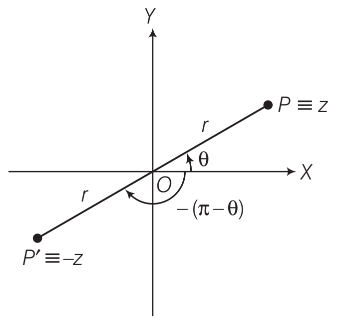
Session 3

amp(z) − amp ($-z$) = $\pm\pi$; According as amp (z) is Positive or Negative, Square Root of a Complex Number, Solution of Complex Equations, De-Moivre's Theorem, Cube Roots of Unity

amp (z) − amp $(-z) = \pm\pi$, According as amp (z) is Positive or Negative

***Case* I amp (z) is positive.**

If amp $(z) = \theta$, we have



$$\text{amp}(-z) = -(\angle P'OX) = -(\pi - \theta)$$

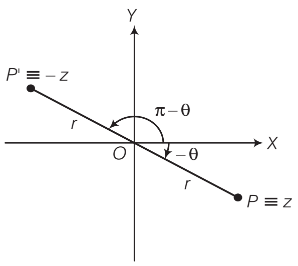
$\therefore$ $\quad$ **amp (z) − amp ($-z$) = π** $\quad$ [here, $OP = OP'$]

***Case* II amp (z) is negative.**

If $\quad$ amp $(z) = -\theta$

We have, $\quad$ amp $(-z) = \angle P'OX = \pi - \theta$

$\therefore$ **amp (z) − amp ($-z$) = $-\pi$** $\quad$ [here, $OP = OP'$]



Example 47. If $|z_1| = |z_2|$ and arg $(z_1/z_2) = \pi$, then find the value of $z_1 + z_2$.

Sol. $\because$ $\arg\left(\dfrac{z_1}{z_2}\right) = \pi$

$\Rightarrow \quad \arg(z_1) - \arg(z_2) = \pi \quad$ ...(i)

$\because \quad z_1 = |z_1|(\cos(\arg z_1) + i\sin(\arg z_1)) \quad$ ...(ii)

and $\quad z_2 = |z_2|(\cos(\arg z_2) + i\sin(\arg z_2)) \quad$ ...(iii)

From Eq. (ii), we get

$$z_1 = |z_2|(\cos(\pi + \arg(z_2)) + i\sin(\pi + \arg(z_2)))$$

[from Eq. (i) and $|z_1| = |z_2|$]

$$= |z_2|(-\cos(\arg z_2) - i\sin(\arg z_2)) = -z_2$$

[from Eq. (iii)]

$\therefore z_1 + z_2 = 0$

Example 48. Let z and w be two non-zero complex numbers, such that $|z| = |w|$ and amp $(z) +$ amp $(w) = \pi$, then find the relation between z and w.

Sol. Given, amp (z) + amp $(w) = \pi$

$\Rightarrow \quad$ amp (z) − amp $(\overline{w}) = \pi$

Here, $\quad |z| = |\overline{w}| = |w| \quad$ [given $|z| = |w|$]

and $\quad$ amp $(z) > 0$

Then, $\quad z + \overline{w} = 0$

Square Root of a Complex Number

Let $\quad z = x + iy$,

where $x, y \in R$ and $i = \sqrt{-1}$.

Suppose $\sqrt{(x + iy)} = a + ib \quad$ ...(i)

On squaring both sides, we get

$$(x + iy) = (a^2 - b^2) + 2iab$$

On comparing the real and imaginary parts, we get

$$a^2 - b^2 = x \quad \text{...(ii)}$$

and $\quad 2ab = y \quad$ ...(iii)

$\therefore \quad a^2 + b^2 = \sqrt{(a^2 - b^2)^2 + 4a^2b^2} = \sqrt{(x^2 + y^2)}$

$$a^2 + b^2 = |z| \quad \text{...(iv)}$$

From Eqs. (ii) and (iv), we get

$$a = \pm\sqrt{\left(\frac{|z| + x}{2}\right)},\ b = \pm\sqrt{\left(\frac{|z| - x}{2}\right)}$$

or $\quad a = \pm\sqrt{\left(\dfrac{|z| + \text{Re}(z)}{2}\right)},\ b = \pm\sqrt{\left(\dfrac{|z| - \text{Re}(z)}{2}\right)}$

Now, from Eq. (i), the required square roots,

$$\text{i.e. } \sqrt{z}=\begin{cases} \pm\left(\sqrt{\dfrac{|z|+\mathbf{Re}\,(z)}{2}}+i\sqrt{\dfrac{|z|-\mathbf{Re}\,(z)}{2}}\right), & \textbf{if Im}\,(z)>0 \\ \pm\left(\sqrt{\dfrac{|z|+\mathbf{Re}\,(z)}{2}}-i\sqrt{\dfrac{|z|-\mathbf{Re}\,(z)}{2}}\right), & \textbf{if Im}\,(z)<0 \end{cases}$$

Aliter

If $\sqrt{(x+iy)}$, where $x, y \in R$ and $i=\sqrt{-1}$, then

(i) If y is not even, then multiply and divide in y by 2, then $\sqrt{(x+iy)}$ convert in

$$\sqrt{x+y\sqrt{-1}}=\sqrt{\left(x+2\sqrt{-\frac{y^2}{4}}\right)}.$$

(ii) Factorise: $-\dfrac{y^2}{4}$ say $\alpha, \beta\ (\alpha<\beta)$.

Take that possible factor which satisfy

$$x=(\alpha i)^2+\beta^2, \text{ if } x>0 \quad\text{or}\quad x=\alpha^2+(i\beta)^2, \text{ if } x<0$$

(iii) Finally, write $x+iy=(\alpha i)^2+\beta^2+2i\alpha\beta$

or $\alpha^2+(i\beta)^2+2i\alpha\beta$

and take their square root.

(iv) $\sqrt{(x+iy)}=\begin{cases}\pm(\alpha i+\beta) \\ \text{or} \pm(\alpha+i\beta)\end{cases}$ and $\sqrt{(x-iy)}=\begin{cases}\pm(\beta-i\alpha) \\ \text{or} \pm(\alpha-i\beta)\end{cases}$

Remark

1. The square root of i is $\pm\left(\dfrac{1+i}{\sqrt{2}}\right)$, where $i=\sqrt{-1}$.
2. The square root of $(-i)$ is $\left(\dfrac{1-i}{\sqrt{2}}\right)$.

Example 49. Find the square roots of the following

(i) $4+3i$ (ii) $-5+12i$

(iii) $-8-15i$ (iv) $7-24i$ (where, $i=\sqrt{-1}$)

Sol. (i) Let $z=4+3i$

$\therefore \quad |z|=5, \text{Re}\,(z)=4, \text{Im}\,(z)=3>0$

$$\because \quad \sqrt{z}=\pm\left(\sqrt{\frac{|z|+\text{Re}\,(z)}{2}}+i\sqrt{\frac{|z|-\text{Re}\,(z)}{2}}\right)$$

$$\therefore\ \sqrt{(4+3i)}=\pm\left(\sqrt{\left(\frac{5+4}{2}\right)}+i\sqrt{\left(\frac{5-4}{2}\right)}\right)=\pm\left(\frac{3+i}{\sqrt{2}}\right)$$

Aliter

$$\sqrt{(4+3i)}=\sqrt{4+3\sqrt{-1}}=\sqrt{4+2\sqrt{\left(-\frac{9}{4}\right)}}$$

$$=\sqrt{\frac{9}{2}-\frac{1}{2}+2\sqrt{\left(-\frac{9}{4}\right)}}$$

$$=\sqrt{\left\{\left(\frac{3}{\sqrt{2}}\right)^2+\left(\frac{i}{\sqrt{2}}\right)^2+2\cdot\frac{3}{\sqrt{2}}\cdot\frac{i}{\sqrt{2}}\right\}}$$

$$=\sqrt{\left(\frac{3+i}{\sqrt{2}}\right)^2}=\pm\left(\frac{3+i}{\sqrt{2}}\right)$$

(ii) Let $z=-5+12i$

$\therefore \quad |z|=13, \text{Re}\,(z)=-5, \text{Im}\,(z)=12>0$

$$\because \quad \sqrt{z}=\pm\left(\sqrt{\frac{|z|+\text{Re}\,(z)}{2}}+i\sqrt{\frac{|z|-\text{Re}\,(z)}{2}}\right)$$

$$\therefore\ \sqrt{(-5+12i)}=\pm\left(\sqrt{\left(\frac{13-5}{2}\right)}+i\sqrt{\left(\frac{13+5}{2}\right)}\right)$$

$$=\pm(2+3i)$$

Aliter

$$\sqrt{(-5+12i)}=\sqrt{(-5+12\sqrt{-1})}$$

$$=\sqrt{(-5+2\sqrt{(-36)}}$$

$$=\sqrt{(-5+2\sqrt{(-9\times 4)})}$$

$$=\sqrt{(-9+4+2\sqrt{(-9\times 4)})}$$

$$=\sqrt{(3i)^2+2^2+2\cdot 3i\cdot 2}$$

$$=\sqrt{(2+3i)^2}=\pm(2+3i)$$

(iii) Let $z=-8-15i$

$\therefore \quad |z|=17, \text{Re}\,(z)=-8, \text{Im}\,(z)=-15<0$

$$\therefore\ \sqrt{(-8-15i)}=\pm\left(\sqrt{\left(\frac{17-8}{2}\right)}-i\sqrt{\left(\frac{17+8}{2}\right)}\right)$$

$$=\pm\left(\frac{3-5i}{\sqrt{2}}\right)$$

Aliter $\sqrt{(-8-15i)}=\sqrt{(-8-15\sqrt{-1})}$

$$=\sqrt{\left(-8-2\sqrt{\left(-\frac{225}{4}\right)}\right)}=\sqrt{\left(-8-2\sqrt{\left(-\frac{25}{2}\times\frac{9}{2}\right)}\right)}$$

$$=\sqrt{\left(\frac{9}{2}-\frac{25}{2}-2\sqrt{\left(-\frac{25}{2}\times\frac{9}{2}\right)}\right)}$$

$$=\sqrt{\left(\frac{3}{\sqrt{2}}\right)^2+\left(\frac{5i}{\sqrt{2}}\right)^2-2\cdot\frac{3}{\sqrt{2}}\cdot\frac{5i}{\sqrt{2}}}$$

$$=\sqrt{\left(\frac{3-5i}{\sqrt{2}}\right)^2}=\pm\left(\frac{3-5i}{\sqrt{2}}\right)$$

(iv) Let $z = 7 - 24i$

$\therefore |z| = 25, \text{Re}(z) = 7, \text{Im}(z) = -24 < 0$

$$\because \sqrt{z} = \pm\left(\sqrt{\frac{|z| + \text{Re}(z)}{2}} - i\sqrt{\frac{|z| - \text{Re}(z)}{2}}\right)$$

$$\therefore \sqrt{(7-24i)} = \pm\left(\sqrt{\left(\frac{25+7}{2}\right)} - i\sqrt{\left(\frac{25-7}{2}\right)}\right)$$

$$= \pm(4 - 3i)$$

Aliter

$$\sqrt{(7-24i)} = \sqrt{(7-24\sqrt{-1})} = \sqrt{7 - 2\sqrt{(-144)}}$$

$$= \sqrt{7 - 2\sqrt{(16 \times -9)}}$$

$$= \sqrt{16 - 9 - 2\sqrt{(16 \times -9)}}$$

$$= \sqrt{(4)^2 + (3i)^2 - 2 \cdot 4 \cdot 3i}$$

$$= \sqrt{(4-3i)^2} = \pm(4 - 3i)$$

Example 50. *Find the square root of* $x + \sqrt{(-x^4 - x^2 - 1)}$.

Sol. Let $z = x + \sqrt{(-x^4 - x^2 - 1)}$

$$= x + i\sqrt{(x^4 + x^2 + 1)} \qquad [\because \sqrt{-1} = i]$$

$$\therefore \quad |z| = \sqrt{x^2 + (x^4 + x^2 + 1)}$$

$$= \sqrt{(x^4 + 2x^2 + 1)} = \sqrt{(x^2+1)^2}$$

$$\therefore \quad |z| = (x^2 + 1)$$

$\text{Re}(z) = x$

$\text{Im}(z) = \sqrt{(x^4 + x^2 + 1)} > 0$

$$\because \quad \sqrt{z} = \pm\left(\sqrt{\frac{|z| + \text{Re}(z)}{2}} + i\sqrt{\frac{|z| - \text{Re}(z)}{2}}\right)$$

$$\therefore \quad \sqrt{x + \sqrt{(-x^4 - x^2 - 1)}}$$

$$= \pm\left(\sqrt{\left(\frac{x^2+1+x}{2}\right)} + i\sqrt{\left(\frac{x^2+1-x}{2}\right)}\right)$$

Aliter

$$\sqrt{x + \sqrt{(-x^4 - x^2 - 1)}} = \sqrt{x + 2\sqrt{\left(\frac{-x^4 - x^2 - 1}{4}\right)}}$$

$$= \sqrt{x + 2\sqrt{\left(\frac{-(x^2+x+1)(x^2-x+1)}{4}\right)}}$$

$$= \sqrt{x + 2\sqrt{\left[\left(\frac{x^2+x+1}{2}\right) \times -\left(\frac{x^2-x+1}{2}\right)\right]}}$$

$$= \sqrt{\left(\frac{x^2+x+1}{2}\right) - \left(\frac{x^2-x+1}{2}\right) + 2\sqrt{\left[\left(\frac{x^2+x+1}{2}\right) \times -\left(\frac{x^2-x+1}{2}\right)\right]}}$$

$$= \sqrt{\left\{\left(\sqrt{\left(\frac{x^2+x+1}{2}\right)}\right)^2 + \left(i\sqrt{\left(\frac{x^2-x+1}{2}\right)}\right)^2 + 2\sqrt{\left(\frac{x^2+x+1}{2}\right)} \cdot i\sqrt{\left(\frac{x^2-x+1}{2}\right)}\right\}}$$

$$= \sqrt{\left(\sqrt{\left(\frac{x^2+x+1}{2}\right)} + i\sqrt{\left(\frac{x^2-x+1}{2}\right)}\right)^2}$$

$$= \pm\left(\sqrt{\left(\frac{x^2+x+1}{2}\right)} + i\sqrt{\left(\frac{x^2-x+1}{2}\right)}\right)$$

Solution of Complex Equations

Putting $z = x + iy$, where $x, y \in R$ and $i = \sqrt{-1}$ in the given equation and equating the real and imaginary parts, we get x and y, then required solution is $z = x + iy$.

Example 51. Solve the equation $z^2 + |z| = 0$.

Sol. Let $z = x + iy$, where $x, y \in R$ and $i = \sqrt{-1}$ ...(i)

$$\Rightarrow \quad z^2 = (x+iy)^2 = x^2 - y^2 + 2ixy$$

and $|z| = \sqrt{(x^2 + y^2)}$

Then, given equation reduces to

$$x^2 - y^2 + 2ixy + \sqrt{(x^2+y^2)} = 0$$

On comparing the real and imaginary parts, we get

$$x^2 - y^2 + \sqrt{(x^2+y^2)} = 0 \qquad \text{...(ii)}$$

and $2xy = 0$ ...(iii)

From Eq. (iii), let $x = 0$ and from Eq. (ii),

$$-y^2 + \sqrt{y^2} = 0$$

$$\Rightarrow \quad -|y|^2 + |y| = 0$$

$$\therefore \quad |y| = 0, 1$$

$$\Rightarrow \quad y = 0, \pm 1$$

From Eq. (iii), let $y = 0$ and from Eq. (ii),

$$x^2 + \sqrt{x^2} = 0$$

$$\Rightarrow \quad x^2 + |x| = 0$$

$$\Rightarrow \quad |x|^2 + |x| = 0 \Rightarrow x = 0$$

$\therefore$ $x + iy$ are $0 + 0 \cdot i,\ 0 + i,\ 0 - i$

i.e. $z = 0, i, -i$ are the solutions of the given equation.

Example 52. Find the number of solutions of the equation $z^2 + |z|^2 = 0$.

Sol. $\because \quad z^2 + |z|^2 = 0$ or $z^2 + z\bar{z} = 0$

$\Rightarrow \quad z(z + \bar{z}) = 0$

$\therefore \quad z = 0$ ...(i)

and $\quad z + \bar{z} = 0 \Rightarrow 2\,\text{Re}(z) = 0$

$\therefore \quad \text{Re}(z) = 0$

If $\quad z = x + iy$ $[\because x = \text{Re}(z)]$

$= 0 + iy, y \in R$

and $\quad i = \sqrt{-1}$ ...(ii)

On combining Eqs. (i) and (ii), then we can say that the given equation has infinite solutions.

Example 53. Find all complex numbers satisfying the equation $2|z|^2 + z^2 - 5 + i\sqrt{3} = 0$, where $i = \sqrt{-1}$.

Sol. Let $z = x + iy$, where $x, y \in R$ and $i = \sqrt{-1}$

$\Rightarrow \quad z^2 = (x + iy)^2 = x^2 - y^2 + 2ixy$

and $\quad |z| = \sqrt{(x^2 + y^2)}$

Then, given equation reduces to

$2(x^2 + y^2) + x^2 - y^2 + 2ixy - 5 + i\sqrt{3} = 0$

$\Rightarrow \quad (3x^2 + y^2 - 5) + i(2xy + \sqrt{3}) = 0 = 0 + i \cdot 0$

On comparing the real and imaginary parts, we get

$3x^2 + y^2 - 5 = 0$ ...(i)

and $\quad 2xy + \sqrt{3} = 0$ ...(ii)

On substituting the value of x from Eq. (ii) in Eq. (i), we get

$$3\left(-\frac{\sqrt{3}}{2y}\right)^2 + y^2 - 5 = 0$$

$$\Rightarrow \quad \frac{9}{4y^2} + y^2 = 5$$

or $\quad 4y^4 - 20y^2 + 9 = 0$

$\Rightarrow \quad (2y^2 - 9)(2y^2 - 1) = 0$

$$\therefore \quad y^2 = \frac{9}{2}, y^2 = \frac{1}{2} \text{ or } y = \pm\frac{3}{\sqrt{2}}, y = \pm\frac{1}{\sqrt{2}}$$

$$\text{or} \quad y = -\frac{3}{\sqrt{2}}, \frac{3}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}$$

From Eq. (ii), we get

$$x = \frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, \sqrt{\frac{3}{2}}, -\sqrt{\frac{3}{2}}$$

$\therefore \quad z = x + iy$

$$= \frac{1}{\sqrt{6}} - \frac{3i}{\sqrt{2}}, -\frac{1}{\sqrt{6}} + \frac{3i}{\sqrt{2}}, \sqrt{\frac{3}{2}} - \frac{i}{\sqrt{2}}, -\sqrt{\frac{3}{2}} + \frac{i}{\sqrt{2}}$$

are the solutions of the given equation.

De-Moivre's Theorem

Statements

(i) If $\theta_1, \theta_2, \theta_3, \ldots, \theta_n \in R$ and $i = \sqrt{-1}$, then

$$(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)(\cos\theta_3 + i\sin\theta_3)\ldots(\cos\theta_n + i\sin\theta_n)$$
$$= \cos(\theta_1 + \theta_2 + \theta_3 + \ldots + \theta_n) + i\sin(\theta_1 + \theta_2 + \theta_3 + \ldots + \theta_n)$$

(ii) If $\theta \in R, n \in I$ (set of integers) and $i = \sqrt{-1}$, then

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

(iii) If $\theta \in R$, $n \in Q$ (set of rational numbers) and $i = \sqrt{-1}$, then $\cos n\theta + i\sin n\theta$ is one of the values of $(\cos\theta + i\sin\theta)^n$.

Proof

(i) By Euler's formula, $e^{i\theta} = \cos\theta + i\sin\theta$

$$\text{LHS} = (\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)(\cos\theta_3 + i\sin\theta_3)\ldots(\cos\theta_n + i\sin\theta_n)$$
$$= e^{i\theta_1} \cdot e^{i\theta_2} \cdot e^{i\theta_3} \ldots e^{i\theta_n} = e^{i(\theta_1 + \theta_2 + \theta_3 + \ldots + \theta_n)}$$
$$= \cos(\theta_1 + \theta_2 + \theta_3 + \ldots + \theta_n) + i\sin(\theta_1 + \theta_2 + \theta_3 + \ldots + \theta_n) = \text{RHS}$$

(ii) If $\theta_1 = \theta_2 = \theta_3 = \ldots = \theta_n = \theta$, then from the above result (i), $(\cos\theta + i\sin\theta)(\cos\theta + i\sin\theta)(\cos\theta + i\sin\theta)\ldots$ upto n factors

$$= \cos(\theta + \theta + \theta + \ldots \text{ upto } n \text{ times}) + i\sin(\theta + \theta + \theta + \ldots \text{ upto } n \text{ times})$$

i.e., $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

(iii) Let $n = \dfrac{p}{q}$, where $p, q \in I$ and $q \neq 0$, from above result (ii),

we have $\left(\cos\left(\dfrac{p}{q}\theta\right) + i\sin\left(\dfrac{p}{q}\theta\right)\right)^q$

$$= \cos\left(\left(\frac{p}{q}\theta\right)q\right) + i\sin\left(\left(\frac{p}{q}\theta\right)q\right) = \cos p\theta + i\sin p\theta$$

$\Rightarrow \cos\left(\dfrac{p\theta}{q}\right) + i\sin\left(\dfrac{p\theta}{q}\right)$ is one of the values of $(\cos p\theta + i\sin p\theta)^{1/q}$

$\Rightarrow \cos\left(\dfrac{p\theta}{q}\right) + i\sin\left(\dfrac{p\theta}{q}\right)$ is one of the values of

$$[(\cos\theta + i\sin\theta)^p]^{1/q}$$

$\Rightarrow \cos\left(\dfrac{p\theta}{q}\right) + i\sin\left(\dfrac{p\theta}{q}\right)$ is one of the values of $(\cos\theta + i\sin\theta)^{p/q}$

Other Forms of De-Moivre's Theorem

1. $(\cos\theta - i\sin\theta)^n = \cos n\theta - i\sin n\theta, \forall\ n \in I$

   Proof $(\cos\theta - i\sin\theta)^n = (\cos(-\theta) + i\sin(-\theta))^n$
   $= \cos(-n\theta) + i\sin(-n\theta) = \cos n\theta - i\sin n\theta$
2. $(\sin\theta + i\cos\theta)^n = (i)^n(\cos n\theta - i\sin n\theta), \forall\ n \in I$

   Proof $(\sin\theta + i\cos\theta)^n = (i(\cos\theta - i\sin\theta))^n$
   $= i^n(\cos\theta - i\sin\theta)^n = (i)^n(\cos n\theta - i\sin n\theta)$
   [from remark (1)]
3. $(\sin\theta - i\cos\theta)^n = (-i)^n(\cos n\theta + i\sin n\theta), \forall\ n \in I$

   Proof $(\sin\theta - i\cos\theta)^n = (-i(\cos\theta + i\sin\theta))^n$
   $= (-i)^n(\cos\theta + i\sin\theta)^n$
   $= (-i)^n(\cos n\theta + i\sin n\theta)$
4. $(\cos\theta + i\sin\phi)^n \neq \cos n\theta + i\sin n\phi, \forall\ n \in I$

   [here, $\theta \neq \phi$ ∴ De-Moivre's theorem is not applicable]
5. $\dfrac{1}{\cos\theta + i\sin\theta} = (\cos\theta + i\sin\theta)^{-1}$
   $= \cos(-\theta) + i\sin(-\theta) = \cos\theta - i\sin\theta$

Example 54. If $z_r = \cos\left(\dfrac{\pi}{3^r}\right) + i\sin\left(\dfrac{\pi}{3^r}\right)$, where $i = \sqrt{-1}$, prove that $z_1 z_2 z_3 \ldots$ upto infinity $= i$.

Sol. We have, $z_r = \cos\left(\dfrac{\pi}{3^r}\right) + i\sin\left(\dfrac{\pi}{3^r}\right)$

$$\therefore z_1 z_2 z_3 \ldots\infty = \cos\left(\frac{\pi}{3} + \frac{\pi}{3^2} + \frac{\pi}{3^3} + \ldots + \infty\right) + i\sin\left(\frac{\pi}{3} + \frac{\pi}{3^2} + \frac{\pi}{3^3} + \ldots + \infty\right)$$

$$= \cos\left(\frac{\frac{\pi}{3}}{1 - \frac{1}{3}}\right) + i\sin\left(\frac{\frac{\pi}{3}}{1 - \frac{1}{3}}\right) = \cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)$$

$$= 0 + i \cdot 1 = i$$

Example 55. Express $\dfrac{(\cos\theta + i\sin\theta)^4}{(\sin\theta + i\cos\theta)^5}$ in $a + ib$ form, where $i = \sqrt{-1}$.

Sol. ∵ $(\sin\theta + i\cos\theta)^5 = (i)^5(\cos\theta - i\sin\theta)^5$
$= i(\cos\theta + i\sin\theta)^{-5}$

$$\therefore \frac{(\cos\theta + i\sin\theta)^4}{(\sin\theta + i\cos\theta)^5} = \frac{(\cos\theta + i\sin\theta)^4}{i(\cos\theta + i\sin\theta)^{-5}}$$

$$= \frac{(\cos\theta + i\sin\theta)^9}{i}$$

$$= \frac{\cos 9\theta + i\sin 9\theta}{i} = -i\cos 9\theta + \sin 9\theta$$

$$= \sin 9\theta - i\cos 9\theta$$

To Find the Roots of $(a + ib)^{p/q}$, where $a, b \in R$; $p, q \in I, q \neq 0$ and $i = \sqrt{-1}$

Let $a + ib = r(\cos\theta + i\sin\theta)$ [polar form]

$$\therefore (a + ib)^{p/q} = \{r(\cos(2n\pi + \theta) + i\sin(2n\pi + \theta))\}^{p/q}, n \in I$$

$$= r^{p/q}(\cos(2n\pi + \theta) + i\sin(2n\pi + \theta))^{p/q}$$

$$= r^{p/q}\left(\cos\left(\frac{p}{q}(2n\pi + \theta)\right) + i\sin\left(\frac{p}{q}(2n\pi + \theta)\right)\right),$$

where, $n = 0, 1, 2, 3, \ldots, q - 1$

Example 56. Find all roots of $x^5 - 1 = 0$.

Sol. ∵ $x^5 - 1 = 0 \Rightarrow x^5 = 1$

$$\therefore x = (1)^{1/5} = (\cos 0 + i\sin 0)^{1/5},$$

where $i = \sqrt{-1}$

$$= [\cos(2n\pi + 0) + i\sin(2n\pi + 0)]^{1/5}$$

$$= \cos\left(\frac{2n\pi}{5}\right) + i\sin\left(\frac{2n\pi}{5}\right),$$

where, $n = 0, 1, 2, 3, 4$

∴ Roots are

$1, \cos\left(\dfrac{2\pi}{5}\right) + i\sin\left(\dfrac{2\pi}{5}\right), \cos\left(\dfrac{4\pi}{5}\right) + i\sin\left(\dfrac{4\pi}{5}\right),$

$\cos\left(\dfrac{6\pi}{5}\right) + i\sin\left(\dfrac{6\pi}{5}\right), \cos\left(\dfrac{8\pi}{5}\right) + i\sin\left(\dfrac{8\pi}{5}\right)$

Now, $\cos\left(\dfrac{6\pi}{5}\right) + i\sin\left(\dfrac{6\pi}{5}\right)$

$$= \cos\left(2\pi - \frac{4\pi}{5}\right) + i\sin\left(2\pi - \frac{4\pi}{5}\right)$$

$$= \cos\left(\frac{4\pi}{5}\right) - i\sin\left(\frac{4\pi}{5}\right)$$

and $\cos\left(\dfrac{8\pi}{5}\right) + i\sin\left(\dfrac{8\pi}{5}\right)$

$$= \cos\left(2\pi - \frac{2\pi}{5}\right) + i\sin\left(2\pi - \frac{2\pi}{5}\right)$$

$$= \cos\left(\frac{2\pi}{5}\right) - i\sin\left(\frac{2\pi}{5}\right)$$

Hence, roots are $1, \cos\left(\dfrac{2\pi}{5}\right) \pm i\sin\left(\dfrac{2\pi}{5}\right)$

and $\cos\left(\dfrac{4\pi}{5}\right) \pm i\sin\left(\dfrac{4\pi}{5}\right)$.

Remark

Five roots are $1, z_1, z_2, \bar{z}_1, \bar{z}_2$ (one real, two complex and two conjugate of complex roots).

Example 57. Find all roots of the equation $x^6 - x^5 + x^4 - x^3 + x^2 - x + 1 = 0$.

Sol. $\because \quad 1 - x + x^2 - x^3 + x^4 - x^5 + x^6 = 0$

$$\Rightarrow \quad 1 \cdot \frac{[1-(-x)^7]}{1-(-x)} = 0, 1 + x \neq 0$$

or $\quad 1 + x^7 = 0, \; x \neq -1 \;$ or $\; x^7 = -1$

$$\therefore \quad x = (-1)^{1/7} = (\cos \pi + i \sin \pi)^{1/7}, i = \sqrt{-1}$$

$$= [\cos(2n+1)\pi + i \sin(2n+1)\pi]^{1/7}$$

$$= \cos\left(\frac{(2n+1)\pi}{7}\right) + i \sin\left(\frac{(2n+1)\pi}{7}\right)$$

for $n = 0, 1, 2, 4, 5, 6$.

Remark

$\because$ For $n = 3$, $x = -1$ but here $x \neq -1$

$\therefore \quad n \neq 3$

Cube Roots of Unity

Let $z = (1)^{1/3} \;\Rightarrow\; z^3 = 1 \;\Rightarrow\; z^3 - 1 = 0$

$\Rightarrow (z-1)(z^2 + z + 1) = 0 \;\Rightarrow\; z - 1 = 0$ or $z^2 + z + 1 = 0$

$$\therefore \quad z = 1 \text{ or } z = \frac{-1 \pm \sqrt{(1-4)}}{2} = \frac{-1 \pm i\sqrt{3}}{2}$$

Therefore, $\boldsymbol{z = 1, \frac{-1 + i\sqrt{3}}{2}, \frac{-1 - i\sqrt{3}}{2}}$, where $i = \sqrt{-1}$.

If second root is represented by ω (omega), third root will be ω^2.

$\therefore$ Cube roots of unity are $1, \omega, \omega^2$ and ω, ω^2 are called non-real complex cube roots of unity.

Remark

1. $\overline{\omega} = \omega^2, (\overline{\omega})^2 = \omega$
2. $\sqrt{\omega} = \pm\, \omega^2, \sqrt{\omega^2} = \pm\, \omega$
3. $|\omega| = |\omega^2| = 1$

Aliter

Let $\quad z = (1)^{1/3} = (\cos 0 + i \sin 0)^{1/3}, i = \sqrt{-1}$

$$= [\cos(2n\pi + 0) + i \sin(2n\pi + 0)]^{1/3}$$

$$= \cos\left(\frac{2n\pi}{3}\right) + i \sin\left(\frac{2n\pi}{3}\right), \text{ where, } n = 0, 1, 2$$

Therefore, roots are

$$1, \cos\left(\frac{2\pi}{3}\right) + i \sin\left(\frac{2\pi}{3}\right), \cos\left(\frac{4\pi}{3}\right) + i \sin\left(\frac{4\pi}{3}\right)$$

or $\quad 1, e^{2\pi i/3}, e^{4\pi i/3}$

If second root is represented by ω, then third root will be ω^2 or if third root is represented by ω, then second root will be ω^2.

Properties of Cube Roots of Unity

(i) $1 + \omega + \omega^2 = 0$ and $\omega^3 = 1$

(ii) **To find the value of $\boldsymbol{\omega^n (n > 3)}$.**

First divide n by 3. Let q be the quotient and r be the remainder.

$$3 \overline{)\, n \,(} q$$
$$\underline{-3q}$$
$$r$$

i.e. $\quad n = 3q + r$, where $0 \le r \le 2$

$$\therefore \quad \omega^n = \omega^{3q+r} = (\omega^3)^q \cdot \omega^r = \omega^r$$

In general, $\boldsymbol{\omega^{3n} = 1, \omega^{3n+1} = \omega, \omega^{3n+2} = \omega^2}$

(iii) $1 + \omega^r + \omega^{2r} = \begin{cases} 3, \text{when } n \text{ is a multiple of 3} \\ 0, \text{when } n \text{ is not a multiple of 3} \end{cases}$

(iv) Cube roots of -1 are $-1, -\omega$ and $-\omega^2$.

(v) $a + b\omega + c\omega^2 = 0 \;\Rightarrow\; a = b = c$, if $a, b, c \in R$.

(vi) If a, b, c are non-zero numbers such that $a + b + c = 0 = a^2 + b^2 + c^2$, then $a : b : c = 1 : \omega : \omega^2$.

(vii) A complex number $a + ib$ (where $i = \sqrt{-1}$), for which $|a : b| = 1 : \sqrt{3}$ or $\sqrt{3} : 1$ can always be expressed in terms of ω or ω^2.

For example,

(a) $1 + i\sqrt{3} = -(-1 - i\sqrt{3}) \qquad [\because |1 : \sqrt{3}| = 1 : \sqrt{3}]$

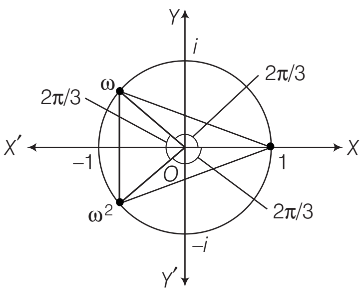
$$= -2\left(\frac{-1 - i\sqrt{3}}{2}\right) = -2\omega^2$$

(b) $\sqrt{3} + i = \dfrac{i(\sqrt{3} + i)}{i} = \dfrac{(-1 + i\sqrt{3})}{i}$

$$= \left(\frac{-1 + i\sqrt{3}}{2}\right)\left(\frac{2}{i}\right) \qquad [\because |\sqrt{3} : 1| = \sqrt{3} : 1]$$

$$= \frac{2\omega}{i} = -2i\omega$$

(viii) The cube roots of unity when represented on complex plane lie on vertices of an equilateral triangle inscribed in a unit circle, having centre at origin. One vertex being on positive real axis.



Important Relations in Terms of Cube Root of Unity

(i) $a^2 + ab + b^2 = (a - b\omega)(a - b\omega^2)$

(ii) $a^2 - ab + b^2 = (a + b\omega)(a + b\omega^2)$

(iii) $a^3 + b^3 = (a + b)(a + b\omega)(a + b\omega^2)$

(iv) $a^3 - b^3 = (a - b)(a - b\omega)(a - b\omega^2)$

(v) $a^2 + b^2 + c^2 - ab - bc - ca$
$$= (a + b\omega + c\omega^2)(a + b\omega^2 + c\omega)$$

(vi) $a^3 + b^3 + c^3 - 3abc$
$$= (a + b + c)(a + b\omega + c\omega^2)(a + b\omega^2 + c\omega)$$

Example 58. If ω is a non-real complex cube root of unity, find the values of the following.

(i) ω^{1999}

(ii) ω^{-998}

(iii) $\left(\frac{-1+i\sqrt{3}}{2}\right)^{3n+2}$, $n \in N$ and $i = \sqrt{-1}$

(iv) $(1+\omega)(1+\omega^2)(1+\omega^4)(1+\omega^8)\ldots$ upto $2n$ factors

(v) $\left(\frac{\alpha + \beta\omega + \gamma\omega^2 + \delta\omega^2}{\beta + \alpha\omega^2 + \gamma\omega + \delta\omega}\right)$, where $\alpha, \beta, \gamma, \delta \in R$

(vi) $1\cdot(2-\omega)(2-\omega^2) + 2\cdot(3-\omega)(3-\omega^2) + 3\cdot(4-\omega)(4-\omega^2) + \ldots + \ldots + (n-1)\cdot(n-\omega)(n-\omega^2)$

Sol. (i) $\omega^{1999} = \omega^{3\times 666 + 1} = \omega$

(ii) $\omega^{-998} = \frac{1}{\omega^{998}} = \frac{\omega}{\omega^{999}} = \omega$

(iii) $\left(\frac{-1+i\sqrt{3}}{2}\right)^{3n+2} = \omega^{3n+2} = \omega^{3n}\cdot\omega^2 = (\omega^3)^n\cdot\omega^2$

$$= (1)^n \cdot \omega^2 = \omega^2$$

(iv) $(1+\omega)(1+\omega^2)(1+\omega^4)(1+\omega^8)\ldots$ upto $2n$ factors

$= (1+\omega)(1+\omega^2)(1+\omega)(1+\omega^2)\ldots$ upto $2n$ factors

$= (-\omega^2)(-\omega)(-\omega^2)(-\omega)\ldots$ upto $2n$ factors

$= (\omega^3)(\omega^3)\ldots$ upto n factors $= 1\cdot 1\cdot 1\cdot\ldots$ upto n factors

$= (1)^n = 1$

(v) $\left(\frac{\alpha + \beta\omega + \gamma\omega^2 + \delta\omega^2}{\beta + \alpha\omega^2 + \gamma\omega + \delta\omega}\right) = \frac{\omega(\alpha + \beta\omega + \gamma\omega^2 + \delta\omega^2)}{(\beta\omega + \alpha\omega^3 + \gamma\omega^2 + \delta\omega^2)}$

$= \frac{\omega(\alpha + \beta\omega + \gamma\omega^2 + \delta\omega^2)}{(\beta\omega + \alpha + \gamma\omega^2 + \delta\omega^2)} = \omega$

(vi) $\Sigma(n-1)(n-\omega)(n-\omega^2) = \Sigma(n^3 - 1) = \Sigma n^3 - \Sigma 1$

$= \left\{\frac{n(n+1)}{2}\right\}^2 - n$

Example 59. If α, β and γ are the roots of $x^3 - 3x^2 + 3x + 7 = 0$, find the value of $\frac{\alpha-1}{\beta-1} + \frac{\beta-1}{\gamma-1} + \frac{\gamma-1}{\alpha-1}$.

Sol. We have, $x^3 - 3x^2 + 3x + 7 = 0$

$\Rightarrow (x-1)^3 + 8 = 0$

$\Rightarrow (x-1)^3 + 2^3 = 0$

$\Rightarrow (x-1+2)(x-1+2\omega)(x-1+2\omega^2) = 0$

$\Rightarrow (x+1)(x-1+2\omega)(x-1+2\omega^2) = 0$

$\therefore x = -1, 1-2\omega, 1-2\omega^2$

$\Rightarrow \alpha = -1, \beta = 1-2\omega, \gamma = 1-2\omega^2$

Then, $\frac{\alpha-1}{\beta-1} + \frac{\beta-1}{\gamma-1} + \frac{\gamma-1}{\alpha-1} = \frac{-2}{-2\omega} + \frac{-2\omega}{-2\omega^2} + \frac{-2\omega^2}{-2}$

$$= \frac{1}{\omega} + \frac{1}{\omega} + \omega^2 = \omega^2 + \omega^2 + \omega^2 = 3\omega^2$$

Example 60. If $z = \frac{\sqrt{3}+i}{2}$, where $i = \sqrt{-1}$, find the value of $(z^{101} + i^{103})^{105}$.

Sol. $\because z = \frac{\sqrt{3}+i}{2} = \frac{1}{i}\left(\frac{i\sqrt{3}+i^2}{2}\right)$ $[\because i^2 = -1]$

$= -i\left(\frac{-1+i\sqrt{3}}{2}\right) = -i\omega$

$\therefore z^{101} = (-i\omega)^{101} = -i^{101}\cdot\omega^{101} = -i\omega^2$ and $i^{103} = i^3 = -i$

Then, $z^{101} + i^{103} = -i\omega^2 - i = -i(\omega^2 + 1)$

$= -i(-\omega) = i\omega$

Hence, $(z^{101} + i^{103})^{105} = (i\omega)^{105} = i^{105}\cdot\omega^{105} = i\cdot 1 = i$

Example 61. If $\left(\frac{3}{2} + \frac{i\sqrt{3}}{2}\right)^{50} = 3^{25}(x - iy)$, where $x, y \in R$ and $i = \sqrt{-1}$, find the ordered pair of (x, y).

Sol. $\because \frac{3}{2} + \frac{i\sqrt{3}}{2} = \sqrt{3}\left(\frac{\sqrt{3}+i}{2}\right) = \frac{\sqrt{3}}{i}\left(\frac{i\sqrt{3}+i^2}{2}\right)$

$= -i\sqrt{3}\left(\frac{-1+i\sqrt{3}}{2}\right) = -i\sqrt{3}\,\omega$

$\therefore \left(\frac{3}{2} + \frac{i\sqrt{3}}{2}\right)^{50} = (-i\sqrt{3}\,\omega)^{50} = i^{50}\cdot 3^{25}\cdot\omega^{50}$

$= -1\cdot 3^{25}\cdot\omega^2 = -3^{25}\cdot\left(\frac{-1-i\sqrt{3}}{2}\right)$

$$= 3^{25}\left(\frac{1}{2}+\frac{i\sqrt{3}}{2}\right) = 3^{25}(x - iy) \qquad \text{[given]}$$

$$\therefore \quad x = \frac{1}{2}, y = -\frac{\sqrt{3}}{2}$$

$\Rightarrow$ Ordered pair is $\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$.

Example 62. If the polynomial $7x^3 + ax + b$ is divisible by $x^2 - x + 1$, find the value of $2a + b$.

Sol. Let $\quad f(x) = 7x^3 + ax + b$

and $x^2 - x + 1 = (x + \omega)(x + \omega^2)$

$\because \; f(x)$ is divisible by $x^2 - x + 1$

Then, $\quad f(-\omega) = 0$ and $f(-\omega^2) = 0$

$\Rightarrow \quad -7\omega^3 - a\omega + b = 0$ and $-7\omega^6 - a\omega^2 + b = 0$

or $\quad -7 - a\omega + b = 0$

and $\quad -7 - a\omega^2 + b = 0$

On adding, we get

$$-14 - a(\omega + \omega^2) + 2b = 0$$

or $\quad -14 + a + 2b = 0$ or $a + 2b = 14 \qquad$ ...(i)

and on subtracting, we get

$$-a(\omega - \omega^2) = 0$$

$\Rightarrow \quad a = 0 \qquad [\because \omega - \omega^2 \neq 0]$

From Eq. (i), we get $b = 7$

$\therefore \quad 2a + b = 7$

Exercise for Session 3

1 The real part of $(1 - i)^{-i}$, where $i = \sqrt{-1}$ is

(a) $e^{-\pi/4}\cos\left(\frac{1}{2}\log_e 2\right)$ (b) $-e^{-\pi/4}\sin\left(\frac{1}{2}\log_e 2\right)$

(c) $e^{\pi/4}\cos\left(\frac{1}{2}\log_e 2\right)$ (d) $e^{-\pi/4}\sin\left(\frac{1}{2}\log_e 2\right)$

2 The amplitude of $e^{e^{-i\theta}}$, where $\theta \in R$ and $i = \sqrt{-1}$ is

(a) $\sin\theta$ (b) $-\sin\theta$

(c) $e^{\cos\theta}$ (d) $e^{\sin\theta}$

3 If $z = i\log_e(2 - \sqrt{3})$, where $i = \sqrt{-1}$, then the $\cos z$ is equal to

(a) i (b) $2i$ (c) 1 (d) 2

4 If $z = i^{i^i}$, where $i = \sqrt{-1}$, then $|z|$ is equal to

(a) 1 (b) $e^{-\pi/2}$ (c) $e^{-\pi}$ (d) e^{π}

5 $\sqrt{(-8 - 6i)}$ is equal to (where, $i = \sqrt{-1}$)

(a) $1 \pm 3i$ (b) $\pm(1 - 3i)$ (c) $\pm(1 + 3i)$ (d) $\pm(3 - i)$

6 $\dfrac{\sqrt{(5 + 12i)} + \sqrt{(5 - 12i)}}{\sqrt{(5 + 12i)} - \sqrt{(5 - 12i)}}$ is equal to (where, $i = \sqrt{-1}$)

(a) $-\frac{3}{2}i$ (b) $\frac{3}{4}i$ (c) $-\frac{3}{4}i$ (d) $-\frac{3}{2}$

7 If $0 < \text{amp}(z) < \pi$, then $\text{amp}(z) - \text{amp}(-z)$ is equal to

(a) 0 (b) $2\,\text{amp}(z)$ (c) π (d) $-\pi$

8 If $|z_1| = |z_2|$ and $\text{amp}(z_1) + \text{amp}(z_2) = 0$, then

(a) $z_1 = z_2$ (b) $\bar{z}_1 = z_2$ (c) $z_1 + z_2 = 0$ (d) $\bar{z}_1 = \bar{z}_2$

9 The solution of the equation $|z| - z = 1 + 2i$, where $i = \sqrt{-1}$, is

(a) $2 - \frac{3}{2}i$ (b) $\frac{3}{2} + 2i$ (c) $\frac{3}{2} - 2i$ (d) $-2 + \frac{3}{2}i$

10 The number of solutions of the equation $z^2 + \bar{z} = 0$, is

(a) 1 (b) 2

(c) 3 (d) 4

11 If $z_r = \cos\left(\frac{r\alpha}{n^2}\right) + i\sin\left(\frac{r\alpha}{n^2}\right)$, where $r = 1, 2, 3, \ldots, n$ and $i = \sqrt{-1}$, then $\lim_{n \to \infty} z_1 z_2 z_3 \ldots z_n$ is equal to

(a) $e^{i\alpha}$ (b) $e^{-i\alpha/2}$

(c) $e^{i\alpha/2}$ (d) $\sqrt[3]{e^{i\alpha}}$

12 If $\theta \in R$ and $i = \sqrt{-1}$, then $\left(\frac{1 + \sin\theta + i\cos\theta}{1 + \sin\theta - i\cos\theta}\right)^n$ is equal to

(a) $\cos\left(\frac{n\pi}{2} - n\theta\right) + i\sin\left(\frac{n\pi}{2} - n\theta\right)$ (b) $\cos\left(\frac{n\pi}{2} + n\theta\right) + i\sin\left(\frac{n\pi}{2} + n\theta\right)$

(c) $\sin\left(\frac{n\pi}{2} - n\theta\right) + i\cos\left(\frac{n\pi}{2} - n\theta\right)$ (d) $\cos\left(n\left(\frac{\pi}{2} + 2\theta\right)\right) + i\sin\left(n\left(\frac{\pi}{2} + 2\theta\right)\right)$

13 If $i z^4 + 1 = 0$, where $i = \sqrt{-1}$, then z can take the value

(a) $\frac{1 + i}{\sqrt{2}}$ (b) $\cos\left(\frac{\pi}{8}\right) + i\sin\left(\frac{\pi}{8}\right)$

(c) $\frac{1}{4i}$ (d) i

14 If $\omega (\neq 1)$ is a cube root of unity, then $(1 - \omega + \omega^2)(1 - \omega^2 + \omega^4)(1 - \omega^4 + \omega^8) \ldots$ upto $2n$ factors, is

(a) 2^n (b) 2^{2n}

(c) 0 (d) 1

15 If α, β and γ are the cube roots of p $(p < 0)$, then for any x, y and z, $\frac{x\alpha + y\beta + z\gamma}{x\beta + y\gamma + z\alpha}$ is equal to

(a) $\frac{1}{2}(-1 - i\sqrt{3}), i = \sqrt{-1}$ (b) $\frac{1}{2}(1 + i\sqrt{3}), i = \sqrt{-1}$

(c) $\frac{1}{2}(1 - i\sqrt{3}), i = \sqrt{-1}$ (d) None of these

Answers

Exercise for Session 3

1. (a) **2.** (b) **3.** (d) **4.** (a) **5.** (b) **6.** (a)
7. (c) **8.** (b) **9.** (c) **10.** (d) **11.** (c) **12.** (a)
13. (b) **14.** (b) **15.** (a)