and C(1, 2) be the vertices of the triangle as shown in the following figure :

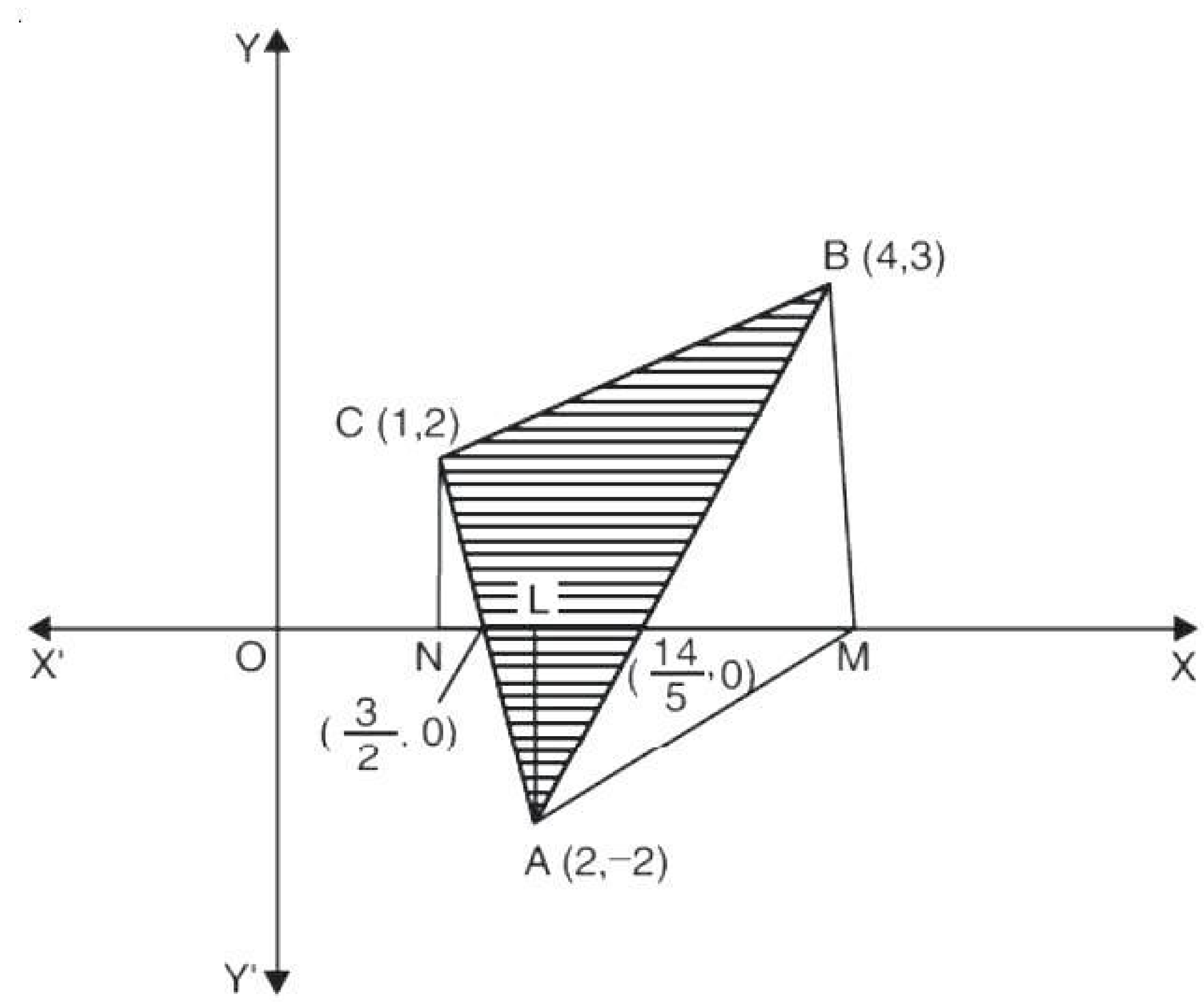


Fig.

Now equation of AB is $y + 2 = \frac{3+2}{4-2}\ (x - 2)$

$\Rightarrow \quad y + 2 = \frac{5}{2}x - 5 \Rightarrow y = \frac{5}{2}x - 7$

$\Rightarrow \quad x = \frac{2y + 14}{5}$ ...(1)

Equation of BC is $y - 2 = \frac{3-2}{4-1}(x - 1)$

$\Rightarrow \quad y - 2 = \frac{1}{3}(x - 1) \Rightarrow y = \frac{x}{3} + \frac{5}{3}$

$\Rightarrow \quad x = 3y - 5$ ...(2)

Equation of CA is $y - 2 = \frac{-2-2}{2-1}(x - 1)$

$\Rightarrow \quad y - 2 = -4x + 4 \Rightarrow y = -4x + 6$

$\Rightarrow \quad x = \frac{6 - y}{4}$ ...(3)

Now Area of ΔABC

$$= \int_2^3 \frac{2y+14}{5}\, dy - \int_2^3 (3y-5)\, dy - \int_{-2}^2 \frac{6-y}{4}\, dy$$

$$= \frac{75}{5} - \frac{5}{2} - \frac{24}{4} = \frac{300 - 50 - 120}{20}$$

$$= \frac{130}{20} = \frac{13}{2} \text{ sq. units.}$$

Example 3. Using the method of integration, find the area of the region bounded by the lines :

3x – 2y + 1 = 0, 2x + 3y – 21 = 0 and x – 5y + 9 = 0.

(C.B.S.E. 2012)

Solution. Let the sides AB, BC and CA of ΔABC be :

$3x - 2y + 1 = 0$ ...(1)

$2x + 3y - 21 = 0$ ...(2)

and $x - 5y + 9 = 0$ ...(3) respectively.

Solving (3) and (1), we get A as (1, 2).

Solving (1) and (2), we get B as (3, 5).

Solving (2) and (3), we get C as (6, 3).

Now ar (ΔABC) = ar (trap ALMB) + ar (trap BMNC) – ar (trap ALNC)

$$= \int_1^3 \frac{3x+1}{2} dx + \int_3^6 \frac{21-2x}{3} dx - \int_1^6 \frac{x+9}{5} dx$$

$$= \frac{1}{2}\left[\frac{3}{2}x^2 + x\right]_1^3 + \frac{1}{3}\left[21x - x^2\right]_3^6 - \frac{1}{5}\left[\frac{x^2}{2} + 9x\right]_1^6$$

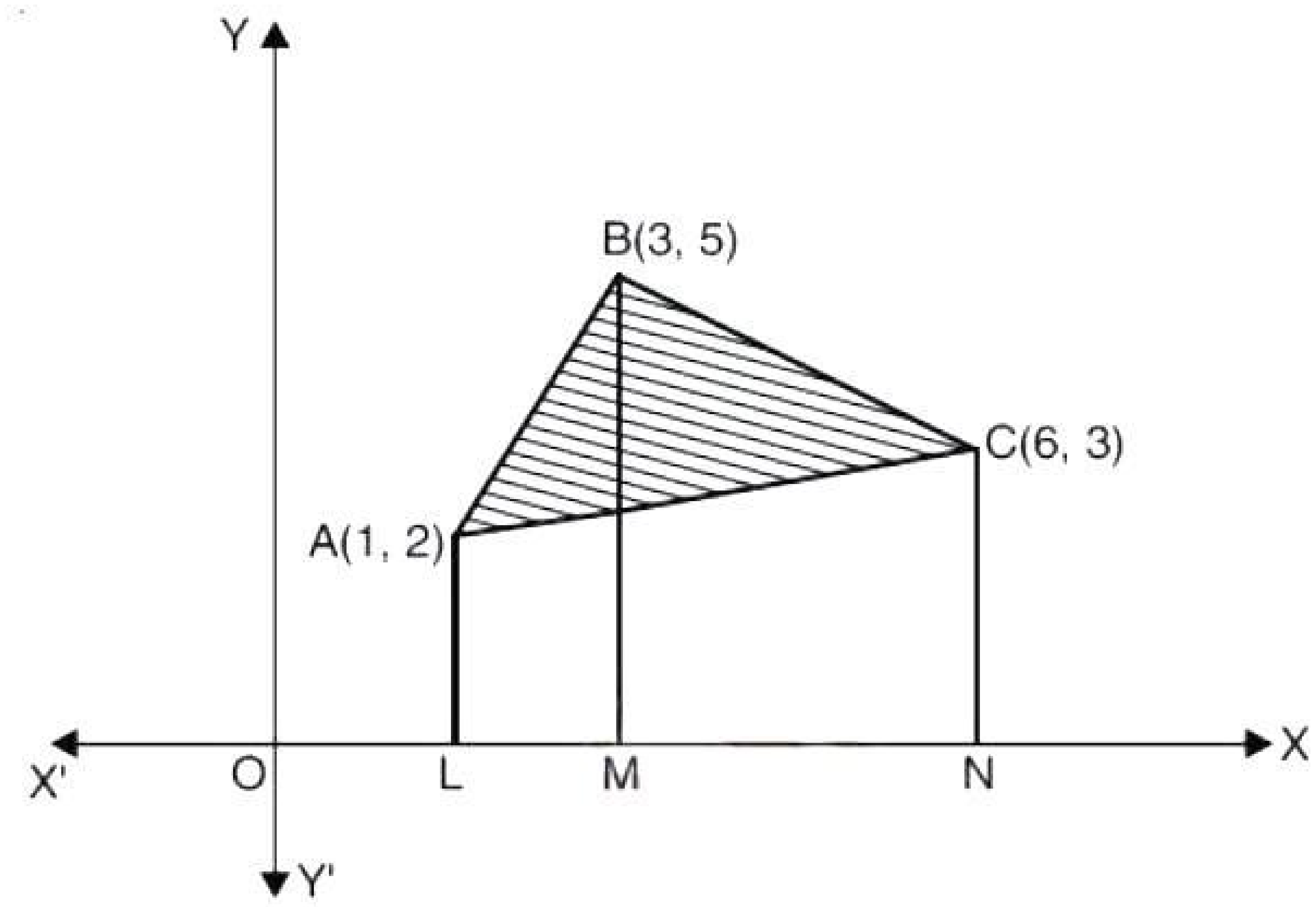


Fig.

$$= \frac{1}{2}\left[\left(\frac{27}{2}+3\right) - \left(\frac{3}{2}+1\right)\right] + \frac{1}{3}[(126-36)-(63-9)]$$
$$-\frac{1}{5}\left[(18+54) - \left(\frac{1}{2}+9\right)\right]$$

$$= \frac{1}{2}[12+2] + \frac{1}{3}[90-54] - \frac{1}{5}\ 72 - \frac{19}{2}$$

$$= 7 + 12 - \frac{125}{10} = 19 - \frac{125}{10}$$

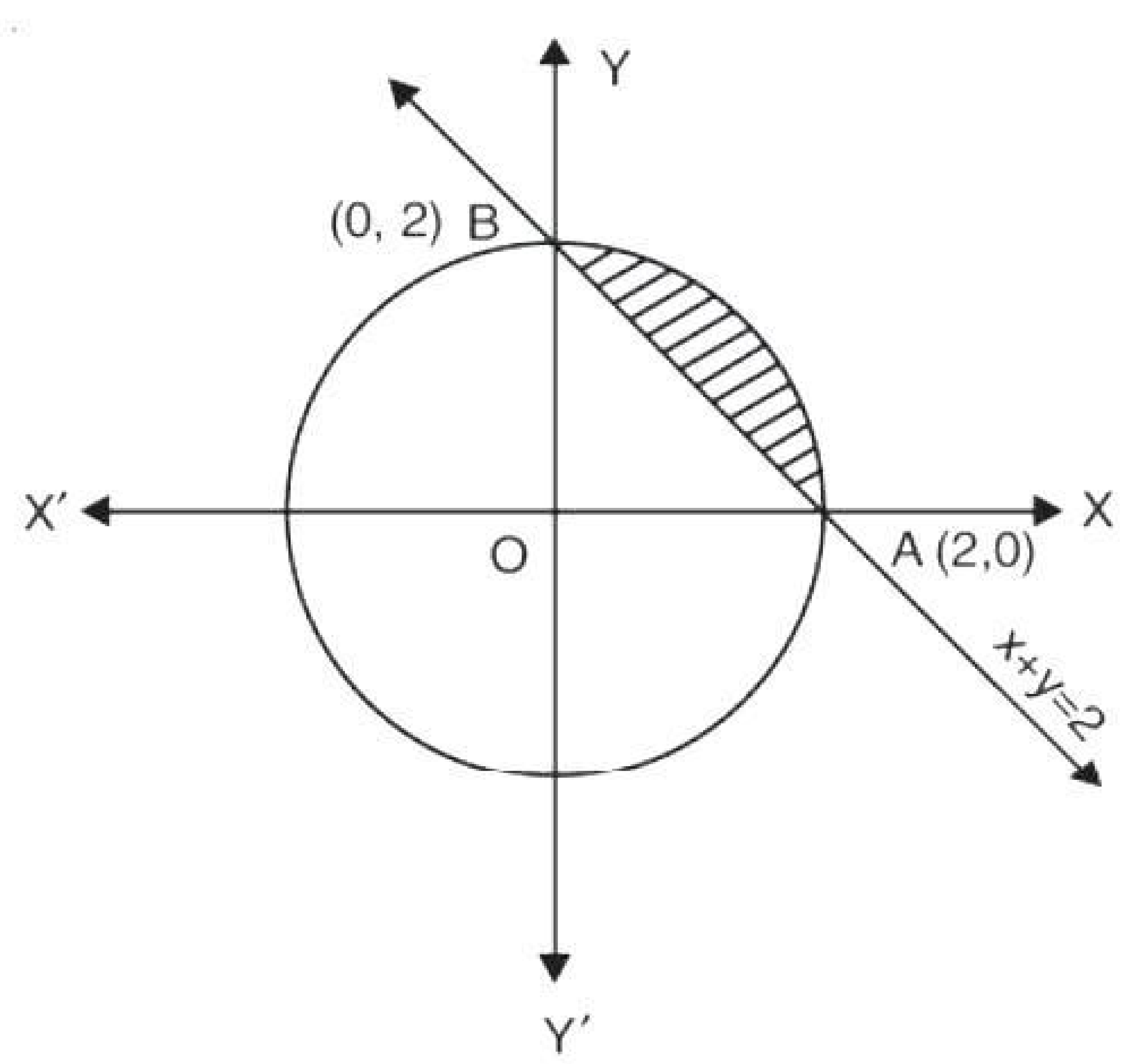
$$= \frac{190-125}{10}$$

$$= \frac{65}{10} = \frac{13}{2} = 6\cdot 5 \text{ sq. units.}$$

Example 4. Find the smaller area enclosed by the circle $x^2 + y^2 = 4$ and the line x + y = 2. ***(H.P.B. 2016)***

Solution. The given circle is $x^2 + y^2 = 4$ ...(1)

and the given line is $x + y = 2$ ...(2)



$\therefore$ Reqd. area = Shaded area

$$= \int_0^2 \sqrt{4-x^2}\,dx - \int_0^2 (2-x)\,dx$$

$$= \left[\frac{x\sqrt{4-x^2}}{2} + \frac{4}{2}\sin^{-1}\frac{x}{2}\right]_0^2 - \left[2x - \frac{x^2}{2}\right]_0^2$$

$$= \left[(0+2\sin^{-1}(1))\right] - \left[\left(4-\frac{4}{2}\right) - (0-0)\right]$$

$$= 2\left(\frac{\pi}{2}\right) - (2) = (\pi - 2) \text{ sq. units.}$$

Example 5. Draw a rough sketch of the region enclosed between the circles :

$x^2 + y^2 = 4$ and $(x - 2)^2 + y^2 = 1$.

Using integration, find the area of the enclosed region.

Solution. The given circles are $x^2 + y^2 = 4$ ...(1)

and $(x - 2)^2 + y^2 = 1$ ...(2)

(1) is a circle with centre (0, 0) and radius 2.

(2) is a circle with centre (2, 0) and radius 1.

The region is shown shaded in the following figure :

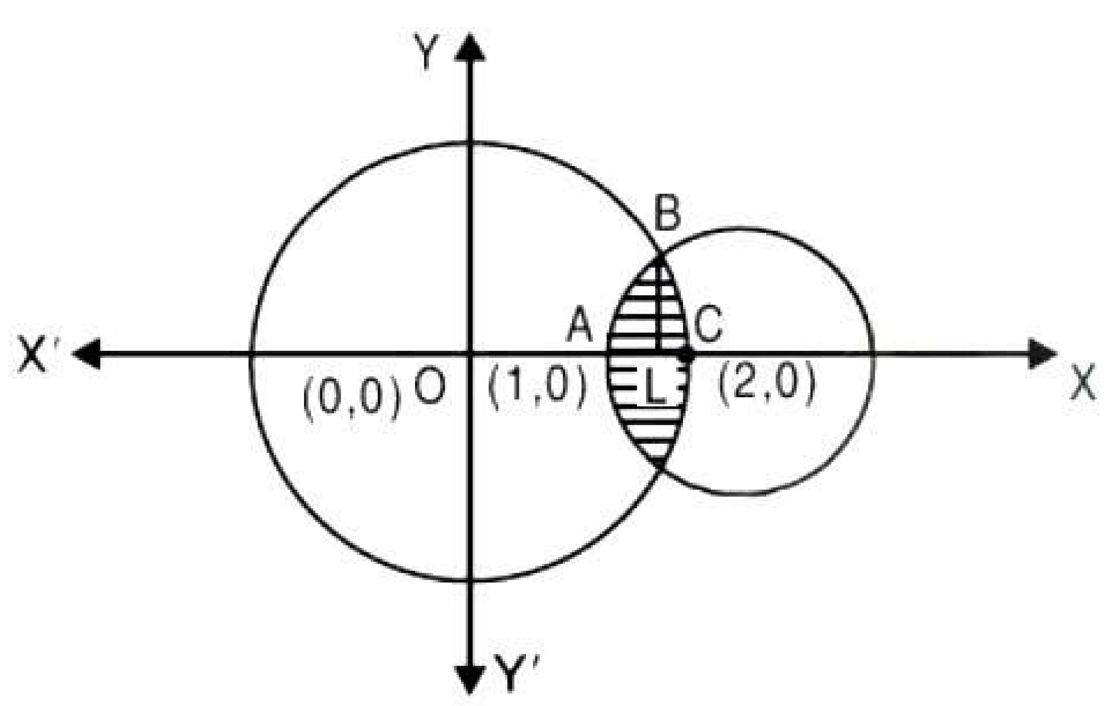


Fig.

Solving (1) and (2) :

Subtracting (1) from (2),

$(x - 2)^2 - x^2 = -3 \Rightarrow -4x + 4 = -3$

$\Rightarrow \quad 4x = 7 \quad \Rightarrow x = \frac{7}{4}.$

$$\therefore \text{ Reqd. area } = 2\left[\int_1^{7/4} \sqrt{1-(x-2)^2}\,dx + \int_{7/4}^{2} \sqrt{4-x^2}\,dx\right] \quad ...(1)$$

$$\text{Now} \quad I_1 = \int_1^{7/4} \sqrt{1-(x-2)^2}\,dx.$$

Put x – 2 = t so that $dx = dt$.

When $x = 1$, $t = -1$. When $x = \frac{7}{4}$, $t = \frac{7}{4} - 2 = -\frac{1}{4}$.

$$\therefore \quad I_1 = \int_{-1}^{-1/4} \sqrt{1-t^2}\,dt$$

$$= \left[\frac{t\sqrt{1-t^2}}{2} + \frac{1}{2}\sin^{-1}t\right]_{-1}^{-1/4}$$

$$= \frac{1}{2}\left[t\sqrt{1-t^2} + \sin^{-1}t\right]_{-1}^{-1/4}$$

$$= \frac{1}{2}\left[\left\{-\frac{1}{4}\sqrt{1-\frac{1}{16}} + \sin^{-1}\left(-\frac{1}{4}\right)\right\} - \left\{0 + \sin^{-1}(-1)\right\}\right]$$

$$= \frac{1}{2}\left[-\frac{\sqrt{15}}{16} - \sin^{-1}\left(\frac{1}{4}\right) + \frac{\pi}{2}\right]$$

$$= -\frac{\sqrt{15}}{32} - \frac{1}{2}\sin^{-1}\frac{1}{4} + \frac{\pi}{4}.$$

$$\text{And} \quad I_2 = \int_{7/4}^{2} \sqrt{4-x^2}\,dx = \int_{7/4}^{2} \sqrt{2^2-x^2}\,dx$$

$$= \left[\frac{x\sqrt{4-x^2}}{2} + \frac{4}{2}\sin^{-1}\frac{x}{2}\right]_{7/4}^{2}$$

$$= \left[\left\{0 + 2\sin^{-1}(1)\right\} - \left\{\frac{7}{8}\sqrt{4-\frac{49}{16}} + 2\sin^{-1}\frac{7}{8}\right\}\right]$$

$$= 2.\frac{\pi}{2} - \frac{7}{32}\sqrt{15} - 2\sin^{-1}\frac{7}{8}$$

$$= \pi - \frac{7}{32}\sqrt{15} - 2\sin^{-1}\frac{7}{8}.$$

$$\therefore \text{ From (1), Reqd. area} = 2\left[\left(\frac{-\sqrt{15}}{32} - \frac{1}{2}\sin^{-1}\frac{1}{4} + \frac{\pi}{4}\right) + \left(\pi - \frac{7}{32}\sqrt{15} - 2\sin^{-1}\frac{7}{8}\right)\right]$$

$$= -\frac{\sqrt{15}}{16} - \sin^{-1}\frac{1}{4} + \frac{\pi}{2} + 2\pi - \frac{7}{16}\sqrt{15} - 4\sin^{-1}\frac{7}{8}$$

$$= \left[\frac{5\pi}{2} - \frac{\sqrt{15}}{2} - 4\sin^{-1}\left(\frac{7}{8}\right) - \sin^{-1}\left(\frac{1}{4}\right)\right] \text{ sq. units.}$$

Example 6. Make a rough sketch of the region given below and find its area, using integration :

$\{(x, y) : y^2 \le 4x, 4x^2 + 4y^2 \le 9\}$.

Or

Using integration, find the area of the region bounded by the parabola $y^2 = 4x$ and the circle :
$4x^2 + 4y^2 = 9$.

Solution. We have :

$$y^2 = 4x \quad ....(1),$$

which is a right-handed parabola.

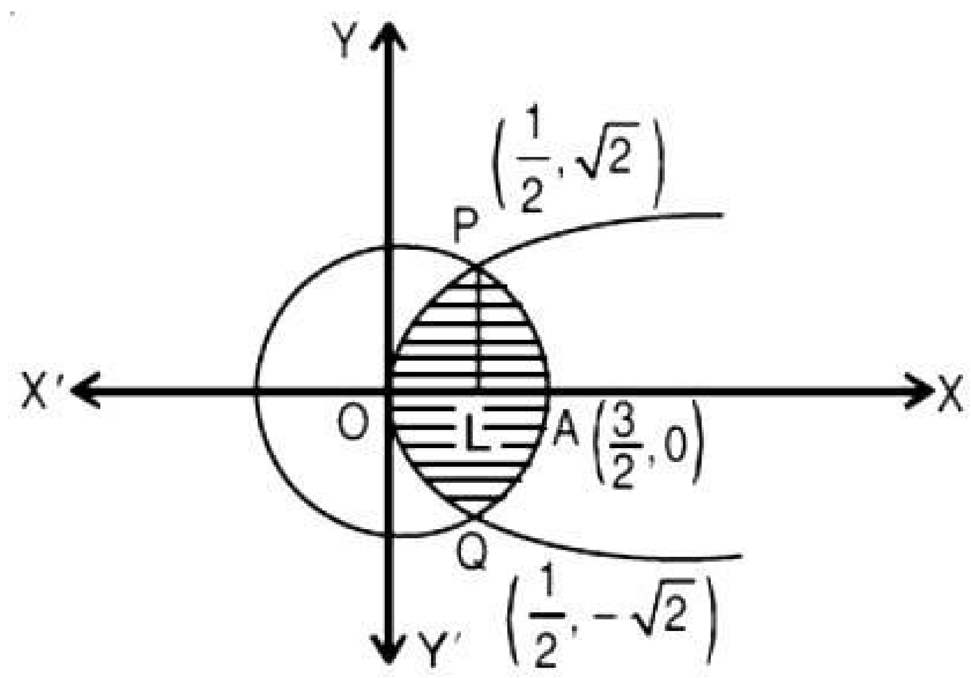


Fig.

And $4x^2 + 4y^2 = 9$

$$\Rightarrow \quad x^2 + y^2 = \left(\frac{3}{2}\right)^2 \quad ....(2),$$

which is a circle with centre (0, 0) and radius $\frac{3}{2}$.

Solving (1) and (2) :

$$x^2 + 4x = \frac{9}{4} \Rightarrow 4x^2 + 16x - 9 = 0$$

$$\Rightarrow \quad x = \frac{-16 \pm \sqrt{256+144}}{8} = \frac{-16 \pm 20}{8}$$

$$= -\frac{36}{8}, \frac{4}{8} = -\frac{9}{2}, \frac{1}{2}$$

$$\Rightarrow \quad x = \frac{1}{2}. \quad \left[\because x = -\frac{9}{2} \textit{ is not possible}\right]$$

Putting in (1), $y^2 = 4\left(\frac{1}{2}\right) = 2 \Rightarrow y = \pm\sqrt{2}$.

Thus (1) and (2) intersect at $P\left(\frac{1}{2}, \sqrt{2}\right)$ and $Q\left(\frac{1}{2}, -\sqrt{2}\right)$.

$\therefore$ Reqd. area = 2 (ar OPAO)

$$= 2\left[\int_0^{1/2} 2\sqrt{x}\, dx + \int_{1/2}^{3/2} \frac{\sqrt{9-4x^2}}{2}\, dx\right]$$

$$= 2\left[2\left[\frac{x^{3/2}}{3/2}\right]_0^{1/2} + \frac{2}{2}\int_{1/2}^{3/2} \sqrt{\left(\frac{3}{2}\right)^2 - x^2}\, dx\right]$$

$$= 2\left[\frac{4}{3}\left(\left(\frac{1}{2}\right)^{3/2} - 0\right) + \left\{\frac{x\sqrt{\frac{9}{4} - x^2}}{2} + \frac{9/4}{2}\sin^{-1}\frac{x}{3/2}\right\}_{1/2}^{3/2}\right]$$

$$= 2\left[\frac{4}{3}\frac{1}{2\sqrt{2}} + \left(0 + \frac{9}{8}\sin^{-1}(1)\right) - \left(\frac{1/2}{2}\sqrt{\frac{9}{4} - \frac{1}{4}} + \frac{9}{8}\sin^{-1}\frac{1}{3}\right)\right]$$

$$= 2\left[\frac{\sqrt{2}}{3} + \frac{9}{8}\cdot\frac{\pi}{2} - \frac{1}{4}\sqrt{2} - \frac{9}{8}\sin^{-1}\frac{1}{3}\right]$$

$$= \frac{2\sqrt{2}}{3} + \frac{9}{8}\pi - \frac{1}{2}\sqrt{2} - \frac{9}{4}\sin^{-1}\frac{1}{3}$$

$$= \left(\frac{9\pi}{8} - \frac{9}{4}\sin^{-1}\frac{1}{3} + \frac{\sqrt{2}}{6}\right) \text{ sq. units.}$$

Example 7. Find the area lying above the x-axis and included between the circle $x^2 + y^2 = 8x$ and the parabola $y^2 = 4x$. ***(N.C.E.R.T.; H.B. 2017, 14; Asssam B.2017, 13; H.P.B. 2012, 11)***

Solution. The given circle is $x^2 + y^2 - 8x = 0$

i.e. $\quad (x-4)^2 + y^2 = 16 \quad ...(1)$

It has centre (4, 0) and radius 4.

The given parabola is $y^2 = 4x \quad ...(2)$

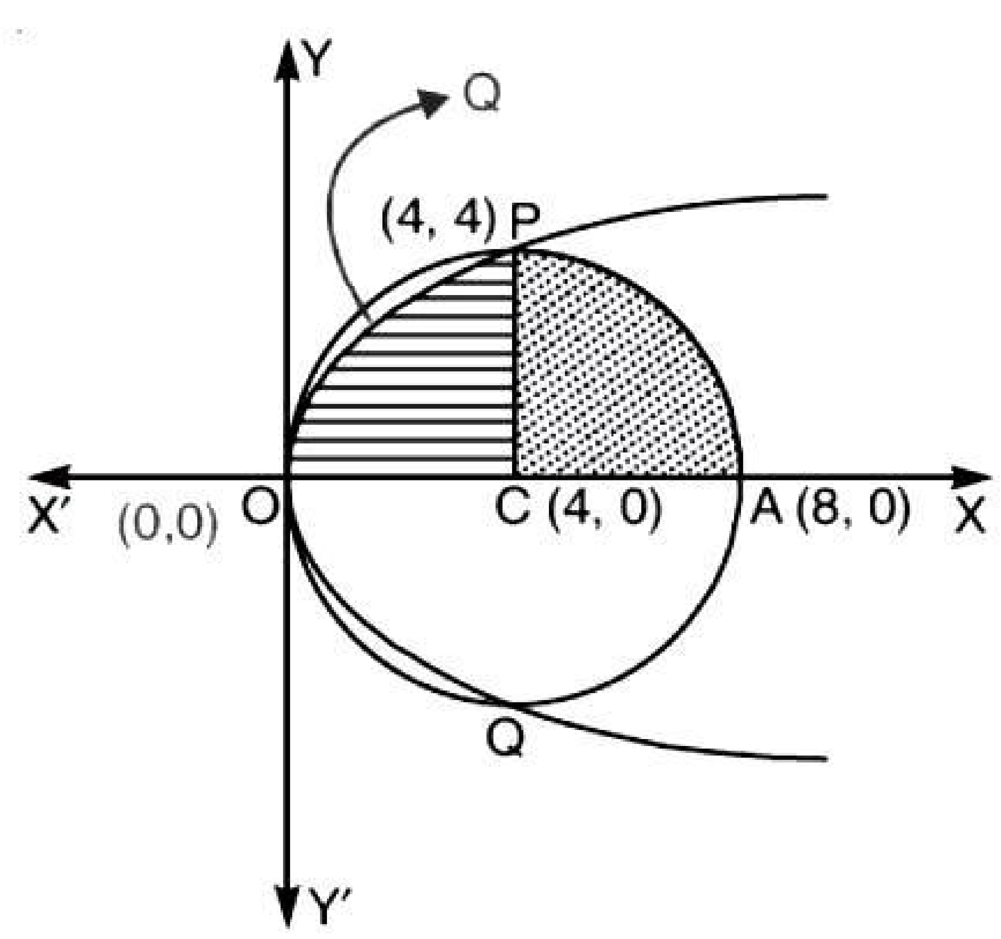


Fig.

Solving (1) and (2) :

$x^2 - 8x + 4x = 0 \quad \Rightarrow \quad x^2 - 4x = 0$

$\Rightarrow \quad x(x-4) = 0 \quad \Rightarrow \quad x = 0, 4.$

When $x = 0$, $y = 0$. When $x = 4$, $y^2 = 16 \Rightarrow y = \pm 4$.

Thus (1) intersects (2) at O (0, 0) and P (4, 4) above the x-axis.

$\therefore$ Area of the region OCAPQO

= Area of the region OCPQO

+ Area of the region CAPC

$$= \int_0^4 y_1\, dx + \int_4^8 y_2\, dx,$$ where y_1, y_2 are ordinates of points on (2) and (1) respectively

$$= \int_0^4 \sqrt{4x}\, dx + \int_4^8 \sqrt{4^2 - (x-4)^2}\, dx$$

[∵ *Taking +ve values as region lies above x-axis*]

$$= 2\int_0^4 x^{1/2}\, dx + \int_4^8 \sqrt{4^2 - (x-4)^2}\, dx$$

$$= 2\left[\frac{x^{3/2}}{3/2}\right]_0^4 + \int_0^4 \sqrt{4^2 - t^2}\, dt$$

[*Putting x – 4 = t in 2nd integral so that dx = dt.*

When x = 4, t = 0; when x = 8, t = 4]

$$= \frac{4}{3}[4^{3/2} - 0] + \left[\frac{t}{2}\sqrt{4^2 - t^2} + \frac{4^2}{2}\sin^{-1}\frac{t}{4}\right]_0^4$$

$$= \frac{32}{3} + \left[\left\{\frac{4}{2}\times 0 + 8\sin^{-1}(1)\right\} - \{0 + 0\}\right]$$

$$= \frac{32}{3} + 8\left(\frac{\pi}{2}\right) = \frac{32}{3} + 4\pi$$

$$= \frac{4}{3}(8 + 3\pi) \text{ sq. units.}$$

Example 8. Find the area included between the curves $y^2 = 4ax$ and $x^2 = 4ay$, $a > 0$.

(H.B. 2018; P.B. 2010, ; A.I.C.B.S.E. 2009)

Solution. The given curves *viz.* parabolas are :

$y^2 = 4ax$ ...(1) and $x^2 = 4ay$ ...(2)

These two intersect at O (0, 0) and B ($4a$, $4a$).

[*Solve !*]

∴ Reqd. area = Shaded area OABCO

= area OABL – area OCBL

= (area under parabola $y^2 = 4ax$) –

(area under parabola $x^2 = 4ay$)

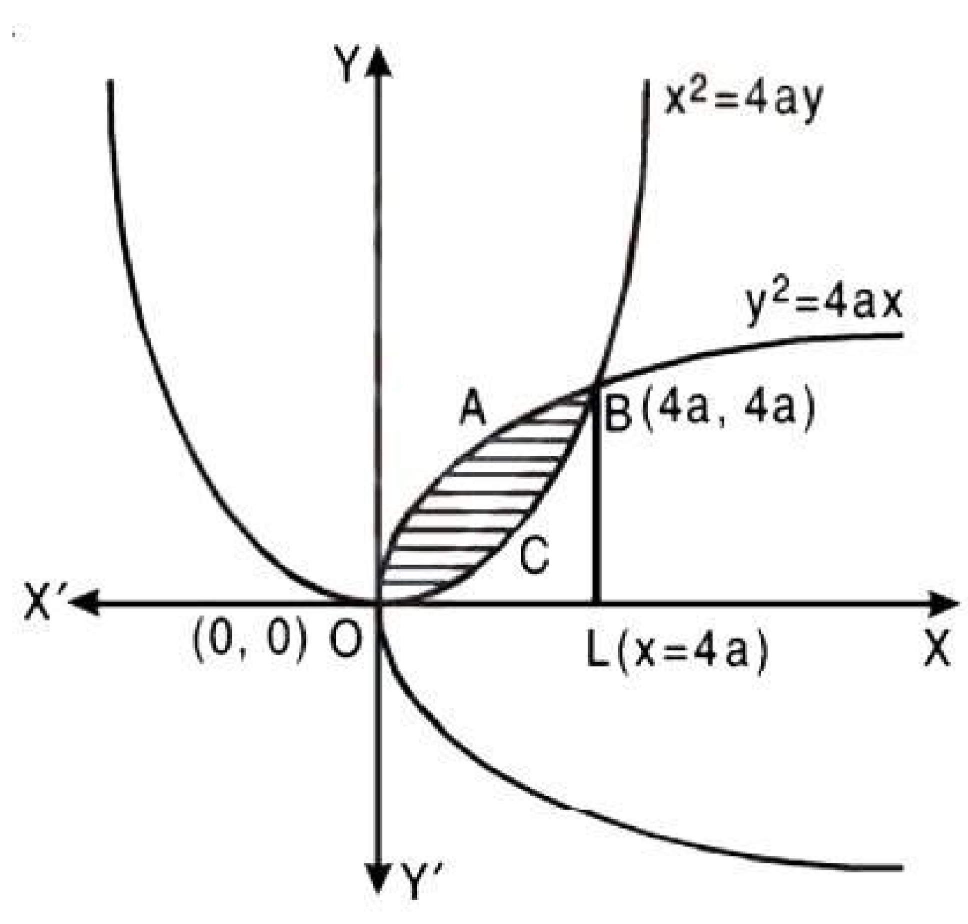


Fig.

$$= \int_0^{4a} \sqrt{4ax}\, dx - \int_0^{4a} \frac{x^2}{4a}\, dx$$

$$= 2\sqrt{a}\int_0^{4a} x^{1/2}\, dx - \frac{1}{4a}\int_0^{4a} x^2 dx$$

$$= 2\sqrt{a}\left[\frac{x^{3/2}}{3/2}\right]_0^{4a} - \frac{1}{4a}\left[\frac{x^3}{3}\right]_0^{4a}$$

$$= \frac{4\sqrt{a}}{3}[(4a)^{3/2} - 0] - \frac{1}{12a}[64a^3 - 0]$$

$$= \frac{32}{3}a^2 - \frac{16}{3}a^2 = \frac{16a^2}{3} \text{ sq. units.}$$

Example 9. Draw the rough sketch of $y^2 = x + 1$ and $y^2 = -x + 1$ and determine the area enclosed by the two curves.

Solution. The given curves are :

$y^2 = (x + 1)$ ...(1)

and $y^2 = -(x - 1)$ ...(2)

These two parabolas meet at A (0, 1) and B (0, – 1).

[*Solve : (1) and (2)*]

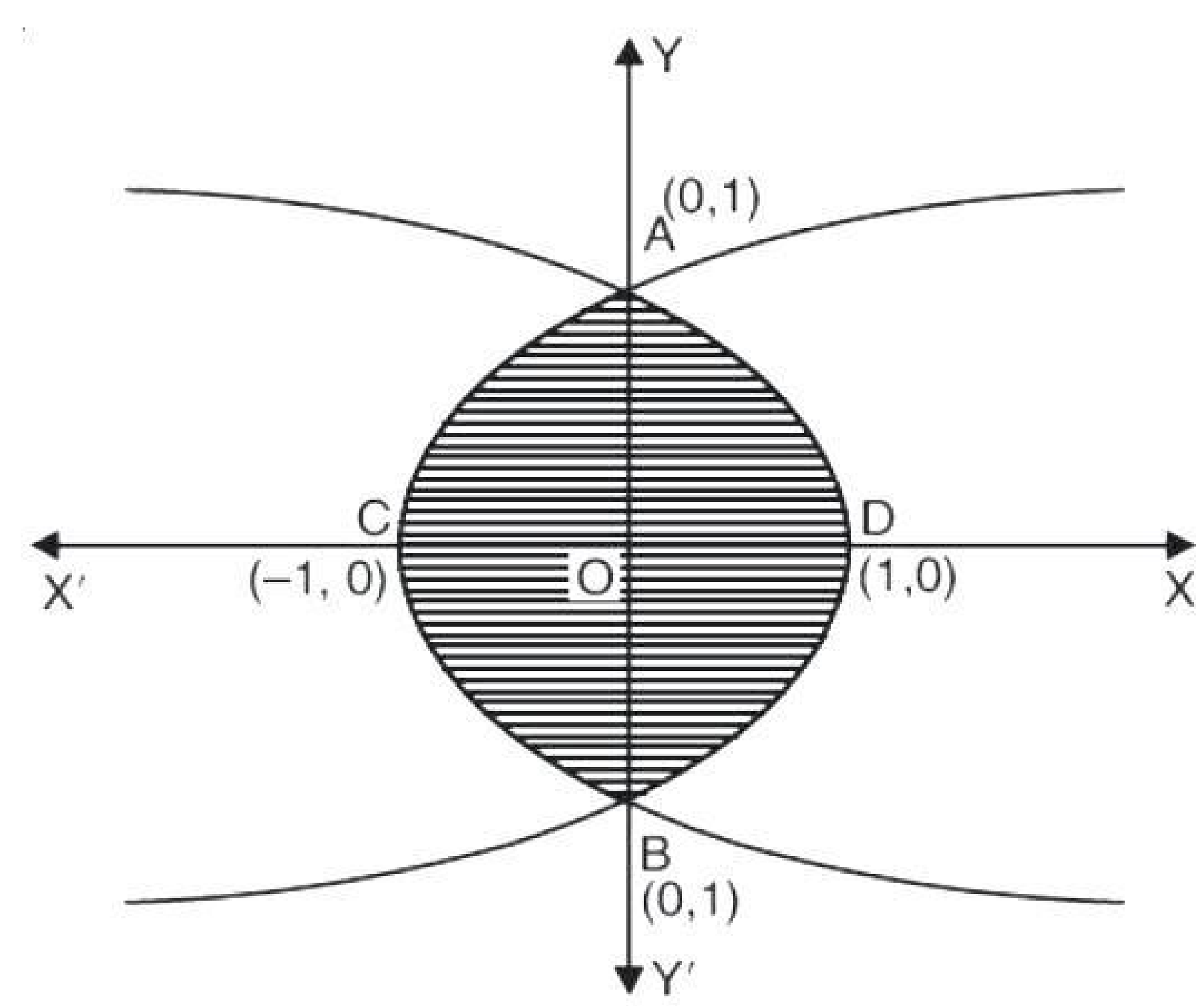


Fig.

∴ Reqd. area = 2 [ar. (CAO) + ar. (OAD)]

$$= 2\left[\int_{-1}^{0}\sqrt{x+1}\, dx + \int_0^1 \sqrt{1-x}\, dx\right]$$

$$= 2\left[\left[\frac{(x+1)^{3/2}}{3/2}\right]_{-1}^{0} + \left[\frac{(1-x)^{3/2}}{\frac{3}{2}(-1)}\right]_0^1\right]$$

$$= \frac{4}{3}\left[\left[(x+1)^{3/2}\right]_{-1}^{0} + \left[-(1-x)^{3/2}\right]_0^1\right]$$

$$= \frac{4}{3}[(1-0) + (-0+1)]$$

$$= \frac{4}{3}(2) = \frac{8}{3} = 2\frac{2}{3} \text{ sq. units.}$$

Example 10. Find the area of the region bounded by the curves :

$y = 6x - x^2$ and $y = x^2 - 2x$.

Solution. The given curves are $y = 6x - x^2$ ....(1)

and $y = x^2 - 2x$ ....(2)

Solving (1) and (2) : $6x - x^2 = x^2 - 2x$

$\Rightarrow \quad 2x^2 - 8x = 0 \Rightarrow x^2 - 4x = 0$

$\Rightarrow \quad x(x-4) = 0 \Rightarrow x = 0, 4.$

When $x = 0$, from (1), $y = 6(0) - (0) = 0$.

When $x = 4$, from (1), $y = 6(4) - (4)^2 = 24 - 16 = 8$.

Thus (1) and (2) intersect at O (0, 0) and A (4, 8).

Also (1) meets x-axis, where $0 = 6x - x^2 \Rightarrow x = 0, 6$.

Thus (1) meets x-axis at O (0, 0) and B (6, 0).

Similarly (2) meets x-axis at O (0, 0) and C (2, 0).

From (1), $y - 9 = -(x^2 - 6x + 9)$

$\Rightarrow \quad y - 9 = -(x-3)^2$

$\Rightarrow \quad (x-3)^2 = -(y-9)$,

which is downward parabola with vertex P (3, 9).

From (2), $y + 1 = x^2 - 2x + 1$

$\Rightarrow (x-1)^2 = y + 1$,

which is upward parabola with vertex Q (1, – 1).

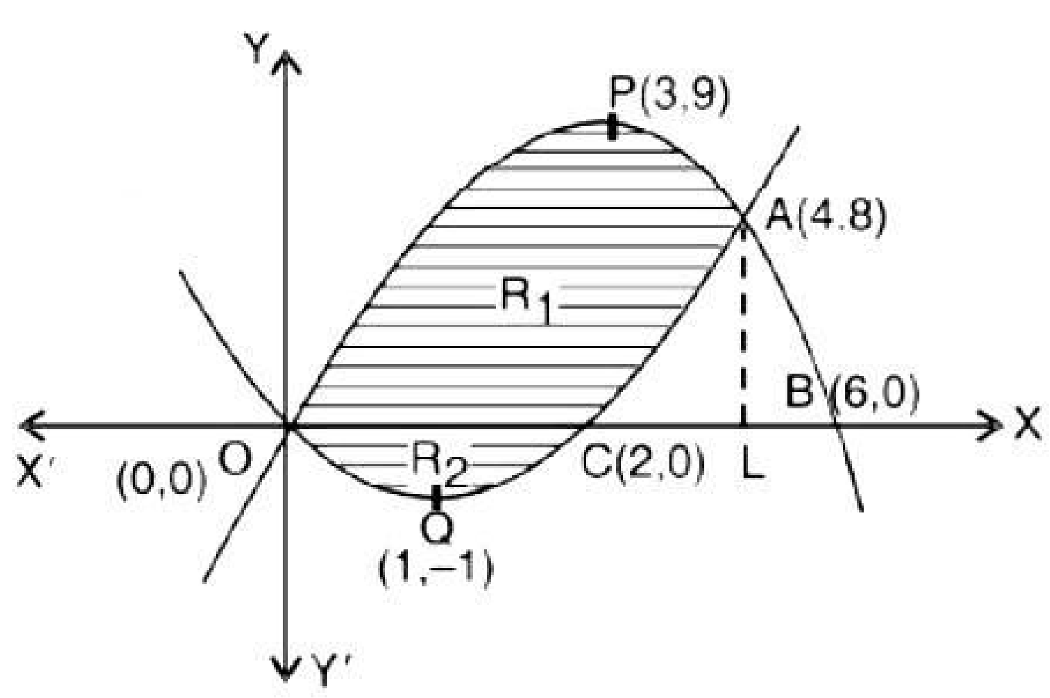


Fig.

$\therefore$ Reqd. area = $R_1 \cap R_2$

$$= \left[\int_0^4 (6x - x^2)\,dx - \int_2^4 (x^2 - 2x)\,dx\right] + \left|\int_0^2 (x^2 - 2x)\,dx\right|$$

$$= \left[\left\{3x^2 - \frac{x^3}{3}\right\}_0^4 - \left\{\frac{x^3}{3} - x^2\right\}_2^4\right] + \left|\left\{\frac{x^3}{3} - x^2\right\}_0^2\right|$$

$$= \left(48 - \frac{64}{3}\right) - \left(\left(\frac{64}{3} - 16\right) - \left(\frac{8}{3} - 4\right)\right) + \left|\frac{8}{3} - 4\right|$$

$$= \frac{80}{3} - \frac{20}{3} + \frac{4}{3} = 20 + \frac{4}{3} = \frac{64}{3}$$

$= 31\frac{1}{3}$ sq. units.

Example 11. Find the area enclosed between the parabola $4y = 3x^2$ and the straight line $3x - 2y + 12 = 0$.

(H.P.B. 2018; A.I.C.B.S.E. 2017)

Solution. The given parabola is $4y = 3x^2$ *i.e.* $x^2 = \frac{4y}{3}$ ....(1),

which is an upward parabola.

The given line is $3x - 2y + 12 = 0$ ...(2)

Solving (1) and (2) :

From (1), $y = \frac{3x^2}{4}$ ...(3)

Putting in (2), $3x - 2\left(\frac{3x^2}{4}\right) + 12 = 0$

$\Rightarrow \quad x - \frac{x^2}{2} + 4 = 0 \quad \Rightarrow x^2 - 2x - 8 = 0$

$\Rightarrow \quad (x-4)(x+2) = 0 \quad \Rightarrow x = -2, 4.$

When $x = -2$, then from (3), $y = \frac{3}{4}(4) = 3$.

When $x = 4$, then from (3), $y = \frac{3}{4}(16) = 12$.

Thus parabola (1) and line (2) meet each other at A(–2, 3) and B(4, 12).

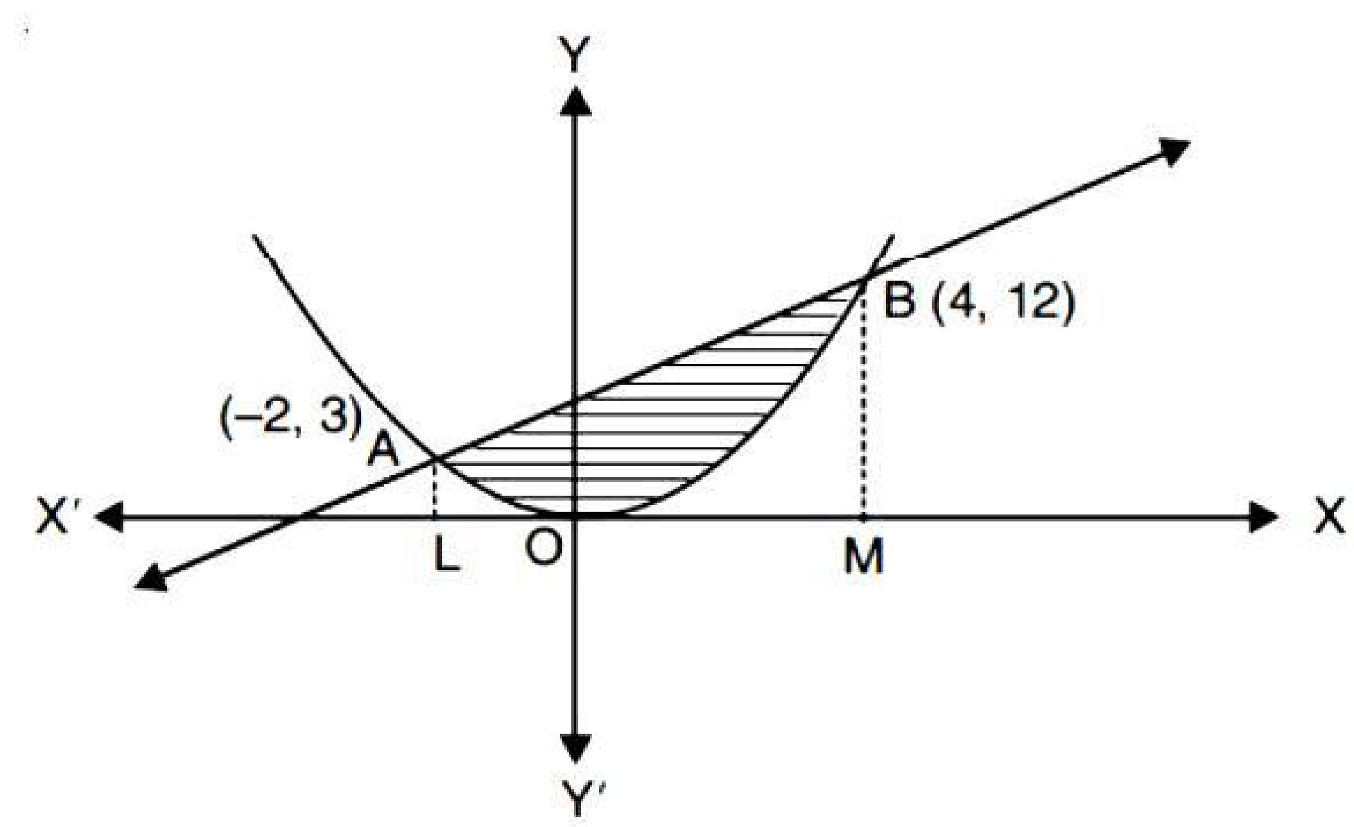


Fig.

$\therefore$ Reqd. area = area ALMB – (area ALO + area OMB)

$$= \int_{-2}^{4} \frac{3x+12}{2}dx - \left(\int_{-2}^{0} \frac{3}{4}x^2 dx + \int_0^4 \frac{3}{4}x^2 dx\right)$$

$$\left[\because \textit{From (2)},\ y = \frac{3x+12}{2}\right]$$

$$= \frac{1}{2}\left[\frac{3x^2}{2} + 12x\right]_{-2}^{4} - \left[\frac{3}{4}\left\{\frac{x^3}{3}\right\}_{-2}^{0} + \frac{3}{4}\left\{\frac{x^3}{3}\right\}_0^4\right]$$

$$= \frac{1}{2}[(24+48) - (6-24)] - \left[\frac{3}{4}\left(0 + \frac{8}{3}\right) + \frac{3}{4}\left(\frac{64}{3} - 0\right)\right]$$

$$= \frac{1}{2}[72 + 18] - [2 + 16] = 45 - 18$$

= 27 sq. units.

Example 12. Find the area bounded by the curves :

$y = \sqrt{x}$, $2y + 3 = x$ and x–axis.

(C.B.S.E. Sample Paper 2019)

Solution. The given curves are :

$$y = \sqrt{x} \quad ...(1)$$

and $$2y + 3 = x \quad ...(2)$$

Solving (1) and (2), we get :

$$2y + 3 = y^2 \quad \Rightarrow \quad y^2 - 2y - 3 = 0$$

$$\Rightarrow \ (y + 1)(y - 3) = 0 \quad \Rightarrow \quad y = -1, 3.$$

Thus, $y = 3$. $[\because y > 0]$

Putting in (2), $2(3) + 3 = x \Rightarrow x = 9$.

Thus, (1) and (2) intersect at (9, 3).

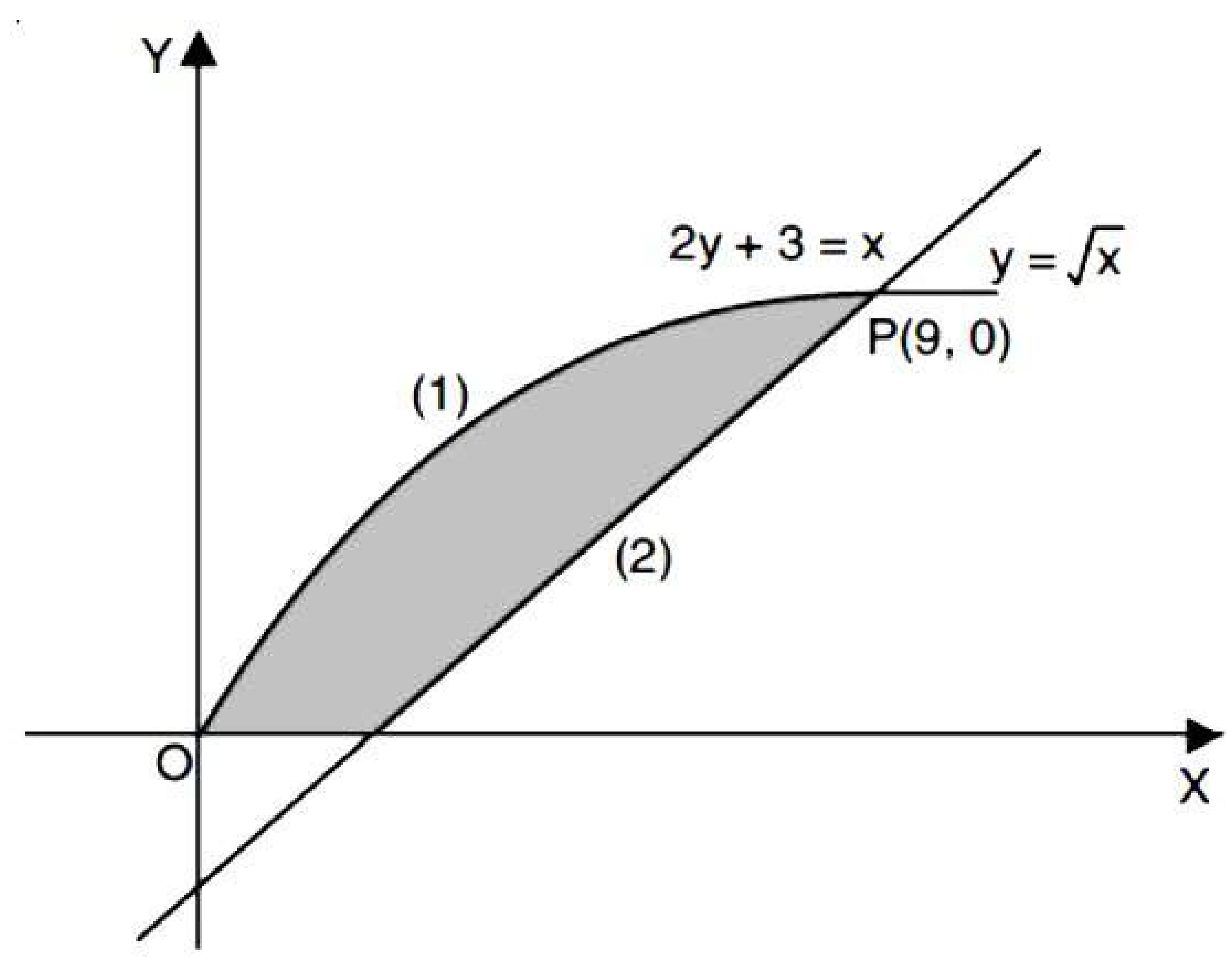


Fig.

$\therefore$ Reqd. area

$$= \int_0^3 (2y+3)\, dy - \int_0^3 y^2 dy = [y^2 + 3y]_0^3 - \left[\frac{y^3}{3}\right]_0^3$$

$$= [3^2 + 3(3)] - \left[\frac{3^3}{3}\right] = (9 + 9) - (9) = 9 \text{ sq. units.}$$

Example 13. Find the area of the region bounded by the curves $y = x^2 + 2$, $y = x$, $x = 0$ and $x = 3$. *(N.C.E.R.T.)*

Solution. $y = x^2 + 2$ *i.e.* $x^2 = y - 2$,

which is an upward parabola.

For tracing, we have the table :

$x =$	...	0	3	...
$y =$	...	2	11	...

$y = x$ is a line, passing through the origin.

This lies below the parabola. $[\because x^2 + 2 > x \textit{ for all } x]$

The region is shown shaded in the following figure :

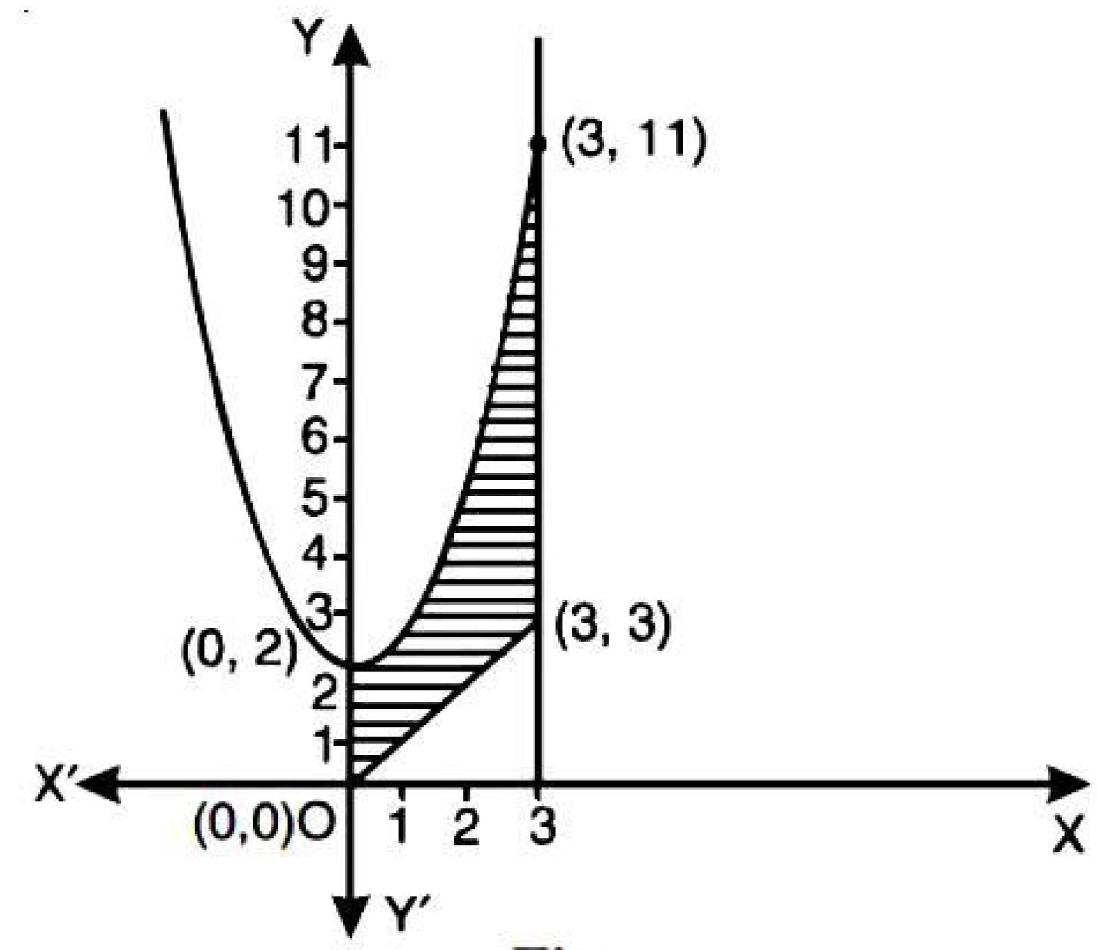


Fig.

$\therefore$ Reqd. area = Area under the parabola – Area under the line $y = x$

$$= \int_0^3 (x^2 + 2)\, dx - \int_0^3 x\, dx = \left[\frac{x^3}{3} + 2x\right]_0^3 - \left[\frac{x^2}{2}\right]_0^3$$

$$= (9 + 6) - \left(\frac{9}{2}\right) = \frac{21}{2} = 10{\cdot}5 \text{ sq. units.}$$

Example 14. Using integration, find the area of the region :

$\{(x, y) : |x - 1| \le y \le \sqrt{5 - x^2}\}$. *(C.B.S.E. 2010)*

Or

Sketch the region bounded by the curves :

$y = \sqrt{5 - x^2}$ and $y = |x - 1|$

and find its area, using integration.

(A.I.C.B.S.E.2015)

Solution. The given curves are :

$$x^2 + y^2 = 5$$

$$[\because y = \sqrt{5 - x^2} \Rightarrow y^2 = 5 - x^2 \Rightarrow x^2 + y^2 = 5]$$

and $$y = \begin{cases} 1 - x, \text{ if } x < 1 \\ x - 1, \text{ if } x > 1. \end{cases}$$

The reqd. region is shown as shaded in the following figure :

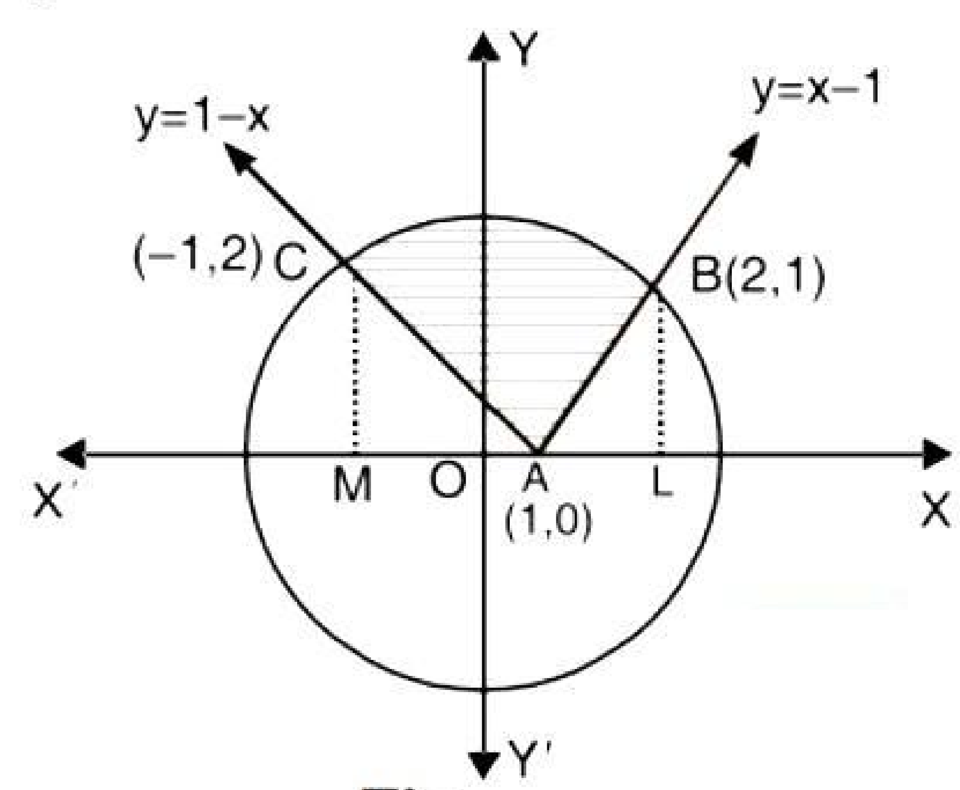


Fig.

$y = x - 1$ meets $x^2 + y^2 = 5$ at B (2, 1).

$y = 1 - x$ meets $x^2 + y^2 = 5$ at C (– 1, 2)

$y = x - 1$ and $y = 1 - x$ meet at A (1, 0).

$\therefore$ Reqd. area = ar (MCBLM) – ar (CMAC) – ar (ALBA)

$$= \int_{-1}^{2} \sqrt{5 - x^2}\, dx - \int_{-1}^{1} (1 - x)\, dx - \int_{1}^{2} (x - 1)\, dx$$

$$= \left[\frac{x\sqrt{5-x^2}}{2} + \frac{5}{2}\sin^{-1}\frac{x}{\sqrt{5}}\right]_{-1}^{2}$$

$$-\left[x - \frac{x^2}{2}\right]_{-1}^{1} - \left[\frac{x^2}{2} - x\right]_{1}^{2}$$

$$= \left[\left(1 + \frac{5}{2}\sin^{-1}\frac{2}{\sqrt{5}}\right) - \left(-\frac{1}{2}\times 2 + \frac{5}{2}\sin^{-1}\left(-\frac{1}{\sqrt{5}}\right)\right)\right]$$

$$-\left[\left(1-\frac{1}{2}\right) - \left(-1-\frac{1}{2}\right)\right] - \left[(2-2) - \left(\frac{1}{2}-1\right)\right]$$

$$= 1 + \frac{5}{2}\sin^{-1}\frac{2}{\sqrt{5}} + 1 - \frac{5}{2}\sin^{-1}\left(-\frac{1}{\sqrt{5}}\right) - 2 - \frac{1}{2}$$

$$= -\frac{1}{2} + \frac{5}{2}\left[\sin^{-1}\frac{2}{\sqrt{5}} - \sin^{-1}\left(-\frac{1}{\sqrt{5}}\right)\right] \text{ sq. units.}$$

Example 15. Using integration, find the area of the triangle formed by positive *x*-axis and tangent and normal to the circle $x^2 + y^2 = 4$ at $(1, \sqrt{3})$. *(C.B.S.E. 2015)*

Solution. The given circle is $x^2 + y^2 = 4$ ...(1)

Diff. *w.r.t.* *x*, $\quad 2x + 2y\frac{dy}{dx} = 0$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{-x}{y}.$$

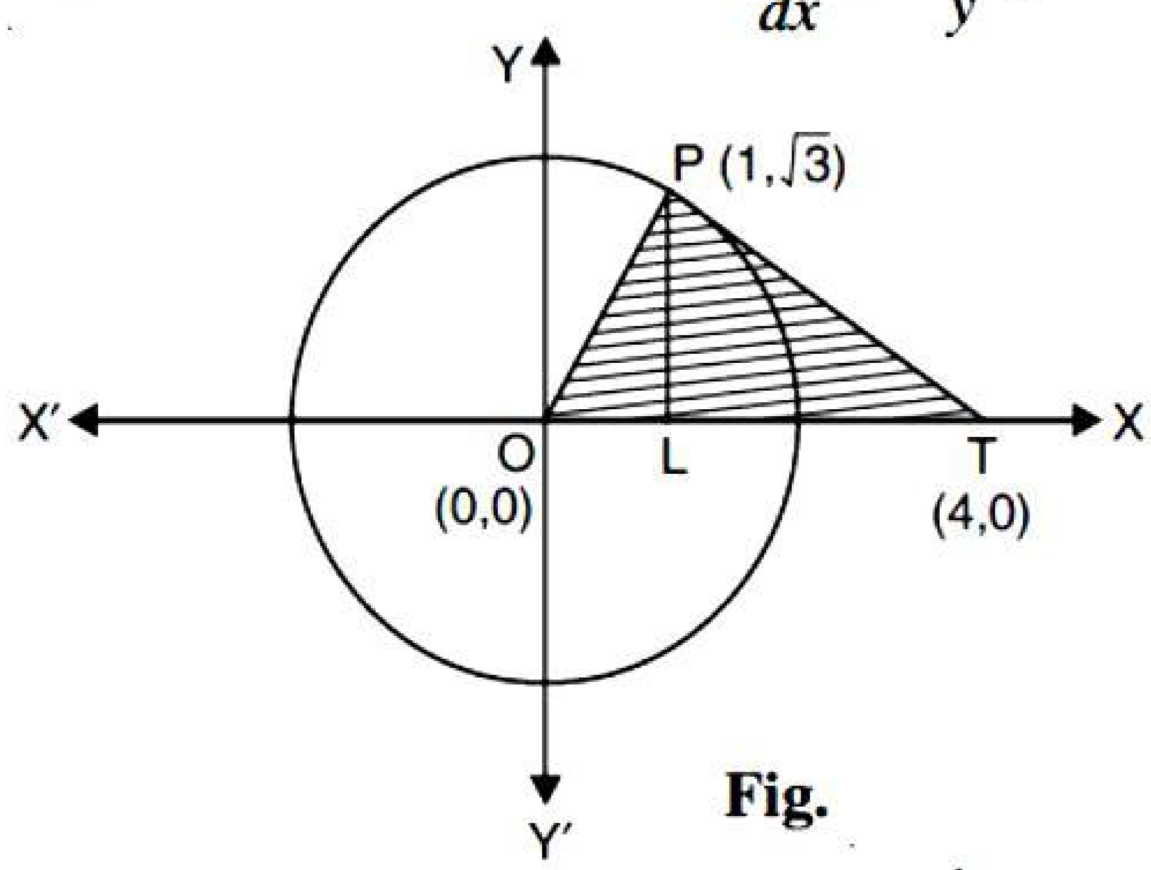


Fig.

Slope of tangent at P $(1, \sqrt{3}) = -\frac{1}{\sqrt{3}}$.

Slope of normal at P $(1, \sqrt{3}) = \sqrt{3}$.

∴ The equation of the tangent at P is :

$$y - \sqrt{3} = -\frac{1}{\sqrt{3}}(x-1)$$

$$\Rightarrow \quad y = -\frac{1}{\sqrt{3}}x + \sqrt{3} + \frac{1}{\sqrt{3}}.$$

This meets *x*-axis, where $0 = -\frac{1}{\sqrt{3}}x + \sqrt{3} + \frac{1}{\sqrt{3}}$

$\Rightarrow \quad 0 = -x + 3 + 1 \quad \Rightarrow \quad x = 4.$

Thus T is (4, 0).

The equation of the normal at P is :

$$y - \sqrt{3} = \sqrt{3}(x-1)$$

$$\Rightarrow \quad y = \sqrt{3}x.$$

This meets *x*-axis *i.e.* $y = 0$, where $x = 0$.

Thus O is (0, 0).

Now ar (ΔOPT)

= ar (OPL) + ar (PLT)

$$= \int_0^1 \sqrt{3}x\,dx + \int_1^4 \left(-\frac{1}{\sqrt{3}}x + \sqrt{3} + \frac{1}{\sqrt{3}}\right)dx$$

$$= \sqrt{3}\int_0^1 x\,dx - \frac{1}{\sqrt{3}}\int_1^4 x\,dx + \left(\sqrt{3} + \frac{1}{\sqrt{3}}\right)\int_1^4 1.dx$$

$$= \sqrt{3}\left[\frac{x^2}{2}\right]_0^1 - \frac{1}{\sqrt{3}}\left[\frac{x^2}{2}\right]_1^4 + \frac{3+1}{\sqrt{3}}[x]_1^4$$

$$= \sqrt{3}\left(\frac{1}{2} - 0\right) - \frac{1}{\sqrt{3}}\left(8 - \frac{1}{2}\right) + \frac{4}{\sqrt{3}}(4-1)$$

$$= \frac{\sqrt{3}}{2} - \frac{8}{\sqrt{3}} + \frac{1}{2\sqrt{3}} + 4\sqrt{3}$$

$$= \frac{9}{2}\sqrt{3} + \frac{-16+1}{2\sqrt{3}} = \frac{9}{2}\sqrt{3} - \frac{15}{2\sqrt{3}}$$

$$= \frac{9}{2}\sqrt{3} - \frac{5}{2}\sqrt{3} = \left(\frac{9}{2} - \frac{5}{2}\right)\sqrt{3} = 2\sqrt{3} \text{ sq units.}$$

Example 16. Sketch the graph of :

$$\mathbf{f(x)} = \begin{cases} |x-2| + 2, & x \le 2 \\ x^2 - 2, & x > 2. \end{cases}$$

Evaluate $\int_0^4 f(x)\,dx$. What does the value of this integral represent on the graph ?

Solution. Graph : *For* $x < 2$, $f(x) = |x - 2| + 2$

$= -(x-2) + 2 \qquad [\because x - 2 < 0]$

$= 4 - x.$

For $x = 2$, $f(x) = 2. \qquad [\because x - 2 = 0]$

For $x > 2$, $f(x) = x^2 - 2$.

TABLE :

x :	3	4	5	.....
y :	7	14	23	.....

The graph is shown as below :

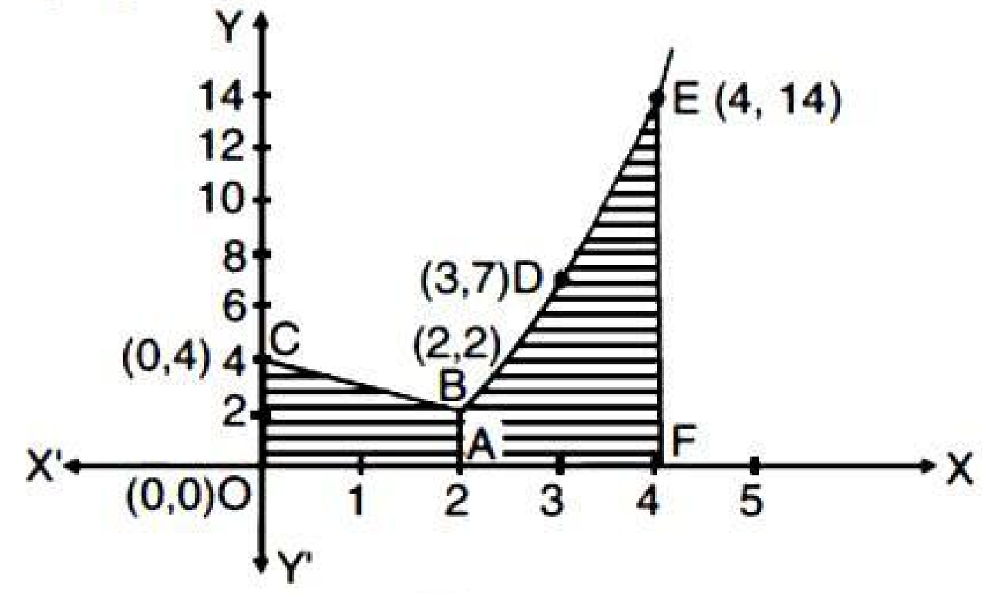


Fig.

$\therefore$ Reqd. area $= \int_0^4 f(x)\,dx$

$= \int_0^2 (4-x)\,dx + \int_2^4 (x^2-2)\,dx$

$= \left[4x - \frac{x^2}{2}\right]_0^2 + \left[\frac{x^3}{3} - 2x\right]_2^4$

$= \left[\left(8-\frac{4}{2}\right)-(0-0)\right] + \left[\left(\frac{64}{3}-8\right)-\left(\frac{8}{3}-4\right)\right]$

$= 6 + \left[\frac{40}{3}+\frac{4}{3}\right] = 6 + \frac{44}{3} = \frac{62}{3} = 20\frac{2}{3}$ sq. units.

$\int_0^4 f(x)\,dx$ represents the sum of the area of trapezium OABC and the area ABDEF.

EXERCISE 8 (b)

Short Answer Type Questions

SATQ

1. Draw a rough sketch of the curves $y = \sin x$ and $y = \cos x$ as x varies from 0 to $\frac{\pi}{2}$ and find the area of the region enclosed by them and the x-axis.

2. Using integration, prove that the area bounded by : $|x| + |y| = 1$ is 2 sq. units. ***(N.C.E.R.T.)***

3. Using integration, find the area of the region bounded by :

(*i*) (2, 0), (4, 5) and (6, 3) ***(N.C.E.R.T. ; H.P.B. 2016, 14, 13; P.B. 2013, 10)***

(*ii*) (1, 0), (2, 2) and (3, 1) ***(N.C.E.R.T. ; Jammu B. 2017; H.P.B. 2016, 14; 2014; A.I.C.B.S.E. Kashmir B. 2011; P.B. 2010)***

(*iii*) (–1, 2), (1, 5) and (3, 4) ***(A.I.C.B.S.E. 2014)***

(*iv*) (3, 0), (4, 5) and (5, 1) ***(P.B. 2013)***

(*v*) (– 1, 0), (1, 3) and (3, 2) ***(N.C.E.R.T.; H.P.B. 2018, 16, 14; Kashmir B. 2017; Jammu B. 2016, 15; Assam B. 2013)***

(*vi*) (1, 3), (2, 5) and (3, 4) ***(Bihar B. 2014)***

(*vii*) (4, 1), (6, 6) and (8, 4) ***(A.I.C.B.S.E. 2017, 10)***

(*viii*) (2, 5), (4, 7) and (6, 2) ***(H.B. 2010)***

(*ix*) (–2, 1), (0, 4) and (2, 3) ***(C.B.S.E. 2017)***

(*x*) (2, 1), (3, 4) and (5, 2). ***(Mizoram B. 2017)***

4. (*a*) Using integration, find the area of the region bounded by the triangle whose sides are :

(*i*) $3x - y - 3 = 0,\ 2x + y - 12 = 0,\ x - 2y - 1 = 0$

(*ii*) $5x - 2y - 10 = 0,\ x + y - 9 = 0,\ 2x - 5y - 4 = 0$ ***(C.B.S.E. 2012)***

(*iii*) $y = 2x + 1,\ y = 3x + 1,\ x = 4$ ***(N.C.E.R.T.; H.P.B. 2018; Kernataka B. 2017; C.B.S.E. 2011; H.B. 2010)***

(*iv*) $2x + y = 4,\ 3x - 2y = 6$ and $x - 3y + 5 = 0$ ***(H.P.B. 2018, 13 ; H.B. 2014; C.B.S.E. (F) 2011; C.B.S.E. 2009)***

(*v*) $y = 2x + 1,\ y = 5x + 1,\ x = 6$. ***(P.B. 2017)***

(*b*) If a triangular field is bounded by the lines $x + 2y = 2$, $y - x = 1$ and $2x + y = 7$.

Using integration, compute the area of the field.

If in each square unit area 4 trees may be planted. Find the number of trees which can be planted in the field.

5. Find the area of the region bounded by the line $y = 3x + 2$, the x-axis and the ordinates $x = -1$ and $x = 1$. ***(N.C.E.R.T.)***

6. Find the area of the region :

(*i*) $\{(x, y) : x^2 \le y \le x\}$ ***(H.P.B. 2013)***

(*ii*) $\{(x, y) : x^2 \le y \le |x|\}$. ***(H.B. 2011)***

Long Answer Type Questions

LATQ

7. Consider the functions :

$f(x) = |x| - 1$ and $g(x) = 1 - |x|$.

(*a*) Sketch their graphs and shade the closed region between them

(*b*) Find the area of their shaded region. ***(Kerala B. 2015)***

8. Using integration, find the area of the region bounded between :

(*i*) the line $x = 2$ and the parabola $y^2 = 8x$

(*ii*) the line $x = 3$ and the parabola $y^2 = 4x$. ***(N.C.E.R.T.)***

9. Find the area of the region bounded by :

(*i*) the parabola $y = x^2$ and the line $y = x$ ***(N.C.E.R.T. ; H.P.B. 2010)***

(*ii*) the parabola $y^2 = x$ and the line $x + y = 2$ ***(A.I.C.B.S.E. 2009)***

(*iii*) the curve $x^2 = 4y$ and the straight line $x = 4y - 2$ ***(N.C.E.R.T.; Nagaland B. 2018; H.P.B. 2017; H.B. 2016)***

(*iv*) the parabola $y^2 = 4ax$ and the chord $y = mx$ ***(N.C.E.R.T. ; H.P.B. 2013 S)***

(*v*) the parabola $y^2 = 4ax$ and its latus-rectum ***(N.C.E.R.T.; H.P.B. 2010 S)***

(*vi*) the parabola (I) $y^2 = 8x$ (II) $y^2 = 6x$ and the latus-rectum. ***(H.B. 2013)***

10. Find the area of the region bounded by the parabola $x^2 = y$, the line $y = x + 2$ and the x-axis. ***(N.C.E.R.T.; H.P.B. 2018, 10)***

11. The area between $x = y^2$ and $x = 4$ is divided into two equal parts by the line $x = a$, find the value of 'a'. ***(N.C.E.R.T.)***

12. Draw a rough sketch of the region enclosed between the curve $y^2 = 4x$ and the line $y = 2x$. Also determine the area of the region. *(H.P.B. 2017)*

13. Find the area of the region bounded by the curve $y = x^2$ and the line $y = 4$. *(N.C.E.R.T.)*

14. (*i*) Find the area enclosed between the straight line $y = x + 2$ and the curve $x^2 = y$.

(Assam B. 2018; H.B. 2013 S ; Rajasthan B. 2013)

(*ii*) Using integration, find the area of the region bounded by the curve $x^2 = 4y$ and the line $x = 4y - 2$.

(C.B.S.E. 2013, 10)

15. (*i*) Find the area of the smaller region bounded by the (*i*) ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and the straight line $\frac{x}{a}+\frac{y}{b}=1$

(N.C.E.R.T. ; H.B. 2016; Nagaland B. 2015; H.P.B. 2010 S)

(*ii*) $\frac{x^2}{16}+\frac{y^2}{9}=1$ and the straight line $3x + 4y = 12$.

(Meghalaya B. 2016)

16. Draw the rough sketch and find the area of the region :

(*i*) $\{(x, y) : 4x^2 + y^2 \le 4,\ 2x + y \ge 2\}$

(*ii*) $\{(x, y) : 16x^2 + y^2 \le 16,\ 4x + y \ge 4\}$.

(P.B. 2011)

17. (*a*) Draw the rough sketch and find the area of the region included between the parabolas :

(*i*) $y^2 = 4x$ and $x^2 = 4y$

(W. Bengal B. 2017; Mizoram B. 2017; H.B. 2015, 11; P.B. 2014, 11)

(*ii*) $y^2 = 9x$ and $x^2 = 9y$ *(P.B. 2014 S, 11)*

(*iii*) $y^2 = 16x$ and $x^2 = 16y$.

(P.B. 2014, 11, ; H.B. 2011)

(*b*) Find the ratio in which the area bounded by the curves $y^2 = 12x$ and $x^2 = 12y$ is divided by the line $x = 3$.

(Assam B. 2015)

18. Calculate the area between the parabolas :

(*i*) $y = x^2$ and $x = y^2$

(N.C.E.R.T.; H.P.B. 2017, 10; P.B. 2014, 10; Kerala B. 2014; Meghalaya B. 2014)

(*ii*) $y^2 = ax$ and $x^2 = ay$, $a > 0$

(*iii*) $4y^2 = 9x$ and $3x^2 = 16y$.

19. Calculate the area of the region enclosed between the circles :

(*i*) $x^2 + y^2 = 1$ and $(x - 1)^2 + y^2 = 1$

(N.C.E.R.T.; Nagaland B. 2018 ; Assam B. 2016 ; Jammu B. 2015 ; H.P.B. 2013, 10 S, 10 ; H.B. 2010)

(*ii*) $x^2 + y^2 = 4$ and $(x - 2)^2 + y^2 = 4$

(N.C.E.R.T.; H.P.B. 2013 S ; C.B.S.E. 2013)

(*iii*) $x^2 + y^2 = 9$, $(x - 3)^2 + y^2 = 9$ *(H.B. 2013)*

(*iv*) $(x - 6)^2 + y^2 = 36$ and $x^2 + y^2 = 36$.

(C.B.S.E. 2009 C)

20. (*i*) Show that the areas under the curves $f(x) = \cos^2 x$ and $f(x) = \sin^2 x$ between $x = 0$ and $x = \pi$ are 1 : 1.

(*ii*) Compare the areas under the curves $y = \cos^2 x$ and $y = \sin^2 x$ between $x = 0$ and $x = \pi$.

21. (*a*) (*i*) Find the area of the circle $x^2 + y^2 = 16$, which is exterior to the parabola $y^2 = 6x$.

(Type : Meghalaya B. 2017; H.P.B. 2011)

(*ii*) Find the area of the circle $4x^2 + 4y^2 = 9$, which is interior to the parabola $y^2 = 4x$. *(N.C.E.R.T.; H.P.B. 2011)*

(*b*) Find the area of the region bounded by the circle $x^2 + y^2 = 16$ and the parabola $x^2 = 6y$.

(Meghalaya B. 2018; Nagaland B. 2016)

(*c*) Using integration, find the area in the first quadrant bounded by the curve $y = x|x|$, the circle $x^2 + y^2 = 2$ and the y-axis. *(C.B.S.E. Sample Paper 2018)*

22. Calculate the area enclosed in the region :

(*i*) $\{(x, y) : x^2 + y^2 \le 1 \le x + y\}$

(Meghalaya B. 2015; C.B.S.E. 2010 C)

(*ii*) $\{(x, y) : y^2 \le 4x,\ 4x^2 + 4y^2 \le 9\}$

(H.P.B. 2015, 11 ; A.I.C.B.S.E. 2013; Nagaland B. 2016)

(*iii*) $\{(x, y) : x^2 + y^2 \le 16,\ x^2 \le 6y\}$

(Mizoram B. 2015; C.B.S.E. 2010 C)

(*iv*) $\{(x, y) : y^2 \le 6ax,\ x^2 + y^2 \le 16a^2\}$

(A.I.C.B.S.E. 2013)

(*v*) $\{(x, y) : x^2 + y^2 \le 8,\ x^2 \le 2y\}$.

(C.B.S.E. Sample Paper 2019)

23. Draw the rough sketch and find the area of the region :

(*i*) $\{(x, y) : x^2 < y < x + 2\}$ *(Kashmir B. 2013)*

(*ii*) $\{(x, y) : x^2 + y^2 \le 4,\ x + y \ge 2\}$

(A.I.C.B.S.E. 2012 ; P.B. 2010 S)

(*iii*) $\{(x, y) : 0 \le y \le x^2 + 1,\ 0 \le y \le x + 1,\ 0 \le x \le 2\}$

(H.B. 2016, 15, 11; H.P.B. 2015)

(*iv*) $\{(x, y) : x^2 + y^2 \le 2ax,\ y^2 \ge ax,\ x, y \ge 0\}$.

(C.B.S.E. 2016)

24. (*i*) Find the area of the region given by :

$\{(x, y) : x^2 \le y \le |x|\}$. *(A.I.C.B.S.E. 2009 C)*

(*ii*) Find the area bounded by the curves :

$\{(x, y) : y \ge x^2$ and $y = |x|\}$.

(N.C.E.R.T.; Assam B. 2015; H.P.B. 2011)

(*iii*) Find the area of the region bounded by the parabola $y = x^2$ and $y = |x|$. *(A.I.C.B.S.E. 2013)*

25. Using integration, find the area of the region bounded by the following curves, af5ter making a rough sketch :

(*i*) $y = 1 + |x + 1|$; $x = -3$, $x = 3$, $y = 0$

(*ii*) $y = 1 + |x + 1|$; $x = -2$, $x = 3$, $y = 0$.

Answers

1. $2-\sqrt{2}$. 2. 2.

3. (i) 7 (ii) $\frac{3}{2}$ (iii) 4 (iv) $\frac{9}{2}$

(v) 4 (vi) $\frac{3}{2}$ (vii) 7 (viii) $\frac{11}{2}$

(ix) 4 (x) 4.

4. (a) (i) 10 (ii) 10·5 (iii) 8 (iv) $\frac{7}{2}$

(v) 54

(b) Area of the field = 6 sq. units

24 trees.

5. $\frac{13}{3}$. 6. (i) $\frac{1}{6}$ (ii) $\frac{1}{3}$.

7. (b) 2. 8. (i) $\frac{32}{3}$ (ii) $8\sqrt{3}$.

9. (i) $\frac{1}{6}$ (ii) $\frac{9}{2}$ (iii) $\frac{9}{8}$ (iv) $\frac{8a^2}{3m^3}$ (v) $\frac{8}{3}a^2$

(vi) (I) $\frac{32}{3}$ (II) 6.

10. $\frac{9}{2}$. 11. $a = 4^{2/3}$.

12. $\frac{1}{3}$. 13. $\frac{32}{3}$. 14. (i) $\frac{9}{2}$ (ii) $\frac{9}{8}$.

15. (i) $\frac{ab}{4}(\pi - 2)$ (ii) $3(\pi - 2)$.

16. (i) $\frac{1}{2}(\pi - 1)$ (ii) $\pi - 2$.

17. (a) (i) $\frac{16}{3}$ (ii) 27 (iii) $\frac{256}{3}$

(b) 15 : 49.

18. (i) $\frac{1}{3}$ (ii) $\frac{a^2}{3}$ (iii) 4.

19. (i) $\left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2}\right)$ (ii) $\frac{8\pi}{3} - 2\sqrt{3}$

(iii) $\left(3\pi - \frac{9\sqrt{3}}{2}\right)$ (iv) $6(4\pi - 3\sqrt{3})$.

20. (ii) 1 : 1.

21. (a) (i) $\frac{4}{3}(8\pi + 3\sqrt{3} - 3\sqrt{6})$

(ii) $\frac{9\pi}{8} - \frac{9}{4}\sin^{-1}\frac{1}{3} + \frac{\sqrt{2}}{6}$.

(b) $\frac{4}{\sqrt{3}} + \frac{8\pi}{3}$ (c) $\left(\frac{\pi}{4} + \frac{1}{6}\right)$.

22. (i) $\left(\frac{\pi}{4} - \frac{1}{2}\right)$ (ii) $\left(\frac{2\sqrt{2}}{3} - \frac{1}{\sqrt{2}} + \frac{9\pi}{8} - \frac{9}{4}\sin^{-1}\frac{1}{3}\right)$

(iii) $\frac{4}{3}(4\pi + \sqrt{3})$ (iv) $\frac{4a^2}{3}(4\pi + \sqrt{3})$

(v) $\left(2\pi + \frac{4}{3}\right)$

23. (i) $\frac{9}{2}$ (ii) $\pi - 2$ (iii) $\frac{23}{6}$ (iv) $\left(\frac{\pi}{4} - \frac{2}{3}\right)a^2$.

24. (i) – (iii) $\frac{1}{3}$. 25. (i) 16 (ii) 13·5.

Hints to Selected Questions

2. Reqd. region is bounded by the lines :

$x + y = 1,\ x - y = 1,\ -x + y = 1$ and $-x - y = 1$.

9. (ii) The parabola $y^2 = x$ meets the line $x + y = 2$ in the points A (1, 1) and B (4, – 2).

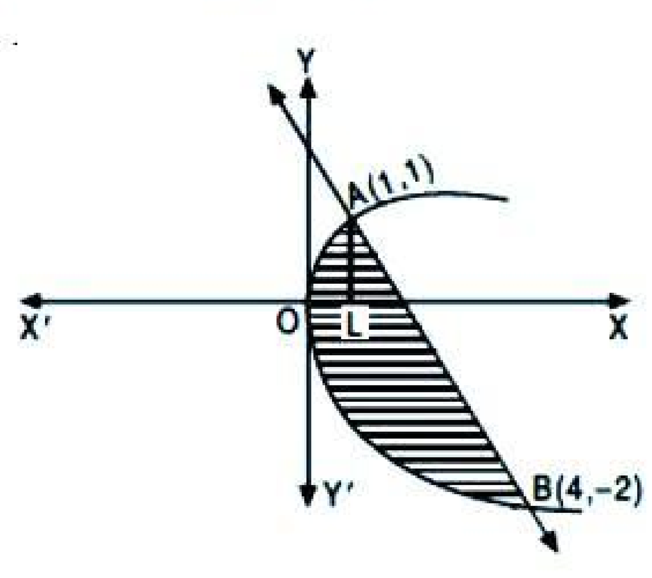


Fig.

Area above x-axis $= \int_0^1 \sqrt{x}\,dx + \int_1^2 (2 - x)\,dx$.

Area below x-axis $= \left|\int_0^4 \sqrt{x}\,dx - \int_2^4 (2 - x)\,dx\right|$.

NCERT-FILE

Questions from NCERT Book

(For each unsolved question, refer : "Solution of Modern's abc of Mathematics")

Exercise 8.1

1. Find the area of the region bounded by the curve $y^2 = x$ and the lines $x = 1$, $x = 4$ and the x-axis in the first quadrant. *(Kerala B. 2018)*

Solution : $y^2 = x$ is right-handed parabola.

$$\therefore \text{ Reqd. area, ABCD } = \int_1^4 y\,dx$$

[*Taking vertical strips*]

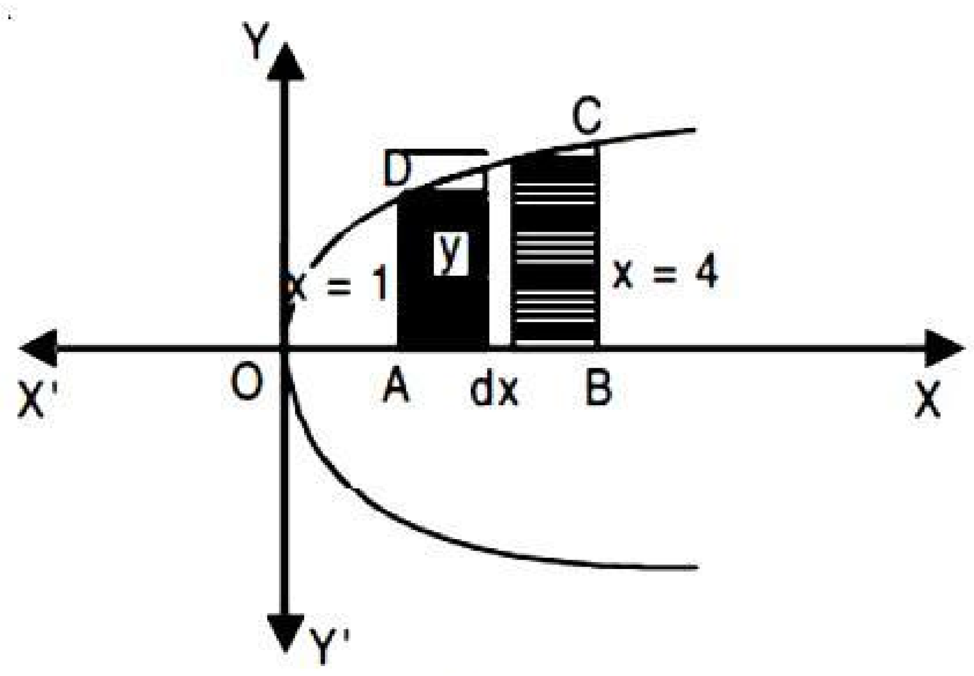


Fig.

$$= \int_1^4 \sqrt{x}\,dx \qquad \left[\begin{array}{l}\because y^2 = x \Rightarrow y = \pm\sqrt{x} \\ \Rightarrow y = \sqrt{x}\text{, taking + ve sign}\end{array}\right]$$

$$= \int_1^4 x^{1/2}\,dx = \left[\frac{x^{3/2}}{3/2}\right]_1^4 = \frac{2}{3}\left[4^{3/2} - 1\right]$$

$$= \frac{2}{3}[8-1] = \frac{14}{3}\text{ sq. units.}$$

2. Find the area of the region bounded by $y^2 = 9x$, $x = 2$, $x = 4$ and the x-axis in the first quadrant.

[**Solution.** Refer Q. 7; Ex. 8(a)]

3. Find the area of the region bounded by $x^2 = 4y$, $y = 2$, $y = 4$ and the y-axis in the first quadrant. *(H.P.B. 2017)*

Solution : $x^2 = 4y$ is an upward parabola.

$$\therefore \text{ Reqd. area ABCD } = \int_2^4 x\,dy$$

[*Taking horizontal strips*]

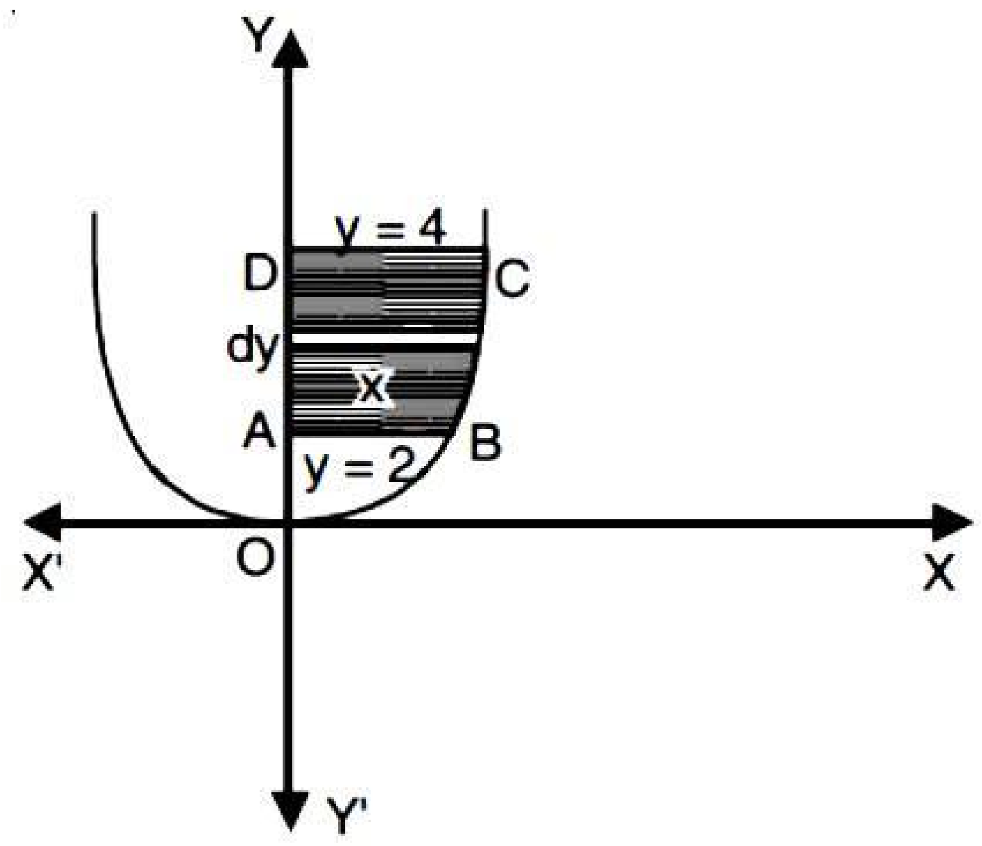


Fig.

$$= \int_2^4 2\sqrt{y}\,dy \qquad [\because x^2 = 4y \Rightarrow x = \pm 2\sqrt{y}.$$

But region ABCD lies in Ist. quadrant, $\therefore$ x is + ve]

$$= 2\int_2^4 y^{1/2}\,dy = 2\left[\frac{y^{3/2}}{3/2}\right]_2^4 = \frac{4}{3}\left[4^{3/2} - 2^{3/2}\right]$$

$$= \frac{4}{3}\left[8 - 2\sqrt{2}\right] = \frac{32 - 8\sqrt{2}}{3}\text{ sq. units.}$$

4. Find the area of the region bounded by the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1.$ *(Kashmir B. 2016)*

Solution : The given ellipse is $\frac{x^2}{16} + \frac{y^2}{9} = 1$ ...(1)

Since (1) is symmetrical about both axes,
$\therefore$ area of the ellipse = 4 (Shaded area).

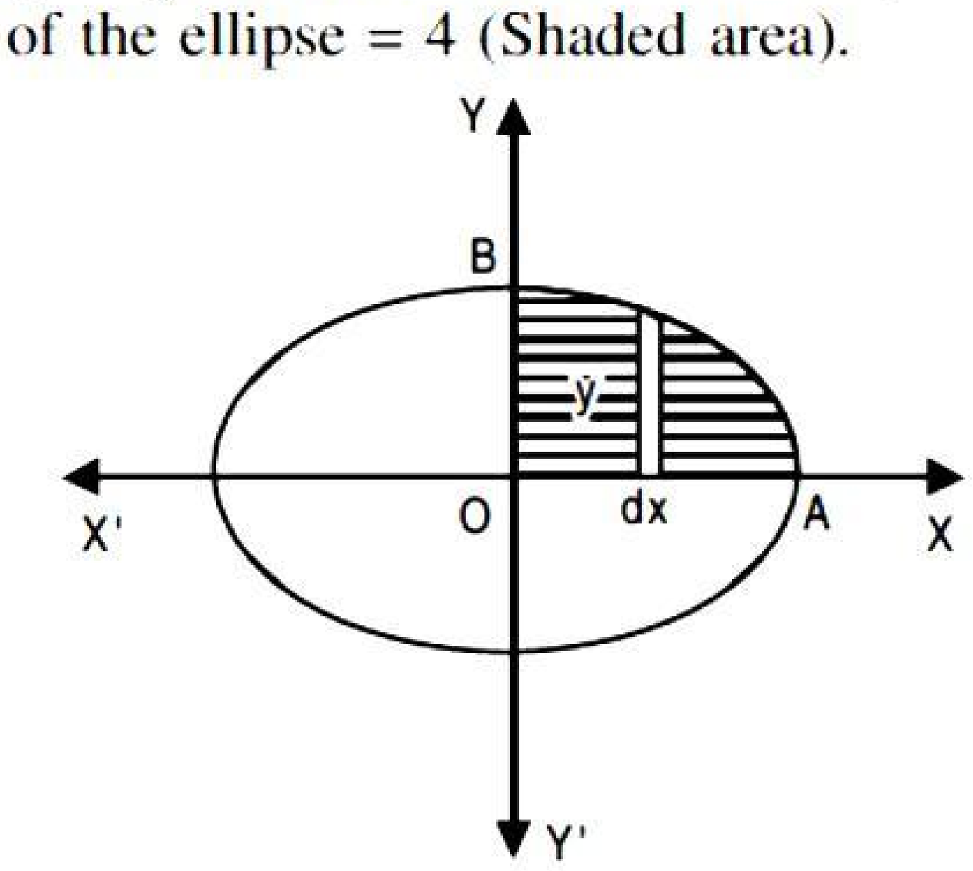


Fig.

= 4 (area OAB) ....(2)

But area OAB $= \int_0^4 y\,dx$ [*Taking vertical strips*]

$$= \int_0^4 \frac{3}{4}\sqrt{16 - x^2}\,dx$$

$\left[\because \frac{x^2}{16} + \frac{y^2}{9} = 1 \Rightarrow \frac{y^2}{9} = 1 - \frac{x^2}{16}\right.$

$\left.\Rightarrow \quad y = \frac{3}{4}\sqrt{16 - x^2} \quad (\because y > 0)\right]$

$$= \frac{3}{4}\left[\frac{x\sqrt{16 - x^2}}{2} + \frac{16}{2}\sin^{-1}\frac{x}{4}\right]_0^4$$

$$= \frac{3}{4}\,[[2\,(0) + 8\sin^{-1}(1)] - [0 - 0]]$$

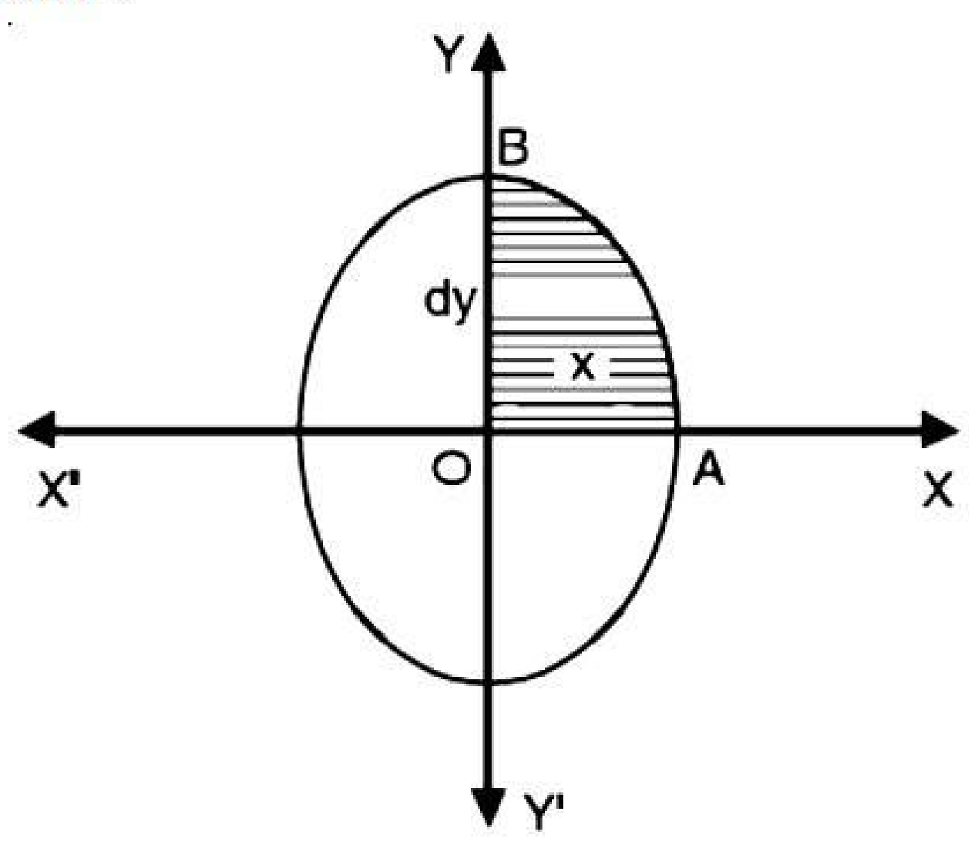
$$= \frac{3}{4}\left[8\,\frac{\pi}{2}\right] = 3\pi.$$

$\therefore$ From (2), area of the ellipse

$= 4\,(3\pi) = 12\pi$ sq. units.

5. Find the area of the region bounded by the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1.$ ***(H.P.B. 2016, 10; Jammu B. 2015W)***

Solution :



Solution. The given ellipse is $\frac{x^2}{4} + \frac{y^2}{9} = 1$ ....(1)

Since (1) is symmetrical about both the axes,

$\therefore$ area of the ellipse = 4 (Shaded area)

= 4 (area OAB) ...(2)

But area OAB $= \int_0^3 x\,dy$

[*Taking horizontal strips*]

$$= \int_0^3 \frac{2}{3}\sqrt{9 - y^2}\,dy$$

$\left[\because \frac{x^2}{4} + \frac{y^2}{9} = 1 \Rightarrow \frac{x^2}{4} = 1 - \frac{y^2}{9}\right.$

$\left.\Rightarrow x = \frac{2}{3}\sqrt{9 - y^2} \quad (\because x > 0)\right]$

$$= \frac{2}{3}\left[\frac{y\sqrt{9 - y^2}}{2} + \frac{9}{2}\sin^{-1}\frac{y}{3}\right]_0^3$$

$$= \frac{2}{3}\left[\left[\frac{3}{2}(0) + \frac{9}{2}\sin^{-1}(1)\right] - [0 - 0]\right]$$

$$= \frac{2}{3}\left[\frac{9}{2}\left(\frac{\pi}{2}\right)\right] = \frac{3\pi}{2}.$$

$\therefore$ From (2), area of the ellipse $= 4\left(\frac{3\pi}{2}\right)$

$= 6\pi$ sq. units.

6. Find the area of the region in the first quadrant enclosed by x-axis, line $x = \sqrt{3}y$ and the circle $x^2 + y^2 = 4$.

[**Solution.** Refer Q. 10; Ex. 8(*a*)]

7. Find the area of the smaller part of the circle $x^2 + y^2 = a^2$ cut off by the line $x = \frac{a}{\sqrt{2}}$.

[**Solution.** Refer Q. 21; Ex. 8(*a*)]

8. The area between $x = y^2$ and $x = 4$ is divided into two equal parts by the line $x = a$, find the value of 'a'.

[Solution. Refer Q. 11; Ex. 8(*b*)]

9. Find the area of the region bounded by the parabola $y = x^2$ and $y = |x|$.

Solution :

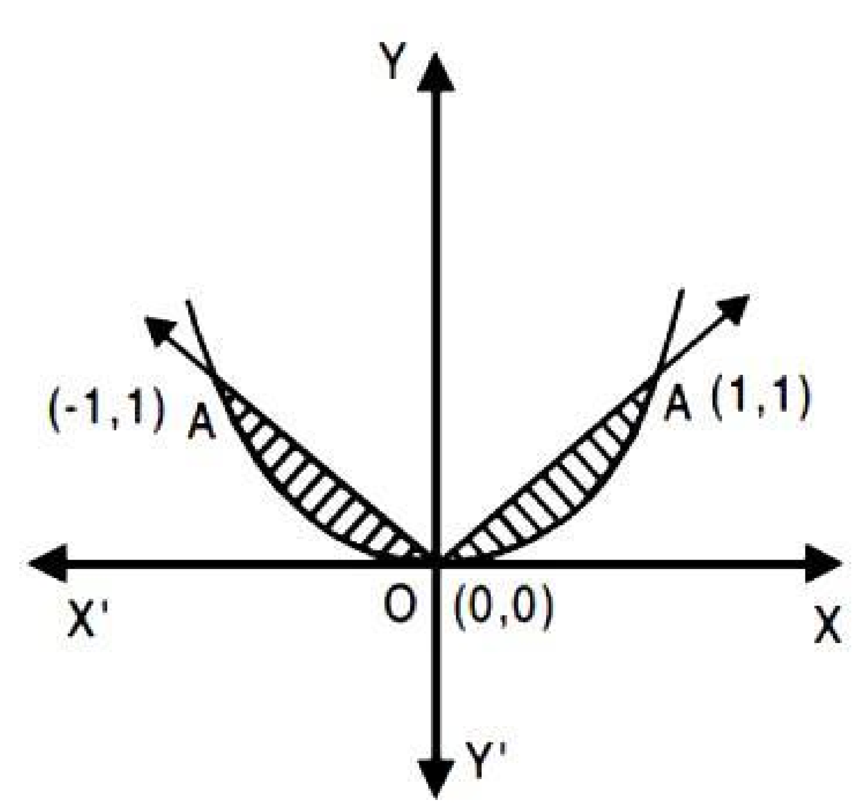


Fig.

The given parabola is $x^2 = y$ ...(1)

This is an upward parabola with vertex (0, 0).

$y = |x|$ represents the st. lines :

$y = x$ and $y = -x$ ....(2)

$y = x$ meets (1) at O (0, 0) and A (1, 1)

$y = -x$ meets (1) at O (0, 0) and A′ (– 1, 1).

∴ Reqd. area = 2 (Shaded area in first quadrant)

$$= 2\left(\int_0^1 x\,dx - \int_0^1 x^2\,dx\right)$$

$$= 2\left(\left[\frac{x^2}{2}\right]_0^1 - \left[\frac{x^3}{3}\right]_0^1\right)$$

$$= 2\left(\left(\frac{1}{2}-0\right)-\left(\frac{1}{3}-0\right)\right) = 2\left(\frac{1}{2}-\frac{1}{3}\right)$$

$$= 2\left(\frac{1}{6}\right) = \frac{1}{3} \text{ sq. unit.}$$

10. Find the area bounded by the curve $x^2 = 4y$ and the line $x = 4y - 2$.

[Solution. Refer Q. 9(*iii*); Ex. 8(*b*)**]**

11. Find the area of the region bounded by the curve $y^2 = 4x$ and the line $x = 3$.

[Solution. Refer Q. 2; Ex. 8(*a*)**]**

Choose the correct answer in the following Exercises 12 and 13.

12. Area lying in the first quadrant and bounded by the circle $x^2 + y^2 = 4$ and the lines $x = 0$ and $x = 2$ is

(A) π (B) $\frac{\pi}{2}$

(C) $\frac{\pi}{3}$ (D) $\frac{\pi}{4}$. **[Ans. (A)]**

13. Area of the region bounded by the curve $y^2 = 4x$, y-axis and the line $y = 3$ is :

(A) 2 (B) $\frac{9}{4}$

(C) $\frac{9}{3}$ (D) $\frac{9}{2}$. **[Ans. (B)]**

Exercise 8.2

1. Find the area of the circle $4x^2 + 4y^2 = 9$ which is interior to the parabola $x^2 = 4y$.

Solution : The given circle is $4x^2 + 4y^2 = 9$

i.e. $x^2 + y^2 = \frac{9}{4}$ ..(1),

which has centre (0, 0) and radius $\frac{3}{2}$.

The given parabola is $x^2 = 4y$ ...(2),

which is an upward parabola with vertex (0, 0).

The reqd. area is shown shaded in the figure.

Solving (1) and (2) :

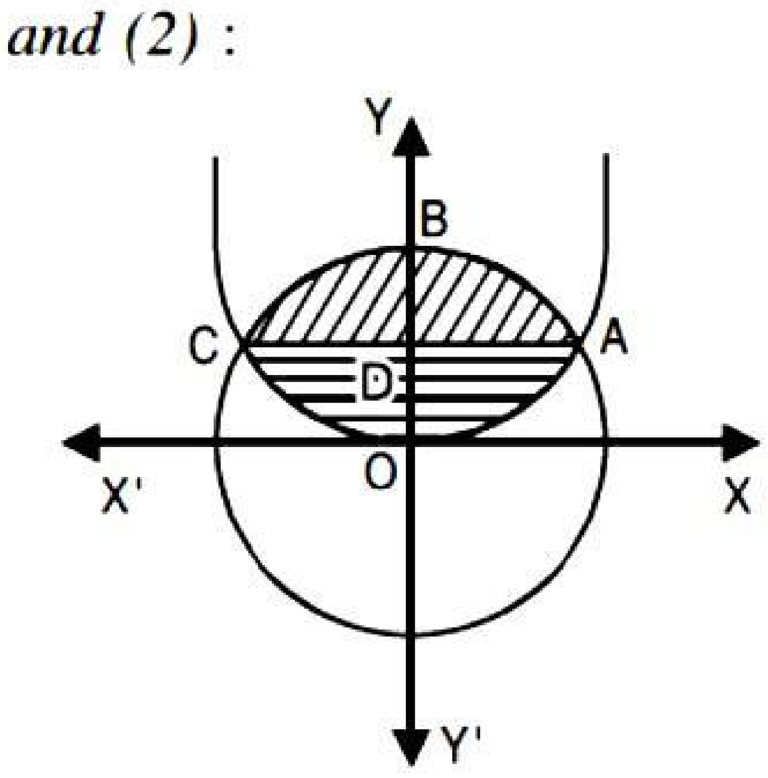


Fig.

Putting the value of x^2 from (2) in (1), we get :

$$4y + y^2 = \frac{9}{4} \Rightarrow 4y^2 + 16y - 9 = 0$$

$$\Rightarrow (2y + 9)(2y - 1) = 0 \Rightarrow y = -\frac{9}{2}, \frac{1}{2}.$$

Thus $y = \frac{1}{2}$. $\left[\because y \neq -\frac{9}{2}\right]$

∴ Reqd. area = ar (OABC)

= ar (OAC) + ar (ACB)

= 2 [ar (OAD) + ar (DAB)]

$$= 2\left[2\int_0^{1/2} \sqrt{y}\,dx + \int_{1/2}^{3/2} \frac{\sqrt{9-4y^2}}{2}\,dy\right]$$

$$\left[\because x^2 + y^2 = \frac{9}{4} \Rightarrow x = \frac{\sqrt{9-4y^2}}{2}\right]$$

$$= 4\int_0^{1/2} y^{1/2}\,dx + 2\int_{1/2}^{3/2} \sqrt{\frac{9}{4} - y^2}\,dy$$

$$= 4\left[\frac{y^{3/2}}{3/2}\right]_0^{1/2} + 2\left[\frac{y}{2}\sqrt{\frac{9}{4}-y^2} + \frac{9}{8}\sin^{-1}\left(\frac{y}{3/2}\right)\right]_{1/2}^{3/2}$$

$$= \frac{8}{3}\left(\frac{1}{2}\right)^{3/2} + 2\left[\frac{y}{2}\sqrt{\frac{9}{4}-y^2} + \frac{9}{8}\sin^{-1}\left(\frac{2y}{3}\right)\right]_{1/2}^{3/2}$$

$$= \frac{8}{3} \times \frac{1}{2\sqrt{2}} + 2\left[\left(0 + \frac{9}{8}\sin^{-1}(1)\right) - \left(\frac{1}{4}\sqrt{2} + \frac{9}{8}\sin^{-1}\frac{1}{3}\right)\right]$$

$$= \frac{2\sqrt{2}}{3} - \frac{\sqrt{2}}{2} + \frac{9}{4}\left(\sin^{-1}1 - \sin^{-1}\frac{1}{3}\right)$$

$$= \frac{\sqrt{2}}{6} + \frac{9}{4}\sin^{-1}\left((1)\sqrt{1-\frac{1}{9}} - \frac{1}{3}\sqrt{1-1}\right)$$

$$= \frac{\sqrt{2}}{6} + \frac{9}{4}\sin^{-1}\left(\frac{2\sqrt{2}}{3} - 0\right)$$

$$= \left(\frac{\sqrt{2}}{6} + \frac{9}{4}\sin^{-1}\frac{2\sqrt{2}}{3}\right) \text{ sq. units.}$$

2. Find the area bounded by curves $(x-1)^2 + y^2 = 1$ and $x^2 + y^2 = 1$.

[**Solution.** Refer Q. 19(*i*) Ex. 8(*b*)]

3. Find the area of the region bounded by the curves $y = x^2 + 2$, $y = x$, $x = 0$ and $x = 3$.

[**Solution.** Refer Ex. 13; Page 8/16]

4. Using integration find the area of region bounded by the triangle whose vertices are $(-1, 0)$, $(1, 3)$ and $(3, 2)$.

[**Solution.** Refer Q. 3(*v*) Ex. 8(*b*)]

5. Using integration find the area of the triangular region whose sides have the equations $y = 2x + 1$, $y = 3x + 1$ and $x = 4$.

[**Solution.** Refer Q. 4(*a*)(*iii*) Ex. 8(*b*)]

Choose the correct answer in the following exercises 6 to 7.

6. Smaller area enclosed by the circle $x^2 + y^2 = 4$ and the line $x + y = 2$ is :

(A) $2(\pi - 2)$ (B) $\pi - 2$

(C) $2\pi - 1$ (D) $2(\pi + 2)$. **[Ans. (B)]**

7. Area lying between the curves $y^2 = 4x$ and $y = 2x$ is :

(A) $\frac{2}{3}$ (B) $\frac{1}{3}$

(C) $\frac{1}{4}$ (D) $\frac{3}{4}$. **[Ans. (B)]**

Miscellaneous Exercise on Chapter 8

1. Find the area under the given curves and given lines :

(*i*) $y = x^2$, $x = 1$, $x = 2$ and x-axis.

[**Solution.** Refer Q. 22; Ex. 8(*a*)]

(*ii*) $y = x^4$, $x = 1$, $x = 5$ and x-axis.

[**Solution.** Refer Q. 6(*i*); Ex. 8(*a*)]

2. Find the area between the curves $y = x$ and $y = x^2$.

[**Solution.** Refer Q. 9(*i*); Ex. 8(*b*)]

3. Find the area of the region lying in the first quadrant and bounded by $y = 4x^2$, $x = 0$, $y = 1$ and $y = 4$.

[**Solution.** Refer Q. 8; Ex. 8(*a*)]

4. Sketch the graph of $y = |x + 3|$ and evaluate

$$\int_{-6}^{0} |x+3|\,dx.$$

Solution : Graph : We have :

$$y = |x+3|$$
$$= x + 3 \quad \text{if } x \ge -3$$
$$= -x - 3 \quad \text{if } x < -3$$

When x ≥ – 3

$x =$	-3	-2
$y =$	0	1

When x < – 3

$x =$	-4	-5
$y =$	1	2

Plot the points A (– 3, 0), B (– 2, 1) and C (– 4, 1), D (– 5, 2) and portion of the graph is as shown :

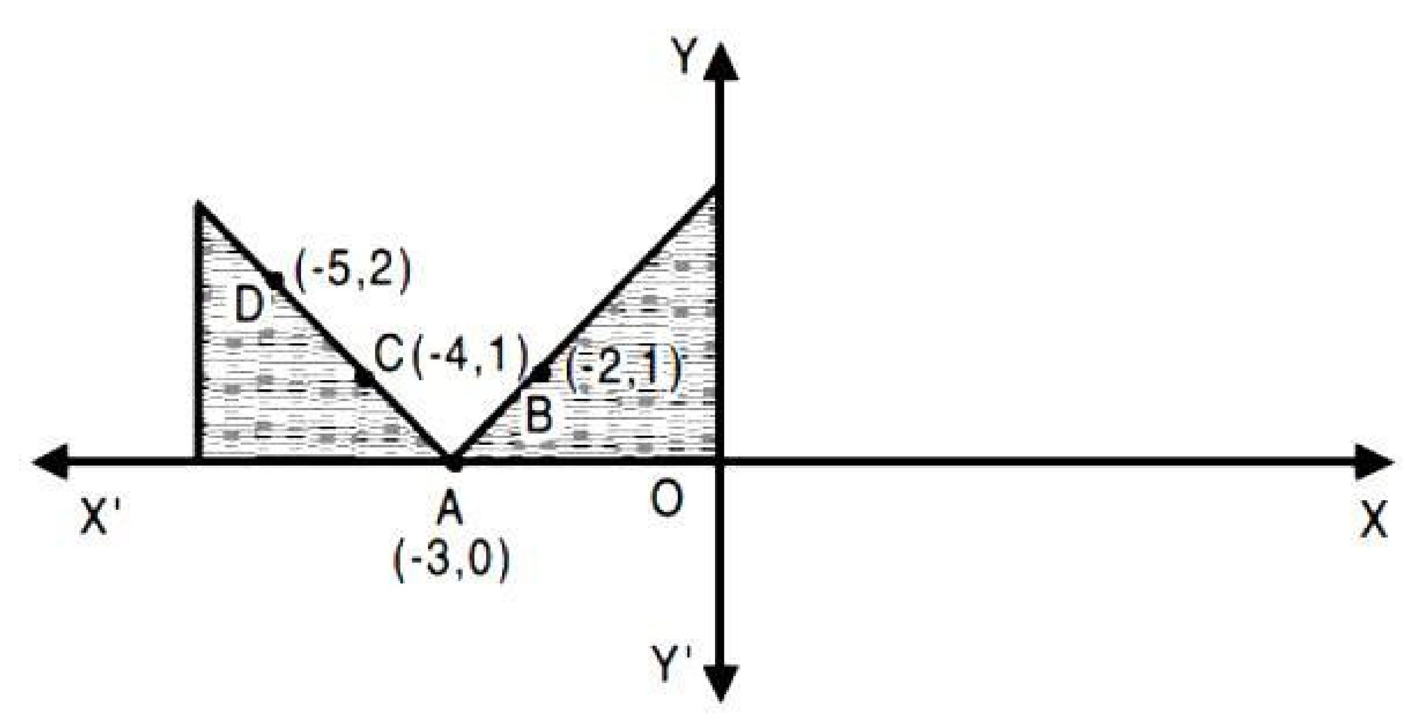


Fig.

Evaluation : $\int_{-6}^{0} |x+3|\,dx$

$$= \int_{-6}^{-3} (-x-3)\,dx + \int_{-3}^{0} (x+3)\,dx$$

$$= \left[-\frac{x^2}{2} - 3x\right]_{-6}^{-3} + \left[\frac{x^2}{2} + 3x\right]_{-3}^{0}$$

$$= \left[\left(-\frac{9}{2} + 9\right) - \left(-\frac{36}{2} + 18\right)\right] + \left[(0+0) - \left(\frac{9}{2} - 9\right)\right]$$

$$= \left(\frac{9}{2} + 0\right) + \left(0 + \frac{9}{2}\right) = 9 \text{ sq. units.}$$

5. Find the area bounded by the curve $y = \sin x$ between $x = 0$ and $x = 2\pi$.

[**Solution.** Refer Q. 16(*ii*) (I); Ex. 8(*a*)]

6. Find the area enclosed between the parabola $y^2 = 4ax$ and the line $y = mx$.

[**Solution.** Refer Q. 9(*iv*); Ex. 8(*b*)]

7. Find the area enclosed by the parabola $4y = 3x^2$ and the line $2y = 3x + 12$.

(Mizoram B. 2016; Nagaland B. 2016)

Solution : The given parabola is $4y = 3x^2$

i.e. $y = \frac{3}{4}x^2$ ...(1)

and the given line is $3x - 2y + 12 = 0$ ...(2)

Putting the value of y from (1) in (2), we get :

$$3x - 2\left(\frac{3}{4}x^2\right) + 12 = 0 \Rightarrow 3x - \frac{3}{2}x^2 + 12 = 0$$

$$\Rightarrow 6x - 3x^2 + 24 = 0 \quad \Rightarrow x^2 - 2x - 8 = 0$$

$$\Rightarrow (x - 4)(x + 2) = 0 \quad \Rightarrow x = 4, -2.$$

Putting in (1), $y = \frac{3}{4}(4)^2 = 12$ and $y = \frac{3}{4}(-2)^2 = 3.$

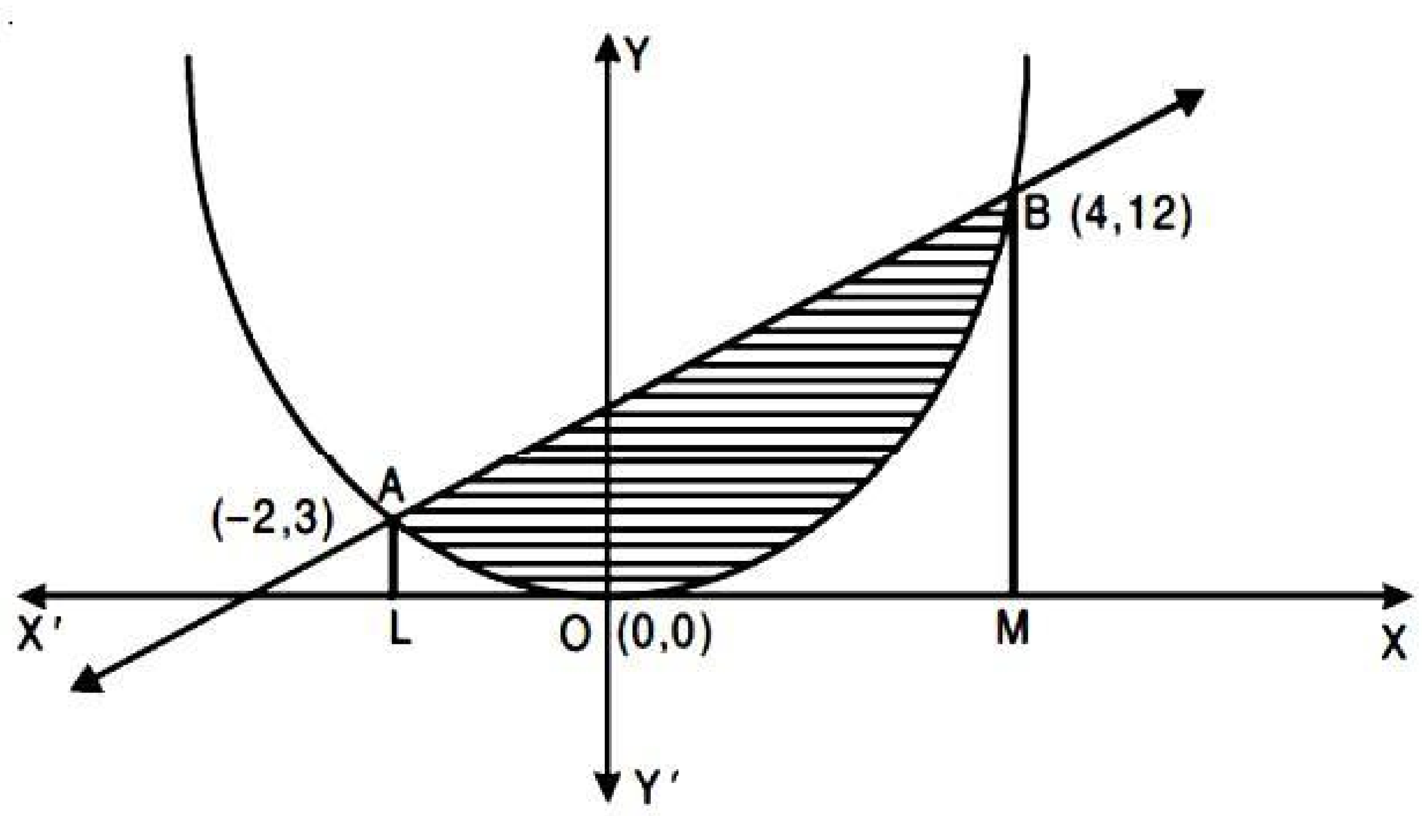


Fig.

Hence, the line (2) intersects parabola (1) in the points A (– 2, 3) and B (4, 12).

∴ Reqd. area = area ALMB – (area ALO + area OMB)

$$= \int_{-2}^{4} \frac{3x + 12}{2}\, dx - \left(\int_{-2}^{0} \frac{3}{4}x^2 dx + \int_{0}^{4} \frac{3}{4}x^2 dx\right)$$

$$\left[\because \textit{From (2)}, y = \frac{3x + 12}{2}\right]$$

$$= \frac{1}{2}\left[\frac{3x^2}{2} + 12x\right]_{-2}^{4} - \left[\frac{3}{4}\left\{\frac{x^3}{3}\right\}_{-2}^{0} + \frac{3}{4}\left\{\frac{x^3}{3}\right\}_{0}^{4}\right]$$

$$= \frac{1}{2}[(24 + 48) - (6 - 24)] - \left[\frac{3}{4}\left(0 + \frac{8}{3}\right) + \frac{3}{4}\left(\frac{64}{3} - 0\right)\right]$$

$$= \frac{1}{2}[72 + 18] - [2 + 16] = 45 - 18$$

$= 27$ sq. units.

8. Find the area of the smaller region bounded by the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$ and the line $\frac{x}{3} + \frac{y}{2} = 1.$

(H.P.B. 2016, 14, 13)

Solution : The given ellipse is :

$$\frac{x^2}{9} + \frac{y^2}{4} = 1 \quad ...(1)$$

and the st. line is $\frac{x}{3} + \frac{y}{2} = 1$ ...(2)

Reqd. area is the shaded area, as shown in the figure.

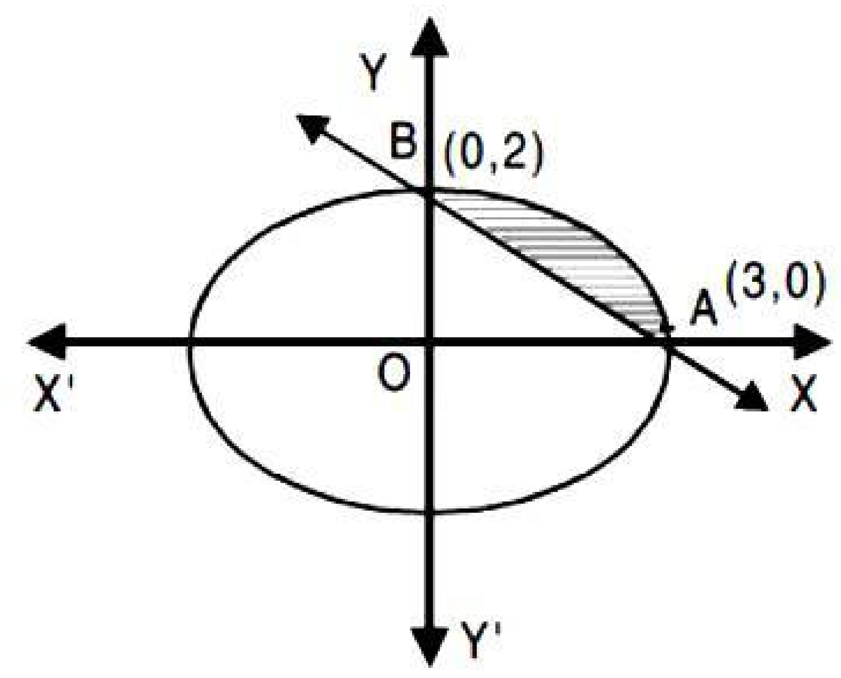


Fig.

$$\therefore \text{ Shaded area} = \int_0^3 \frac{2}{3}\sqrt{9 - x^2}\, dx - \int_0^3 \frac{2}{3}(3 - x)\, dx$$

$$= \frac{2}{3}\int_0^3 \sqrt{3^2 - x^2}\, dx - \frac{2}{3}\int_0^3 (3 - x)\, dx$$

$$= \frac{2}{3}\left[\frac{x}{2}\sqrt{9 - x^2} + \frac{9}{2}\sin^{-1}\frac{x}{3} - 3x + \frac{x^2}{2}\right]_0^3$$

$$= \frac{2}{3}\left[\left\{0 + \frac{9}{2}\left(\sin^{-1} 1\right) - 3(3) + \frac{9}{2}\right\} - 0\right]$$

$$= \frac{2}{3}\left[\frac{9}{2}\frac{\pi}{2} - 9 + \frac{9}{2}\right]$$

$$= \frac{2}{3}\left[\frac{9\pi}{4} - \frac{9}{2}\right]$$

$$= \frac{3\pi}{2} - 3$$

$$= \frac{3}{2}(\pi - 2) \text{ sq. units.}$$

9. Find the area of the smaller region bounded by the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and the line $\frac{x}{a}+\frac{y}{b}=1$.

[**Solution.** Refer Q. 15 (*i*); Ex. 8(*b*)]

10. Find the area of the region enclosed by the parabola $x^2 = y$, the line $y = x + 2$ and the x-axis.

[**Solution.** Refer Q. 10; Ex. 8(*b*)]

11. Using the method of integration find the area bounded by the curve $|x| + |y| = 1$. {**Hint :** The required region is bounded by lines $x + y = 1$, $x - y = 1$ and $-x + y = 1$ and $-x - y = 1$].

[**Solution :** Refer Q. 2; Ex. 8(*b*)]

12. Find the area bounded by curves $\{(x, y) : y \geq x^2$ and $y = |x|\}$.

Solution : Let us first sketch the region whose area is to be found out. The required area is the area included between the curves $x^2 = y$ and $y = |x|$.

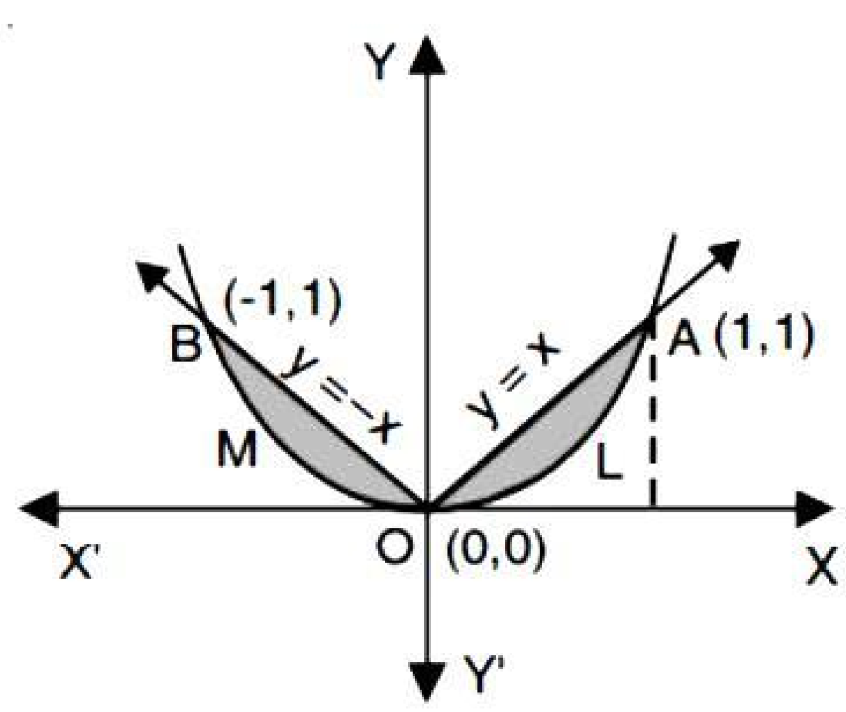


Fig.

The graph of $x^2 = y$ is parabola with vertex (0, 0) and axis as y-axis.

The graph of $y = |x|$ is the union of lines $y = x$, $x \geq 0$ and $y = -x$, $x < 0$.

Solving, $x^2 = y$ and $y = x$, we get the points of intersection as : O (0, 0) and A (1, 1).

Solving, $x^2 = y$ and $y = -x$, we get the points of intersection as O (0, 0) and B (– 1, 1).

$\therefore$ Required area = area OAL + area OBM

= 2 area OAL

$$= 2\left[\int_0^1 x\,dx - \int_0^1 x^2\,dx\right]$$

$$= 2\left[\left|\frac{x^2}{2}\right|_0^1 - \left|\frac{x^3}{3}\right|_0^1\right]$$

$$= 2\left[\left(\frac{1}{2}-0\right)-\left(\frac{1}{3}-0\right)\right]$$

$$= 2\left[\frac{1}{2}-\frac{1}{3}\right]$$

$$= 2\left(\frac{1}{6}\right) = \frac{1}{3} \text{ sq. unit.}$$

13. Using the method of integration find the area of the triangle ABC, coordinates of whose vertices are A(2, 0), B(4, 5) and C(6, 3).

[**Solution :** Refer Q. 3(*i*); Ex. 8(*b*)]

14. Using the method of integration find the area of the region bounded by lines :

$2x + y = 4$, $3x - 2y = 6$ and $x - 3y + 5 = 0$.

[**Solution.** Refer Q. 4(*iv*); Ex. 8(*b*)]

15. Find the area of the region $\{(x, y) : y^2 \leq 4x,\ 4x^2 + 4y^2 \leq 9\}$

Solution : $y^2 = 4x$ ...(1) is a parabola with vertex (0, 0)

$4x^2 + 4y^2 = 9$ ...(2) is a circle with centre (0, 0) and radius $\frac{3}{2}$.

Solving (1) and (2) :

$$4x^2 + 16x = 9$$

$$\Rightarrow \quad 4x^2 + 16x - 9 = 0$$

$$\Rightarrow (2x - 1)(2x + 9) = 0$$

$$\Rightarrow \quad x = \frac{1}{2}, -\frac{9}{2}.$$

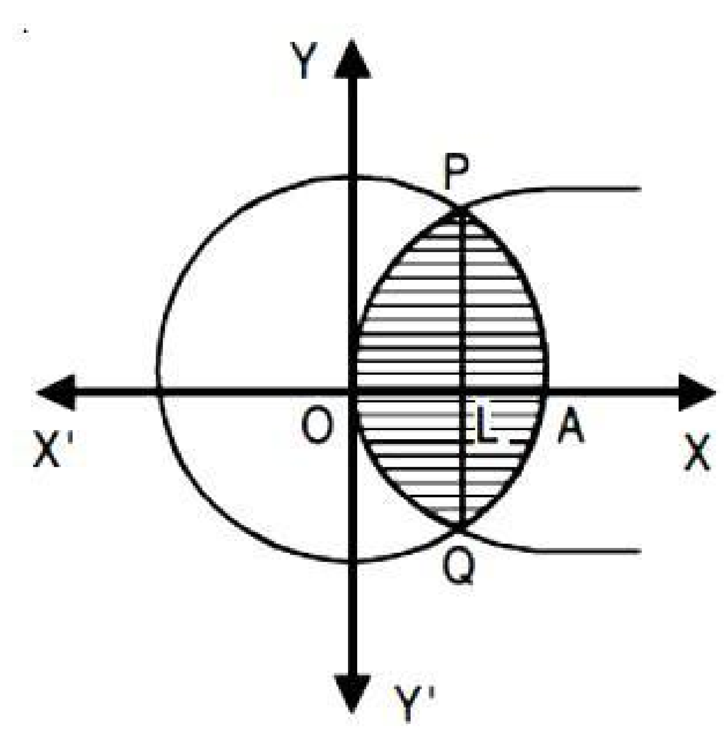


Fig.

$$\Rightarrow \quad x = \frac{1}{2}. \quad [\because x \neq -\frac{9}{2}]$$

Putting in (1), $y^2 = 4\left(\frac{1}{2}\right)$

$\Rightarrow \qquad y^2 = 2$

$\Rightarrow \qquad y = \pm\sqrt{2}$.

Thus (1) and (2) intersect at P $\left(\frac{1}{2}, \sqrt{2}\right)$ and Q $\left(\frac{1}{2}, -\sqrt{2}\right)$.

$\therefore$ Reqd. area = Shaded area

= 2 (ar OAPO)

= 2 [ar (OLPO) + ar (LAPL)]

$$= 2\left[\int_1^{\frac{1}{2}} 2\sqrt{x}\, dx + \int_{\frac{1}{2}}^{\frac{3}{2}} \sqrt{\frac{9}{4} - x^2}\, dx\right]$$

$$= 2\left[2\left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^{\frac{1}{2}}\right] + 2\left[\frac{x}{2}\sqrt{\frac{9}{4} - x^2} + \frac{\frac{9}{4}}{2}\sin^{-1}\frac{x}{\frac{3}{2}}\right]_{\frac{1}{2}}^{\frac{3}{2}}$$

$$= \frac{8}{3}\left[\left(\frac{1}{2}\right)^{\frac{3}{2}} - 0\right] + \left[\left(\frac{3}{2}(0) + \frac{9}{4}\sin^{-1}(1)\right)\right.$$

$$\left. - \left(\frac{1}{2}\sqrt{\frac{9}{4} - \frac{1}{4}} + \frac{9}{4}\sin^{-1}\frac{1}{3}\right)\right]$$

$$= \frac{8}{3}\frac{1}{2\sqrt{2}} + \frac{9}{4}\frac{\pi}{2} - \frac{1}{2}\frac{1}{\sqrt{2}} - \frac{9}{4}\sin^{-1}\left(\frac{1}{3}\right)$$

$$= \frac{2\sqrt{2}}{3} + \frac{9\pi}{8} - \frac{1}{\sqrt{2}} - \frac{9}{4}\sin^{-1}\left(\frac{1}{3}\right)$$

$$= \left(\frac{2\sqrt{2}}{3} - \frac{\sqrt{2}}{2}\right) + \frac{9\pi}{8} - \frac{9}{4}\sin^{-1}\left(\frac{1}{3}\right)$$

$$= \left[\frac{\sqrt{2}}{6} + \frac{9\pi}{8} - \frac{9}{4}\sin^{-1}\left(\frac{1}{3}\right)\right]$$

$$= \left(\frac{9\pi}{8} - \frac{9}{4}\sin^{-1}\left(\frac{1}{3}\right) + \frac{1}{3\sqrt{2}}\right) \text{ sq. units.}$$

Choose the correct answer in the following Exercises from 16 to 19.

16. Area bounded by the curve $y = x^3$, the x-axis and the ordinates $x = -2$ and $x = 1$ is :

(A) -9 (B) $\frac{-15}{4}$

(C) $\frac{15}{4}$ (D) $\frac{17}{4}$. **[Ans. (D)]**

17. The area bounded by the curve $y = x|x|$, x-axis and the ordinates $x = -1$ and $x = 1$ is given by :

(A) 0 (B) $\frac{1}{3}$

(C) $\frac{2}{3}$ (D) $\frac{4}{3}$.

[**Hint :** $y = x^2$ if $x > 0$ and $y = -x^2$ if $x < 0$].

[Ans. (C)]

18. The area of the circle $x^2 + y^2 = 16$ exterior to the parabola $y^2 = 6x$ is :

(A) $\frac{4}{3}(4\pi - \sqrt{3})$

(B) $\frac{4}{3}(4\pi + \sqrt{3})$

(C) $\frac{4}{3}(8\pi - \sqrt{3})$

(D) $\frac{4}{3}(8\pi + \sqrt{3})$.

[Ans. (C)]

19. The area bounded by the y-axis, $y = \cos x$ and $y = \sin x$ when $0 \le x \le \frac{\pi}{2}$ is :

(A) $2(\sqrt{2} - 1)$ (B) $\sqrt{2} - 1$

(C) $\sqrt{2} + 1$ (D) $\sqrt{2}$. **[Ans. (B)]**

Questions From NCERT Exemplar

Example 1. Find the area of the region bounded by the curve $ay^2 = x^3$, the y-axis and the lines y = a and y = 2a.

Solution. The given curve is $ay^2 = x^3$ ...(1)

The region is shown as shaded in the figure :

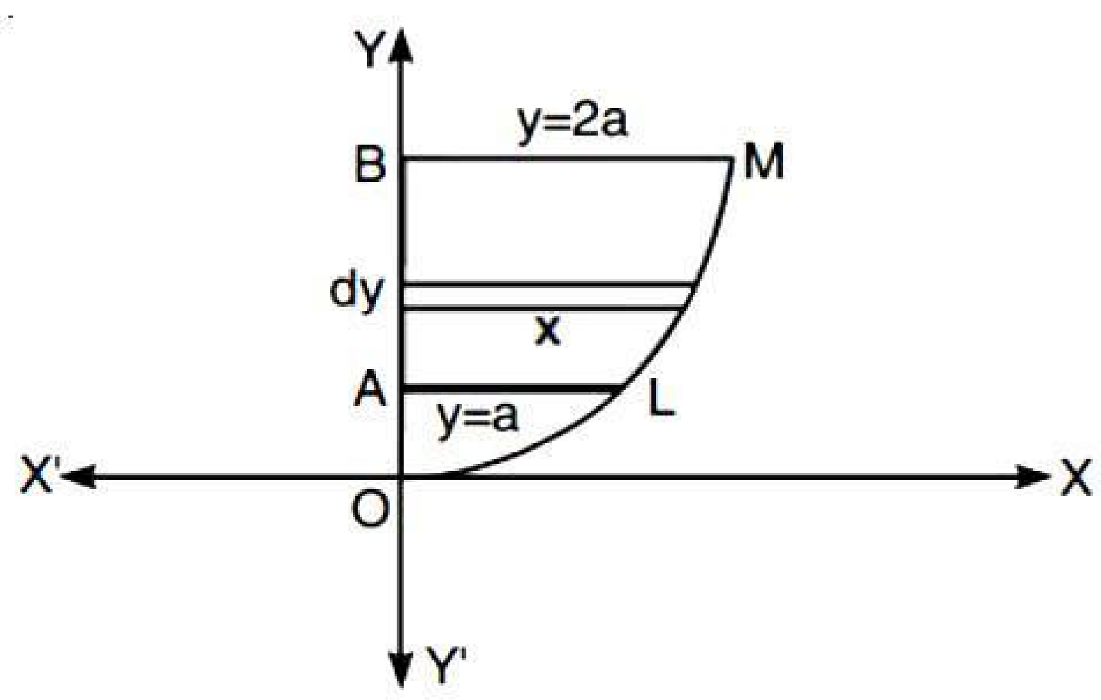


Fig.

∴ Reqd. area, ALMB

$$= \int_a^{2a} x\,dy = \int_a^{2a} a^{1/3}y^{2/3}\,dy$$

$$= a^{1/3}\left[\frac{y^{5/3}}{5/3}\right]_a^{2a} = \frac{3}{5}a^{1/3}\left[(2a)^{5/3} - a^{5/3}\right]$$

$$= \frac{3}{5}a^{1/3}a^{5/3}\left[2.2^{2/3} - 1\right] = \frac{3}{5}a^2(2.2^{2/3} - 1) \text{ sq. units.}$$

Example 2. Find the area of the region bounded by the parabola $y^2 = 2x$ and the straight line x – y = 4.

Solution. The given parabola is $y^2 = 2x$ ...(1)

and the given st. line is $x - y = 4$ ...(2)

Solving (1) and (2) :

From (2), $x = 4 + y$ ...(3)

Putting in (1), $y^2 = 8 + 2y \Rightarrow y^2 - 2y - 8 = 0$

$\Rightarrow (y - 4)(y + 2) = 0 \Rightarrow y = 4, -2.$

When $y = 4$, then from (3), $x = 4 + 4 = 8.$

When $y = -2$, then from (3), $x = 4 - 2 = 2.$

Thus the line (2) cuts parabola (2) in the points A (2, – 2) and B (8, 4).

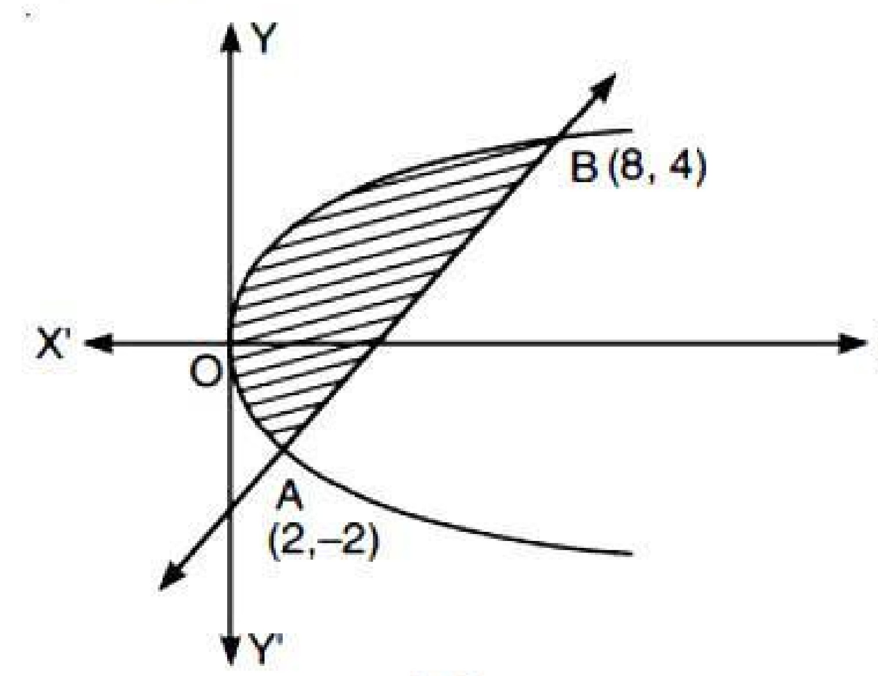


Fig.

The region is shown as shaded in the above figure.

∴ Reqd. area

$$= \int_{-2}^{4}\left(4 + y - \frac{y^2}{2}\right)dy = \left[4y + \frac{y^2}{2} - \frac{y^3}{6}\right]_{-2}^{4}$$

$$= \left(4(4) + \frac{16}{2} - \frac{64}{6}\right) - \left(-8 + \frac{4}{2} + \frac{8}{6}\right)$$

$$= \left(16 + 8 - \frac{64}{6}\right) - \left(-8 + 2 + \frac{8}{6}\right)$$

$$= \left(24 - \frac{64}{6}\right) + \left(6 - \frac{8}{6}\right) = 30 - \frac{72}{6} = 30 - 12 = 18 \text{ sq. units.}$$

Exercise

1. Find the area of the region bounded by the parabolas $y^2 = 6x$ and $x^2 = 6y$.

2. Find the area of the region bounded by the curves $x = at^2$ and $y = 2at$ between the ordinates corresponding to $t = 1$ and $t = 2$.

3. Find the area enclosed by the curve : $x = 3 \cos t$, $y = 2 \sin t$.

Answers

1. 12 sq. units. **2.** $\frac{56}{3}a^2$ sq. units. **3.** 6π sq. units.

Revision Exercise

1. Find the area enclosed by the parabola $4y = 3x^2$ and the line $2y = 3x + 12$. *(N.C.E.R.T.)*

2. Find the area of the region bounded by the curve $y^2 = 4x$ and the line $x = 3$.

3. Prove that the curves $y^2 = 4x$ and $x^2 = 4y$ divide the area of the square bounded by x = 0, x = 4, y = 0 and y = 4 into three equal parts.

(N.C.E.R.T.; H.B. 2018 ; Assam B. 2017; H.P.B. 2015; C.B.S.E. 2009)

Solution. The lines $x = 0$, $x = 4$, $y = 4$ and $y = 0$ represent the square OACB.

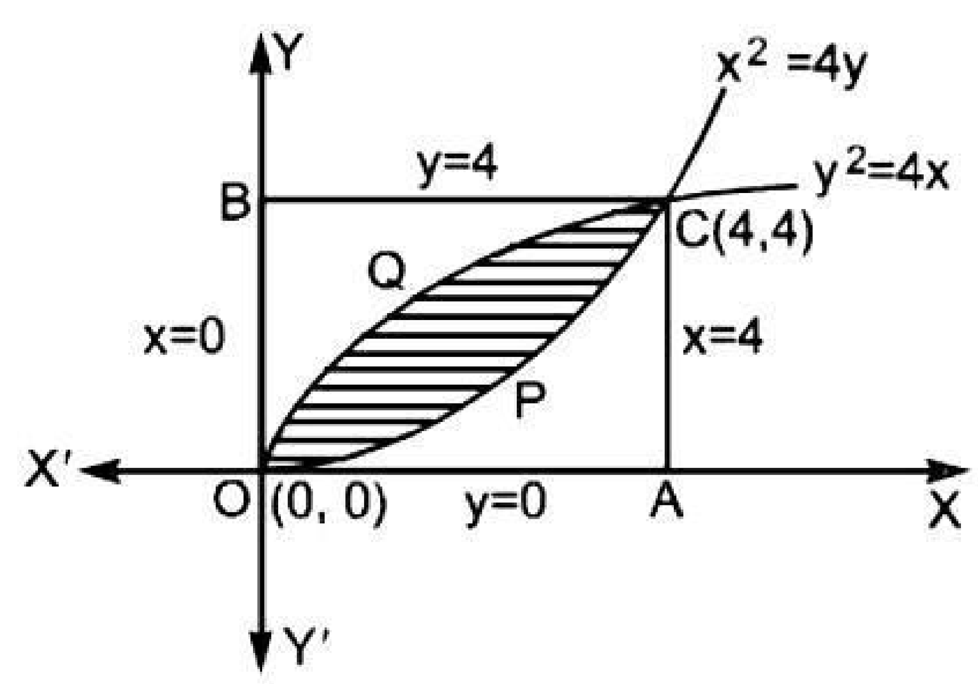


Fig.

The given parabolas $y^2 = 4x$ and $x^2 = 4y$ meet at O (0, 0) and C (4, 4).

Now area of the region OPCQO, bounded by $y^2 = 4x$ and $x^2 = 4y$

$$= \int_0^4 \left(2\sqrt{x} - \frac{x^2}{4}\right) dx = \left[2\,\frac{x^{3/2}}{3/2} - \frac{1}{4}\,\frac{x^3}{3}\right]_0^4$$

$$= \left[\frac{4}{3}(4)^{3/2} - \frac{1}{12}(64)\right] - (0 - 0)$$

$$= \frac{32}{3} - \frac{16}{3} = \frac{16}{3} \qquad ...(1)$$

Again area of the region OACPO, bounded by $x^2 = 4y$, $x = 0$, $x = 4$ and x-axis

$$= \int_0^4 \frac{x^2}{4}\,dx = \frac{1}{4}\left[\frac{x^3}{3}\right]_0^4$$

$$= \frac{1}{12}[64 - 0] = \frac{16}{3} \qquad ...(2)$$

Lastly area of the region OQCBO, bounded by $y^2 = 4x$, $y = 0$, $y = 4$ and y-axis

$$= \int_0^4 x\,dy = \int_0^4 \frac{y^2}{4}\,dy$$

$$= \frac{1}{4}\left[\frac{y^3}{3}\right]_0^4 = \frac{1}{12}[64 - 0] = \frac{16}{3} \qquad ...(3)$$

From (1), (2) and (3), the required result follows.

4. Draw the diagram to show the area enclosed by the curves :

$y^2 = 16x$ and $x^2 = 16y$.

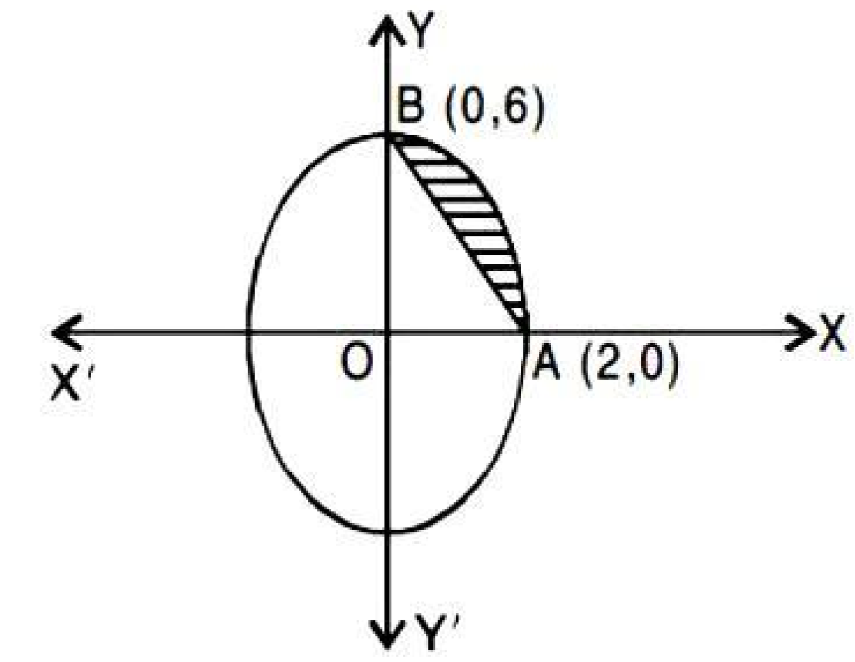
The straight line $x = 4$ divides the area into two parts. Find the area of the larger portion by integration.

(Tripura B. 2016)

5. AOBA is the part of the ellipse $9x^2 + y^2 = 36$ in the first quadrant such that OA = 2 and OB = 6. Find the area between the arc AB and the chord AB.

(N.C.E.R.T.)

Solution. The given equation of the ellipse can be written as



$$\frac{x^2}{4} + \frac{y^2}{36} = 1 \text{ i.e. } \frac{x^2}{2^2} + \frac{y^2}{6^2} = 1.$$

∴ A is (2, 0) and B is (0, 6).

∴ The equation of chord AB is :

$$y - 0 = \frac{6 - 0}{0 - 2}(x - 2)$$

⇒ $y = -3x + 6.$

∴ Reqd. area (shown shaded)

$$= \int_0^2 3\sqrt{4 - x^2}\,dx - \int_0^2 (6 - 3x)\,dx$$

$$= 3\left[\frac{x\sqrt{4 - x^2}}{2} + \frac{4}{2}\sin^{-1}\frac{x}{2}\right]_0^2 - \left[6x - \frac{3x^2}{2}\right]_0^2$$

$$= 3\left[\frac{2}{2}(0) + 2\sin^{-1}(1)\right] - \left[6(2) - \frac{3(4)}{2}\right]$$

$$= 3\left[2 \times \frac{\pi}{2}\right] - [12 - 6]$$

$= (3\pi - 6)$ sq. units.

6. Sketch the region enclosed between the circles $x^2 + y^2 = 1$ and $(x - 1)^2 + y^2 = 1$, which lies in the first quadrant. Also, find the area of the region.

7. Draw a rough sketch of the following region and find the area enclosed by the region, using method of integration :

$\{(x, y) : y^2 \le 5x,\ 5x^2 + 5y^2 \le 36\}$.

8. Find the area bounded by :

$y = 1 + 2\sin^2 x$, x-axis, $x = 0$ and $x = \pi$.

Answers

1. 27.

2. $8\sqrt{3}$.

4. $\frac{196}{3}$.

6. $\frac{\pi}{3} - \frac{\sqrt{3}}{4}$.

7. $2\left[\frac{\sqrt{5}}{3}(1\cdot17)^{3/2} + \frac{9\pi}{10} - \frac{1\cdot7\sqrt{\frac{36}{5} - (1\cdot17)^2}}{4} - \frac{9}{5}\sin^{-1}\frac{1\cdot17}{6/\sqrt{5}}\right]$.

8. 2π.

CHECK YOUR UNDERSTANDING

1. Is the parabola $y^2 = 4x$ symmetrical about x-axis ?

Ans. Yes.

2. Is the circle $x^2 + y^2 = r^2$ symmetrical about the line $y = x$?

Ans. Yes.

3. Find the area enclosed by the circle $x^2 + y^2 = 9$.

Ans. 9π sq. units.

4. Find the area of the semi-portion of the circle $x^2 + y^2 = 4$.

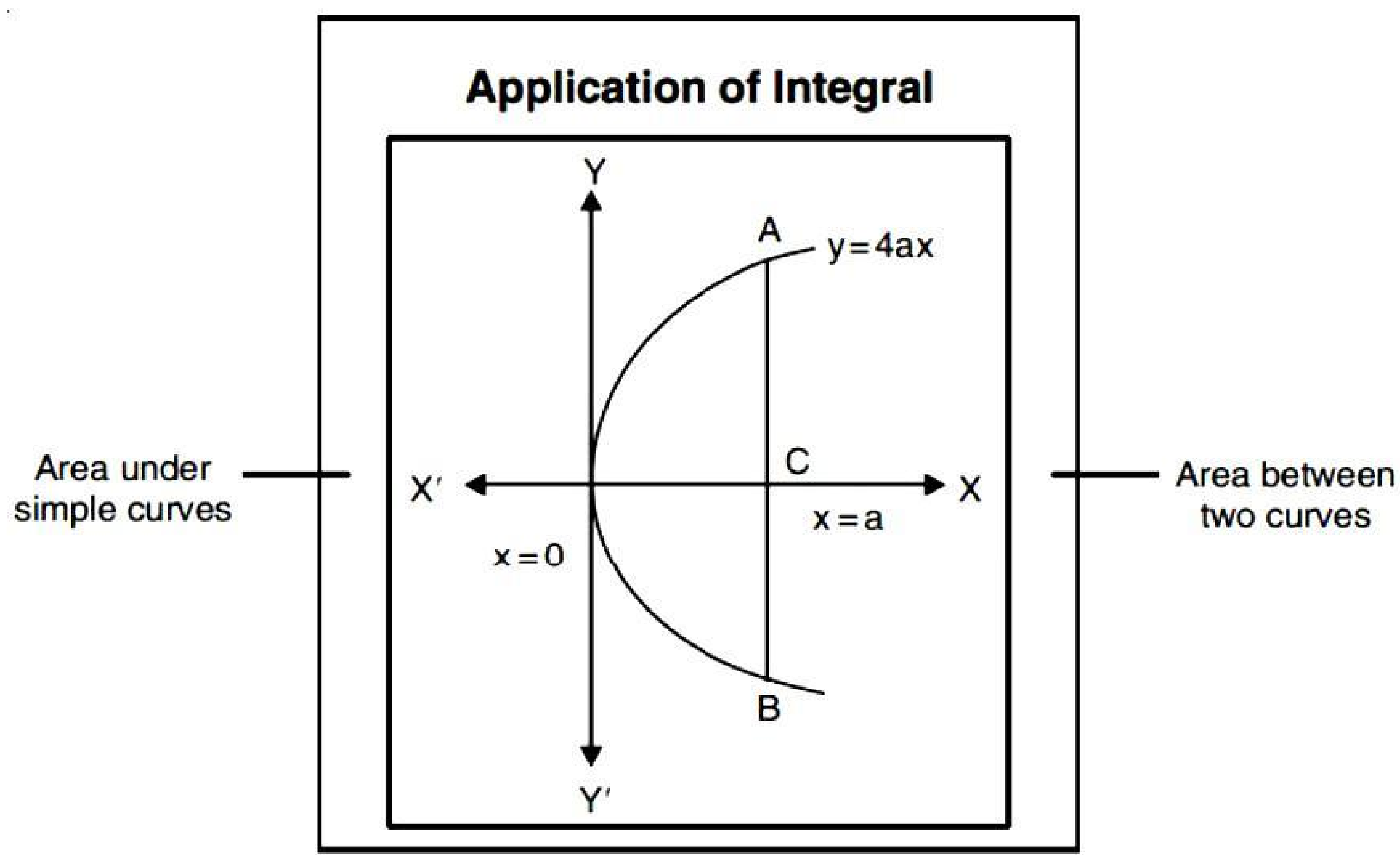
Ans. 2π sq. units.

5. Find the area of the region bounded by $y = x^4$, $x = 1$, $x = 5$ and x-axis.

Ans. 624·8 sq. units.

SUMMARY — APPLICATIONS OF THE INTEGRALS

DEFINITIONS AND IMPORTANT RESULTS



Area under simple curves

- Area of the region bounded by the curve $y = f(x)$, x-axis and the lines $x = a$ and $x = b$ $(b > a)$ is given by the formula :

$$= \int_a^b y\,dx = \int_a^b f(x)\,dx.$$

- Area of the region bounded by the curve $x = f(y)$, y-axis and the lines $y = c$, $y = d$ is given by the formula :

$$= \int_c^d x\,dy = \int_c^d f(y)\,dy.$$

Area between two curves

- Area of the region enclosed between two curves $y = f(x)$, $y = g(x)$ and the lines $x = a$, $x = b$ is $\int_a^b [g(x) - f(x)]\,dx$, where $g(x) \geq f(x)$ in $[a, b]$.
- If $f(x) \geq g(x)$ in $[a, c]$ and $f(x) \leq g(x)$ in $[c, b]$, $a < c < b$, then we write the area as :

$$\text{Area} = \int_a^c [f(x) - g(x)]\,dx + \int_c^b [g(x) - f(x)]\,dx.$$

MULTIPLE CHOICE QUESTIONS

For Board Examinations

1. The area bounded by the curve $y = f(x)$, above the x-axis, between $x = a$ and $x = b$ is :

(A) $\int_{f(a)}^{b} y\, dy$ (B) $\int_{a}^{f(b)} x\, dx$

(C) $\int_{a}^{b} x\, dy$ (D) $\int_{a}^{b} y\, dx$. **(Kerala B. 2016)**

2. The area of the cicle $x^2 + y^2 = a^2$ is :

(A) πa^2 (B) $2\pi a$

(C) $2\pi a^2$ (D) None of these. **(P.B. 2012)**

3. The area between the curve $y = x^2$, x-axis and the lines $x = 0$ and $x = 2$ is :

(A) $\frac{2}{3}$ sq. units (B) 4 sq. units

(C) $\frac{8}{3}$ sq. units (D) $\frac{4}{3}$ sq. units. **(H.B. 2011)**

4. The area of the region bounded by the parabola $y^2 = 9x$ and the line $y = 3x$ is :

(A) $\frac{1}{2}$ sq. units (B) $\frac{1}{3}$ sq. units

(C) $\frac{1}{4}$ sq. units (D) $\frac{2}{3}$ sq. units. **(H.B. 2011)**

5. The area bounded by the curve $y = 4 \sin x$, x-axis from $x = 0$ to $x = \pi$ is equal to :

(A) 1 sq. units (B) 2 sq. units

(C) 4 sq. units (D) 8 sq. units. **(H.B. 2011)**

RCQ Pocket

(Single Correct Answer Type)

(JEE-Main and Advanced)

6. The area of the plane region bounded by the curves : $x + 2y^2 = 0$ and $x + 3y^2 = 1$ is equal to :

(A) $\frac{4}{3}$ (B) $\frac{5}{3}$

(C) $\frac{1}{3}$ (D) $\frac{2}{3}$ **(A.I.E.E.E. 2008)**

7. The area of the region bounded by the parabola $(y - 2)^2 = x - 1$, the tangent to the parabola at the point (2, 3) and the x-axis is :

(A) 6 (B) 9

(C) 12 (D) 3. **(A.I.E.E.E. 2009)**

8. The area bounded by the curves $y = \cos x$ and $y = \sin x$ between the ordinates $x = 0$ and $x = \frac{3}{2}\pi$ is :

(A) $4\sqrt{2} - 2$ (B) $4\sqrt{2} + 2$

(C) $4\sqrt{2} - 1$ (D) $4\sqrt{2} + 1$. **(A.I.E.E.E. 2010)**

9. The area of the region enclosed by the curves : $y = x$, $x = e$, $y = \frac{1}{x}$ and the positive x-axis is :

(A) $\frac{1}{2}$ square units (B) 1 square units

(C) $\frac{3}{2}$ square units (D) $\frac{5}{2}$ square units. **(A.I.E.E.E. 2011)**

10. The area bounded by the curves $y^2 = 4x$ and $x^2 = 4y$ is :

(A) $\frac{32}{3}$ (B) $\frac{16}{3}$

(C) $\frac{8}{3}$ (D) 0. **(A.I.E.E.E. 2011 S)**

11. The area bounded between the parabolas : $x^2 = \frac{y}{4}$ and $x^2 = 9y$ and the straight line $y = 2$ is :

(A) $20\sqrt{2}$ (B) $\frac{10\sqrt{2}}{3}$

(C) $\frac{20\sqrt{2}}{3}$ (D) $10\sqrt{2}$. **(A.I.E.E.E. 2012)**

12. The area (in square units) bounded by the curves : $y = \sqrt{x}$, $2y - x + 3 = 0$, x-axis, and lying in the first quadrant is :

(A) 36 (B) 18

(C) $\frac{27}{4}$ (D) 9. **(J.E.E. (Main) 2013))**

13. The area enclosed by the curves : $y = \sin x + \cos x$ and $y = |\cos x - \sin x|$ over the interval $\left[0, \frac{\pi}{2}\right]$ is :

(A) $4(\sqrt{2} - 1)$ (B) $2\sqrt{2}(\sqrt{2} - 1)$

(C) $2(\sqrt{2} + 1)$ (D) $2\sqrt{2}(\sqrt{2} + 1)$. **(J.E.E. (Advanced) 2013))**

14. The area of the region described by : $A = \{(x, y) : x^2 + y^2 \le 1 \text{ and } y^2 \le 1 - x\}$ is :

(A) $\frac{\pi}{2} - \frac{4}{3}$ (B) $\frac{\pi}{2} - \frac{2}{3}$

(C) $\frac{\pi}{2} + \frac{2}{3}$ (D) $\frac{\pi}{2} + \frac{4}{3}$. **(J.E.E. (Main) 2014)**

15. The area (in sq. units) of the region described by : $\{(x, y) : y^2 \le 2x \text{ and } y \ge 4x - 1\}$ is :

(A) $\frac{4}{32}$ (B) $\frac{5}{64}$

(C) $\frac{15}{64}$ (D) $\frac{9}{32}$.

(J.E.E. (Main) 2015)

16. The area (in sq. units) of the region : $\{(x, y) : y^2 \ge 2x \text{ and } x^2 + y^2 \le 4x, x \ge 0, y \ge 0\}$ is :

(A) $\pi - \frac{8}{3}$ (B) $\pi - \frac{4\sqrt{2}}{3}$

(C) $\frac{\pi}{2} - \frac{2\sqrt{2}}{3}$ (D) $\pi - \frac{4}{3}$.

(J.E.E. (Main) 2016)

17. The area (in sq. units) of the region : $\{(x, y) : x \ge 0, x + y \le 3, x^2 \le 4y \text{ and } y \le 1 + \sqrt{x}\}$ is :

(A) $\frac{7}{3}$ (B) $\frac{5}{2}$

(C) $\frac{59}{12}$ (D) $\frac{3}{2}$.

(J.E.E. (Main) 2017)

18. Let $g(x) = x^2, f(x) = \sqrt{x}$; and α, β $(\alpha < \beta)$ be the roots of the quadratic equation $18x^2 - 9\pi x + \pi^2 = 0$. Then the area (is sq. units) bounded by curve $y = (gof)(x)$ and the lines $x = \alpha$, $x = \beta$ and $y = 0$, is :

(A) $\frac{1}{2}(\sqrt{3} - 1)$ (B) $\frac{1}{2}(\sqrt{3} + 1)$

(C) $\frac{1}{2}(\sqrt{3} - \sqrt{2})$ (D) $\frac{1}{2}(\sqrt{2} - 1)$.

(J.E.E. (Main) 2018)

Answers

1. (D) **2.** (A) **3.** (C) **4.** (A) **5.** (D) **6.** (A) **7.** (B) **8.** (A) **9.** (C) **10.** (B)
11. (C) **12.** (D) **13.** (B) **14.** (D) **15.** (D) **16.** (A) **17.** (B) **18.** (A).

Hints/Solutions

RCQ Pocket

6. (A) $x + 2y^2 = 0 \Rightarrow y^2 = -\frac{x}{2}$

(Left-handed parabola)

$x + 3y^2 = 1 \Rightarrow y^2 = -\frac{1}{3}(x-1)$

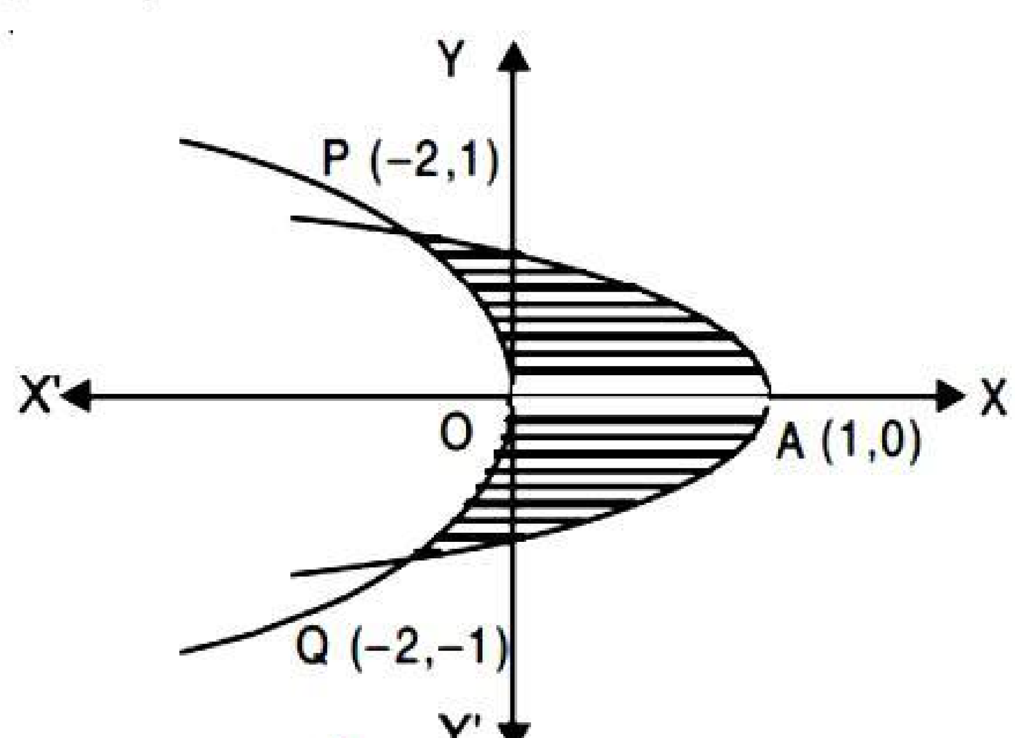
(Left–handed parabola)

$1 - 3y^2 = -2y^2 \Rightarrow y^2 = 1 \Rightarrow y = \pm 1.$

When $y = 1$, then $x = -2$.

When $y = -1$, then $x = -2$.

Thus the given parabolas meet at P (– 2, 1) and Q (– 2, – 1).



$\therefore$ Reqd. area $= 2\int_0^1 \left[\left(1 - 3y^2\right) - \left(-2y^2\right)\right] dy$

$= 2\int_0^1 \left(1 - y^2\right) dy = 2\left[y - \frac{y^3}{3}\right]_0^1$

$= 2\left[1 - \frac{1}{3}\right] = 2\left[\frac{2}{3}\right] = \frac{4}{3}$ sq. units.

7. (C) The given parabola is $(y - 2)^2 = x - 1$ ....(1)

Its vertex is (1, 2).

This meets x-axis at (5, 0).

(1) is $y^2 - 4y - x - 5 = 0$.

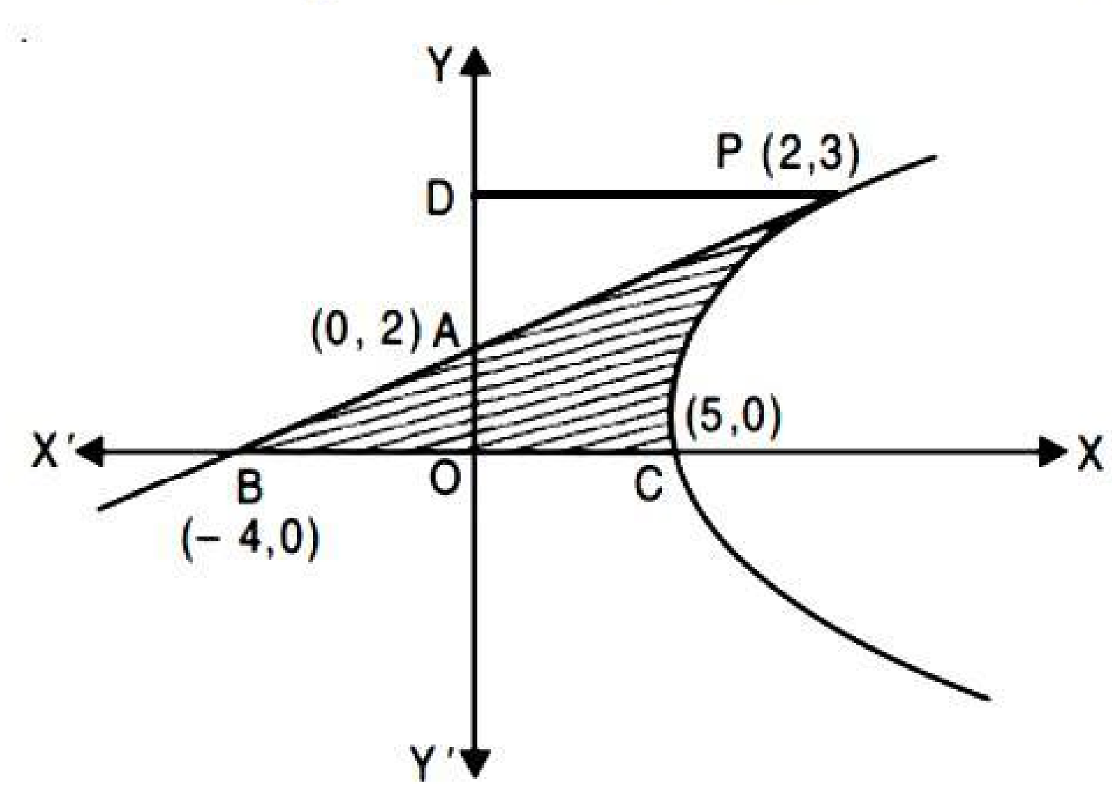
Tangent at (2, 3) is :

$y - 3 - 2(y + 3) - \frac{1}{2}(x + 2) + 5 = 0$

$\Rightarrow x - 2y + 4 = 0.$

This meets x-axis at (– 4, 0).

$\therefore$ Reqd. area = ar (BOA) + ar (OPCD)



$= \frac{1}{2}(4)(2) + \int_0^3 x\, dy$

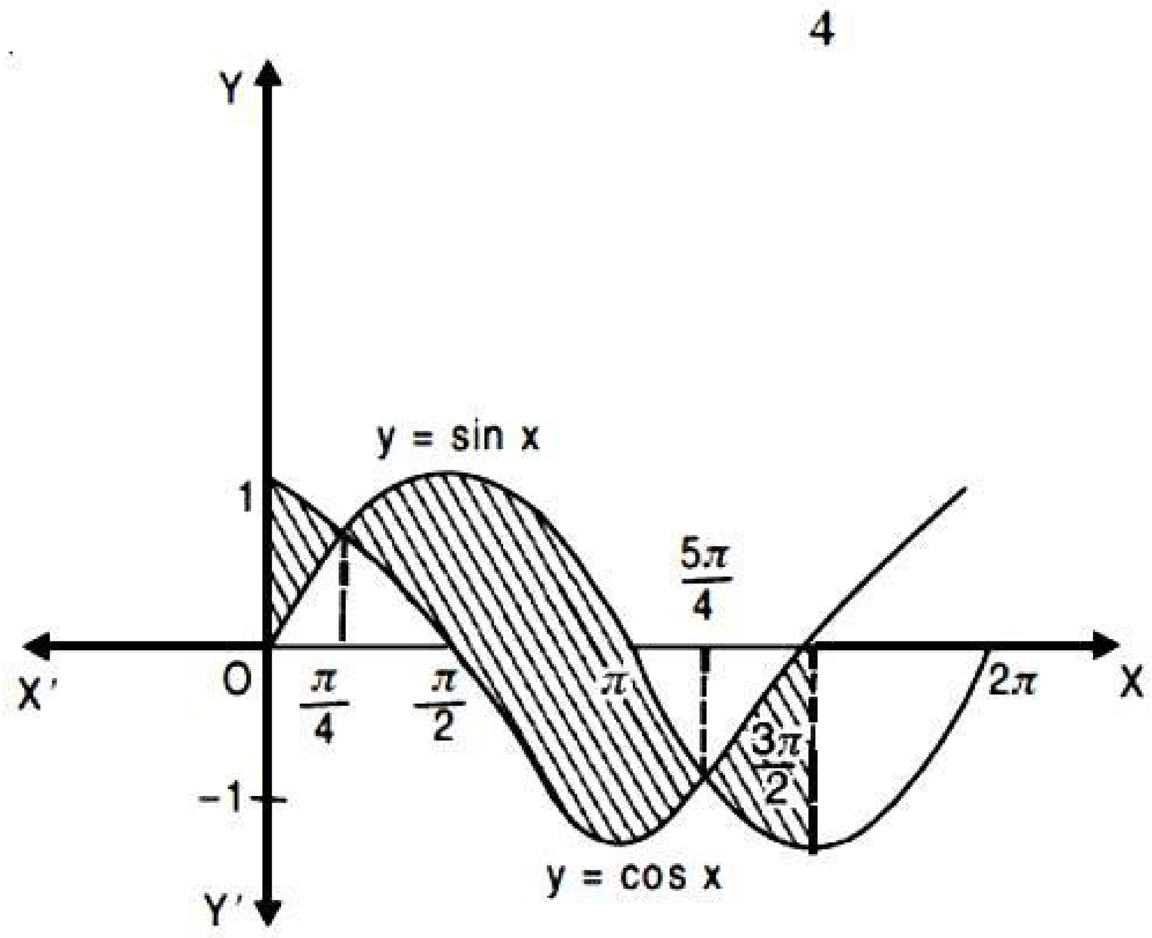
$= 3 + \int_0^3 [(y-2)^2 + 1]\, dy$

$$= 3 + \left[\frac{(y-2)^3}{3} + y\right]_0^3 = 3 + \left[\frac{1}{3} + 3 + \frac{8}{3}\right]$$

$= 3 + 6 = 9$ sq. units.

8. (A) Reqd. area = Shaded Area

$$= \int_0^{\frac{\pi}{4}} (\cos x - \sin x)\, dx + \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} (\sin x - \cos x)\, dx + \int_{\frac{5\pi}{4}}^{\frac{3\pi}{2}} (\cos x - \sin x) dx$$



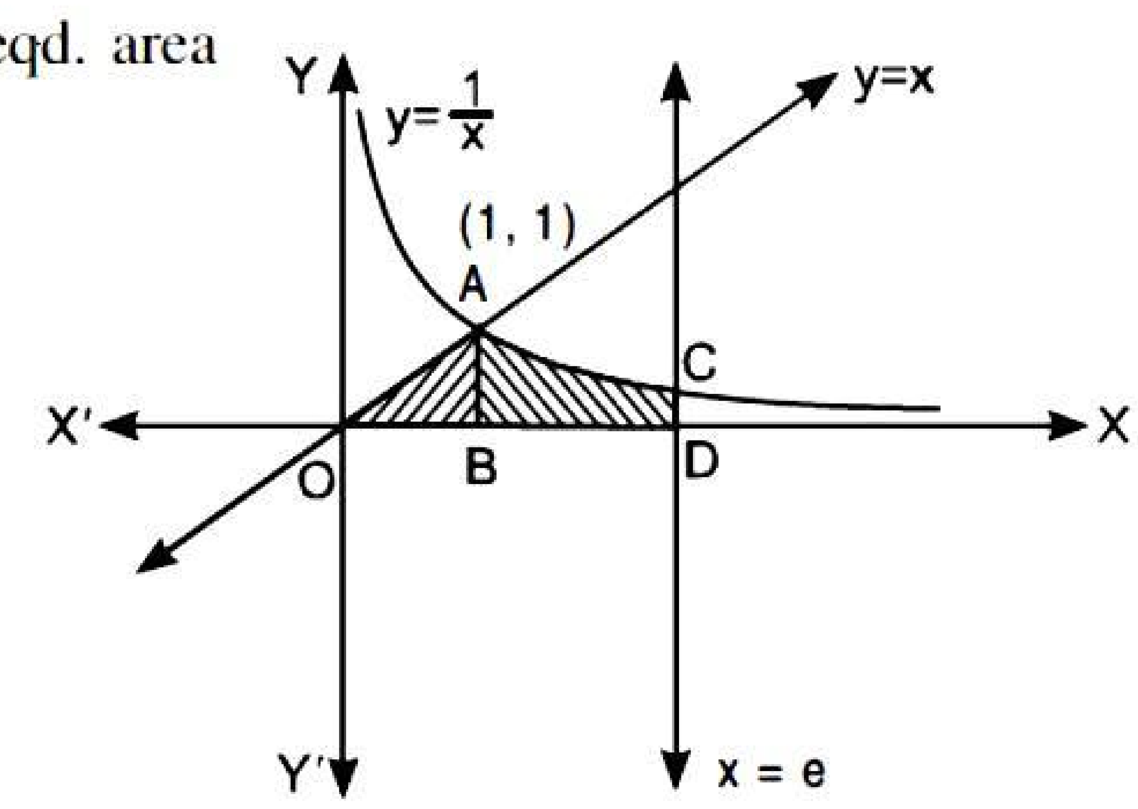
$$= [\sin x + \cos x]_0^{\frac{\pi}{4}} + [-\cos x - \sin x]_{\frac{\pi}{4}}^{\frac{5\pi}{4}} + [\sin x + \cos x]_{\frac{5\pi}{4}}^{\frac{3\pi}{2}}$$

$$= \left[\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) - (0+1)\right] - \left[\left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right) - \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right)\right] + \left[(-1+0) - \left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right)\right]$$

$$= \left(\sqrt{2} - 1\right) - \left(-\sqrt{2} - \sqrt{2}\right) + \left(-1 + \sqrt{2}\right)$$

$$= \sqrt{2} - 1 + 2\sqrt{2} - 1 + \sqrt{2} = 4\sqrt{2} - 2.$$

9. (C) Reqd. area



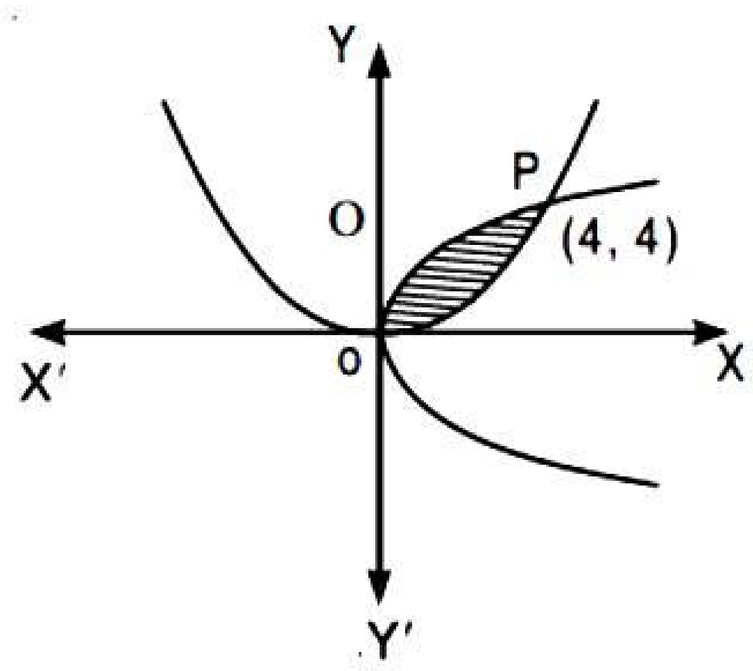
$$= \text{ar (OAB)} + \text{ar (ABDC)}$$

$$= \frac{1}{2}(1)(1) + \int_1^e \frac{1}{x}\, dx \qquad [\because A \text{ is } (1, 1)]$$

$$= \frac{1}{2} + [\ln x]_1^e = \frac{1}{2} + [\ln e - \ln 1]$$

$$= \frac{1}{2} + [1 - 0] = \frac{3}{2} \text{ square units.}$$

10. (B) The given curves are :
$y^2 = 4x$ and $x^2 = 4y$.
These meet at P (4, 4).

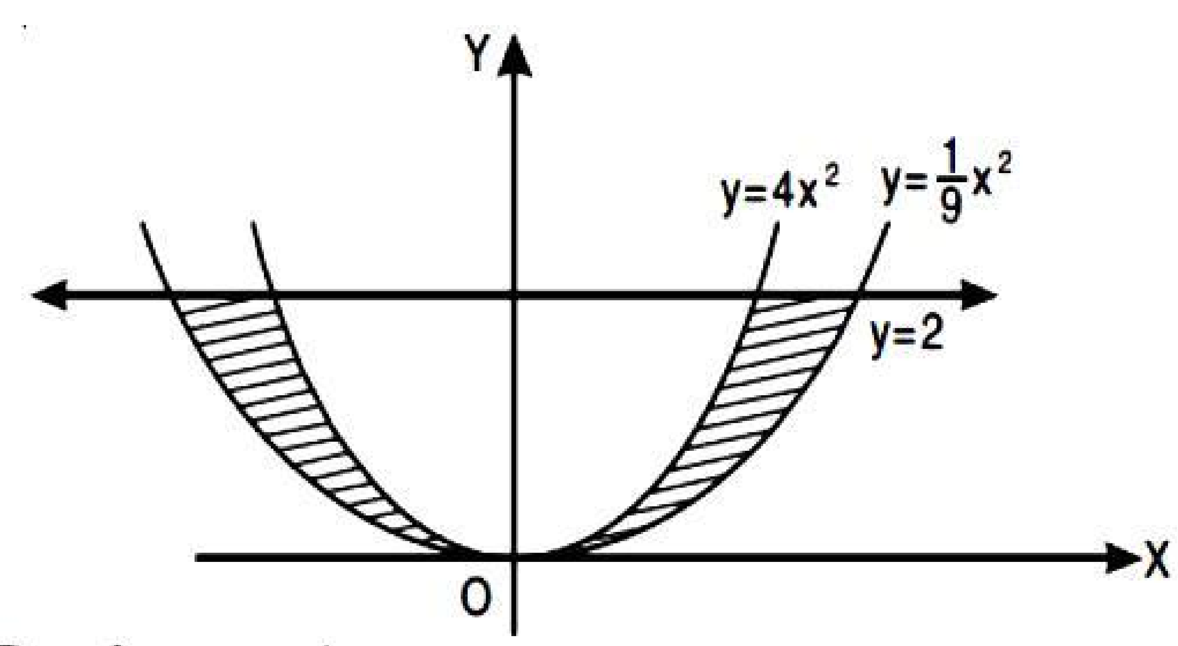


$$\therefore \text{ Reqd. area} = \int_0^4 \left(2\sqrt{x} - \frac{x^2}{4}\right) dx$$

$$= \left[2\frac{x^{3/2}}{3/2} - \frac{x^3}{12}\right]_0^4$$

$$= \frac{4}{3}(4)^{3/2} - \frac{1}{12}(64) = \frac{32}{3} - \frac{16}{3} = \frac{16}{3}.$$

11. (C)



Reqd. area, A

$$= 2\left[\int_0^2 \left(3\sqrt{y} - \frac{\sqrt{y}}{2}\right) dy\right] = 2\int_0^2 \frac{5\sqrt{y}}{2} dy$$

$$= 5\left[\frac{y^{3/2}}{3/2}\right]_0^2 = \frac{10}{3}[2^{3/2} - 0] = \frac{20\sqrt{2}}{3}$$

12. (D) The given curve is $y = \sqrt{x}$ ...(1)
and the given line is $2y - x + 3 = 0$ ...(2)
Putting the value of y in (1), we get :

$$2\sqrt{x} - x + 3 = 0 \Rightarrow 2\sqrt{x} = x - 3.$$

Squaring, $4x = x^2 - 6x + 9$

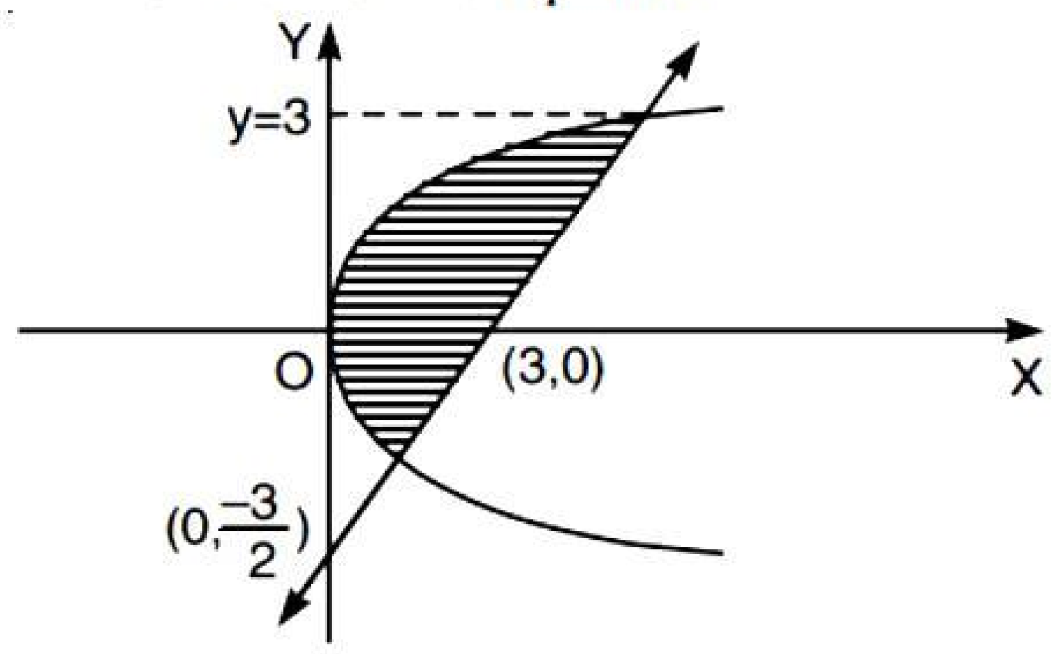
$\Rightarrow \quad x^2 - 10x + 9 = 0$

$\Rightarrow (x-9)(x-1) = 0 \Rightarrow x = 9, 1.$

$\therefore$ Reqd. area

$$= \int_0^3 \left[(2y+3) - y^2\right] dy = \left[y^2 + 3y - \frac{y^3}{3}\right]_0^3$$

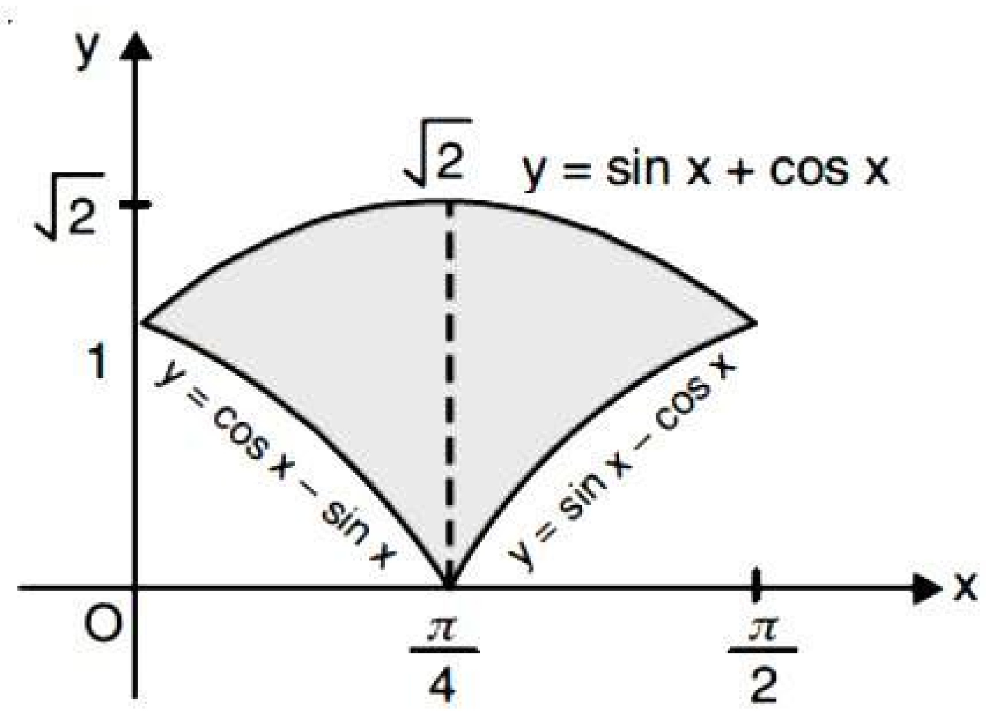
$= 9 + 9 - 9 = 9$ sq. units.



13. (B) The given curves are :

$$y = \sin x + \cos x, \; x \in \left[0, \frac{\pi}{2}\right]$$

$$\text{and } y = \begin{cases} \cos x - \sin x, & x \in \left[0, \frac{\pi}{4}\right) \\ \sin x - \cos x & x \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right]. \end{cases}$$



$\therefore$ Reqd. area

$$= \int_0^{\pi/4} \left[(\sin x + \cos x) - (\cos x - \sin x)\right] dx$$

$$+ \int_{\pi/4}^{\pi/2} \left[(\sin x + \cos x) - (\sin x - \cos x)\right] dx$$

$$= 2\int_0^{\pi/4} \sin x \, dx + 2\int_{\pi/4}^{\pi/2} \cos x \, dx$$

$$= 2\left[-\cos x\right]_0^{\pi/4} + 2\left[\sin x\right]_{\pi/4}^{\pi/2}$$

$$= 2\left[-\cos\frac{\pi}{4} + \cos 0\right] + 2\left[\sin\frac{\pi}{2} - \sin\frac{\pi}{4}\right]$$

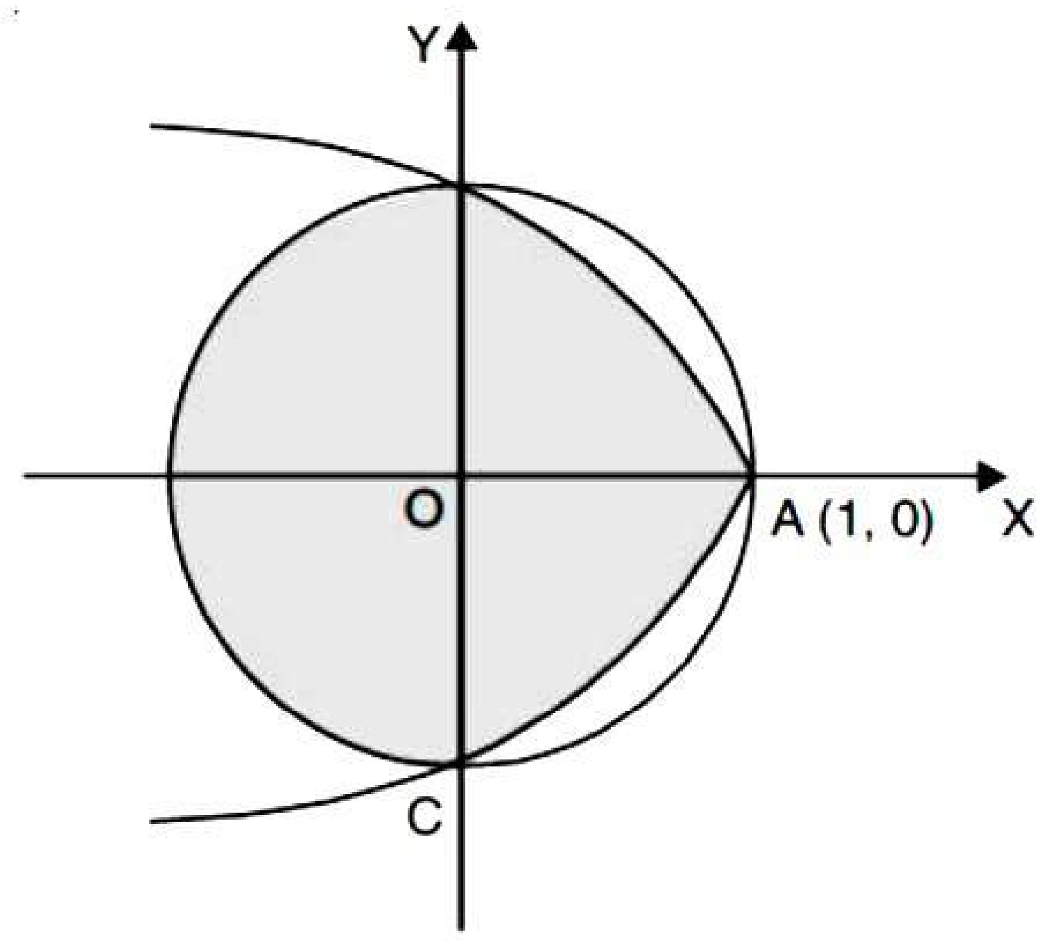
$$= 2\left[-\frac{1}{\sqrt{2}} + 1\right] + 2\left[1 - \frac{1}{\sqrt{2}}\right]$$

$$= -\sqrt{2} + 2 + 2 - \sqrt{2}$$

$$= 4 - 2\sqrt{2} = 2\sqrt{2}(\sqrt{2} - 1).$$

14. (D) We have : $x^2 + y^2 = 1$ ...(1)

and $\quad y^2 = -(x-1)$ ...(2)



$x^2 - x + 1 = 1 \Rightarrow x^2 - x = 0$

$\Rightarrow \quad x(x-1) = 0 \quad \Rightarrow x = 0, 1.$

When $x = 0$, $y = \pm 1$.

When $x = 1$, $y = 0$.

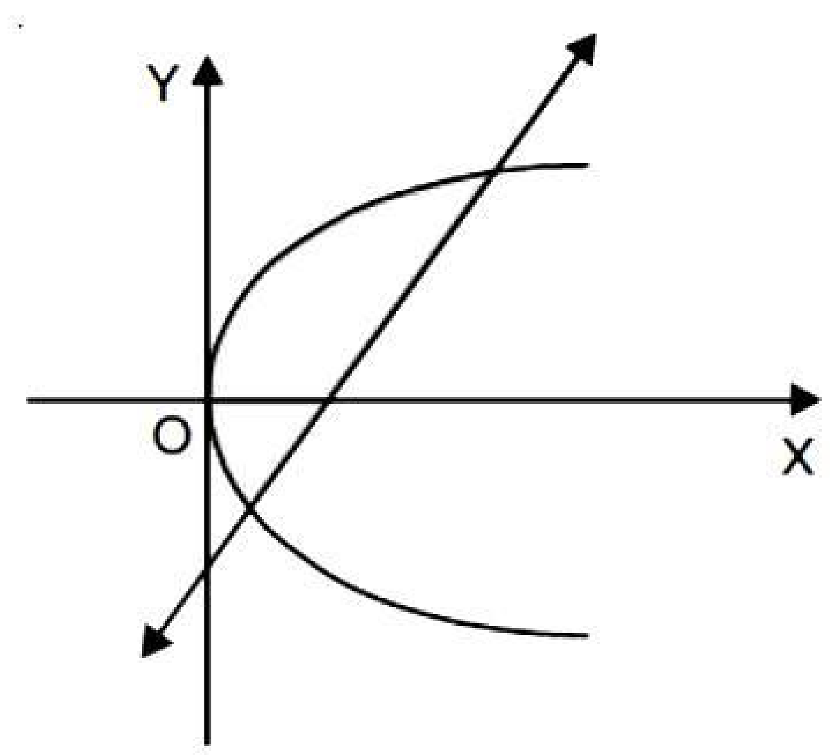
$\therefore$ B is (0, 1) and is (0, – 1).

$$\therefore \quad \text{Required area} = 2\left(\frac{\pi}{4}\right) + 2\left(\int_0^1 \sqrt{1-x}\, dx\right)$$

$$= \frac{\pi}{2} + 2\left[\frac{(1-x)^{3/2}}{\frac{3}{2}(-1)}\right]_0^1$$

$$= \frac{\pi}{2} - \frac{4}{3}[0-1] = \frac{\pi}{2} + \frac{4}{3}.$$

15. (D)



We have : $y^2 \leq 2x$ ...(1)

and $\quad y \geq 4x - 1$ ...(2)

Solving, $y^2 = 2x$ and $y = 4x - 1$,

we get $y = 1, -\frac{1}{2}$.

$$\therefore \quad \text{Reqd. area} = \int_{-1/2}^{1} \left[\frac{y+1}{4} - \left(\frac{y^2}{2}\right)\right] dy$$

$$= \left[\frac{y^2}{8} + \frac{y}{4} - \frac{y^3}{6}\right]_{-1/2}^{1}$$

$$= \left(\frac{1}{8}+\frac{1}{4}-\frac{1}{6}\right)-\left(\frac{1}{32}-\frac{1}{8}+\frac{1}{48}\right)$$

$$= \frac{3+6-4}{24}-\frac{3-12+2}{96} = \frac{5}{24}+\frac{7}{96}$$

$$= \frac{20+7}{96} = \frac{27}{96} = \frac{9}{32}.$$

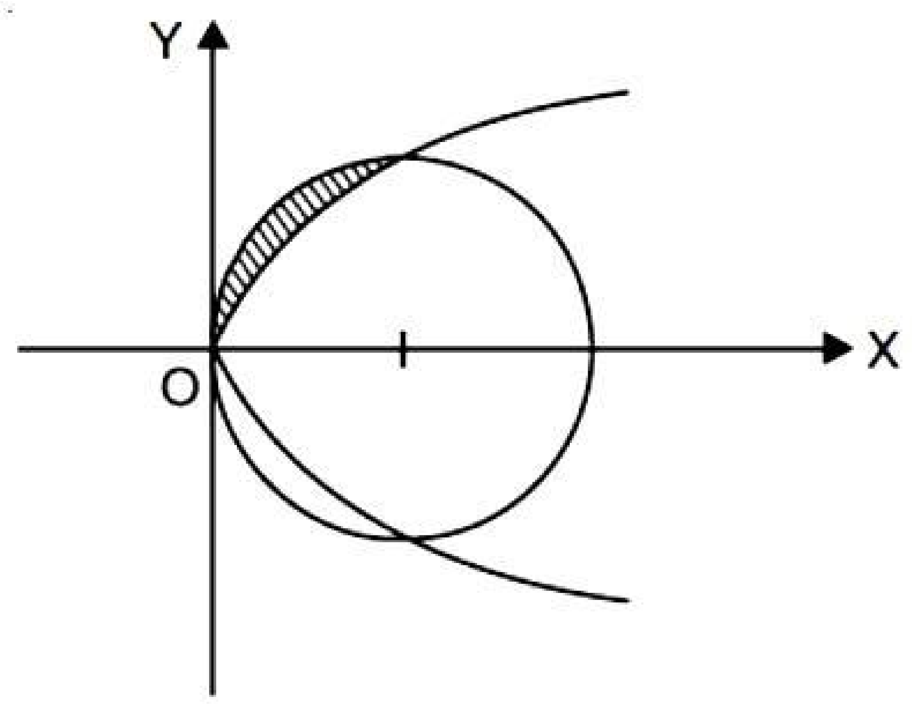
16. (A)

$y^2 = 2x$ is a parabola.

$x^2 + y^2 = 4x \Rightarrow (x-2)^2 + y^2 = 4.$

It is a circle having centre (2, 0) and radius 2.

$\therefore$ Reqd. area = Shaded area



$$= \frac{1}{4}(\pi(2)^2) - \int_0^2 \sqrt{2}\sqrt{x}\,dx$$

$$= \pi - \sqrt{2}\left[\frac{x^{3/2}}{3/2}\right]_0^2$$

$$= \pi - \frac{2\sqrt{2}}{3}\left[2^{3/2}-0\right] = \pi - \frac{8}{3}.$$

17. (B) Solving $x^2 = 4y$ and $x + y = 3$, we get :

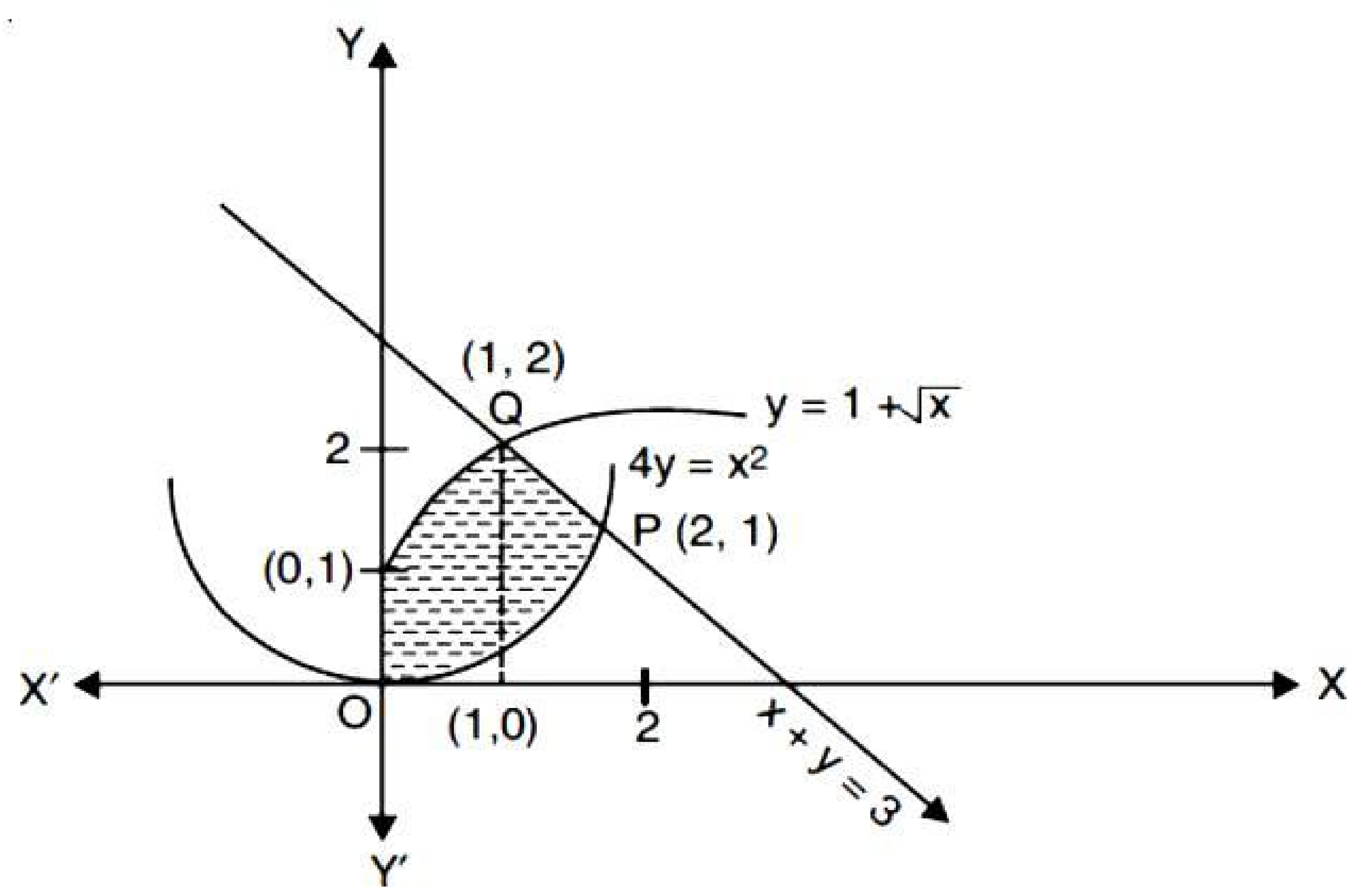
$$\frac{x^2}{4}+x = 3$$

$\Rightarrow \quad x^2 + 4x - 12 = 0$

$\Rightarrow \quad (x+6)(x-2) = 0$

$\Rightarrow \quad x = 2.$

$\therefore \quad y = 3 - 2 = 1.$



Thus P is (2, 1).

Solving, $y = 1+\sqrt{x}$ and $x + y = 3$, we get :

$$1+\sqrt{x}+x = 3$$

$\Rightarrow \quad \sqrt{x}+x = 2$

$\Rightarrow \quad x = 1.$

$\therefore$ Q is (1, 2).

$\therefore$ Requied area

$$= \int_0^1 (1+\sqrt{x})dx + \int_1^2 (3-x)dx - \int_0^2 \frac{x^2}{4}dx$$

$$= \left[x+\frac{2}{3}x^{3/2}\right]_0^1 + \left[3x-\frac{x^2}{2}\right]_1^2 - \left[\frac{x^3}{12}\right]_0^2$$

$$= \left(1+\frac{2}{3}\right)+\left((6-2)-\left(3-\frac{1}{2}\right)\right)-\frac{8}{12}$$

$$= \frac{5}{3}+4-\frac{5}{2}-\frac{2}{3} = 5-\frac{5}{2}$$

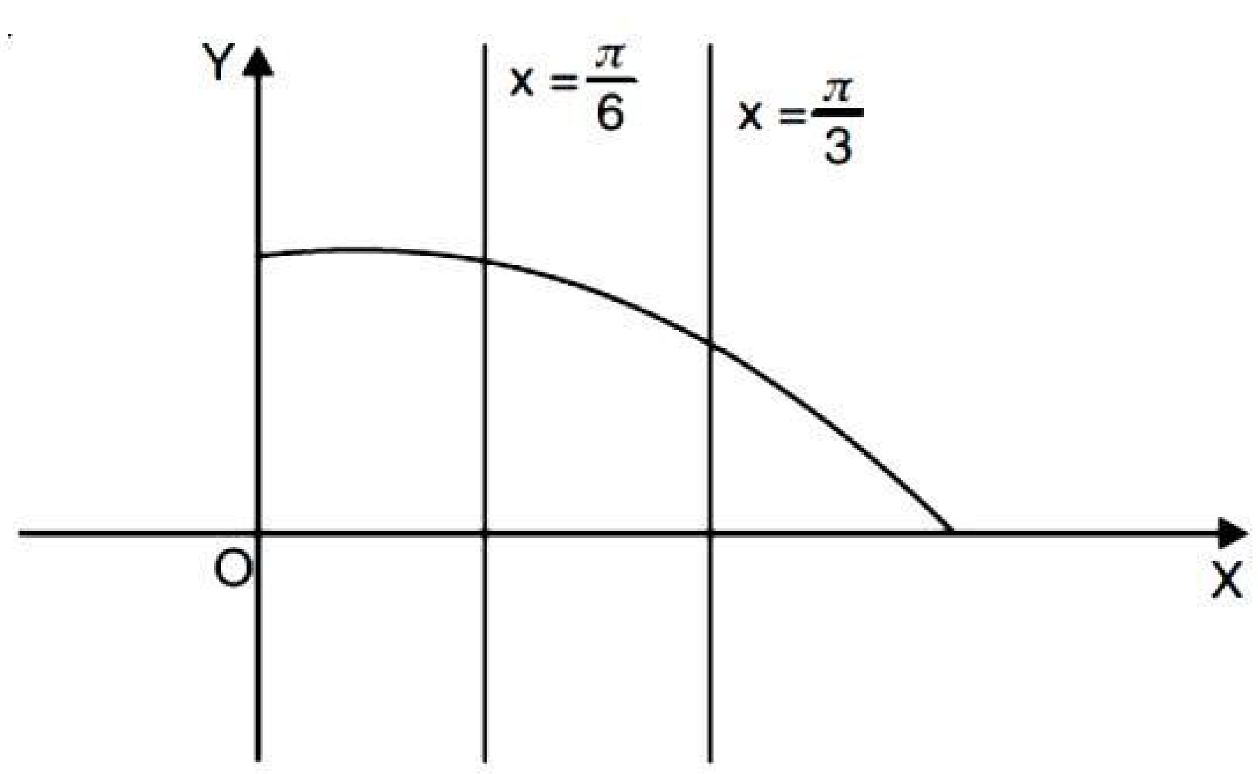
$$= \frac{5}{2}.$$

18. (A) Here, $18x^2 - 9\pi x + \pi^2 = 0$

$\Rightarrow (6x-\pi)(3x-\pi) = 0$

$\Rightarrow \quad x = \frac{\pi}{6}, \frac{\pi}{3}.$

Let $\alpha = \frac{\pi}{6}$ and $\beta = \frac{\pi}{3}$.



Here, $y = (gof)(x) = \cos x$.

$$\therefore \text{ Reqd. area} = \int_{\pi/6}^{\pi/3} \cos x\,dx$$

$$= [\sin x]_{\pi/6}^{\pi/3} = \sin\frac{\pi}{3} - \sin\frac{\pi}{6}$$

$$= \frac{\sqrt{3}}{2} - \frac{1}{2} = \frac{1}{2}(\sqrt{3}-1) \text{ sq. units.}$$

CHAPTER TEST 8

Time Allowed : 1 Hour **Max. Marks : 34**

Notes : 1. *All questions are compulsory.*

2. *Marks have been indicated against each question.*

1. Using integration, find the area of the quadrant of the circle $x^2 + y^2 = 4$. **(1)**

2. Find the area bounded by $y = x$, the x-axis and the lines $x = -1$ and $x = 2$. **(1)**

3. Calculate the area under the curve :

$y = \sqrt{2x}$ between the ordinates $x = 0$ and $x = 1$. **(2)**

4. Find the area enclosed by the ellipse :

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, a>b.$$ **(2)**

5. Find the area of the region bounded by :

$y^2 = 9x$, $x = 2$, $x = 4$ and the x-axis in the first quadrant. **(4)**

6. Using integration, find the area of the region bounded by :

$(-1, 1)$, $(0, 5)$ and $(3, 2)$. **(4)**

7. Find the area lying above the x-axis and included between the circle $x^2 + y^2 = 8x$ and the parabola $y^2 = 4x$. **(4)**

8. Calculate the area of the region enclosed between the circles $x^2 + y^2 = 1$ and $(x - 1)^2 + y^2 = 1$. **(4)**

9. Find the area of the region in the first quadrant enclosed by the x-axis, the line $y = x$ and the circle $x^2 + y^2 = 32$. **(6)**

10. Find the area of the smaller part of the circle $x^2 + y^2 = a^2$ cut off by the line $x = \frac{a}{\sqrt{2}}$. **(6)**

Answers

1. π sq. units.

2. $\frac{5}{2}$ sq. units.

3. $\frac{8}{3}$ sq. units.

4. πab sq. units.

5. $(16-4\sqrt{2})$ sq. units.

6. 7·5 sq. units.

7. $\frac{4}{3}(8+3\pi)$ sq. units.

8. $\left(\frac{2\pi}{3}-\frac{\sqrt{3}}{2}\right)$ sq. units.

9. 4π sq. units.

10. $\frac{a^2}{2}\left(\frac{\pi}{2}-1\right)$ sq. units.